Lesson 02 Transmission Lines Fundamentals

2.1 Introduction

■ Why discussing transmission lines?

The rapid development of electronic technology in the 20th century lies on the employment of simple but powerful tool called (lumped) circuit theory to accurately predict the performance of sophisticated electrical circuits. Circuit theory considers the effects of **lumped** elements (*R*, *C*, *L*, dependent sources) connected in series and/or shunt, while the conducting wires play no role (space-independent *v*, *i*). In fact, elements and wires provide a framework over which electric charges move, setting up electric and magnetic (vector) fields and determining the circuit behaviors. A full vector analysis based on Maxwell's equations is most complete. The theory of **distributed** circuits (transmission lines) bridges circuit theory and Maxwell's equations. On the one hand, it can describe some wave properties (wavelength, phase velocity, reflection, …) that are absent in circuit theory but critical in power transmission and current integrated circuits. On the other hand, it deals with scalar quantities (*v*, *i*) as in circuit theory (but with one extra spatial variable *z*), free of complicated vector analysis.

■ Criteria to consider distributed circuits

The central difference between lumped and distributed circuit theories is the latter considers time delay t_d when signal (*v*, *i*) travels from one point to another. As will be justified, the signal travels with velocity $v = c/n$, where c is the light velocity in vacuum, n is the refractive index of the medium where EM fields exist. Distributed circuit theory matters when t_d is comparable or longer than the "signal time scale".

Example 2-1: The Taipower distributes electric power via 60-Hz sinusoidal waves traveling in air (oscillating period $T = 1/60$ sec, $v = c$). The source and load signals, i.e., $V_{AA'}(t) = V_0 \cos(2\pi \cdot 60 \cdot t)$ and $V_{BB'}(t) = V_0 \cos[2\pi \cdot 60 \cdot (t - t_d)]$, have a "non-negligible" time delay $t_d = l/v$ (thus lumped circuits model is inadequate) if $t_d > 0.01T$ (rule of thumb), $\Rightarrow$ $l > 50$ km. As a result, the operation of the island-wide power system relies on distributed circuit analysis.

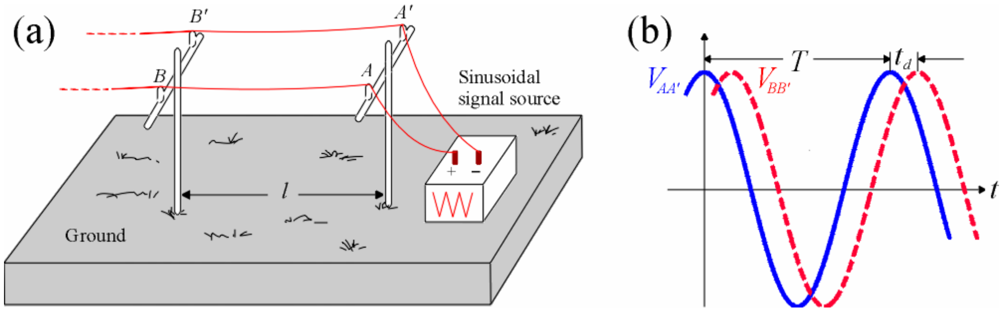

Fig. 2-1. (a) Schematic of the power line. (b) Definitions of oscillating period and delay.

Example 2-2: In digital electronic circuits, rise time t_r is defined as the duration when the signal changes from 10% to 90% of its final value (Fig. 2-2a). For 1-cm on-chip SiO_2 interconnection, $v \approx 0.5c$, $t_d = l/v \approx 67$ ps (1 ps=10^{-12} sec). The source and load signals have a "non-negligible" time delay if $t_r < 2.5t_d \approx 165$ ps (rule of thumb). Rise time of CMOS transistors can be as fast as 100 ps, where distributed circuit theory is required.

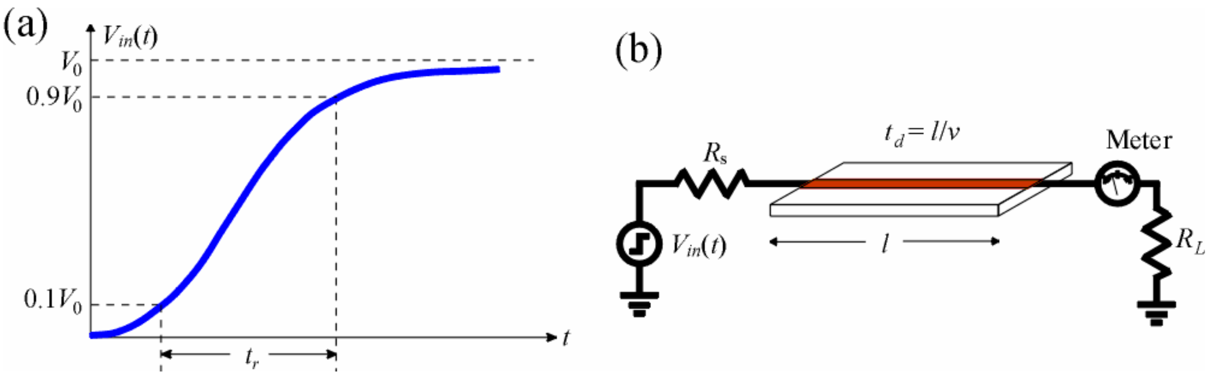

Fig. 2-2. Definitions of (a) rise time, and (b) one-way time delay.

■ Things you have to know in advance

1) Models of linear circuit elements: (a) $v = R \cdot i$, (b) $i = C \cdot \frac{d}{dt} v$, (c) $v = L \cdot \frac{d}{dt} i$. These $i - v$ relations will be justified in Lesson 10, 9, 13, respectively.

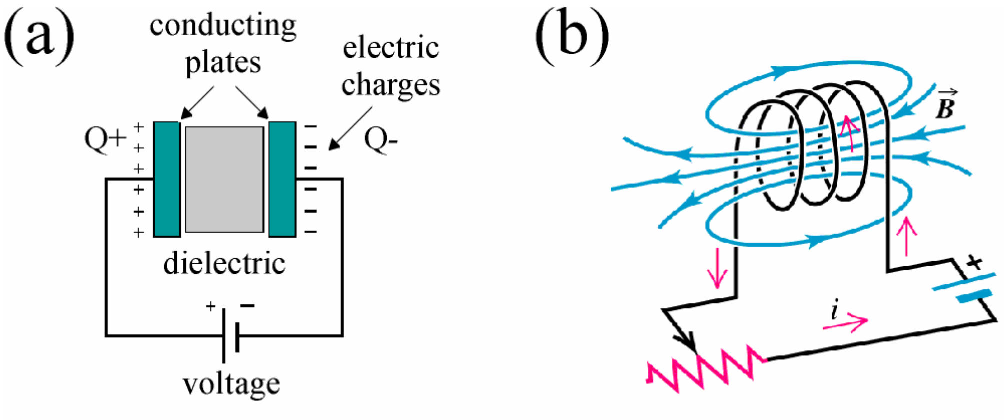

Fig. 2-3. Schematic diagrams of (a) capacitor, and (b) inductor.

2) Kirchhoff's laws: (a) $\sum_k i_k = 0$, (b) $\sum_k v_k = 0$.

3) Phasor representation of time-harmonic (i.e., sinusoidal) functions: $v(t) = V_0 \cos(\omega t + \phi)$ $= \mathrm{Re}\{V \cdot e^{j\omega t}\}$, where phasor $V = V_0 e^{j\phi} = V_0(\cos\phi + j \sin\phi)$. $\frac{d}{dt} v(t) = \mathrm{Re}\left\{V \cdot \frac{d}{dt} e^{j\omega t}\right\} =$ $\mathrm{Re}\{j\omega V \cdot e^{j\omega t}\}$, $\Rightarrow$ phasor of $\frac{d}{dt} v(t)$ becomes $j\omega V$. $\Rightarrow$ Time derivatives of sinusoidal functions are replaced by algebraic multiplications: $\frac{d^n}{dt^n} \to (j\omega)^n$.

2.2 Equivalent Circuit and Equations of Transmission Lines

■ Geometry of transmission lines

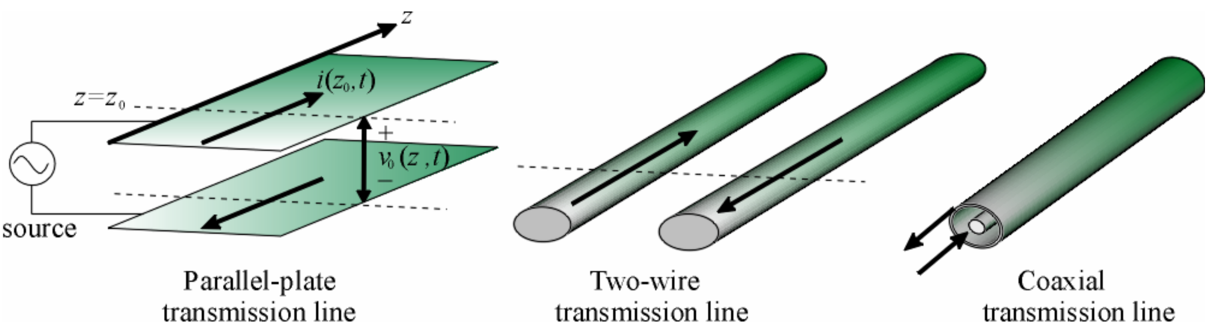

Fig. 2-4. Typical types of transmission lines.

Typical transmission lines consist of two long conductors separated by some insulating material (Fig. 2-4). At any transverse plane $z = z_0$: (1) a voltage drop $v(z_0, t)$ between the two conductors exists, (2) currents $\pm i(z_0, t)$ with equal magnitude but opposite directions flow along the two conductors, when the two electrodes of a (voltage or current) source are connected to the transmission line.

■ Equivalent circuit

Since the voltage and current of a transmission line vary with position z (and time t), we have to characterize it by a "distributed" circuit model. Consider an infinitesimal line of length Δz, the currents set up magnetic field between the conductors (by Ampere's law), causing magnetic flux. When currents are time-varying, so is the magnetic flux, and a voltage variation "along" the conductor (electromotive force) is induced (by Faraday's law) in an attempt to drive the currents oppositely (by Lenz's law). This behavior can be modeled by a **series inductor** $\left(v = L \cdot \frac{d}{dt} i\right)$. Meanwhile, two separated conductors form a capacitor. Since the upper and lower conductors of adjacent infinitesimal lines are connected respectively, the capacitive behavior of an infinitesimal line can be modeled by a **shunt capacitor**. In the presence of imperfect conducting and imperfect insulating materials, voltage drop along the conducting line and leakage current between them exist, which can be modeled by a series resistor and a shunt conductor, respectively.

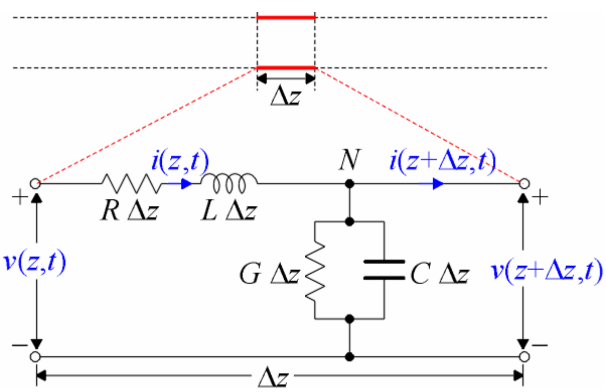

Fig. 2-5. Equivalent circuit of a real transmission line.

The equivalent circuit of a real transmission line is shown in Fig. 2-5, where *R*, *L*, *G*, *C* represent resistance, inductance, conductance, and capacitance per unit length. Transmission line is lossless in the absence of *R* and *G*.

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1) By using the equivalent circuit, analysis of electric and magnetic vector fields is substituted by that of scalar voltage between and current along the line, greatly simplifying the math.
2) Values of *R*, *L*, *G*, *C* depend on geometry and material characteristics of transmission line. We will discuss how to calculate them in the subsequent lessons.

■ Lossless transmission line equations

Assuming $R=0$, $G=0$ (lossless line) in Fig. 2-5:

1) Applying Kirchhoff's voltage law: $v(z+\Delta z,t)=v(z,t)-(L\Delta z)\frac{\partial}{\partial t}i(z,t)$. By taking $\Delta z\to 0$, we arrive at a first-order PDE of two unknown functions $v(z,t)$ and $i(z,t)$:

$$\frac{\partial}{\partial z}v(z,t)=-L\frac{\partial}{\partial t}i(z,t) \qquad (2.1)$$

2) Applying Kirchhoff's current law: $i(z,t)=i(z+\Delta z,t)+(C\Delta z)\frac{\partial}{\partial t}v(z+\Delta z,t)$. By taking $\Delta z\to 0$ and omitting higher-order terms of $v(z+\Delta z,t)$ [i.e., $v(z+\Delta z,t)\approx v(z,t)$], we arrive at another first-order PDE:

$$\frac{\partial}{\partial z}i(z,t)=-C\frac{\partial}{\partial t}v(z,t) \qquad (2.2)$$

By taking $\frac{\partial}{\partial z}$ for both sides of eq. (2.1), and substituting eq. (2.2), we obtain a second-order PDE of single unknown function $v(z,t)$:

$$\frac{\partial^2}{\partial z^2}v(z,t)=LC\frac{\partial^2}{\partial t^2}v(z,t) \tag{2.3}$$

Similarly, $i(z,t)$ is governed by the same PDE:

$$\frac{\partial^2}{\partial z^2}i(z,t)=LC\frac{\partial^2}{\partial t^2}i(z,t) \tag{2.4}$$

■ Solutions for infinite lossless lines

Exact solution of $v(z,t)$ in eq. (2.3) requires boundary and initial conditions [e.g. $v(0,t)=V_1(t)$, $v(z,0)=V_2(z)$]. However, any function $f(\cdot)$ of variable $\tau=t-\frac{z}{v_p}$ is a solution to eq. (2.3), as long as

$$v_p\equiv\frac{1}{\sqrt{LC}}. \tag{2.5}$$

To verify this fact, let $v(z,t)=f(\tau)$, $\Rightarrow$

$$\frac{\partial v}{\partial z}=\frac{df}{d\tau}\cdot\frac{\partial\tau}{\partial z}=-\frac{1}{v_p}\frac{df}{d\tau}, \text{ and } \frac{\partial^2 v}{\partial z^2}=-\frac{1}{v_p}\frac{\partial}{\partial z}\left(\frac{df}{d\tau}\right)=\frac{1}{v_p^2}\frac{\partial^2 f}{\partial\tau^2}=\frac{1}{v_p^2}f''(\tau)\Big|_{\tau=t-z/v_p}.$$

$$\frac{\partial v}{\partial t}=\frac{df}{d\tau}\cdot\frac{\partial\tau}{\partial t}=\frac{df}{d\tau}, \text{ and } \frac{\partial^2 v}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{df}{d\tau}\right)=\frac{d^2 f}{d\tau^2}=f''(\tau)\Big|_{\tau=t-z/v_p}.$$

$\Rightarrow \frac{\partial^2 v}{\partial z^2}=\frac{f''(\tau)}{v_p^2}=\frac{1}{v_p^2}\frac{\partial^2 v}{\partial t^2}$, consistent with eq. (2.3) regardless of the functional form $f(\cdot)$.

In fact, $f\left(t-z/v_p\right)$ represents a distortion-free wave traveling in the +z direction with phase velocity v_p. Fig. 2-6 shows an example when $f(\tau)=\begin{cases}-\tau, \text{if } -1<\tau<0\\ 0, \text{otherwise}\end{cases}$, and $v_p=1$ m/s. At $t=0$, $f\left(t-z/v_p\right)=f(-z)=\begin{cases}z, \text{if } 0<z<1\\ 0, \text{otherwise}\end{cases}$. At $t=0.5$ sec, $f\left(t-z/v_p\right)=f\left(0.5-z\right)=\begin{cases}z-0.5, \text{if } 0.5<z<1.5\\ 0, \text{otherwise}\end{cases}$, i.e. the waveform is displaced by +0.5 m within 0.5 sec (speed equals $v_p=1$ m/s) without changing its shape. The wave propagation property continues for any $t = t_0$ sec, where $f\left(t - z/v_p\right) = f\left(t_0 - z\right) = \begin{cases} z - t_0, \text{if } t_0 < z < 1 + t_0 \\ 0, \text{otherwise} \end{cases}$.

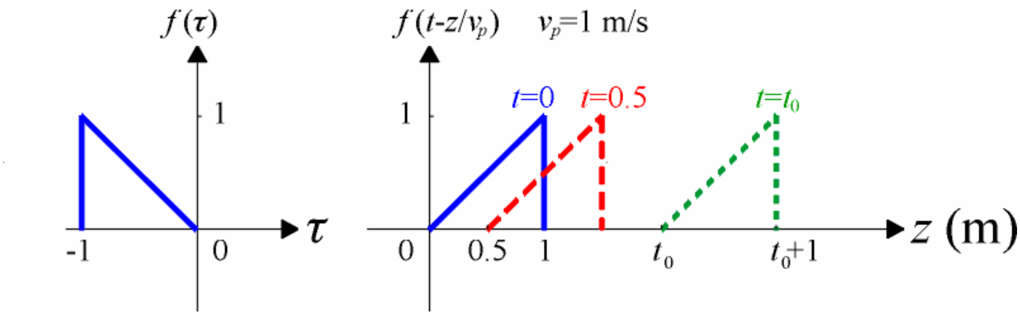

Fig. 2-6. Example of solution to transmission line equation.

In addition to $f\left(t - z/v_p\right)$, we can find that any function $f(\cdot)$ of variable $\tau' = t + \dfrac{z}{v_p}$ is also a solution to eq. (2.3), representing a distortion-free wave traveling in the $-z$ direction with the same phase velocity v_p (check by yourself). So the general (D'Alembart's) solution to eq. (2.3) is a superposition of two counter-propagating waves:

$$v(z,t) = f^{+}\left(t - z/v_p\right) + f^{-}\left(t + z/v_p\right) \tag{2.6}$$

where the superscripts "+", "–" denote the directions of propagation, and $f^{+}(\cdot)$, $f^{-}(\cdot)$ can be completely different functions (determined by external excitation, i.e. initial conditions).

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1) Phase velocity $v_p = \dfrac{1}{\sqrt{LC}}$ of most transmission lines depends only on properties of insulating media, though parameters L, C are associated with the geometry of the line.
2) Eq. (2.6) exhibits properties of "wave", a result when the space and time variations of a physical quantity [$v(z,t)$ in our case] are "coupled" through second-order derivatives.

Once $v(z,t)$ is known [eq. (2.6)], $i(z,t)$ can be uniquely determined by the following steps:

(i) Substitute eq. (2.6) into eq. (2.1): $\dfrac{\partial v}{\partial z} = -\dfrac{1}{v_p}\dfrac{df^{+}(\tau)}{d\tau} + \dfrac{1}{v_p}\dfrac{df^{-}(\tau)}{d\tau} = -L\dfrac{\partial i}{\partial t}$.

(ii) Integrate with respect to t : $i(z,t) = -\frac{1}{L}\int \frac{\partial v}{\partial z}\partial t = \frac{1}{Lv_p}\int\left[\frac{df^+(\tau)}{d\tau} - \frac{df^-(\tau)}{d\tau}\right]\partial t$. Since

$\frac{\partial \tau}{\partial t} = 1$, $\Rightarrow$ $i(z,t) = \sqrt{\frac{C}{L}}\int\left[\frac{df^+(\tau)}{d\tau} - \frac{df^-(\tau)}{d\tau}\right]d\tau$,

$$i(z,t) = \frac{1}{Z_0}\left[f^+\left(t - z/v_p\right) - f^-\left(t + z/v_p\right)\right], \tag{2.7}$$

where

$$Z_0 = \sqrt{\frac{L}{C}} \quad (\Omega) \tag{2.8}$$

is known as the characterization impedance of the transmission line (not the resistance of conductors or insulator). If we denote the voltage component propagating in +*z* and -*z* directions as $v^+(z,t) = f^+\left(t - z/v_p\right)$ and $v^-(z,t) = f^-\left(t + z/v_p\right)$, $\Rightarrow$

$$i^+(z,t) = \frac{1}{Z_0} f^+\left(t - z/v_p\right), \quad i^-(z,t) = -\frac{1}{Z_0} f^-\left(t + z/v_p\right),$$

$$Z_0 = \frac{v^+(z,t)}{i^+(z,t)} = -\frac{v^-(z,t)}{i^-(z,t)} \tag{2.9}$$

The characteristic impedance is the ratio of voltage to current for a "single" wave propagating in the +z direction. However, $\frac{v(z,t)}{i(z,t)}\neq$constant if two counter-propagating waves coexist.

<u>Example 2-3</u>: Consider an infinitely long lossless transmission line of characteristic impedance Z_0 ($\in R$) and phase velocity v_p connected to a step voltage source of amplitude V_0 and internal resistance R_s. Find the voltage, current, and power propagating down the line.

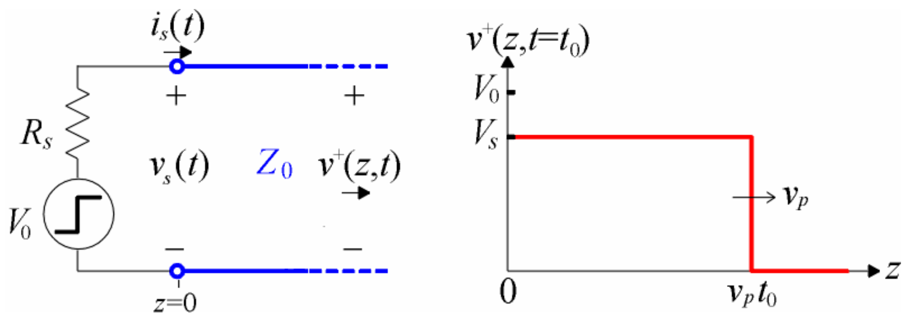

Fig. 2-7. Infinitely long lossless transmission line driven by step voltage source.

Ans: Assume the line is initially at rest: $v(z,t=0^-)=0$, $i(z,t=0^-)=0$. At $t=0$, the source voltage changes from 0 to V_0. In the absence of reflected wave (infinitely long line), the line acts as a load of impedance Z_0 for the source. The initial voltage established at the source ends is: $v_s(t=0^+)=V_s=\frac{Z_0}{Z_0+R_s}V_0$. The voltage signal of constant amplitude V_s propagates in the +z direction with velocity v_p, $\Rightarrow$ $v^+(z,t)=\begin{cases} V_s, \text{if } z/t<v_p, z>0, t>0 \\ 0, \text{otherwise} \end{cases}$. By eq. (2.9), $i^+(z,t)=\frac{v^+(z,t)}{Z_0}=\begin{cases} I_s, \text{if } z/t<v_p, z>0, t>0 \\ 0, \text{otherwise} \end{cases}$, where $I_s=\frac{V_s}{Z_0}$. The total power supplied by the source: $P_{tot}=I_sV_0$, while only a fraction of it is supplied to (stored in) the line: $P^+_{line}=I_sV_s$.