Homework Solutions #6

(Due date: 2011/4/18)

This problem set covers materials of Lesson 7. The full score is 45 points + 20 bonus points.

1) When a metal is in contact with an *n*-type semiconductor (both extend infinitely in the *yz*-plane) at $x = 0$, free electrons of the semiconductor will diffuse into the metal and are deposited on the interface, leaving a positively charged depletion layer $\{0 < x < d\}$ with constant volume charge density ρ_v (C/m^3) and permittivity ε (Fig. 1).

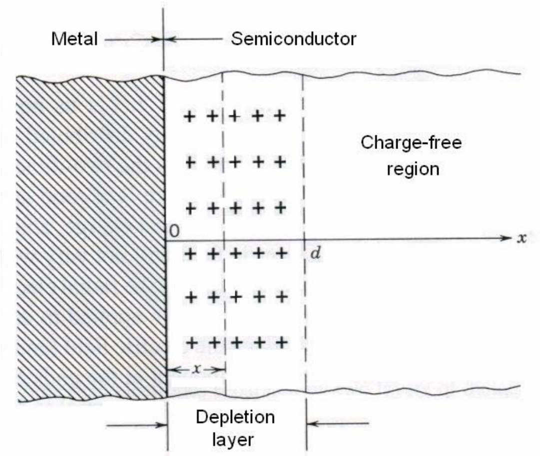

Fig. 1. Metal-semiconductor junction.

1a) (5%) What are the electric field intensities $\vec{E}$ in the regions of: (i) $x < 0$, and (ii) $x > d$, respectively? Justify your answer.

Answer:

There is no electric field $\vec{E} = 0$ inside the conductor ($x < 0$), otherwise work has to be done to move the "free" charges.

There is no charge in the charge-free region ($\rho_v = 0$, $x > d$). According to the Gauss's Law (the Gaussian surface is denoted as **surface 1, red line** in Figure 1), the total charge enclosed is zero, so the electric field must be zero $\vec{E} = 0$ for $x > d$.

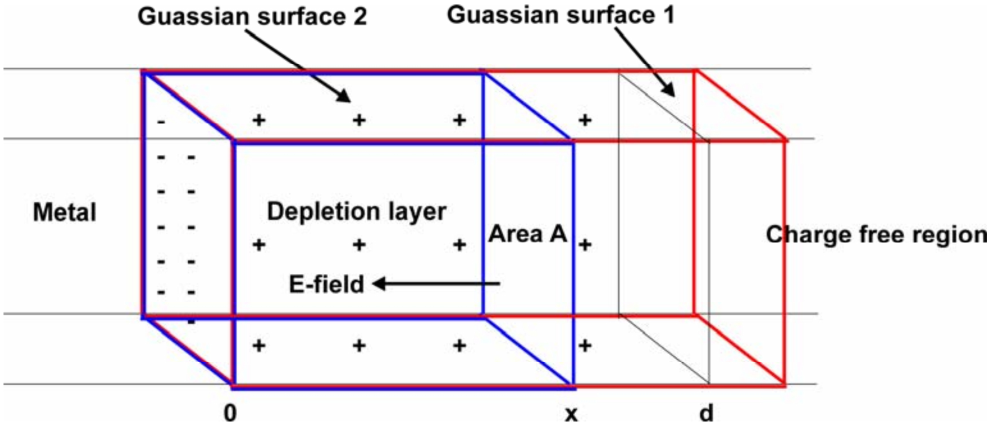

Figure 1.

1b) (5%) What is the electric field intensity $\vec{E}(x)$ in the depletion layer $\{0 < x < d\}$?

(*Hint*: Use Gauss's law.)

Answer:

Since electrons are deposited on the metal surface, the direction of the electric field is $-\vec{a}_x$.

Because of the planar symmetry, $\vec{E}(x) = -\vec{a}_x E_0$.

By Guess Law $\oint_s \vec{E} \cdot d\vec{s} = \dfrac{Q}{\varepsilon}$, assume the Gaussian surface is the **surface 2, blue line** in Figure 1, and the area is *A*.

$$E \cdot A = \frac{\rho_v A(d-x)}{\varepsilon}. \Rightarrow \vec{E} = -\frac{\rho_v (d-x)}{\varepsilon}\vec{a}_x$$

2) Consider a dielectric sphere with radius R_0 and uniform polarization vector $\vec{P} = \vec{a}_z P$.

2a) (5%) Find the surface polarization charge density distribution ρ_{ps} in spherical coordinates.

Answer:

From Eq. (3-88) of the text book, $\rho_{ps} = \vec{P} \cdot \vec{a}_n = \vec{a}_z P \cdot \vec{a}_R = P\cos\theta$.

2b) (10%) Find the electric field intensity $\vec{E}_0$ at the spherical center.

(*Hint*: Apply the result of Problem 1 of HW5.)

Answer:

By Problem 1 of HW5, the *z*-component of the electric field intensity at the spherical center (0,0,0) due to an infinitesimal ring of width $R_0 d\theta$ at azimuthal angle θ (the center of the ring is located at (0,0,*z′*), where $z' = R_0 \cos\theta$) is:

$$dE_z = \frac{\rho_l b(-z')}{2\varepsilon_0 \left(b^2 + (z')^2\right)^{3/2}},$$

where $\rho_l = \rho_{ps}\ R_0 d\theta = P\cos\theta\ R_0 d\theta$, $b = R_0 \sin\theta$, $z' = R_0 \cos\theta$,

$$\Rightarrow\ dE_z = -\frac{P}{2\varepsilon_0}\cos^2\theta \sin\theta d\theta .$$

$$\vec{E}_0\left(r < R_0\right) = \vec{a}_z \int dE_z = -\vec{a}_z \frac{P}{2\varepsilon_0}\int_0^{\pi} \cos^2\theta \sin\theta d\theta = -\vec{a}_z \frac{P}{2\varepsilon_0}\int_0^{\pi}(1-\sin^2\theta)\sin\theta d\theta$$

$$= -\vec{a}_z \frac{P}{2\varepsilon_0}\int_0^{\pi}(\sin\theta - \sin^3\theta)\, d\theta = -\frac{\vec{P}}{3\varepsilon_0}.$$

2c) (Bonus 20 points) Write a program to plot the normalized electric field intensity along the *z*-axis:

$$\left|\frac{\vec{E}(0,0,z)}{\vec{E}_0}\right|, \quad 0 \le z/R_0 \le 5\,.$$

Answer:

The *z*-component of electric field at position (0,0,*z*) due to an infinitesimal ring of width $R_0 d\theta$ at azimuthal angle θ (the center of the ring is located at (0,0,*z′*), where $z' = R_0 \cos\theta$) is:

$$dE_z(z,z') = \frac{\rho_l b(z-z')}{2\varepsilon_0\left[b^2 + (z-z')^2\right]^{3/2}} = \frac{P}{2\varepsilon_0}\frac{\sin\theta\cos\theta(z/R_0 - \cos\theta)}{\left[\sin^2\theta + (z/R_0 - \cos\theta)^2\right]^{3/2}}$$

The total field at (0,0,*z*) can be derived by integration of dE_z with respect to θ from 0 to π.

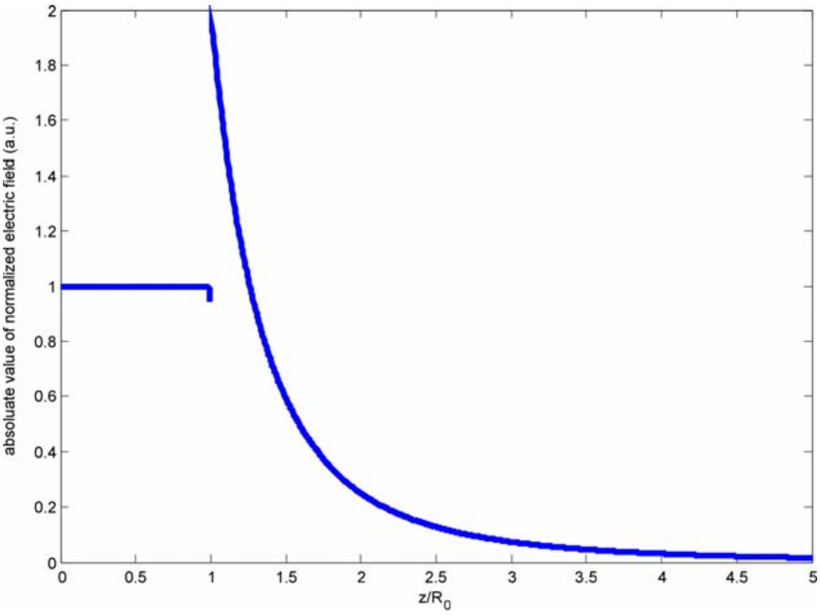

Figure 2.

3) Consider two large parallel conducting plates of area S separated by a distance d. The region between the two conducting plates is filled with two dielectric materials with $\varepsilon_1 = 2\varepsilon_0$, $d_1 = \frac{d}{2}$, $\varepsilon_2 = \varepsilon_0$, $d_2 = \frac{d}{2}$ (Fig. 2). The top and bottom plates are deposited with free charge $+Q$ and $-Q$, respectively.

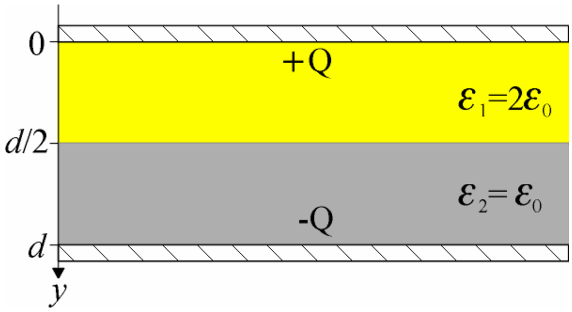

Fig. 2. Parallel-plate capacitor.

3a) (10%) What are the electric flux densities $\vec{D}_1$, $\vec{D}_2$, the electric field intensities $\vec{E}_1$, $\vec{E}_2$, and the polarization vectors $\vec{P}_1$, $\vec{P}_2$ between the two plates?

(*Hint*: Use Gauss's law.)

Answer:

Let $\vec{D}_1 = \vec{a}_y D_1$ and $\vec{D}_2 = \vec{a}_y D_2$, where D_1 and D_2 is the magnitude of $\vec{D}_1$ and $\vec{D}_2$.

Applying Gauss's law at the top plate:

$$\oint_s \vec{D}_1 \cdot d\vec{s} = \int_v \rho dv = +Q \Rightarrow \oint_s \vec{a}_y D_1 \cdot \vec{a}_y ds = +Q \Rightarrow D_1 \times S = +Q \Rightarrow D_1 = \frac{Q}{S} \quad \therefore \vec{D}_1 = \vec{a}_y \frac{Q}{S}.$$

Applying Gauss's law at the bottom plate:

$$\oint_s \vec{D}_2 \cdot d\vec{s} = \int_v \rho dv = -Q \Rightarrow \oint_s \vec{a}_y D_2 \cdot (-\vec{a}_y) ds = -Q \Rightarrow -D_2 \times S = -Q \Rightarrow D_2 = \frac{Q}{S}$$

$$\therefore \vec{D}_2 = \vec{a}_y \frac{Q}{S}.$$

Thus, $\vec{E}_1 = \frac{\vec{D}_1}{2\varepsilon_0} = \vec{a}_y \frac{Q}{2\varepsilon_0 S}$ and $\vec{E}_2 = \frac{\vec{D}_2}{\varepsilon_0} = \vec{a}_y \frac{Q}{\varepsilon_0 S}$.

Because $\vec{P} = \vec{D} - \varepsilon_0 \vec{E}$, we can calculate $\vec{P}_1$ and $\vec{P}_2$ as follows:

$$\vec{P}_1 = \vec{D}_1 - \varepsilon_0 \vec{E}_1 = \vec{a}_y \frac{Q}{S} - \vec{a}_y \frac{Q}{2S} = \vec{a}_y \frac{Q}{2S}.$$

$$\vec{P}_2 = \vec{D}_2 - \varepsilon_0 \vec{E}_2 = \vec{a}_y \frac{Q}{S} - \vec{a}_y \frac{Q}{S} = 0$$

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3b) (10%) What are the polarization surface charge densities ρ_{ps} at the three interfaces $y = 0^+$, $\frac{d}{2}$, and d^-, respectively.

(*Hint*: $\rho_{ps}\left(\frac{d}{2}\right) = \rho_{ps}\left(\frac{d^-}{2}\right) + \rho_{ps}\left(\frac{d^+}{2}\right)$)

Answer:

$$\rho_{ps}\left(0^+\right) = \vec{P}_1 \cdot \vec{a}_y = \vec{a}_y \frac{Q}{2S} \cdot \vec{a}_y = \frac{Q}{2S}.$$

$$\rho_{ps}\left(d^-\right) = \vec{P}_2 \cdot (-\vec{a}_y) = 0 \cdot (-\vec{a}_y) = 0.$$

$$\rho_{ps}\left(\frac{d}{2}\right) = \rho_{ps}\left(\frac{d^-}{2}\right) + \rho_{ps}\left(\frac{d^+}{2}\right) = \vec{P}_2 \cdot (-\vec{a}_y) + \vec{P}_1 \cdot \vec{a}_y = 0 + \frac{Q}{2S} = \frac{Q}{2S}.$$