Chapter 11
Balanced Three-Phase Circuits

11.1-2 Three-Phase Systems

11.3 Analysis of the Y-Y Circuit

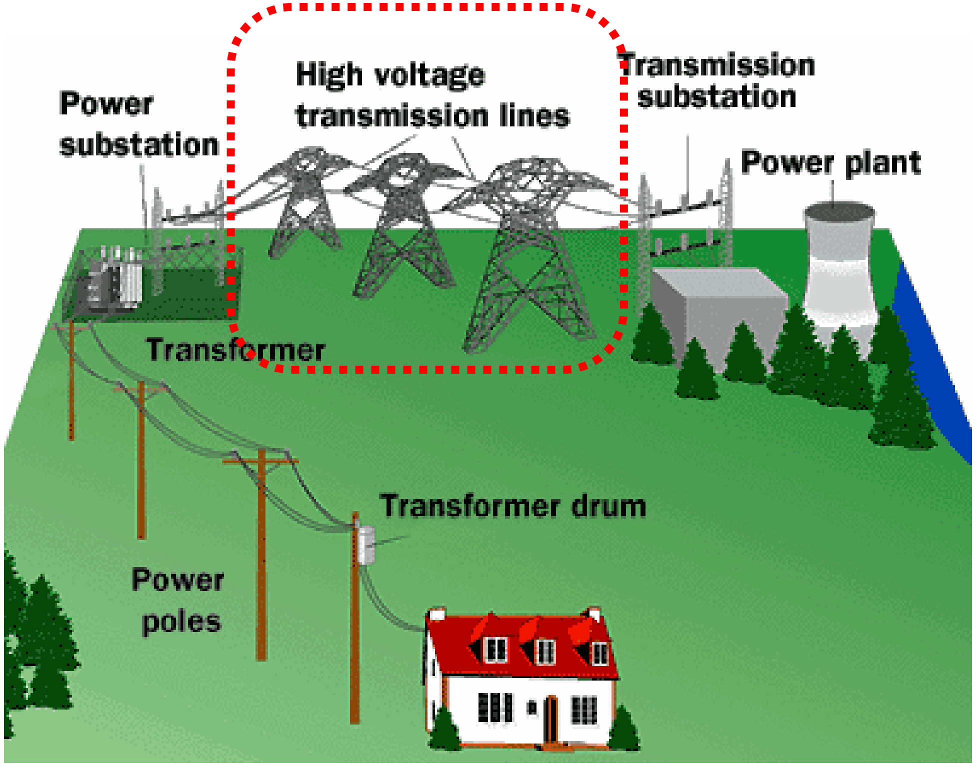
11.4 Analysis of the Y-Δ Circuit

11.5 Power Calculations in Balanced Three-Phase Circuits

11.6 Measuring Average Power in Three-Phase Circuits

Overview

- An electric power distribution system looks like:



where the power transmission uses “balanced three-phase” configuration.

Why three-phase?

- Three-phase generators can be driven by constant force or torque (to be discussed).
- Industrial applications, such as high-power motors, welding equipments, have constant power output if they are three-phase systems (to be discussed).

Key points

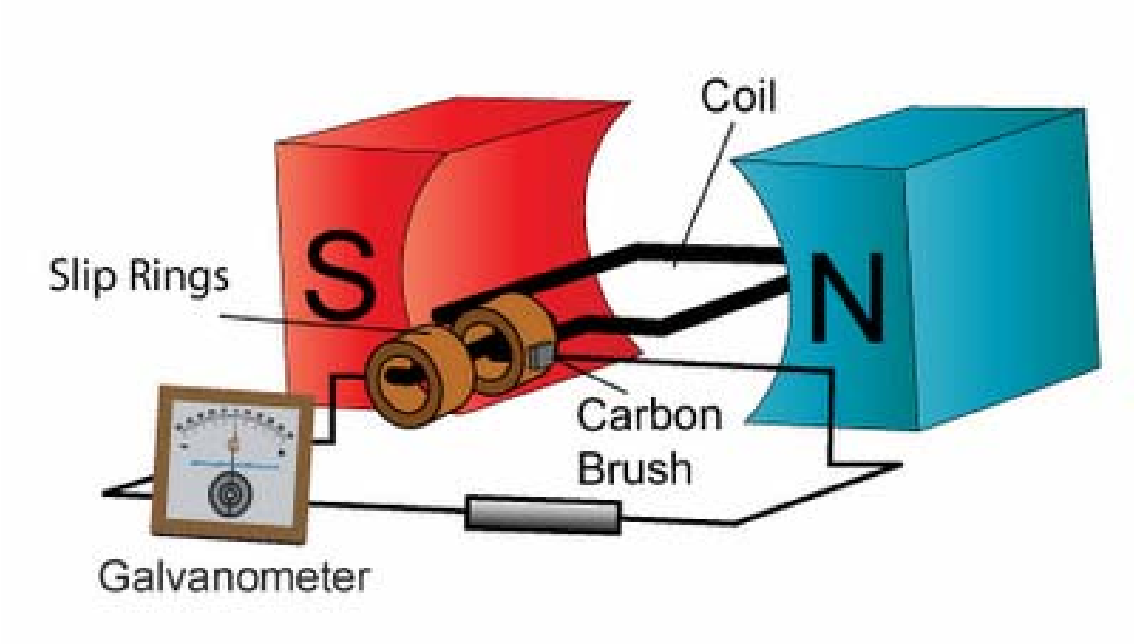
- What is a three-phase circuit (source, line, load)?
- Why a balanced three-phase circuit can be analyzed by an equivalent one-phase circuit?
- How to get all the unknowns (e.g. line voltage of the load) by the result of one-phase circuit analysis?
- Why the total instantaneous power of a balanced three-phase circuit is a constant?

Section 11.1, 11.2 Three-Phase Systems

1. Three-phase sources
2. Three-phase systems

One-phase voltage sources

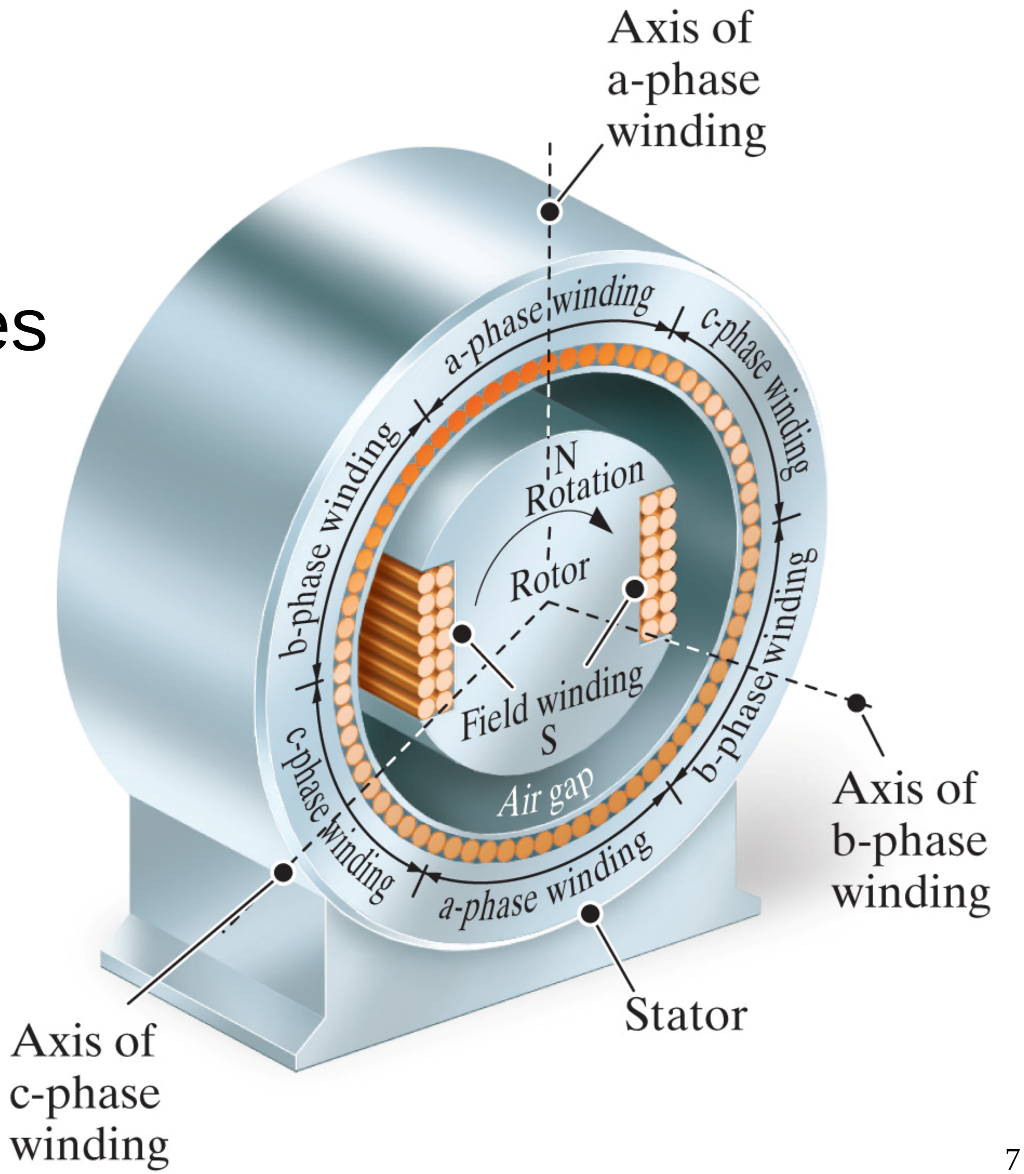
- One-phase ac generator: static magnets, one rotating coil, single output voltage $v(t)=V_m\cos\omega t$.



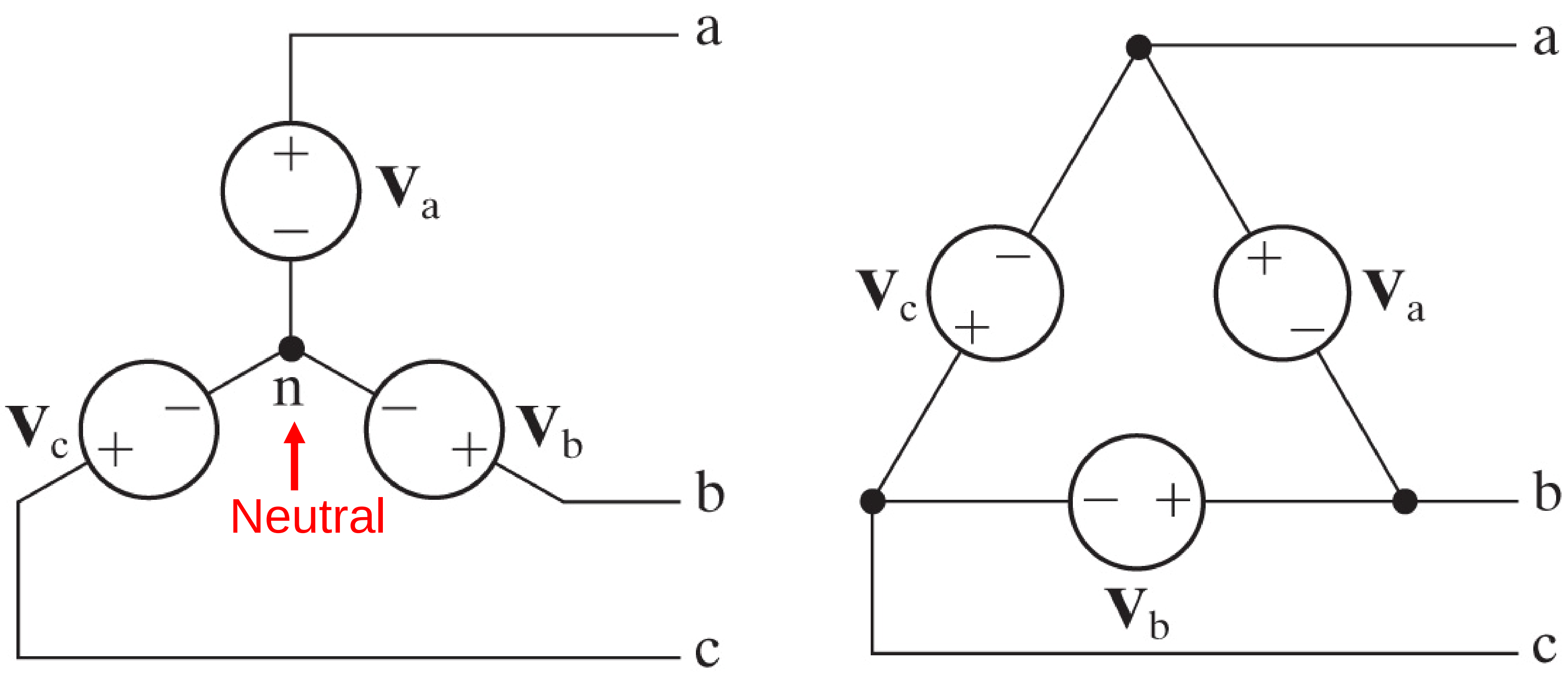
(www.ac-motors.us)

Three-phase voltage sources

- Three static coils, rotating magnets, three output voltages $v_a(t)$, $v_b(t)$, $v_c(t)$.

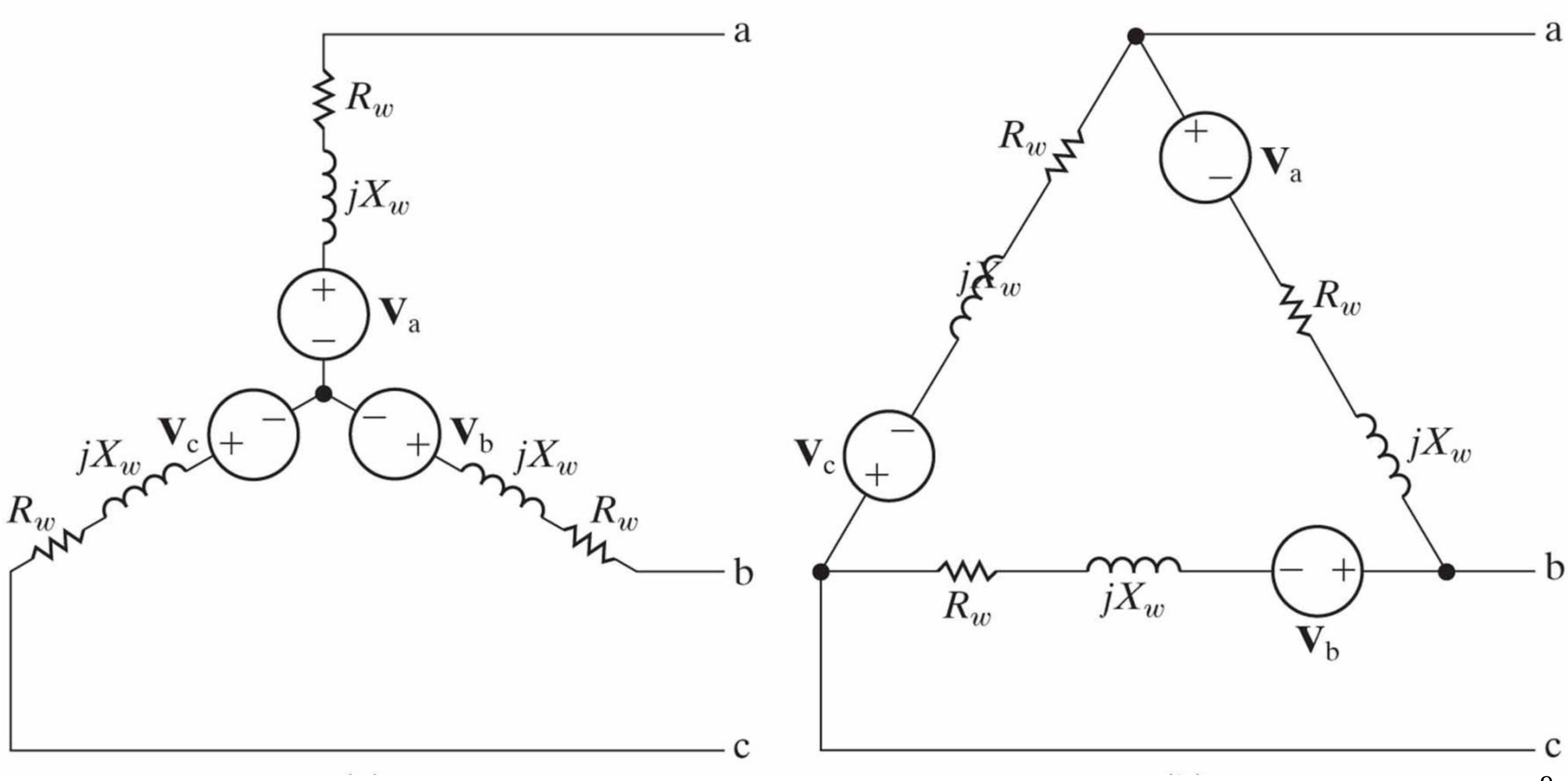


Ideal Y- and Δ-connected voltage sources



Real Y- and Δ-connected voltage sources

- Internal impedance of a generator is usually inductive (due to the use of coils).



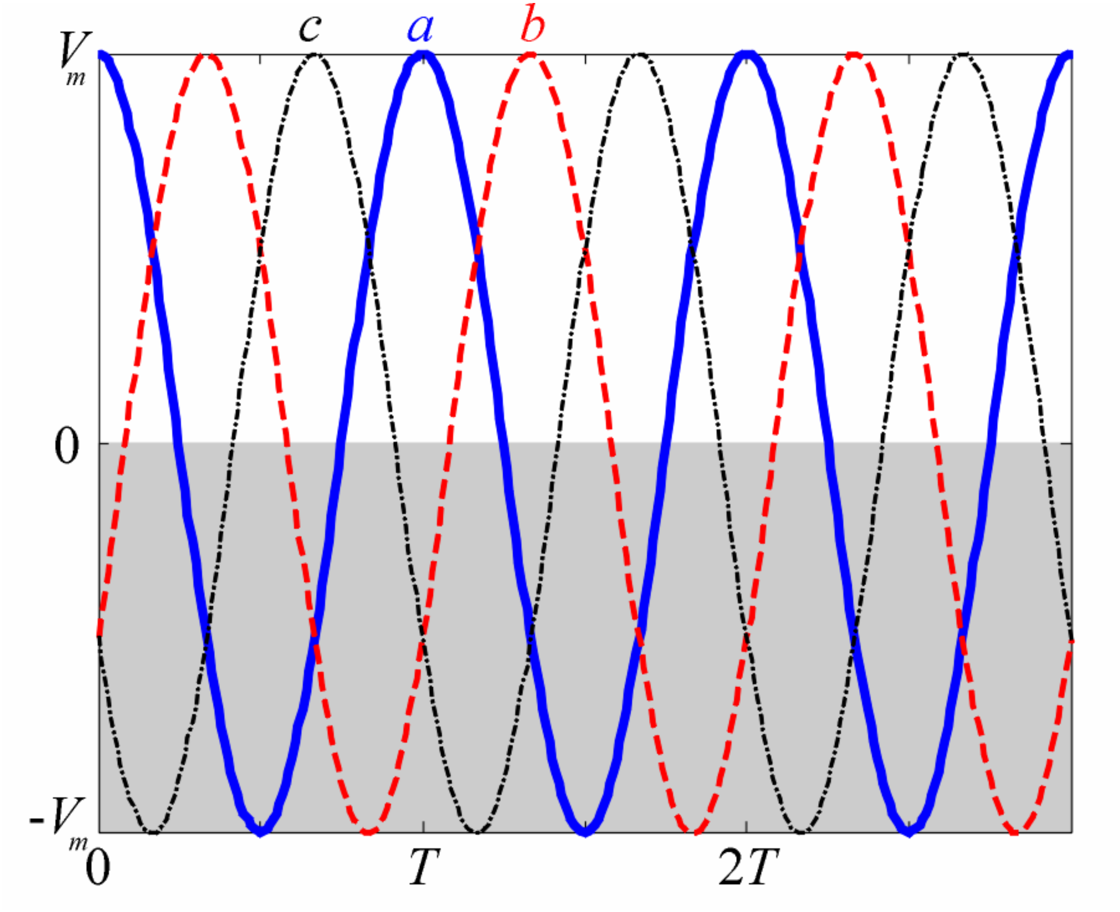
Balanced three-phase voltages

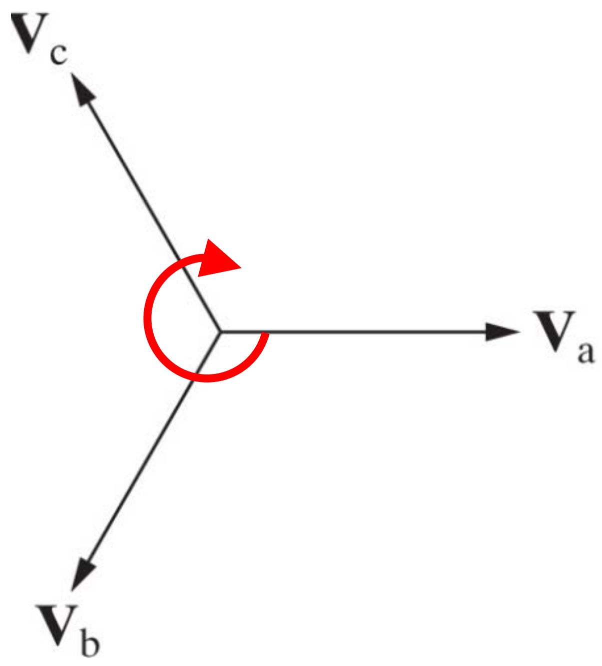
- Three sinusoidal voltages of the same amplitude, frequency, but differing by 120° phase difference with one another.
- There are two possible sequences:

1. abc (positive) sequence: $v_b(t)$ lags $v_a(t)$ by 120°.
2. acb (negative) sequence: $v_b(t)$ leads $v_a(t)$ by 120°.

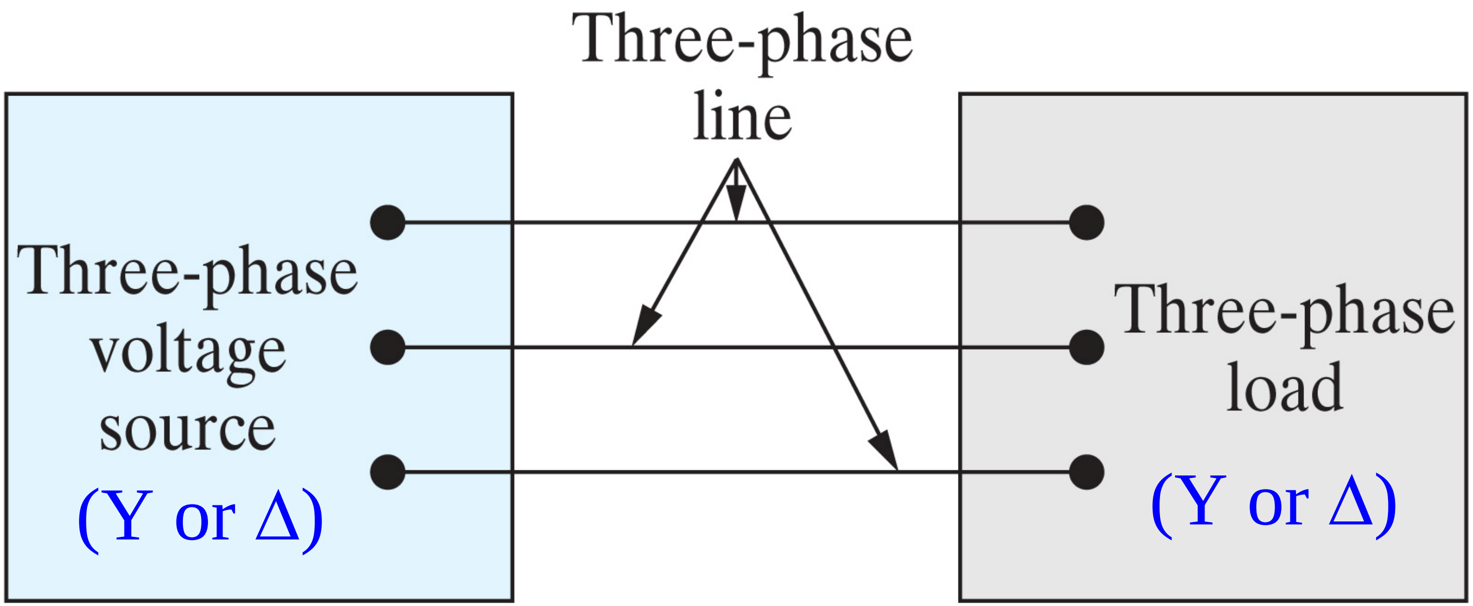
abc sequence

- $v_b(t)$ lags $v_a(t)$ by 120° or $T/3$.
- $\mathbf{V}_a = V_m\angle 0^\circ,\ \mathbf{V}_b = V_m\angle -120^\circ,\ \mathbf{V}_c = V_m\angle +120^\circ.$





Three-phase systems

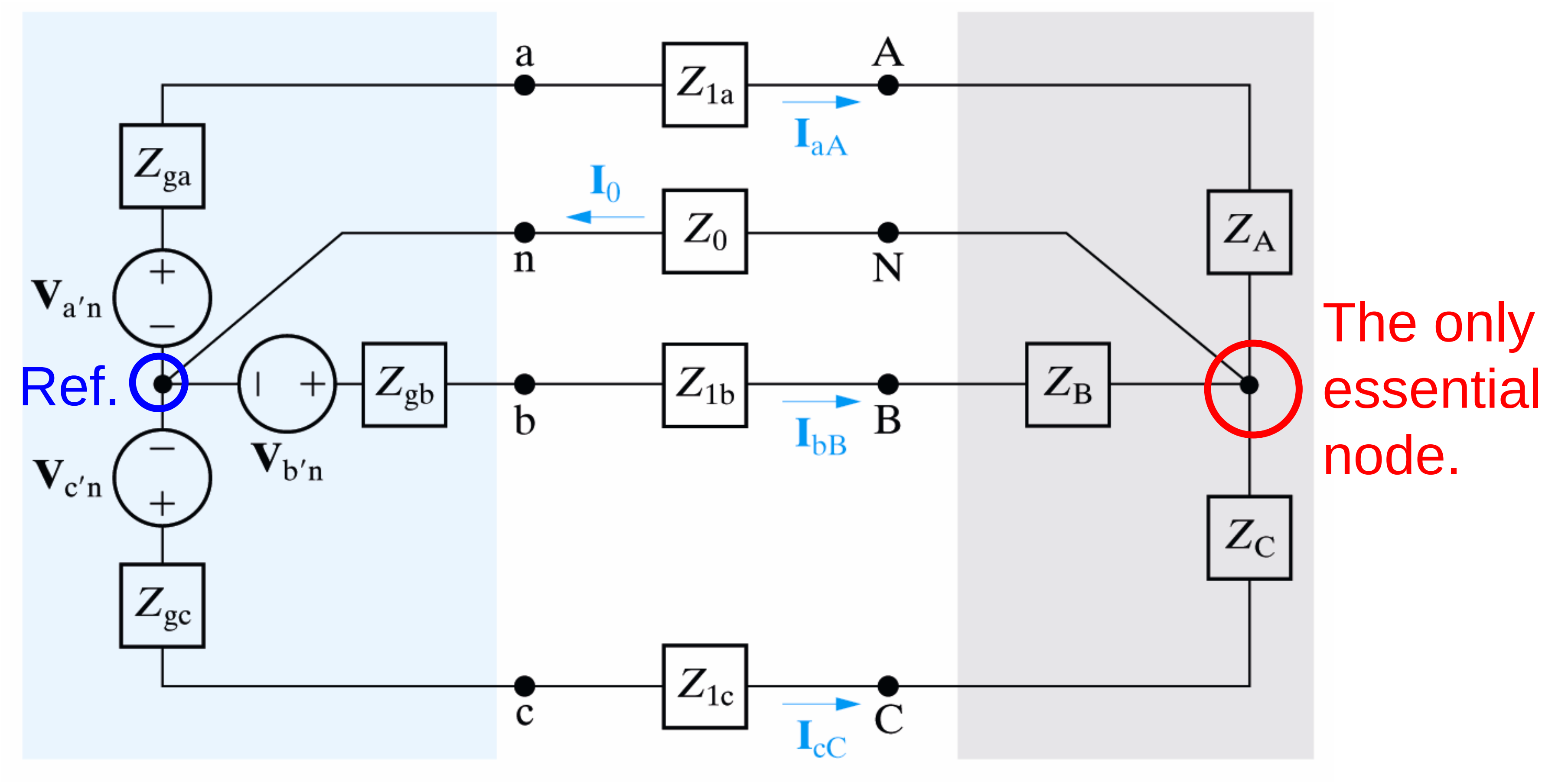


- Source-load can be connected in four configurations: Y-Y, Y-Δ, Δ-Y, Δ-Δ.
- It's sufficient to analyze Y-Y, while the others can be treated by Δ-Y and Y-Δ transformations.

Section 11.3
Analysis of the Y-Y Circuit

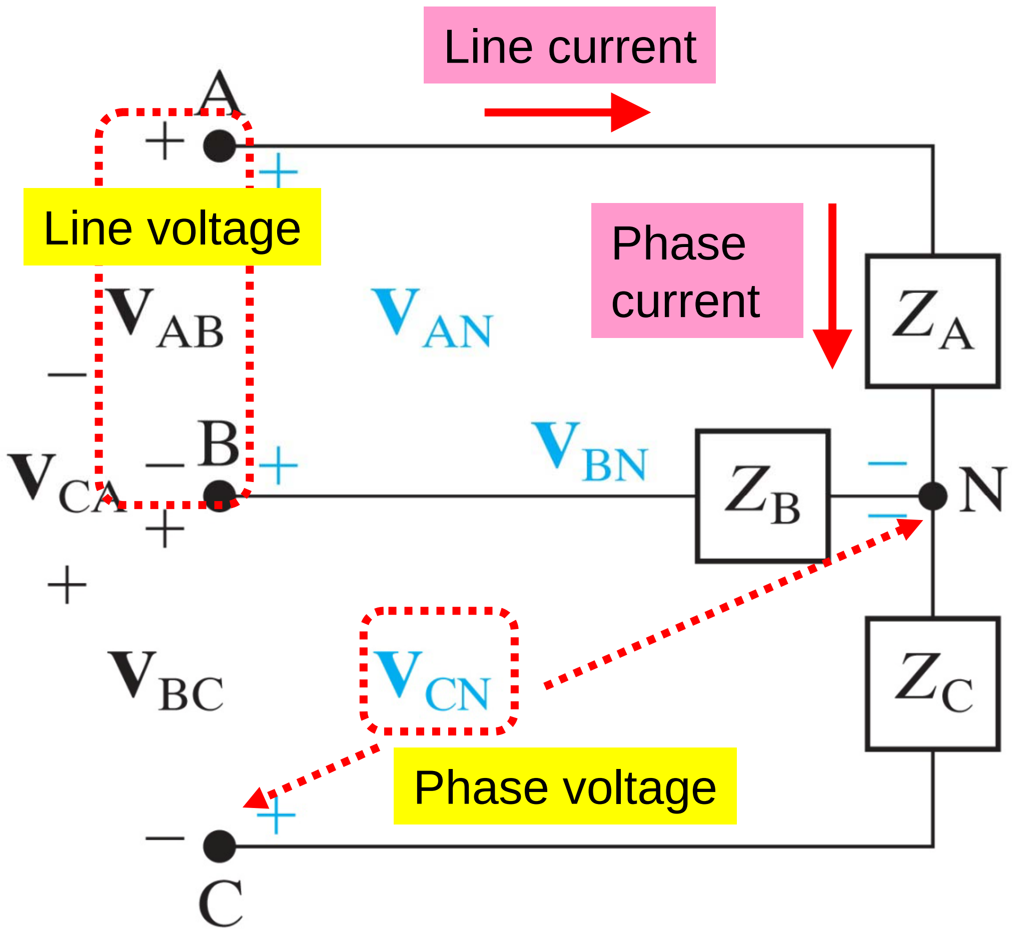
1. Equivalent one-phase circuit for balanced Y-Y circuit
2. Line currents, phase and line voltages

General Y-Y circuit model



Unknowns to be solved

- Line (line-to-line) voltage: voltage across any pair of lines.
- Phase (line-to-neutral) voltage: voltage across a single phase.



- For Y-connected load, line current equals phase current.

Solution to general three-phase circuit

- No matter it's balanced or imbalanced three-phase circuit, KCL leads to one equation:

$$\mathbf{I}_0 = \mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC}, \Rightarrow$$

$$\frac{\mathbf{V}_N}{Z_0} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_N}{Z_{ga} + Z_{1a} + Z_A} + \frac{\mathbf{V}_{b'n} - \mathbf{V}_N}{Z_{gb} + Z_{1b} + Z_B} + \frac{\mathbf{V}_{c'n} - \mathbf{V}_N}{Z_{gc} + Z_{1c} + Z_C} \ldots (1),$$

Z_0	$Z_{ga} + Z_{1a} + Z_A$	$Z_{gb} + Z_{1b} + Z_B$	$Z_{gc} + Z_{1c} + Z_C$
Impedance of neutral line.	Total impedance along line aA.	Total impedance along line bB.	Total impedance along line cC.

which is sufficient to solve $\mathbf{V}_N$ (thus the entire circuit).

Solution to “balanced” three-phase circuit

- For balanced three-phase circuits,

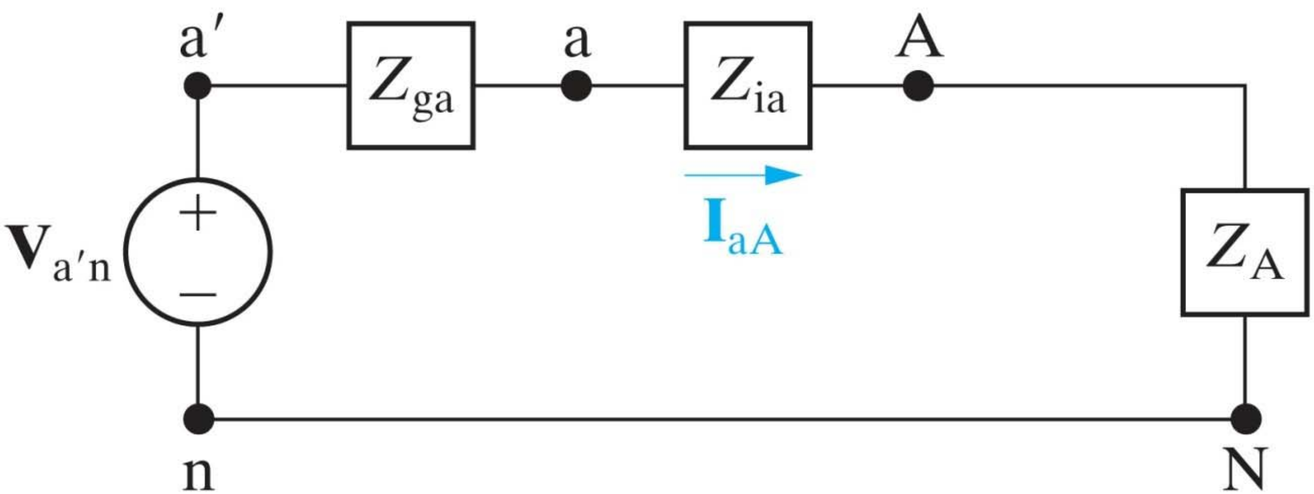
1. $\{\mathbf{V}_{a'n}, \mathbf{V}_{b'n}, \mathbf{V}_{c'n}\}$ have equal magnitude and $120°$ relative phases;
2. $\{Z_{ga} = Z_{gb} = Z_{gc}\}, \{Z_{1\mathrm{a}} = Z_{1\mathrm{b}} = Z_{1\mathrm{c}}\}, \{Z_A = Z_B = Z_C\}$;
   $\Rightarrow$ total impedance along any line is the same
   $Z_{ga} + Z_{1\mathrm{a}} + Z_A = \ldots = Z_\phi$.

- Eq. (1) becomes: $\frac{\mathbf{V}_N}{Z_0} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_N}{Z_\phi} + \frac{\mathbf{V}_{b'n} - \mathbf{V}_N}{Z_\phi} + \frac{\mathbf{V}_{c'n} - \mathbf{V}_N}{Z_\phi}$,

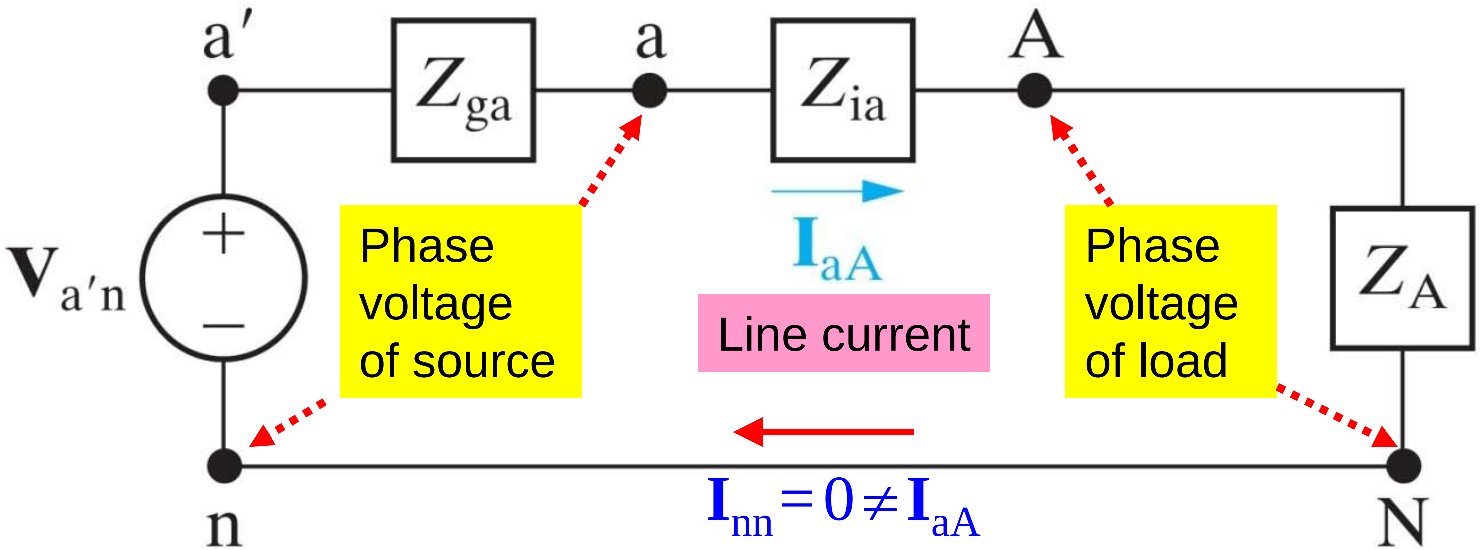
$$\Rightarrow \mathbf{V}_N\left(\frac{1}{Z_0} + \frac{3}{Z_\phi}\right) = \frac{\mathbf{V}_{a'n} + \mathbf{V}_{b'n} + \mathbf{V}_{c'n}}{Z_\phi} = 0, \quad \mathbf{V}_N = 0.$$

Meaning of the solution

- $\mathbf{V}_N = 0$ means no voltage difference between nodes $\boldsymbol{n}$ and $\boldsymbol{N}$ in the presence of Z_0. $\Rightarrow$ Neutral line is both short ($v = 0$) and open ($i = 0$).
- The three-phase circuit can be separated into 3 one-phase circuits (open), while each of them has a short between nodes $\boldsymbol{n}$ and $\boldsymbol{N}$.



Equivalent one-phase circuit



- Directly giving the line current & phase voltages:

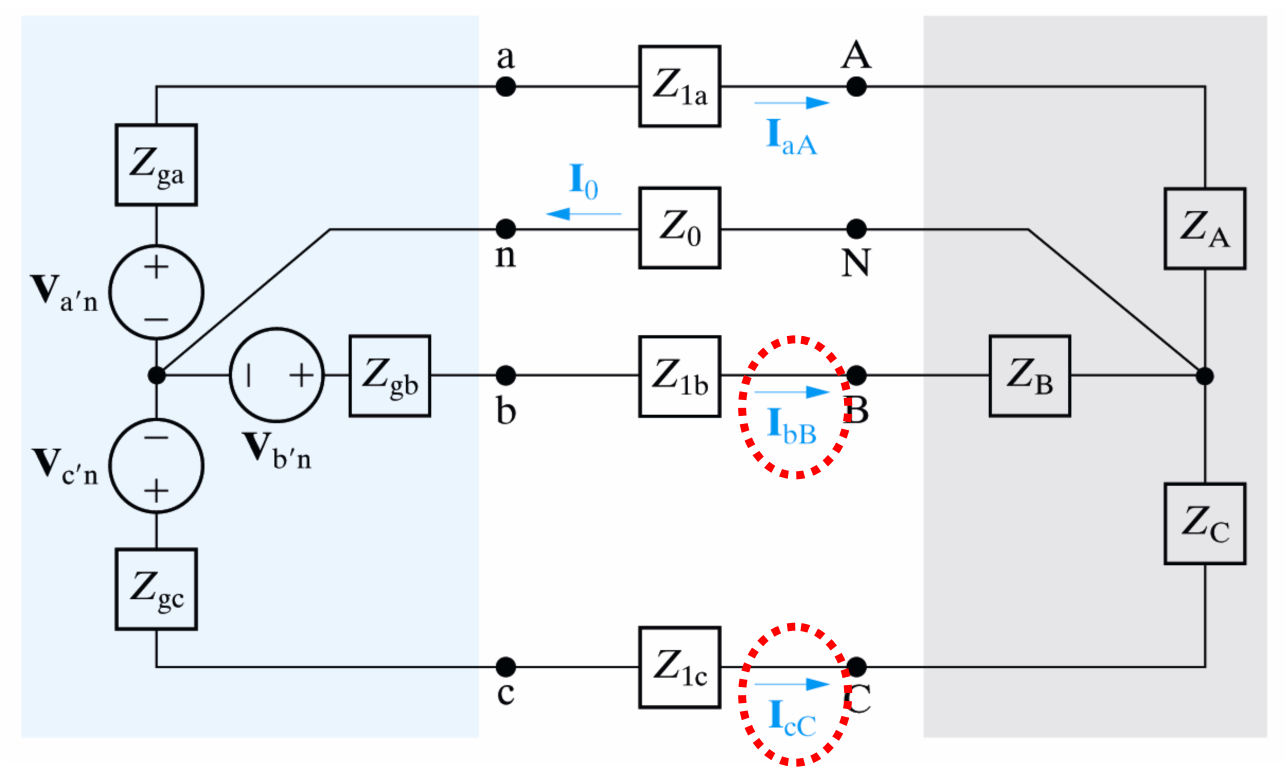
$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_N}{\left(Z_{ga} + Z_{1a} + Z_A\right) = Z_\phi}, \ \mathbf{V}_{AN} = \mathbf{I}_{aA} Z_A, \ \mathbf{V}_{an} = \mathbf{I}_{aA}\left(Z_{1a} + Z_A\right).$$

- Unknowns of phases b, c can be determined by the fixed (abc or acb) sequence relation.

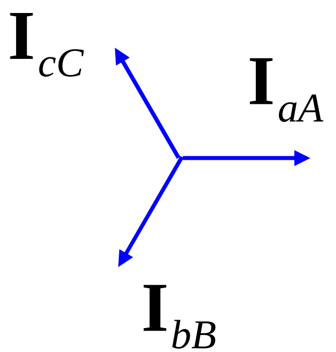
The 3 line and phase currents in abc sequence

- Given $\mathbf{I}_{aA} = \mathbf{V}_{a'n} / Z_{\phi}$, the other 2 line currents are:

$$\mathbf{I}_{bB} = \frac{\mathbf{V}_{b'n}}{Z_{\phi}} = \mathbf{I}_{aA}\angle -120^{\circ}, \quad \mathbf{I}_{cC} = \frac{\mathbf{V}_{c'n}}{Z_{\phi}} = \mathbf{I}_{aA}\angle 120^{\circ},$$



which still follow the abc sequence relation.



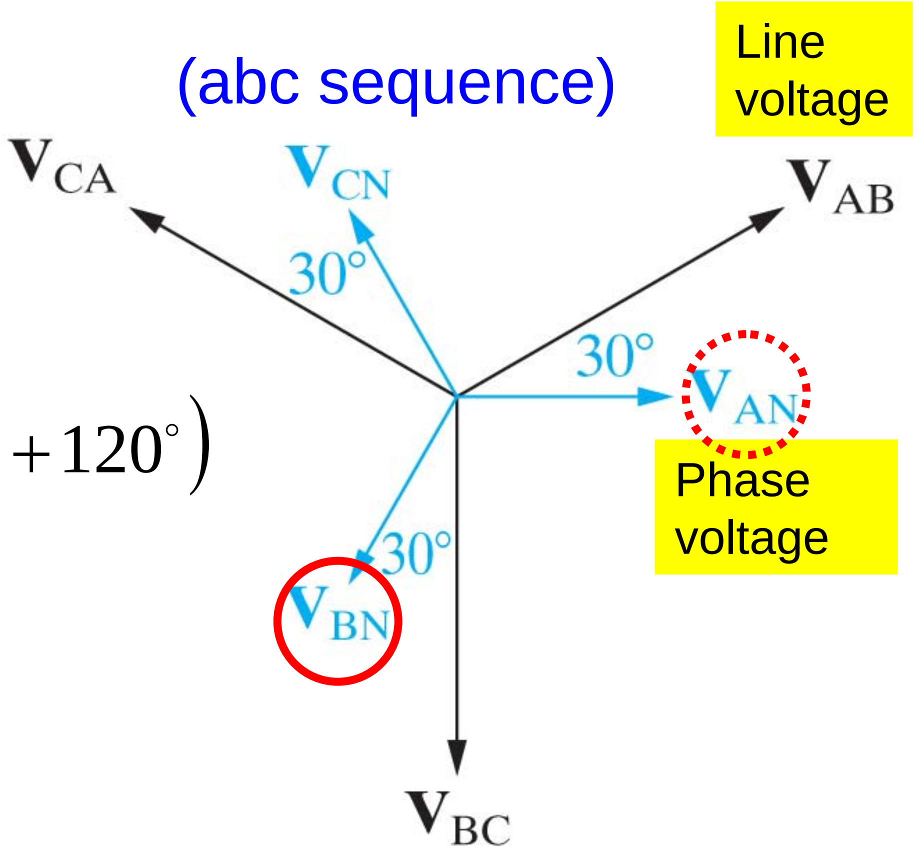
The phase & line voltages of the load in abc seq.

$$\mathbf{V}_{AN} = \mathbf{V}_{a'n}\frac{Z_A}{Z_\phi},\ \mathbf{V}_{BN} = \mathbf{V}_{b'n}\frac{Z_B}{Z_\phi} = \mathbf{V}_{AN}\angle -120^\circ,\ \mathbf{V}_{CN} = \mathbf{V}_{AN}\angle 120^\circ.$$

$$\begin{aligned}\mathbf{V}_{AB} &= \mathbf{V}_{AN} - \mathbf{V}_{BN}\\ &= \mathbf{V}_{AN} - \left(\mathbf{V}_{AN}\angle -120^\circ\right)\\ &= \sqrt{3}\mathbf{V}_{AN}\angle +30^\circ,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_{BC} &= \left(\mathbf{V}_{AN}\angle -120^\circ\right) - \left(\mathbf{V}_{AN}\angle +120^\circ\right)\\ &= \sqrt{3}\mathbf{V}_{AN}\angle -90^\circ,\end{aligned}$$

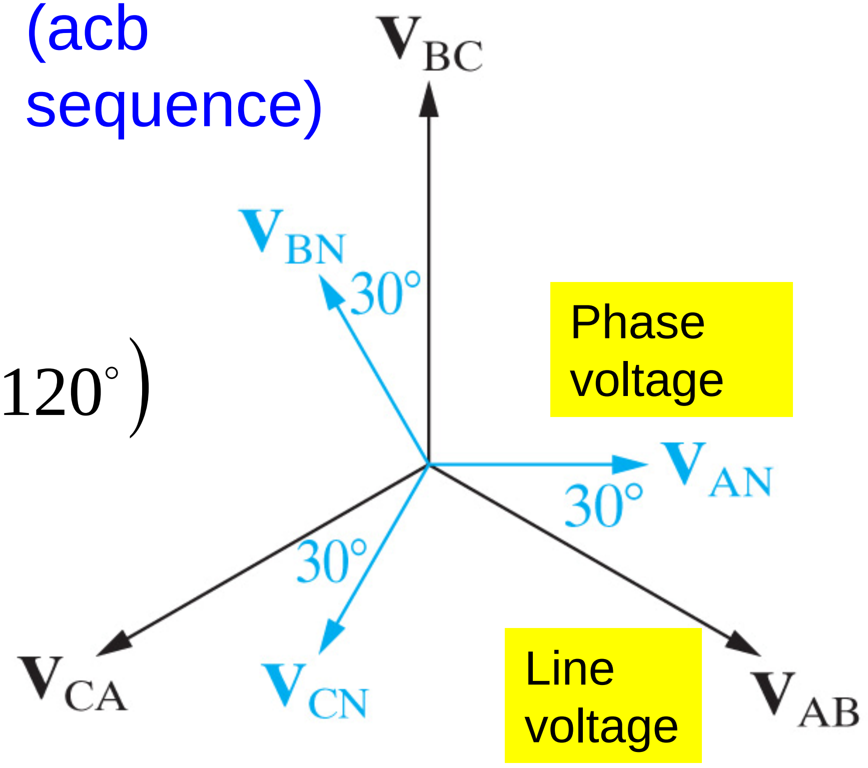
$$\begin{aligned}\mathbf{V}_{CA} &= \left(\mathbf{V}_{AN}\angle +120^\circ\right) - \mathbf{V}_{AN}\\ &= \sqrt{3}\mathbf{V}_{AN}\angle +150^\circ.\end{aligned}$$



The phase & line voltages of the load in acb seq.

$$\mathbf{V}_{AB} = \mathbf{V}_{AN} - \mathbf{V}_{BN}$$
$$= \mathbf{V}_{AN} - \left(\mathbf{V}_{AN}\angle +120^\circ\right)$$
$$= \sqrt{3}\mathbf{V}_{AN}\angle -30^\circ,$$
$$\mathbf{V}_{BC} = \left(\mathbf{V}_{AN}\angle +120^\circ\right) - \left(\mathbf{V}_{AN}\angle -120^\circ\right)$$
$$= \sqrt{3}\mathbf{V}_{AN}\angle +90^\circ,$$
$$\mathbf{V}_{CA} = \left(\mathbf{V}_{AN}\angle -120^\circ\right) - \mathbf{V}_{AN}$$
$$= \sqrt{3}\mathbf{V}_{AN}\angle -150^\circ.$$

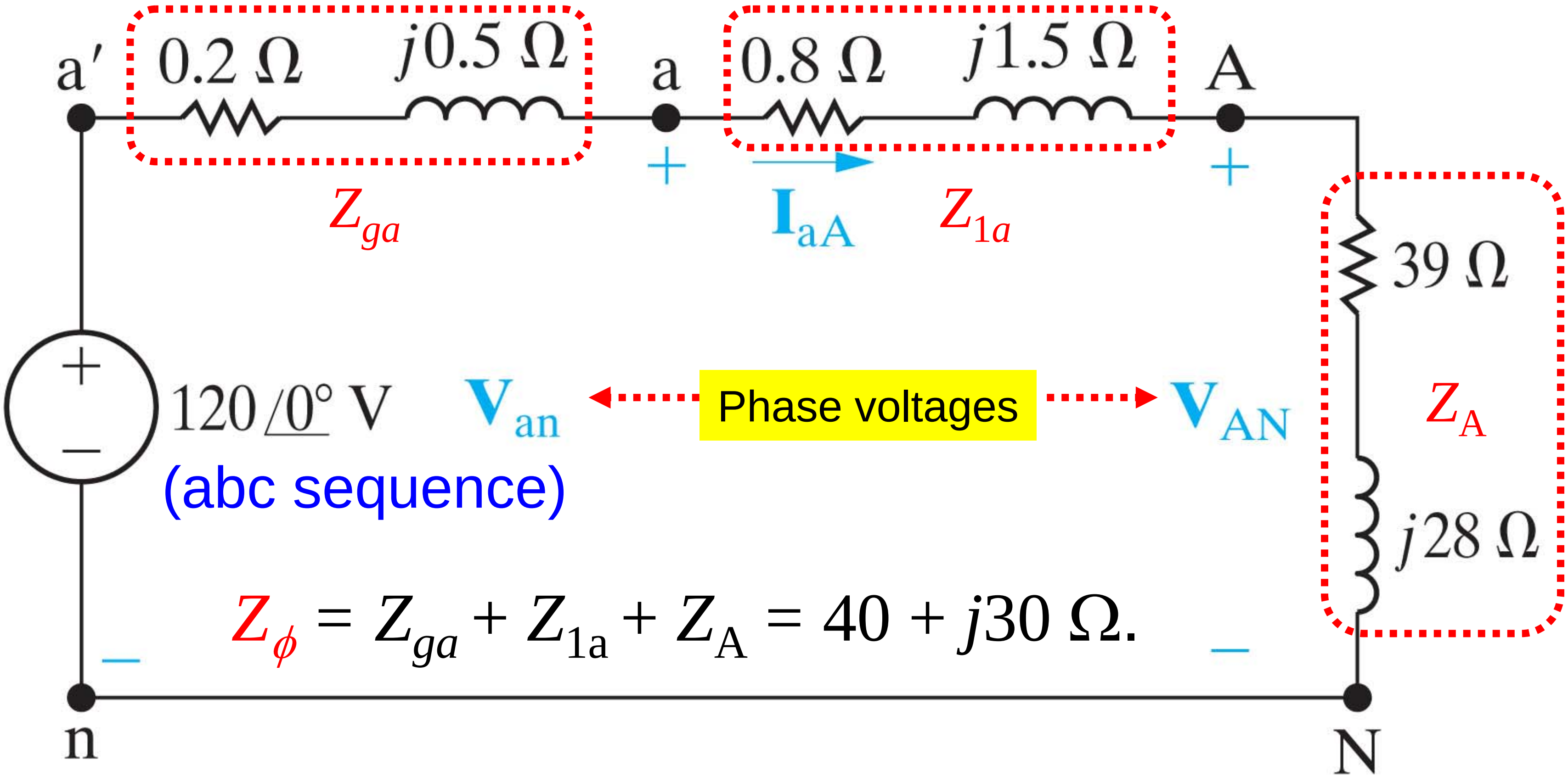
(acb sequence)



- Line voltages are $\sqrt{3}$ times bigger, leading (abc) or lagging (acb) the phase voltages by $30°$.

Example 11.1 (1)

- Q: What are the line currents, phase and line voltages of the load and source, respectively?



$$Z_\phi = Z_{ga} + Z_{1a} + Z_A = 40 + j30\ \Omega.$$

Example 11.1 (2)

- The 3 line currents (of both load & source) are:

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n}}{Z_{ga} + Z_{1a} + Z_A} = \frac{120\angle 0^\circ}{40 + j30} = \left(2.4\angle -36.87^\circ\right)\text{A},$$

$$\mathbf{I}_{bB} = \mathbf{I}_{aA}\angle -120^\circ = \left(2.4\angle -156.87^\circ\right)\text{A},$$

$$\mathbf{I}_{cC} = \mathbf{I}_{aA}\angle +120^\circ = \left(2.4\angle +83.13^\circ\right)\text{A}.$$

- The 3 phase voltages of the load are:

$$\mathbf{V}_{AN} = \mathbf{I}_{aA} Z_A = \left(2.4\angle -36.87^\circ\right)\left(39 + j28\right) = \left(115.22\angle -1.19^\circ\right)\ \text{V}.$$

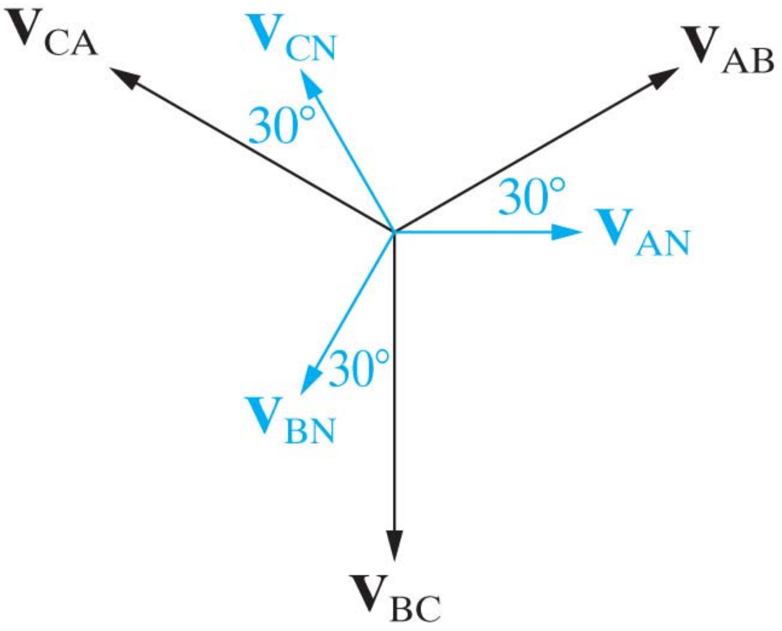
$$\mathbf{V}_{BN} = \mathbf{V}_{AN}\angle -120^\circ = \left(115.22\angle -121.19^\circ\right)\text{V},$$

$$\mathbf{V}_{CN} = \mathbf{V}_{AN}\angle +120^\circ = \left(115.22\angle +118.81^\circ\right)\text{V}.$$

Example 11.1 (3)

- The 3 line voltages of the load are:

$$\mathbf{V}_{AB} = \left(\sqrt{3}\angle 30^\circ\right)\mathbf{V}_{AN}$$
$$= \left(\sqrt{3}\angle 30^\circ\right)\left(115.22\angle -1.19^\circ\right)$$
$$= \left(199.58\angle +28.81^\circ\right)\text{V},$$
$$\mathbf{V}_{BC} = \mathbf{V}_{AB}\angle -120^\circ$$
$$= \left(199.58\angle -91.19^\circ\right)\text{V},$$
$$\mathbf{V}_{CA} = \mathbf{V}_{AB}\angle +120^\circ$$
$$= \left(199.58\angle +148.81^\circ\right)\text{V}.$$



Example 11.1 (4)

- The 3 phase voltages of the source are:

$$\mathbf{V}_{an} = \mathbf{V}_{a'n} - \mathbf{I}_{aA} Z_{ga} = 120 - \left(2.4\angle -36.87^\circ\right)\left(0.2 + j0.5\right)$$

$$= \left(118.9\angle -0.32^\circ\right)\text{V},$$

$$\mathbf{V}_{bn} = \mathbf{V}_{an}\angle -120^\circ = \left(118.9\angle -120.32^\circ\right)\text{V},$$

$$\mathbf{V}_{cn} = \mathbf{V}_{an}\angle +120^\circ = \left(118.9\angle +119.68^\circ\right)\text{V}.$$

- The three line voltages of the source are:

$$\mathbf{V}_{ab} = \left(\sqrt{3}\angle 30^\circ\right)\mathbf{V}_{an} = \left(\sqrt{3}\angle 30^\circ\right)\left(118.9\angle -0.32^\circ\right)$$

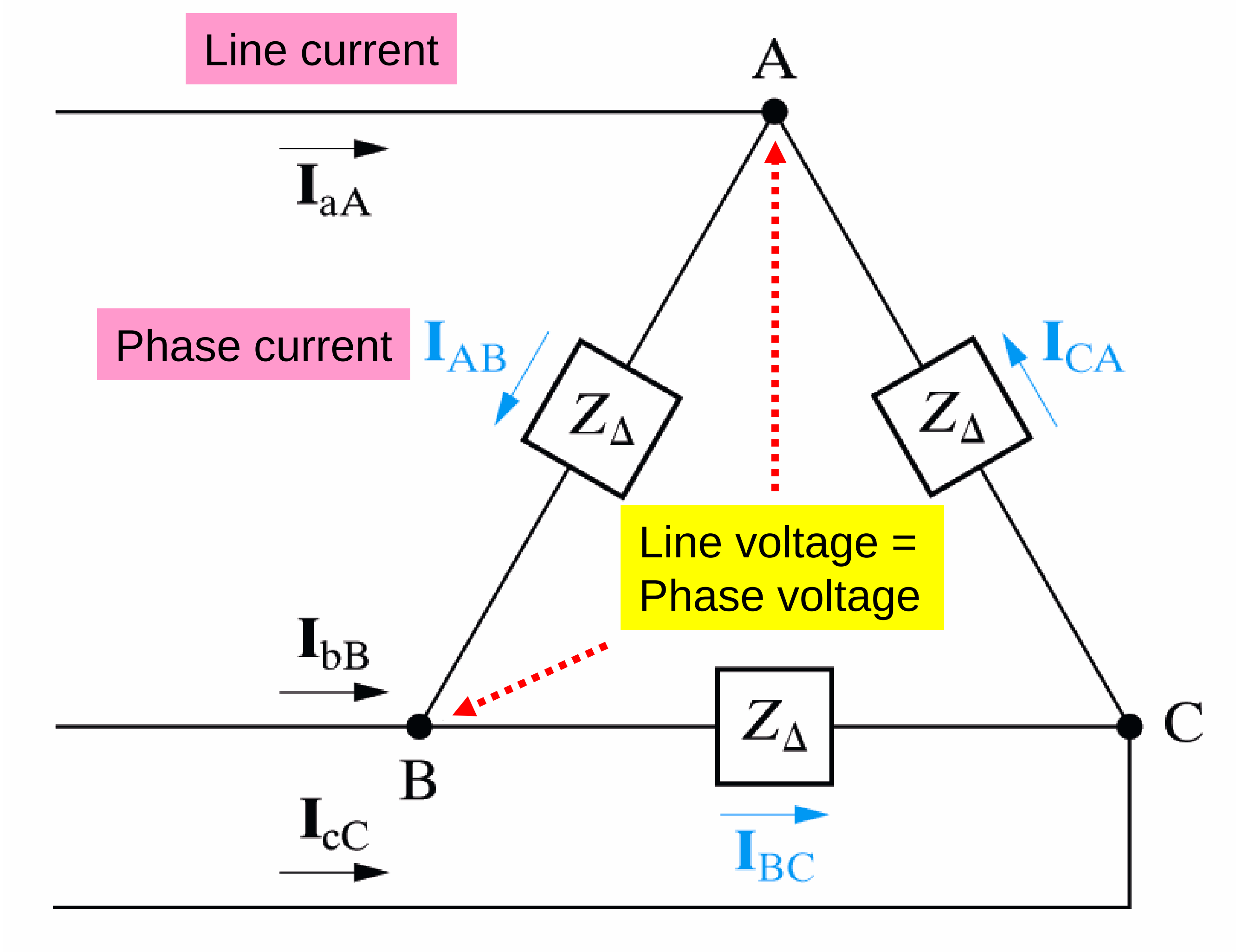
$$= \left(205.94\angle +29.68^\circ\right)\text{V},$$

$$\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle -120^\circ = \left(205.94\angle -90.32^\circ\right)\text{V},$$

$$\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle +120^\circ = \left(205.94\angle +149.68^\circ\right)\text{V}.$$

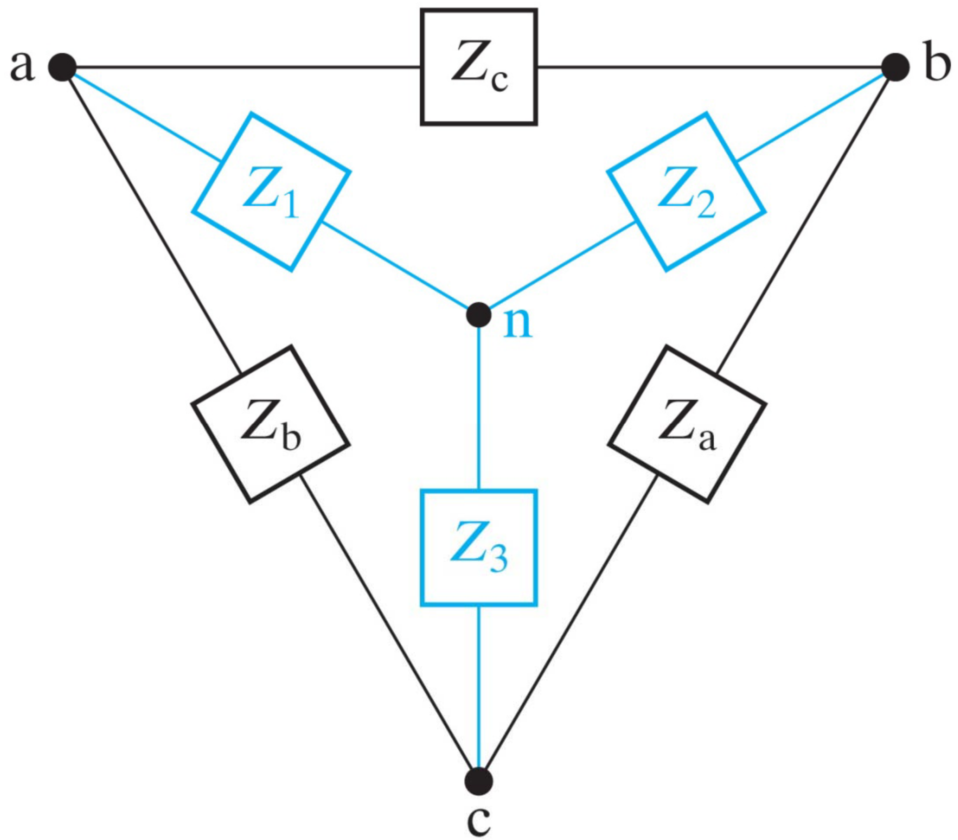
Section 11.4
Analysis of the Y-Δ Circuit

Load in Δ configuration



Δ-Y transformation for balanced 3-phase load

- The impedance of each leg in Y-configuration (Z_Y) is one-third of that in Δ-configuration (Z_Δ):



$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c},$$

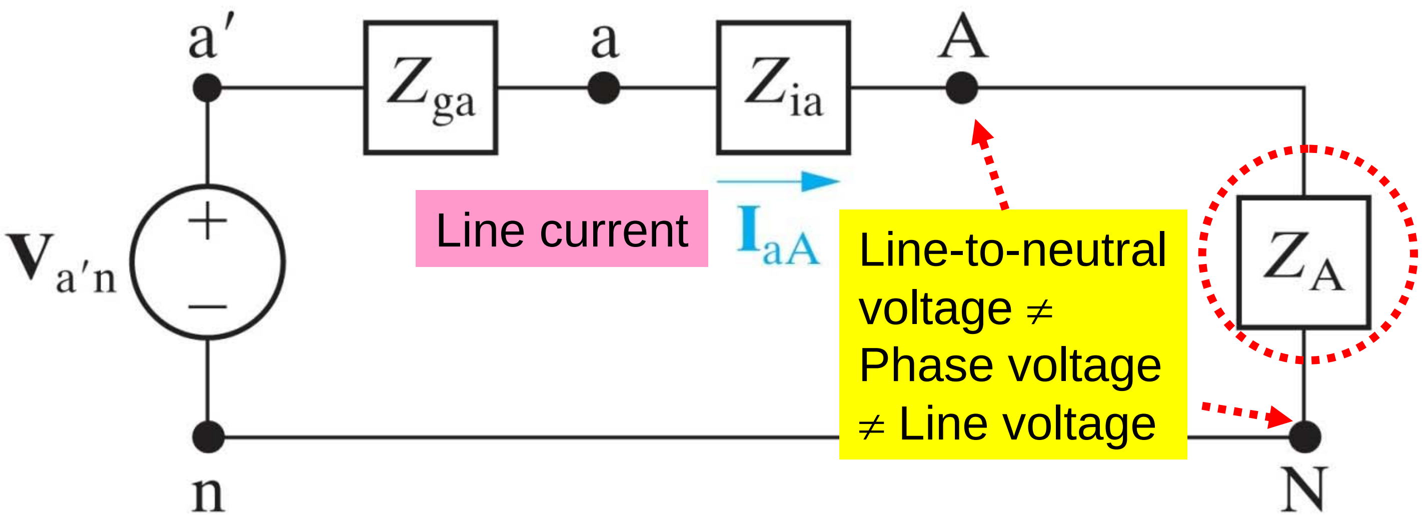
$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c},$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}.$$

$$\Rightarrow Z_Y = \frac{Z_\Delta Z_\Delta}{3Z_\Delta} = \frac{Z_\Delta}{3}.$$

Equivalent one-phase circuit

- The 1-phase equivalent circuit in Y-Y config. continues to work if Z_A is replaced by $Z_\Delta/3$:



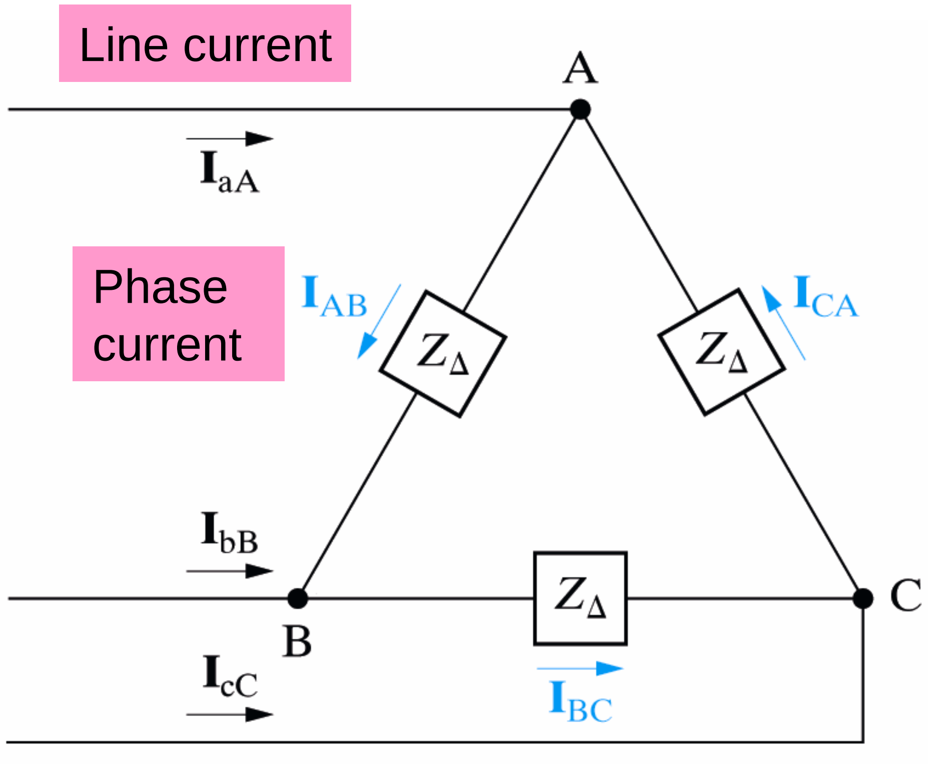
directly giving the line current: $\mathbf{I}_{aA} = \dfrac{\mathbf{V}_{a'n}}{Z_{ga} + Z_{1a} + Z_A}$,

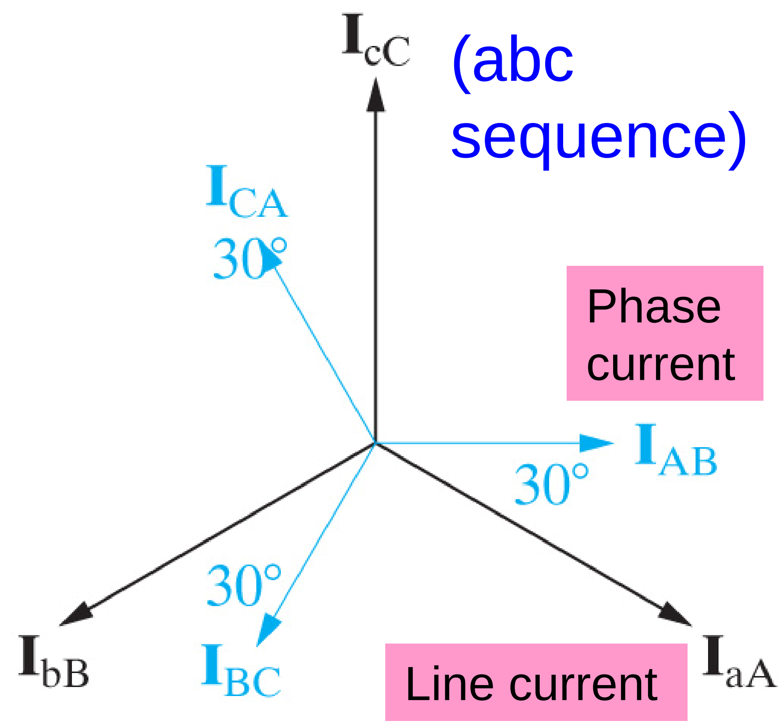
and line-to-neutral voltage: $\mathbf{V}_{AN} = \mathbf{I}_{aA} Z_A$.

The 3 phase currents of the load in abc seq.

- Can be solved by 3 node equations once the 3 line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, $\mathbf{I}_{cC}$ are known:

$$\mathbf{I}_{aA} = \mathbf{I}_{AB} - \mathbf{I}_{CA},\ \mathbf{I}_{bB} = \mathbf{I}_{BC} - \mathbf{I}_{AB},\ \mathbf{I}_{cC} = \mathbf{I}_{CA} - \mathbf{I}_{BC}.$$





Section 11.5
Power Calculations in Balanced Three-Phase Circuits

1. Complex powers of one-phase and the entire Y-Load
2. The total instantaneous power

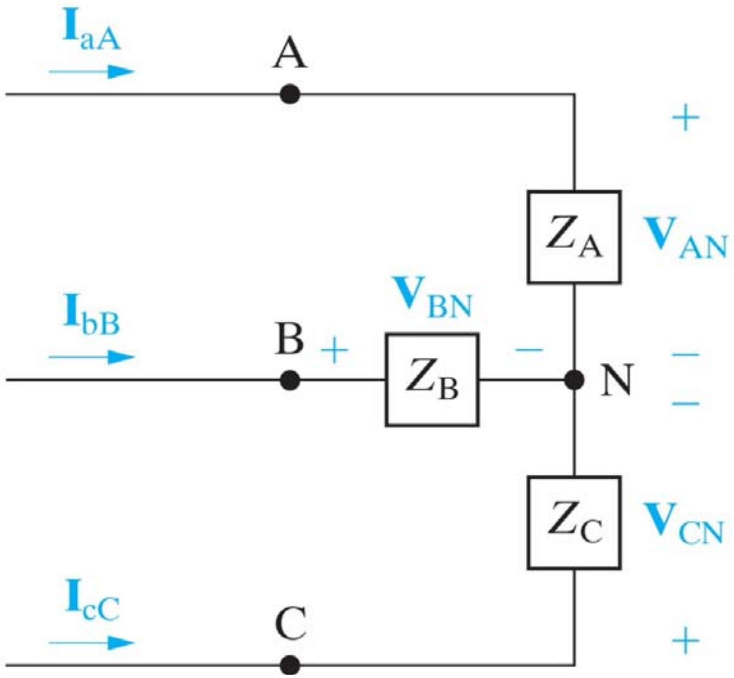
Average power of balanced Y-Load

- The average power delivered to Z_A is:

$$P_A = V_\phi I_\phi \cos\theta_\phi,$$

$$\begin{cases} V_\phi \equiv \left|\mathbf{V}_{AN}\right| = V_L/\sqrt{3}, \\ I_\phi \equiv \left|\mathbf{I}_{aA}\right| = I_L, \\ \theta_\phi \equiv \angle V_\phi - \angle I_\phi = \angle Z_A. \end{cases}$$

(rms value)



- The total power delivered to the Y-Load is:

$$P_{tot} = 3P_A = 3V_\phi I_\phi \cos\theta_\phi = \sqrt{3}V_L I_L \cos\theta_\phi.$$

Complex power of a balanced Y-Load

- The reactive powers of one phase and the entire Y-Load are:

$$\begin{cases} Q_\phi = V_\phi I_\phi \sin\theta_\phi, \\ Q_{tot} = 3V_\phi I_\phi \sin\theta_\phi = \sqrt{3} V_L I_L \sin\theta_\phi. \end{cases}$$

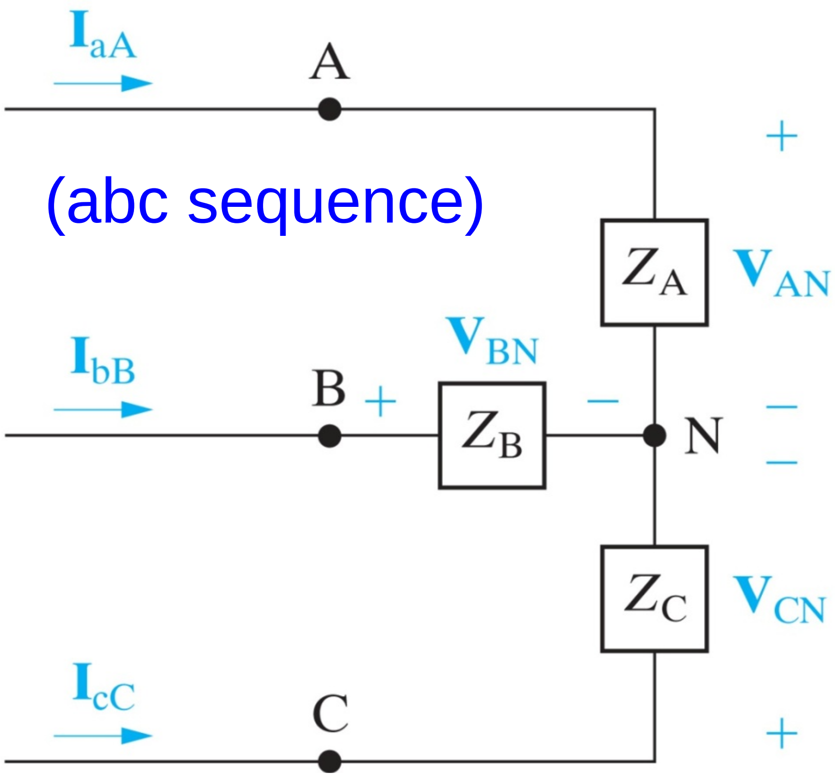
- The complex powers of one phase and the entire Y-Load are:

$$\begin{cases} S_\phi = P_\phi + jQ_\phi = V_\phi I_\phi e^{j\theta_\phi} = \mathbf{V}_\phi \mathbf{I}_\phi^*; \\ S_{tot} = 3S_\phi = 3V_\phi I_\phi e^{j\theta_\phi} = \sqrt{3} V_L I_L e^{j\theta_\phi}. \end{cases}$$

One-phase instantaneous powers

- The instantaneous power of load Z_A is:

$$p_A(t) = v_{AN}(t) i_{aA}(t) = V_m I_m \cos \omega t \cos(\omega t - \theta_\phi).$$



- The instantaneous powers of Z_A, Z_C are:

$$p_B(t) = v_{BN}(t) i_{bB}(t) = V_m I_m \cos(\omega t - 120^\circ) \cos(\omega t - \theta_\phi - 120^\circ),$$

$$p_C(t) = V_m I_m \cos(\omega t + 120^\circ) \cos(\omega t - \theta_\phi + 120^\circ).$$

Total instantaneous power

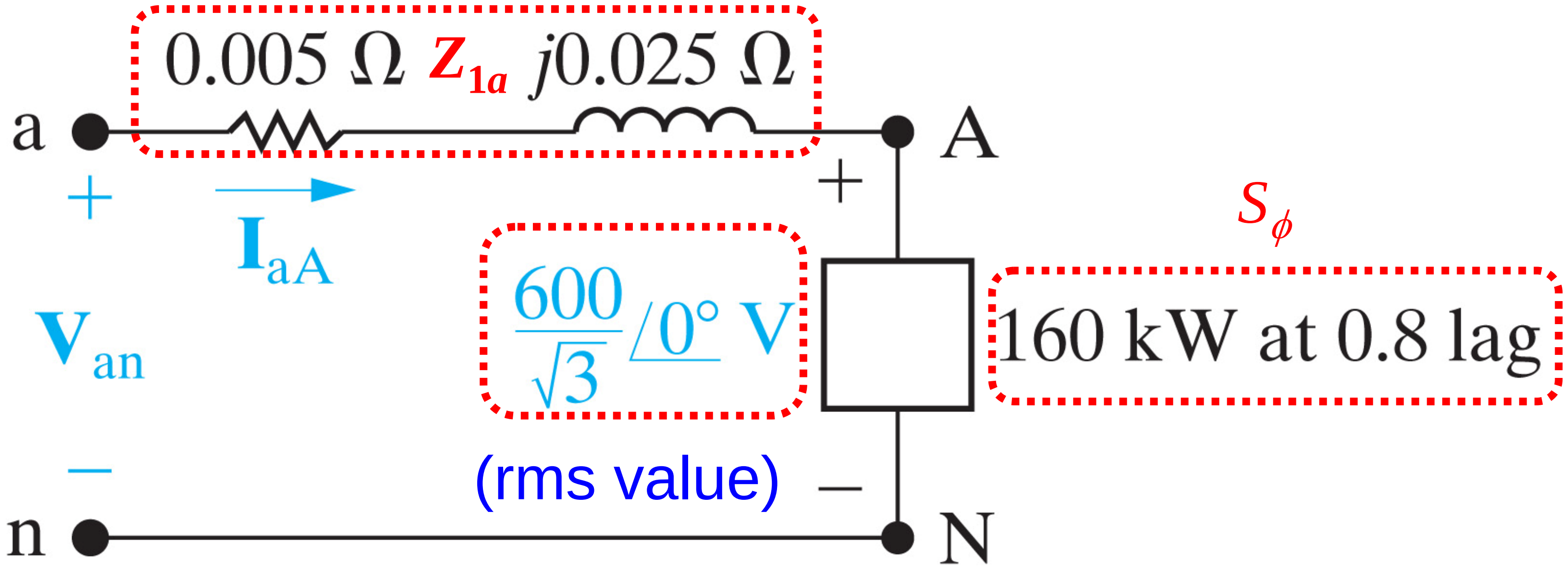
- The instantaneous power of the entire Y-Load is a constant independent of time!

$$p_{tot}(t) = p_A(t) + p_B(t) + p_C(t) = 1.5 V_m I_m \cos\theta_\phi$$
$$= 1.5\left(\sqrt{2}V_\phi\right)\left(\sqrt{2}I_\phi\right)\cos\theta_\phi = 3V_\phi I_\phi \cos\theta_\phi .$$

- The torque developed at the shaft of a 3-phase motor is constant, $\Rightarrow$ less vibration in machinery powered by 3-phase motors.
- The torque required to empower a 3-phase generator is constant, $\Rightarrow$ need steady input.

Example 11.5 (1)

- Q: What are the complex powers provided by the source and dissipated by the line of a-phase?
- The equivalent one-phase circuit in Y-Y configuration is:



Example 11.5 (2)

- The line current of a-phase can be calculated by the complex power is:

$$S_\phi = \mathbf{V}_\phi \mathbf{I}_\phi^*,\ \left(160 + j120\right)10^3 = \frac{600}{\sqrt{3}} \mathbf{I}_{aA}^*,$$

$$\Rightarrow \mathbf{I}_{aA} = \left(577.35\angle -36.87^\circ\right) \text{A}.$$

- The a-phase voltage of the source is:

$$\mathbf{V}_{an} = \mathbf{V}_{AN} + \mathbf{I}_{aA} Z_{1a}$$

$$= 600/\sqrt{3} + \left(577.35\angle -36.87^\circ\right)\left(0.005 + j0.025\right)$$

$$= \left(357.51\angle 1.57^\circ\right) \text{V}.$$

Example 11.5 (3)

- The complex power provided by the source of a-phase is:

$$S_{an} = \mathbf{V}_{an}\mathbf{I}_{aA}^{*} = \left(357.51\angle 1.57^\circ\right)\left(577.35\angle 36.87^\circ\right)$$
$$= \left(206.41\angle 38.44^\circ\right)\text{kVA}.$$

- The complex power dissipated by the line of a-phase is:

$$S_{aA} = \left|\mathbf{I}_{aA}\right|^2 Z_{1a} = \left(577.35\right)^2\left(0.005 + j0.025\right)$$
$$= \left(8.50\angle 78.66^\circ\right)\text{kVA}.$$

Key points

- What is a three-phase circuit (source, line, load)?
- Why a balanced three-phase circuit can be analyzed by an equivalent one-phase circuit?
- How to get all the unknowns (e.g. line voltage of the load) by the result of one-phase circuit analysis?
- Why the total instantaneous power of a balanced three-phase circuit is a constant?