Homework Solutions #4

(Due date: 2011/03/28)

This problem set covers materials of Lesson 5. The full score is 40 points.

1) Please evaluate:

1a) (5%) $\vec{a}_\phi \times \vec{a}_z$ (Represent the result in Cartesian coordinate system.)

Answer:

$$\vec{a}_\phi \times \vec{a}_z = \left[\vec{a}_x(-\sin\phi) + \vec{a}_y(\cos\phi)\right] \times \vec{a}_z = \vec{a}_y(\sin\phi) + \vec{a}_x(\cos\phi)$$

1b) (5%) $\frac{\partial}{\partial\phi}\vec{a}_\phi$ (Represent the result in Spherical coordinate system.)

Answer:

$$\frac{\partial}{\partial\phi}\vec{a}_\phi = \frac{\partial}{\partial\phi}\left[\vec{a}_x(-\sin\phi) + \vec{a}_y(\cos\phi)\right] = \vec{a}_x(-\cos\phi) + \vec{a}_y(-\sin\phi) = -\vec{a}_r = -\vec{a}_R \sin\theta - \vec{a}_\theta \cos\theta$$

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2) (10%) For a scalar function V and a vector function $\vec{A}$, prove that:

$$\nabla \times (V\vec{A}) = V(\nabla \times \vec{A}) + (\nabla V) \times \vec{A} .$$

Note that this is the chain rule of outer product.

Answer:

Let $\vec{A} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$, then $V\vec{A} = \vec{a}_x A_x V + \vec{a}_y A_y V + \vec{a}_z A_z V$

$$\nabla \times (V\vec{A}) = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ VA_x & VA_y & VA_z \end{vmatrix} = \vec{a}_x \left(\frac{\partial VA_z}{\partial y} - \frac{\partial VA_y}{\partial z} \right) = \vec{a}_x \left(A_z \frac{\partial V}{\partial y} + V \frac{\partial A_z}{\partial y} - A_y \frac{\partial V}{\partial z} - V \frac{\partial A_y}{\partial z} \right)$$

(consider only $\vec{a}_x$ component first) $= \vec{a}_x \cdot V \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \vec{a}_x \left(\frac{\partial V}{\partial y} A_z - \frac{\partial V}{\partial z} A_y \right)$

$= V(\nabla \times \vec{A}) + (\nabla V) \times \vec{A}$, the $\vec{a}_y$ and $\vec{a}_z$ components can also be derived by the same procedure.

3) (10%) The position vector of a point $P(x, y, z)$ is $\vec{R} = \vec{a}_R R$, where $R = |\vec{R}| = \sqrt{x^2 + y^2 + z^2}$ is a scalar field. Evaluate $\nabla(R^{-1})$, and represent the result by spherical coordinate system.

Answer:

By eq. 2-97 in DKC's book, $\nabla \equiv \vec{a}_R \frac{\partial}{\partial R} + \vec{a}_\theta \frac{\partial}{R \partial \theta} + \vec{a}_\phi \frac{\partial}{R \sin\theta \partial \phi}$,

so, $\nabla\left(R^{-1}\right) = \vec{a}_R \frac{\partial R^{-1}}{\partial R} = \frac{-\vec{a}_R}{R^2}$.

4) Consider a vector field $\vec{A}(R,\phi)=\vec{a}_R\dfrac{\sin^2\phi}{R^3}$ existing in a volume V bounded by two concentric spherical shells with $R=2$, $R=3$ (center at the origin).

4a) (5%) Evaluate the net outward flux $\oint_S \vec{A}\cdot d\vec{s}$, where S is the surface enclosing V.

Answer:

In the spherical coordinates: $d\vec{s}=\vec{a}_R R^2\sin\theta d\theta d\phi$

$$\oint_S \frac{\sin^2\phi}{R}\sin\theta d\theta d\phi=\frac{1}{R}\int_0^{2\pi}\int_0^{2\pi}\sin\theta d\theta\sin^2\phi d\phi$$

$$=\frac{1}{R}\left[-\cos\theta\,|_0^{2\pi}\right]\left[\frac{1}{2}\left(\phi-\frac{\sin^2\phi}{2}\,|_0^{2\pi}\right)\right]=\frac{1}{R}(-2)(\pi)=\frac{-2\pi}{R}$$

$$\oint_{S,out}\vec{A}\cdot d\vec{s}-\oint_{S,in}\vec{A}\cdot d\vec{s}=\frac{-2\pi}{3}-\frac{-2\pi}{2}=\frac{\pi}{3}$$

4b) (5%) Evaluate $\int_V\left(\nabla\cdot\vec{A}\right)dv$.

Answer:

In the spherical coordinates:

$$\nabla\cdot\vec{A}=\frac{1}{R^2}\frac{\partial}{\partial R}(R^2A_R)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot A_\theta)+\frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi},\ dv=R^2\sin\theta dR d\theta d\phi$$

$$\int_V(\nabla\cdot\vec{A})dv=\int_V\frac{1}{R^2}\frac{\partial}{\partial R}\left(\frac{\sin^2\phi}{R}\right)dv=\int_V-\frac{\sin^2\phi}{R^4}dv=\int_0^{2\pi}\int_0^{2\pi}\int_2^3-\frac{1}{R^2}dR\sin\theta d\theta\sin^2\phi d\phi$$

$$=\left[\frac{1}{R}\,|_2^3\right]\left[-\cos\theta\,|_0^{2\pi}\right]\left[\frac{1}{2}\left(\phi-\frac{\sin^2\phi}{2}\,|_0^{2\pi}\right)\right]=\left[-\frac{1}{6}\right](-2)(\pi)=\frac{\pi}{3}$$