Chapter 10
Sinusoidal Steady–State Power Calculations

Overview

- Nearly all electric energy is supplied in the form of sinusoidal voltages and currents (i.e. AC, alternating currents), because

1. Generators generate AC naturally.
2. Transformers must operate with AC.
3. Transmission relies on AC.
4. It is expensive to transform from DC to AC.

Key points

- How to decompose a sinusoidal instantaneous power into the average power and reactive power components? What are the meanings?
- How to decompose a sinusoidal instantaneous power into the in-phase and quadrature components?
- Why and how to do power factor correction?
- For a specific circuit, how to maximize the average power delivered to a load?

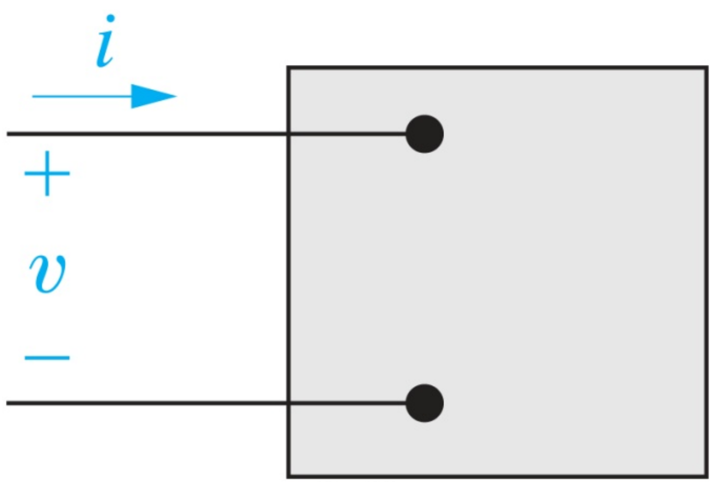
Section 10.1
Instantaneous Power

Definition

- "Instantaneous" power is the product of the instantaneous terminal voltage and current, or

$$p(t) = \pm v(t) \cdot i(t).$$

- Positive sign is used if the passive sign convention is satisfied (current is in the direction of voltage drop).



Sinusoidal power formula

Absolute timing is unimportant

$$\begin{cases} v(t) = V_m \cos(\omega t + \theta_v), \\ i(t) = I_m \cos(\omega t + \theta_i), \end{cases} \Rightarrow \begin{cases} v(t) = V_m \cos(\omega t + \phi), \\ i(t) = I_m \cos(\omega t), \\ \phi = \theta_v - \theta_i; \end{cases}$$

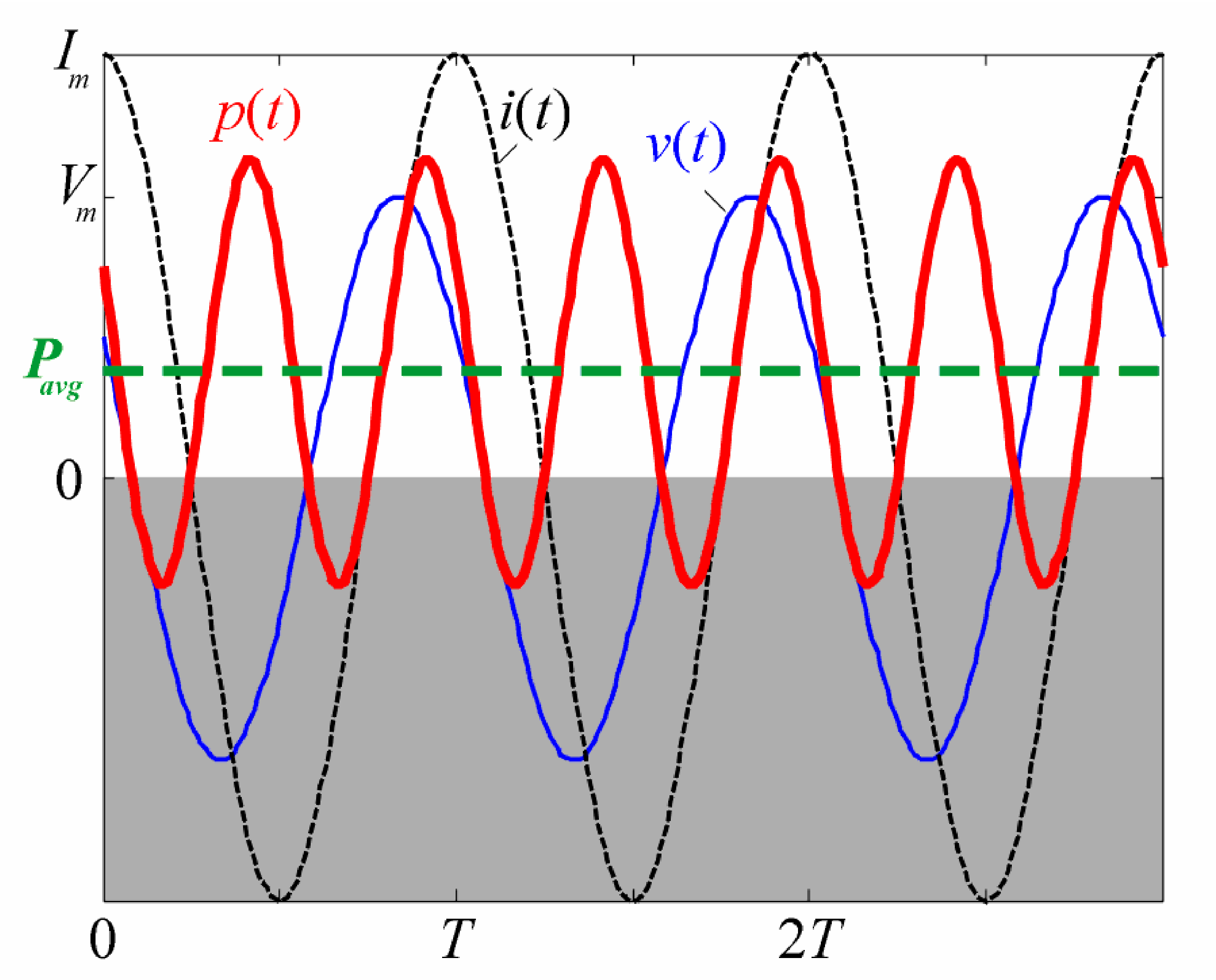
$$\text{By } \cos\alpha \cos\beta = \frac{\cos(\alpha - \beta)}{2} + \frac{\cos(\alpha + \beta)}{2},$$

$$\Rightarrow p(t) = V_m I_m \cos(\omega t + \phi)\cos(\omega t)$$

$$= \frac{V_m I_m}{2}\cos\phi + \frac{V_m I_m}{2}\cos(2\omega t + \phi).$$

Constant, P_{avg} Oscillating at frequency 2ω

Relation among $i(t)$, $v(t)$, $p(t)$



Section 10.2
Average and Reactive Power

1. Decompose the instantaneous power in different ways
2. Instantaneous powers of resistive, inductive, and capacitive loads
3. Power factor and reactive factor

Definitions

$$p(t) = \frac{V_m I_m}{2}\cos\phi + \frac{V_m I_m}{2}\cos(2\omega t + \phi)$$

$$= \frac{V_m I_m}{2}\left(\cos\phi + \cos 2\omega t \cos\phi - \sin 2\omega t \sin\phi\right)$$

$$= P\left(1 + \cos 2\omega t\right) - Q\sin 2\omega t,$$

In-phase quadrature

where

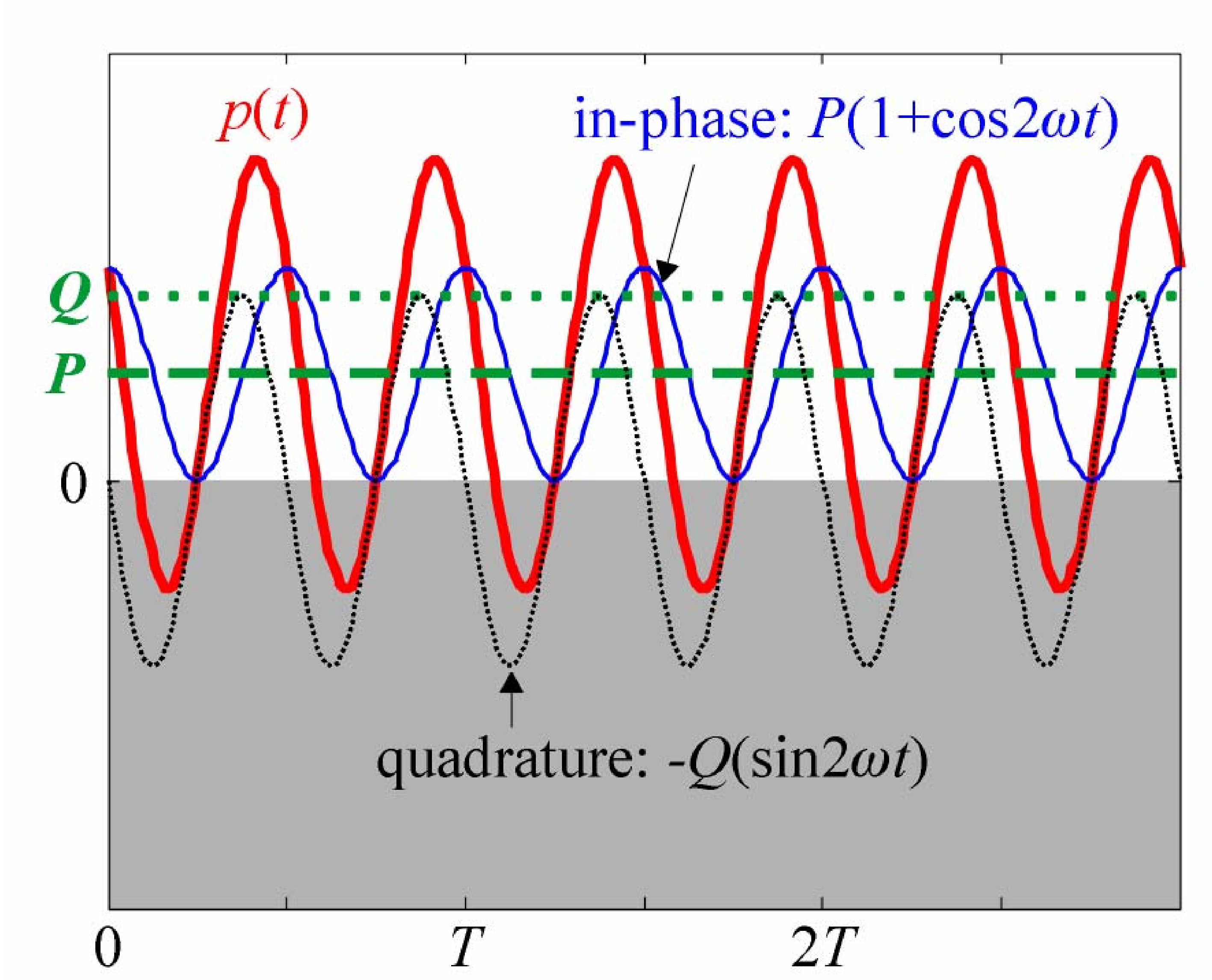
$$P = \frac{V_m I_m}{2}\cos\phi,$$

Average power (watt, or W)

$$Q = \frac{V_m I_m}{2}\sin\phi.$$

Reactive power (volt-ampere reactive, or VAR)

Relation among components of power



What is average power ?

- The power transformed from electric to non-electric energy or vice versa.
- The average of instantaneous power $p(t)$.
- The average of $P(1+\cos 2\omega t)$, which is a power component in-phase with the current $i(t)$.
- The circuit dissipates (delivers) electric energy if $P>0$ ($P<0$).

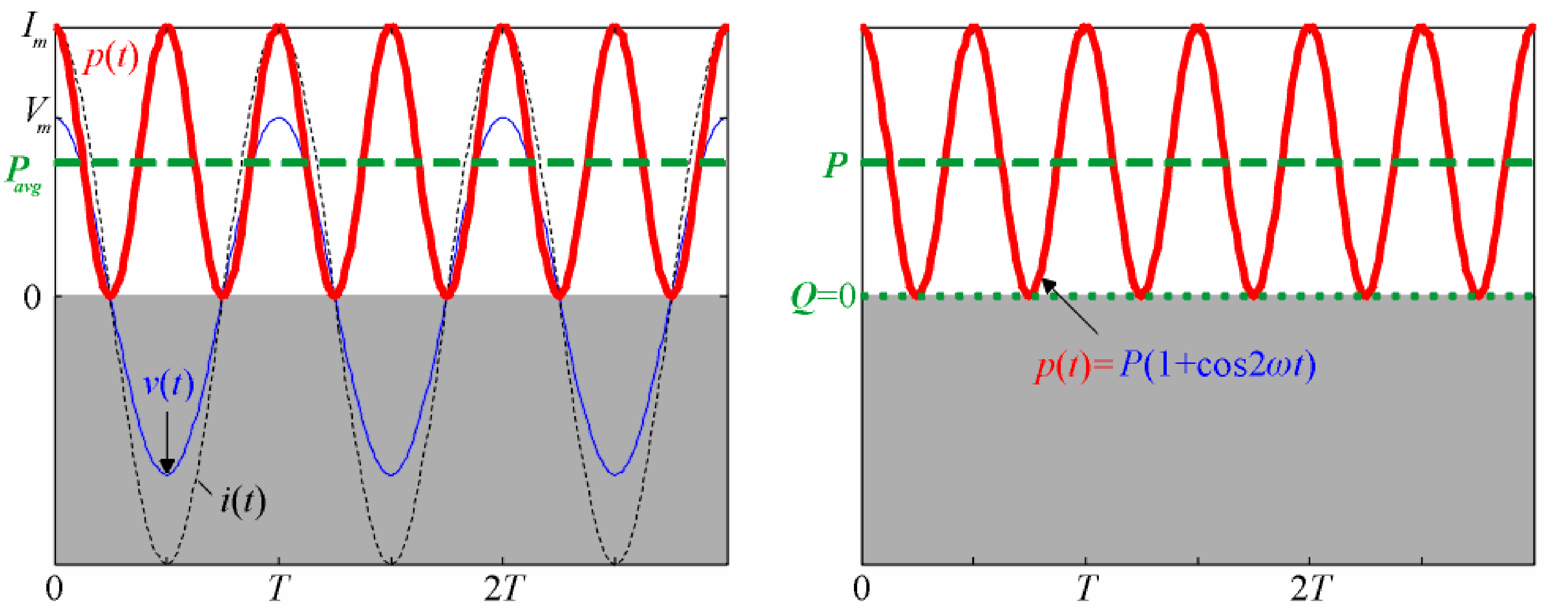
What is reactive power ?

- The power exchanged among (1) the magnetic field in an inductor, (2) the electric field in a capacitor, and (3) the electric source.
- The magnitude of $-Q(\sin 2\omega t)$ (while its average equals zero), which is a power component in quadrature with $i(t)$ (leading or lagging $i(t)$ by 90° or $T/4$ in time).
- Reactive power cannot do work.

Power for resistive loads

$$v(t) = Ri(t), \Rightarrow \phi = 0, \; v(t) \text{ and } i(t) \text{ are } \underline{\text{in - phase}}.$$

$$p(t) = \frac{V_m I_m}{2} + \frac{V_m I_m}{2}\cos 2\omega t = \frac{V_m I_m}{2}\left(1 + \cos 2\omega t\right) + 0.$$

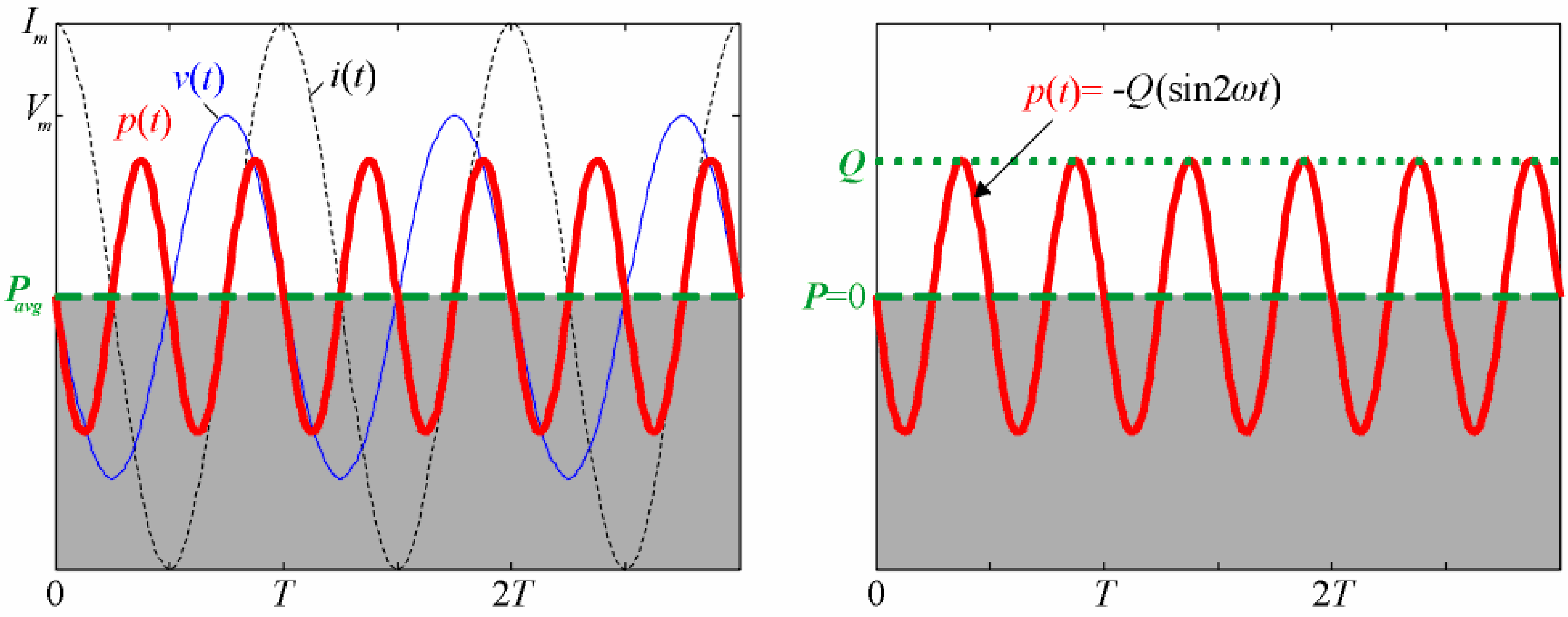


- $p(t) > 0$ at all times, $\Rightarrow$ P is maximized, $Q = 0$.

Power for inductive loads

$$v(t) = Li'(t), \Rightarrow \phi = 90^\circ, \ \underline{i(t) \text{ lags } v(t)} \text{ by } T/4.$$

$$p(t) = 0 + \frac{V_m I_m}{2}\cos(2\omega t + 90^\circ) = 0 - \frac{V_m I_m}{2}\sin 2\omega t.$$

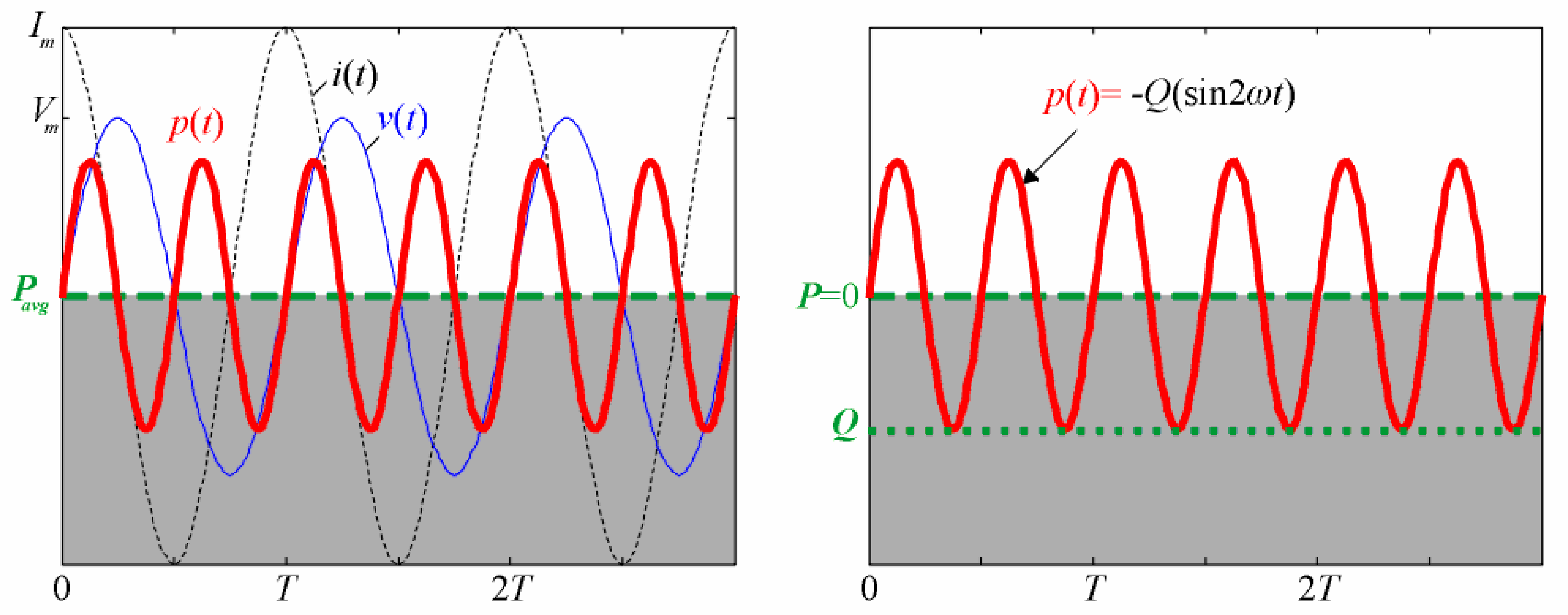


- $p(t)$ is halved by 0-level, $P=0$, $Q>0$ is maximized.

Power for capacitive loads

$$i(t) = Cv'(t), \Rightarrow \phi = -90^\circ, \underline{i(t) \text{ leads } v(t)} \text{ by } T/4.$$

$$p(t) = 0 + \frac{V_m I_m}{2}\cos(2\omega t - 90^\circ) = 0 + \frac{V_m I_m}{2}\sin 2\omega t.$$



- $p(t)$ is halved by 0-level, $\Rightarrow$ $P=0$, $Q<0$.

Power factor & reactive factor

- The above examples show that the relative phase ϕ between $v(t)$ and $i(t)$ determines whether the electric power is delivered to the load or simply exchanges between EM fields.
- Power factor and reactive factor, defined as:

$$\mathbf{pf} \equiv \cos\phi, \quad \mathbf{rf} = \sin\phi,$$

quantitatively describe the impact of ϕ on power delivery.

1. Lagging pf: inductive, $Q > 0$, $0 < \phi < 180°$;
2. Leading pf: capacitive, $Q < 0$, $-180° < \phi < 0$.

Section 10.3
The rms Value and Power Calculations

Definition of root-mean-square (rms) value

- The rms value of any periodic (not necessarily sinusoidal) function $y(t)$ of period T is the "square root of the mean value of the squared function" (Section 9.1).

$$Y_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} y^2(t)dt}$$

- The function square makes it suitable in describing the concept of power.

rms value of sinusoidal functions

- Consider a sinusoidal voltage: $v(t) = V_m \cos(\omega t + \phi)$

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) dt}$$

$$= \sqrt{\frac{V_m^2}{T}\int_{t_0}^{t_0+T} \frac{1+\cos(2\omega t + 2\phi)}{2} dt} = \sqrt{\frac{V_m^2}{T}\frac{T}{2}} = \frac{V_m}{\sqrt{2}}$$

- rms value is $1/\sqrt{2}$ times of the amplitude, independent of frequency ω and phase ϕ.
- The ratio of rms value to the function amplitude changes with the functional shape.

Power formulas in terms of rms values

- The average power P and reactive power Q due to sinusoidal voltage and current are:

$$P = \frac{V_m I_m}{2}\cos\phi = V_{rms} I_{rms} \cos\phi,$$

$$Q = \frac{V_m I_m}{2}\sin\phi = V_{rms} I_{rms} \sin\phi.$$

- A sinusoidal voltage source of rms value V_{rms} and a dc voltage source of constant voltage V_s deliver the same average power to a load resistance R if $V_{rms} = V_s$.

rms values in daily life

- Voltage rating of residential electric wiring 220 V/110 V are given in terms of rms values.
- E.g. A lamp rated by {120 V, 100 W} has:

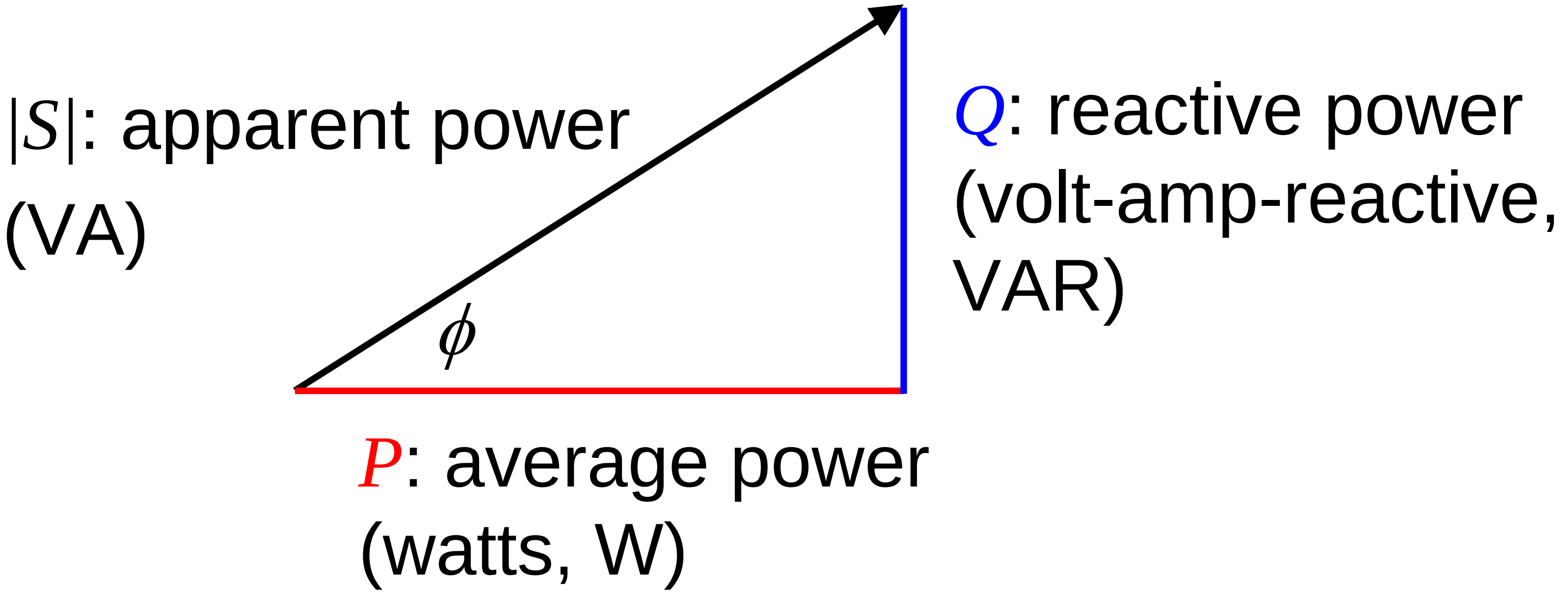
1. resistance $R = (V_{rms})^2/P = 144\ \Omega$;
2. rms current $I_{rms} = V_{rms}/R = 0.83$ A;
3. peak current $I_m = \sqrt{2}I_{rms} = 1.18$ A, which is critical in safety.

Section 10.4
Complex Power

Definition

- The complex power S (volt-amps, VA) is:

$$S = P + jQ$$



Example 10.4

- Q: An electric load has $V_{rms} = 240$ V, $P = 8$ kW, $pf =$ 0.8 (lagging); $\Rightarrow$ (1) $S = ?$ (2) $I_{rms} = ?$ (3) $Z_L = ?$

(1) Lagging pf, $Q > 0$, (2) $P > 0$, $\Rightarrow 0 < \phi < 90°$; (3) $\cos\phi = 0.8$; $\phi = 36.87°$.

$P = |S| \times \cos\phi$, 8 kW $= |S| \times (0.8)$, $\Rightarrow |S| = 10$ kVA.

$S = P + jQ = |S|\angle\phi = 10\angle 36.87° = (8 + j6)$ kVA.

$P = V_{rms} I_{rms} \cos\phi$, $I_{rms} = (8 \text{ kW})/[(240 \text{ V})(0.8)] = 41.67$ A.

$Z_L = \mathbf{V}_L/\mathbf{I}_L = (V_{rms}/I_{rms})\angle\phi = (5.76\angle 36.87°)\ \Omega$.

Section 10.5
Power Calculations

1. Complex powers in a circuit
2. Power factor correction

Power calculations by voltage & current phasors

$$S = P + jQ = \frac{V_m I_m}{2}\cos\phi + j\frac{V_m I_m}{2}\sin\phi$$

$$= \frac{V_m I_m}{2}(\cos\phi + j\sin\phi) = \frac{V_m I_m}{2}\angle\phi = \frac{1}{2}\mathbf{V}\mathbf{I}^*,$$

where $\mathbf{V} = V_m \angle \theta_v$, $\mathbf{I} = I_m \angle \theta_i$, $\phi = \theta_v - \theta_i$.

Beides, $S = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$,

where $\mathbf{V}_{rms} = \frac{V_m}{\sqrt{2}}\angle\theta_v$, $\mathbf{I}_{rms} = \frac{I_m}{\sqrt{2}}\angle\theta_i$.

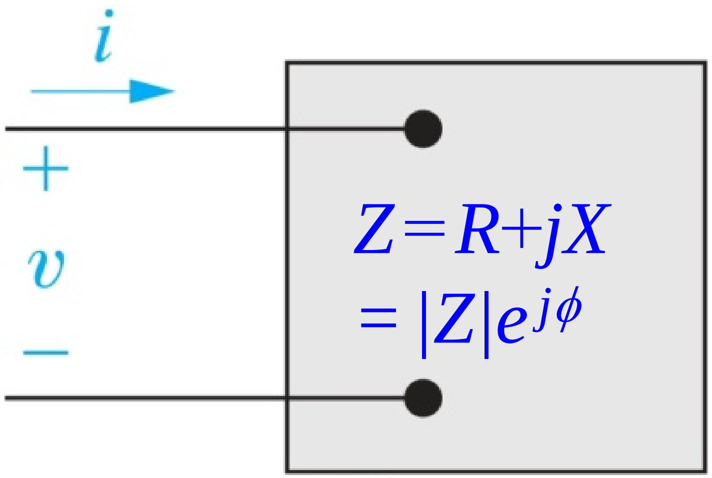
Power calculations by impedance

$$\because \mathbf{V}_{rms} = \mathbf{I}_{rms} Z, \quad Z = R + jX,$$

$$\Rightarrow S = (\mathbf{I}_{rms} Z)\mathbf{I}_{rms}^{*} = |\mathbf{I}_{rms}|^{2} Z$$

$$= I_{rms}^{2}(R + jX) = P + jQ,$$

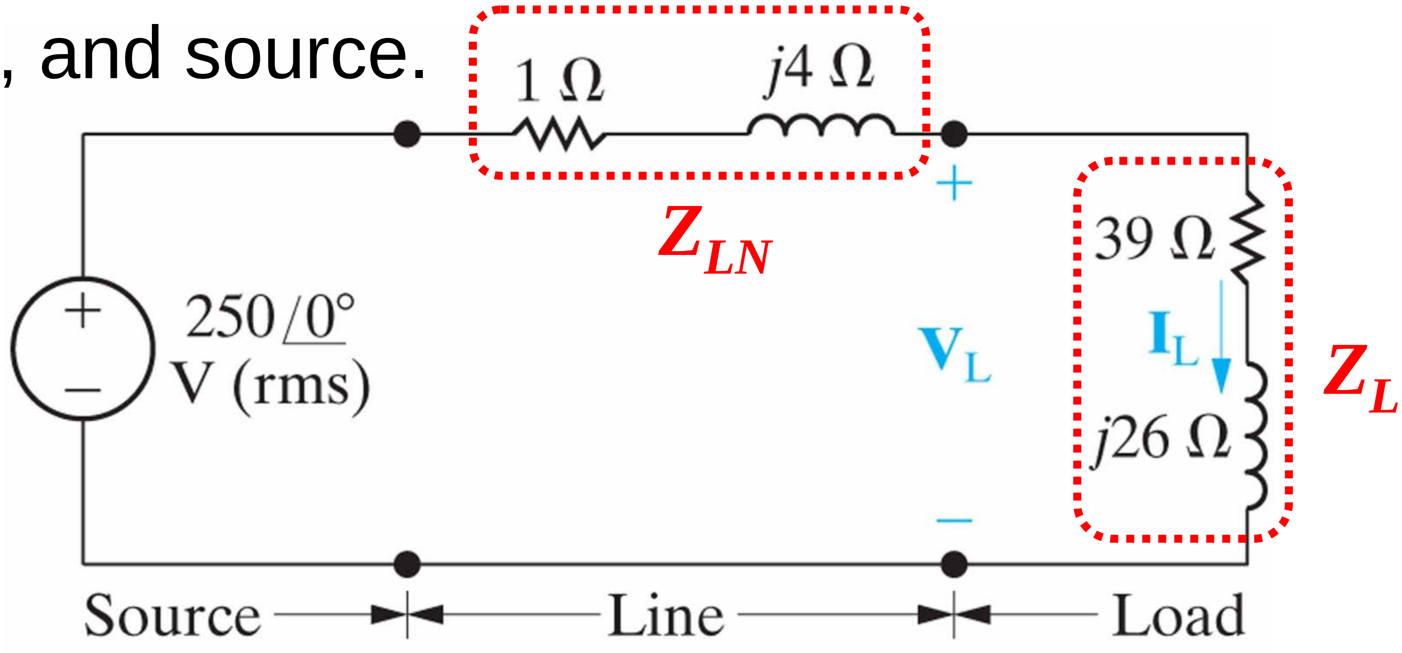
$$\Rightarrow P = I_{rms}^{2} R, \quad Q = I_{rms}^{2} X.$$



- A power consumer has to suppress its load reactance (power factor correction, making $X \approx 0$) such that a smaller apparent power $|S|$ is sufficient to deliver the specified average power.

Example 10.5 (1)

- Q: The complex powers delivered to the load, line, and source.



$$\mathbf{I}_L = \frac{\mathbf{V}_s}{Z_{LN} + Z_L} = \frac{250\angle 0^\circ}{40 + j30} = \left(5\angle -36.87^\circ\right) \text{A (rms)}.$$

$$\mathbf{V}_L = \mathbf{I}_L Z_L = \left(5\angle -36.87^\circ\right)\left(39 + j26\right) = \left(234.4\angle -3.18^\circ\right) \text{V (rms)}.$$

Example 10.5 (2)

$$S_L = \mathbf{V}_L \mathbf{I}_L^* = \left(234.4\angle -3.18^\circ\right)\left(5\angle +36.87^\circ\right) = \left(975 + j650\right)\text{VA}.$$

rms values

$$S_{LN} = \left|\mathbf{I}_L\right|^2 Z_{LN} = 5^2\left(1 + j4\right) = \left(25 + j100\right)\text{VA}.$$

$$S_s = -\mathbf{V}_s \mathbf{I}_L^* = -\left(250\angle 0^\circ\right)\left(5\angle +36.87^\circ\right) = \left(-1000 - j750\right)\text{VA}.$$

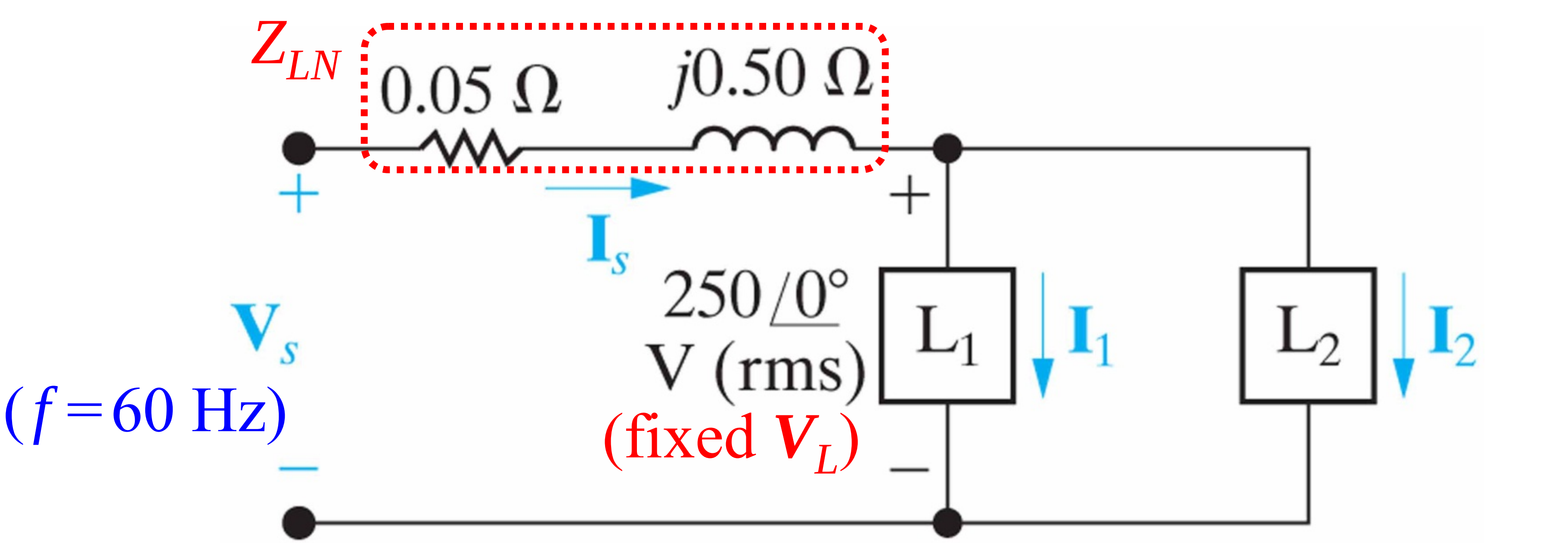
power delivered to the source,
passive sign convention

$$S_L + S_{LN} + S_s$$
$$= \left(975 + j650\right) + \left(25 + j100\right) + \left(-1000 - j750\right)\text{VA} = 0,$$

$\Rightarrow$ the complex powers are conserved.

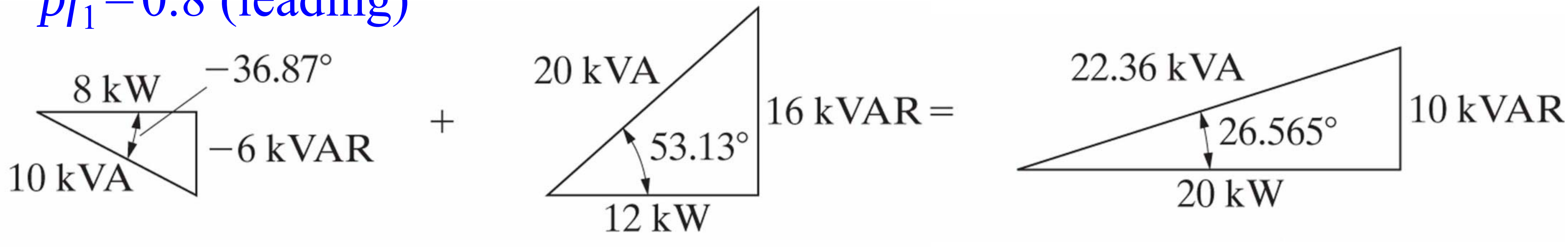
Example 10.6: PF correction (1)

- Q: Let $\{P_1 = 8\text{ kW}, pf_1 = 0.8\text{ (leading)}\}$, $\{|S_2| = 20\text{ kVA}, pf_2 = 0.6\text{ (lagging)}\}$. $\Rightarrow$ (1) How to make the total $pf = 1$? (2) What are the powers lost in the line P_{LN} before and after the pf correction?



Example 10.6: (2)

$pf_1 = 0.8$ (leading) $\quad pf_2 = 0.6$ (lagging) $\quad pf_{tot} = 0.89$ (lagging)



- To make $pf_{tot} = 1$, one has to add a parallel capacitance C that gives $\{P_C = 0,\ Q_C = \text{-}10\ \text{kVAR}\}$.

$$\because Z_C = jX_C,\ \Rightarrow Q_C = I_{rms}^2 X_C = \frac{V_{rms}^2}{X_C^2} X_C = \frac{V_{rms}^2}{X_C} = \frac{V_L^2}{X_C},$$

$$\Rightarrow X_C = \frac{V_L^2}{Q_C} = \frac{250^2}{-10\ \text{kVAR}} = -6.25\,\Omega,$$

$$X_C = -\frac{1}{\omega C},\ \Rightarrow C = -\frac{1}{\omega X_C} = -\frac{1}{2\pi(60)(-6.25)} = 424.4\ \mu\text{F}.$$

Example 10.6 (3)

- The average power lost in the line is:

$$P_{LN} = |\mathbf{I}_s|^2 R_{LN},$$

where the line current phasor (rms) is:

$$\mathbf{I}_s = (S_L / \mathbf{V}_L)^*, \ \because S_L = \mathbf{V}_L \mathbf{I}_s^*.$$

$$\Rightarrow P_{LN} = \frac{|S_L|^2}{|\mathbf{V}_L|^2} R_{LN} = \frac{|S_L|^2}{(250\ \mathrm{V})^2}(0.05\,\Omega)$$

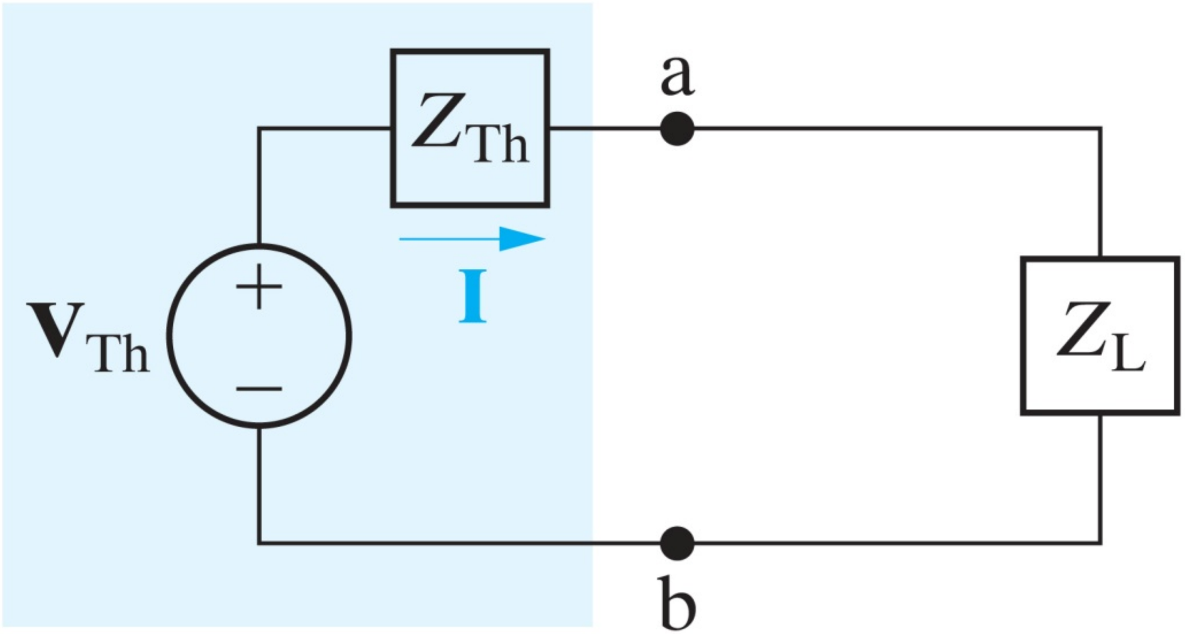
$$= \begin{cases} 400\ \mathrm{W},\ \text{before pf correction}, |S_L| = 22.36\ \mathrm{kVA}, \\ 320\ \mathrm{W},\ \text{after pf correction}, |S_L| = 20\ \mathrm{kVA}. \end{cases}$$

Section 10.6
Maximum Power Transfer

1. Unrestricted optimal load impedance
2. Restricted optimal load impedance

Conclusion

- For a general circuit with 2 output terminals a, b, the optimal load impedance that will consume the maximum average power is $Z_L = Z_{Th}^*$, where Z_{Th} is the Thévenin impedance of the circuit.



Proof (1)

- The rms current phasor **I** through the load is:

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{Z_{Th} + Z_L} = \frac{\mathbf{V}_{Th}}{(R_{Th} + R_L) + j(X_{Th} + X_L)},$$

$$\text{where } Z_{Th} \equiv R_{Th} + jX_{Th},\ Z_L \equiv R_L + jX_L.$$

- The average power P delivered to the load is:

$$P(R_L, X_L) = |\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}.$$

Proof (2)

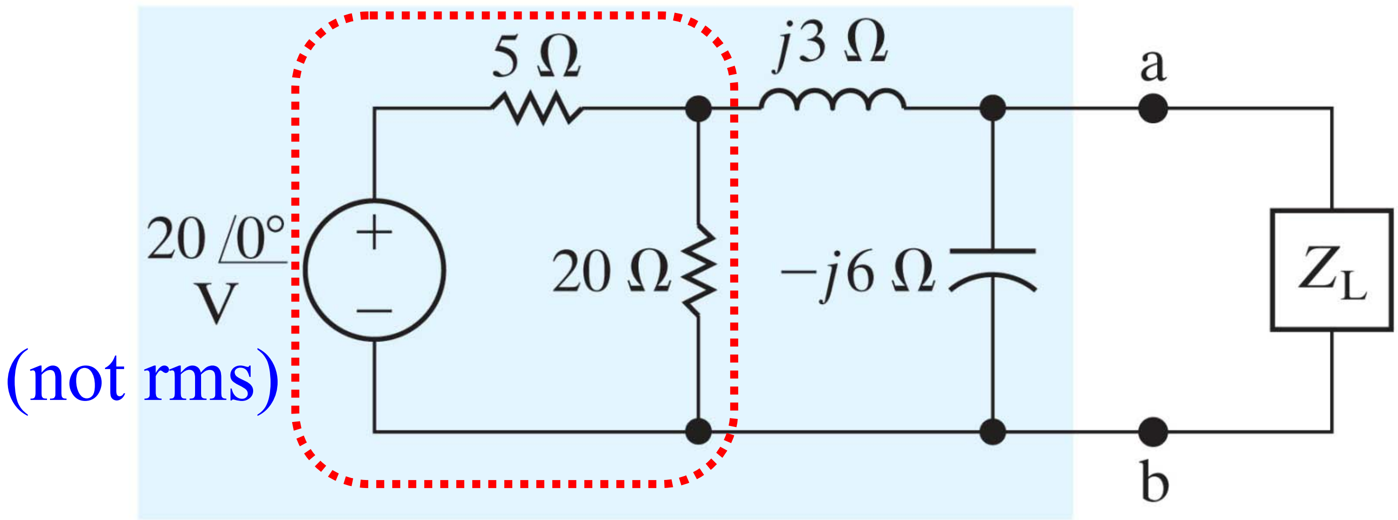
- The maximum average power occurs when the two partial derivatives are zero:

$$\begin{cases} \dfrac{\partial P}{\partial X_L} = \dfrac{-2|\mathbf{V}_{Th}|^2 R_L (X_{Th} + X_L)}{\left[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2\right]^2} = 0, \Rightarrow X_L = -X_{Th} \cdots (1) \\ \left(\left.\dfrac{\partial P}{\partial R_L}\right|_{X_L = -X_{Th}} \right) = \dfrac{|\mathbf{V}_{Th}|^2 (R_{Th}^2 - R_L^2)}{(R_{Th} + R_L)^4} = 0, \Rightarrow R_L = R_{Th} \cdots (2) \end{cases}$$

$$\Rightarrow Z_L = Z_{Th}^*.$$

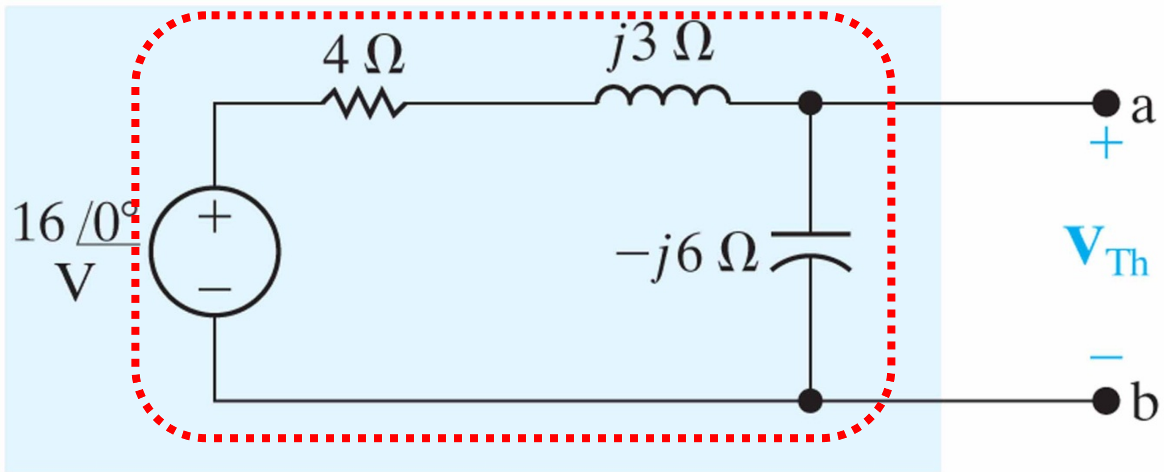
Example 10.8: Max power transfer w/n restriction (1)

- Q: Without restrictions on Z_L, determine (1) Z_L that results in the maximum average power $P_{\max}$ transferred to Z_L, (2) the value of $P_{\max}$.



- Apply source transformation to $\{20\angle 0°$ V, 5 Ω, 20 Ω$\}$, we got a simplified circuit:

Example 10.8 (2)

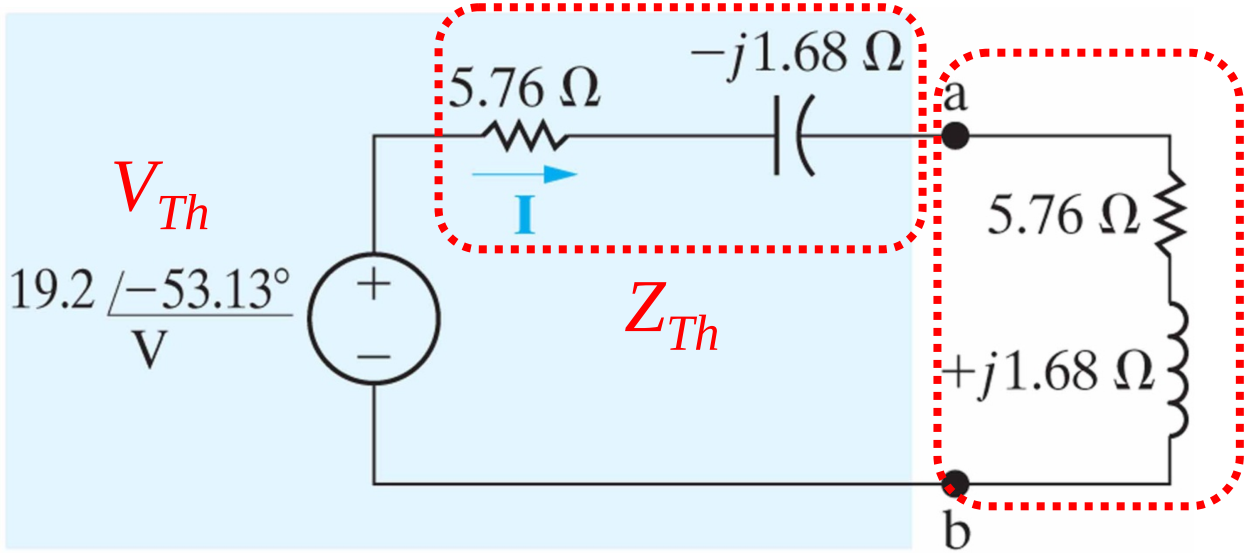


- Apply source transformation to {16∠0° V, 4+*j*3 Ω, -*j*6 Ω}, we got the Thévenin circuit:

$$Z_{Th} = (4 + j3)//(-j6) = (5.76 - j1.68)\Omega.$$

$$\mathbf{I}_{Nt} = \frac{16\angle 0^\circ}{4 + j3}\text{A}, \Rightarrow \mathbf{V}_{Th} = \mathbf{I}_{Nt} Z_{Th} = (19.2\angle -53.13^\circ)\text{V}.$$

Example 10.8 (3)



Max power occurs if $Z_L = (Z_{Th})^*$

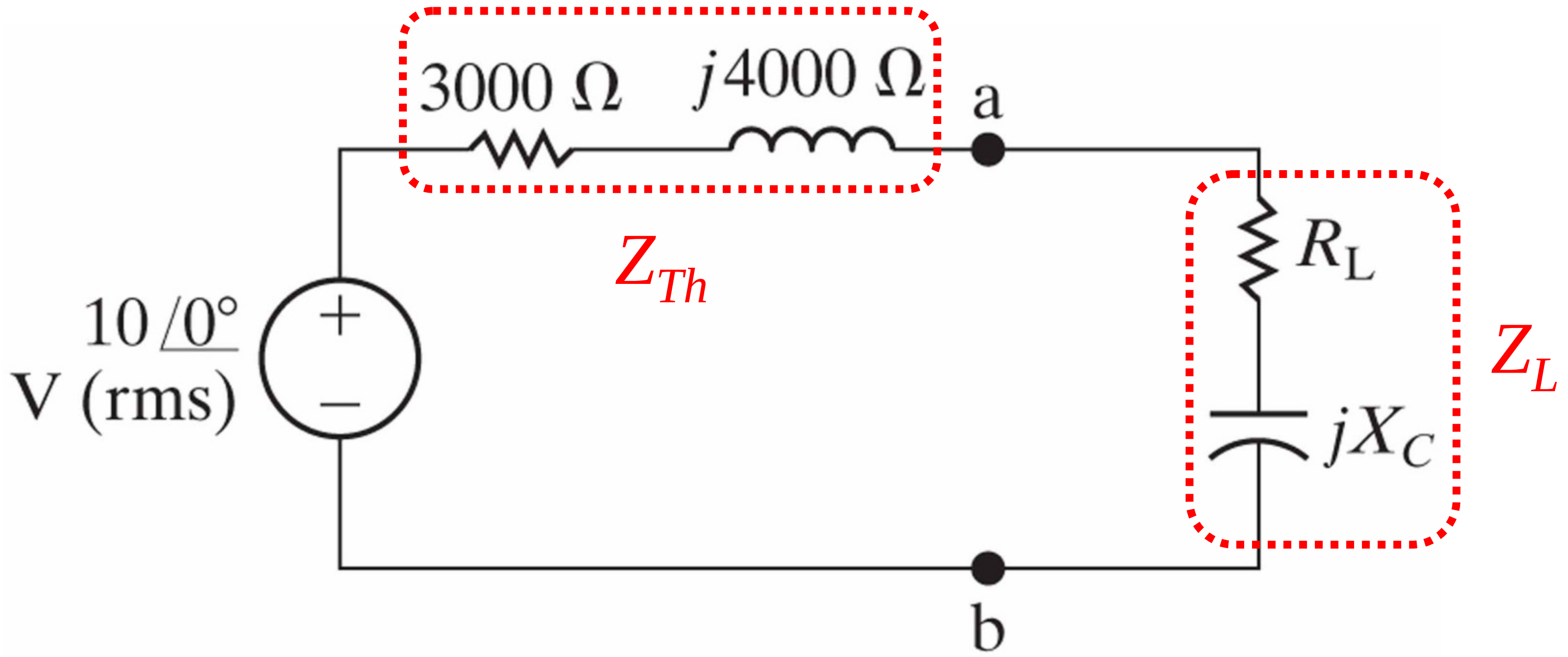
$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{Z_{Th} + Z_L} = \frac{19.2\angle -53.13^\circ}{2(5.76)} = \left(1.67\angle -53.13^\circ\right)\text{A},$$

$$\Rightarrow I_{rms} = |\mathbf{I}|/\sqrt{2} = 1.18\ \text{A},$$

$$\Rightarrow P_{\max} = I_{rms}^2 R_L = (1.18)^2(5.76) = 8\ \text{W}.$$

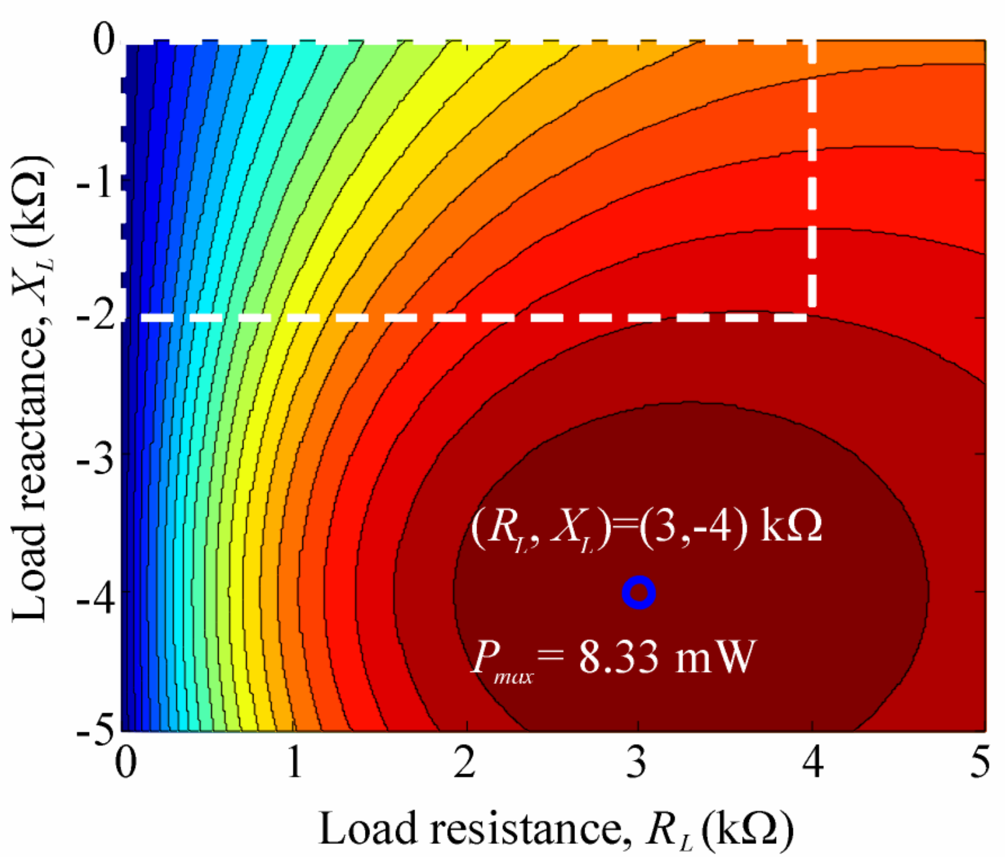
Example 10.9: Max power transfer with restriction (1)

- Q: What are the optimal load impedance Z_L that lead to the maximum average power if $0 \le R_L \le 4$ kΩ, -2 kΩ $\le X_L \le 0$ are required?



Example 10.9 (2)

- Without restriction $Z_L = (Z_{Th})^*$, $\Rightarrow R_L = R_{Th} = 3$ kΩ, $X_L = -X_{Th} = -4$ kΩ, respectively.



- The result can be verified by calculating average power P for possible combinations of (R_L, X_L).

Example 10.9 (3)

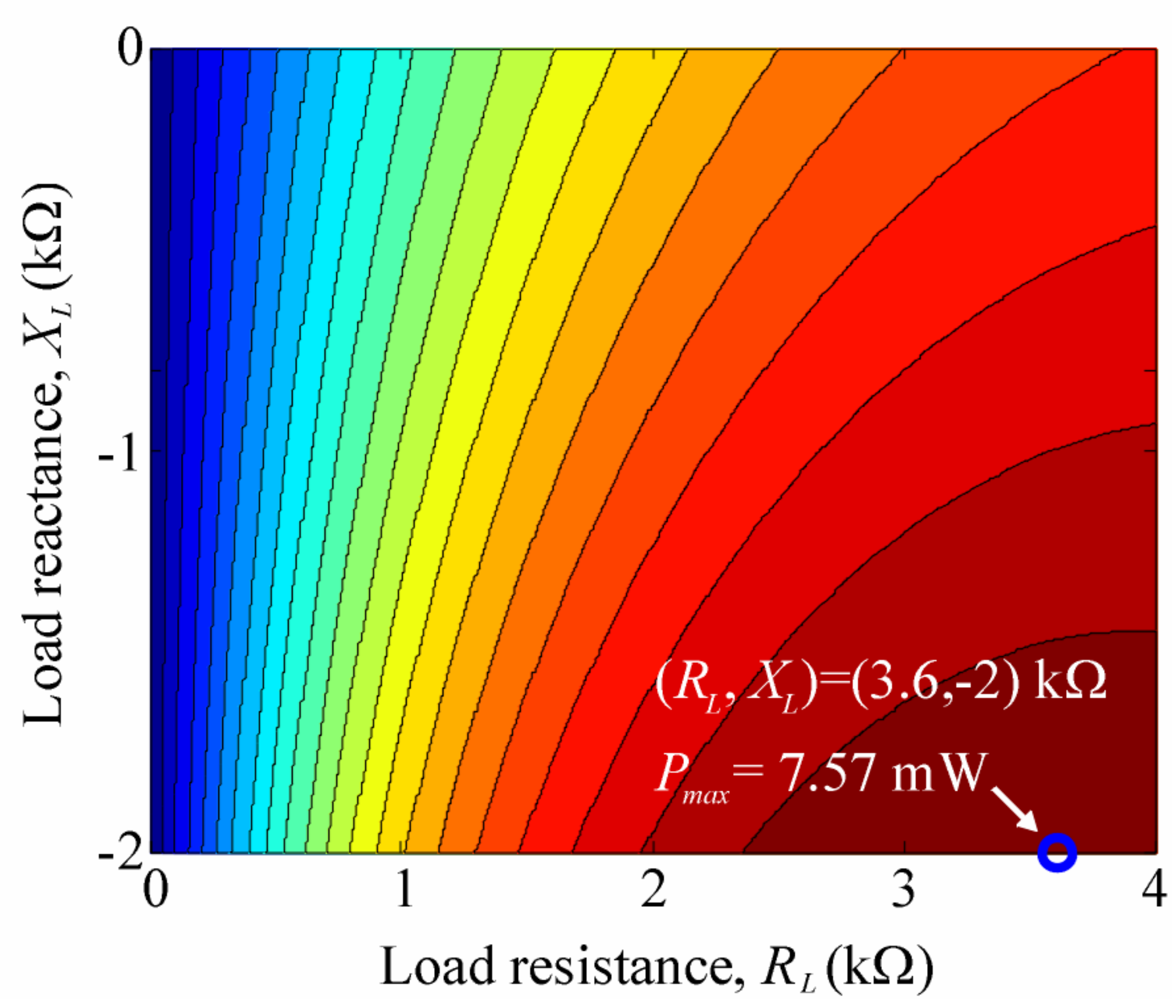
- Since $X_L = \text{-}4\ \mathrm{k\Omega}$ is outside the permitted range of $\text{-}2\ \mathrm{k\Omega} \le X_L \le 0$, one first sets X_L as close to the boundary as possible $\Rightarrow X_L = \text{-}2\ \mathrm{k\Omega}$.
- In this case, one has to modify the optimal R_L formula when $X_L \ne \text{-}X_{Th}$:

$$\left(\left. \frac{\partial P}{\partial R_L} \right|_{X_L \ne -X_{Th}} \right) = \frac{\left| \mathbf{V}_{Th} \right|^2 \left[R_{Th}^2 - R_L^2 + \left(X_{Th} + X_L \right)^2 \right]}{\left(R_{Th} + R_L \right)^4} = 0,$$

$$\Rightarrow R_L = \sqrt{R_{Th}^2 + \left(X_{Th} + X_L \right)^2} > R_{Th}.$$

Example 10.9 (4)

$$\Rightarrow R_L = \sqrt{R_{Th}^2 + \left(X_{Th} + X_L\right)^2} = \sqrt{\left(3\text{k}\right)^2 + \left(4\text{k} - 2\text{k}\right)^2} = 3.6\ \text{k}\Omega > R_{Th}.$$



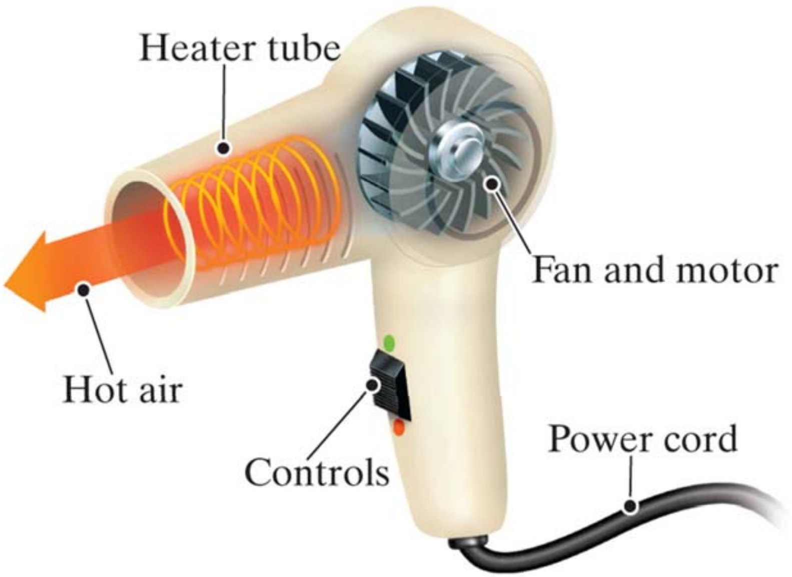
Key points

- How to decompose a sinusoidal instantaneous power into the average power and reactive power components? What are the meanings?
- How to decompose a sinusoidal instantaneous power into the in-phase and quadrature components?
- Why and how to do power factor correction?
- For a specific circuit, how to maximize the average power delivered to a load?

Practical Perspective
Hair Dryer

Physics

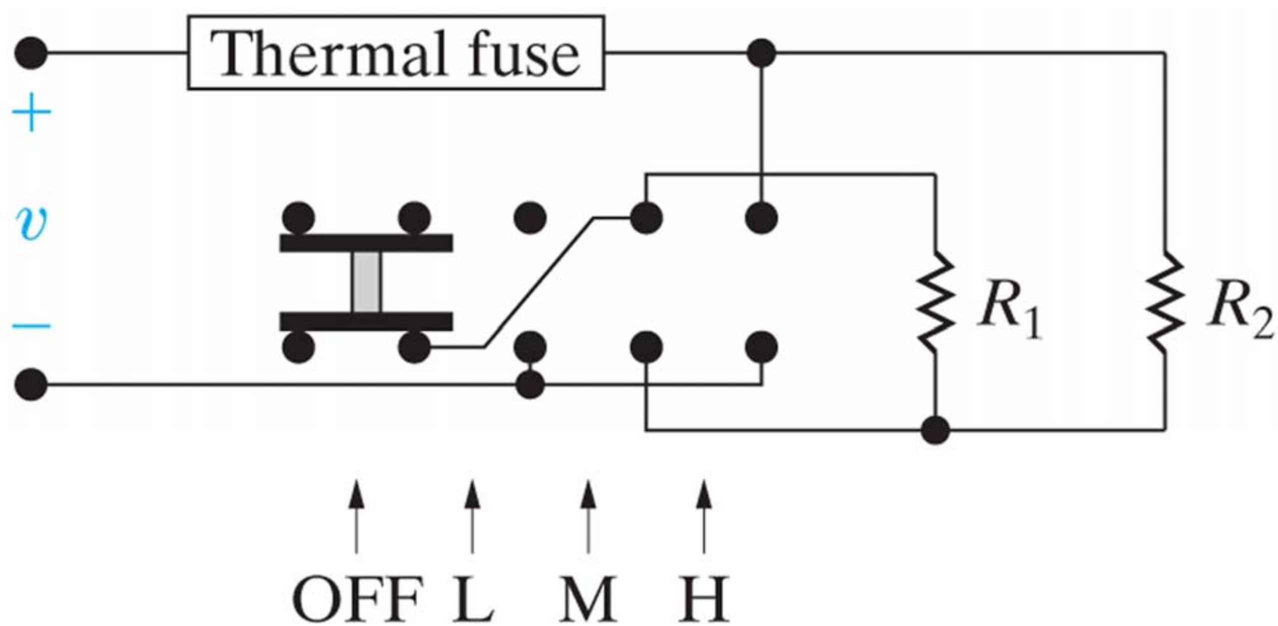
- A resistor heated by the sinusoidal current, and a fan that blows the warm air out.
- Heater tube is made of coiled nichrome (80% nickel, 20% chromium) wire, because of



- High resistivity: $\rho\,(=RA/L) \approx 10^{-6}\ \Omega\text{m}$, while other metals have $\rho \approx 10^{-8}\ \Omega\text{m}$.
- No oxidation when heated red hot (longer life time).

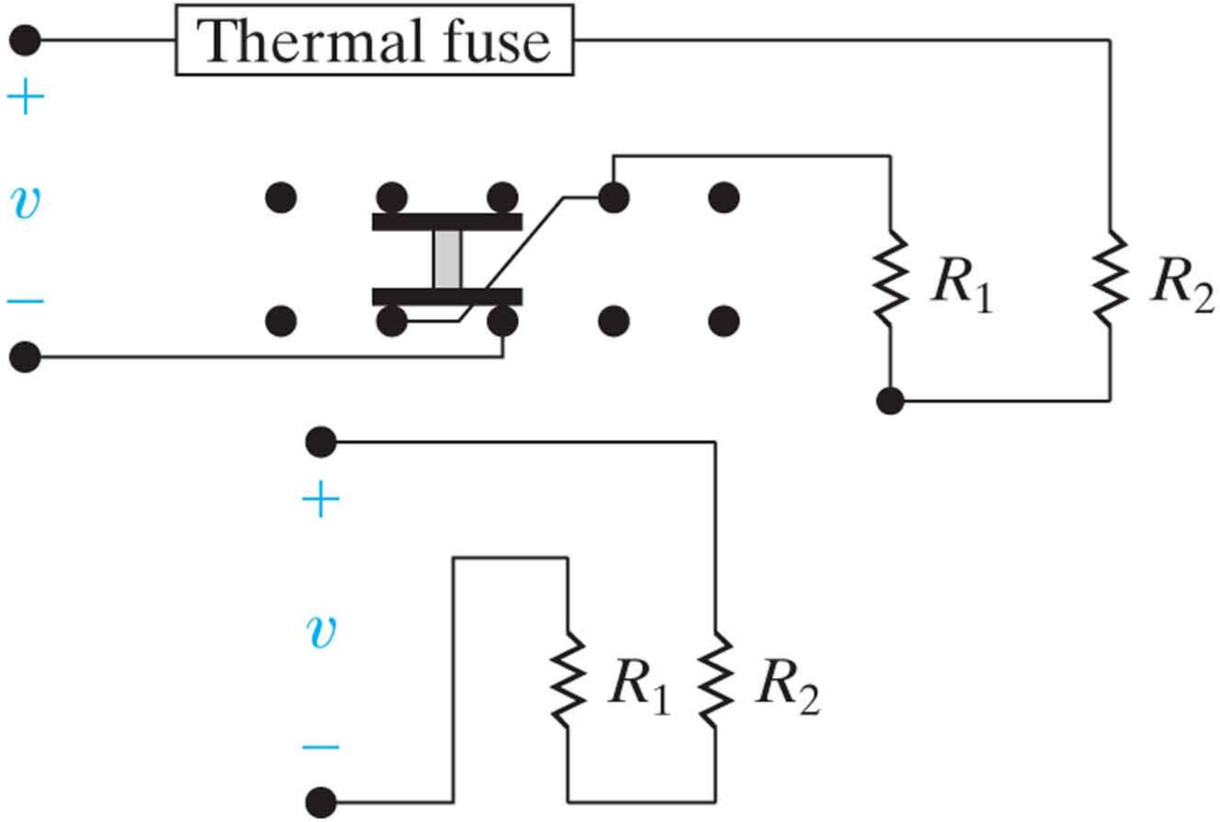
Circuit

- A connection partway divides the heater tube coil into two pieces.
- The position of a four-position switch controls the heat setting.



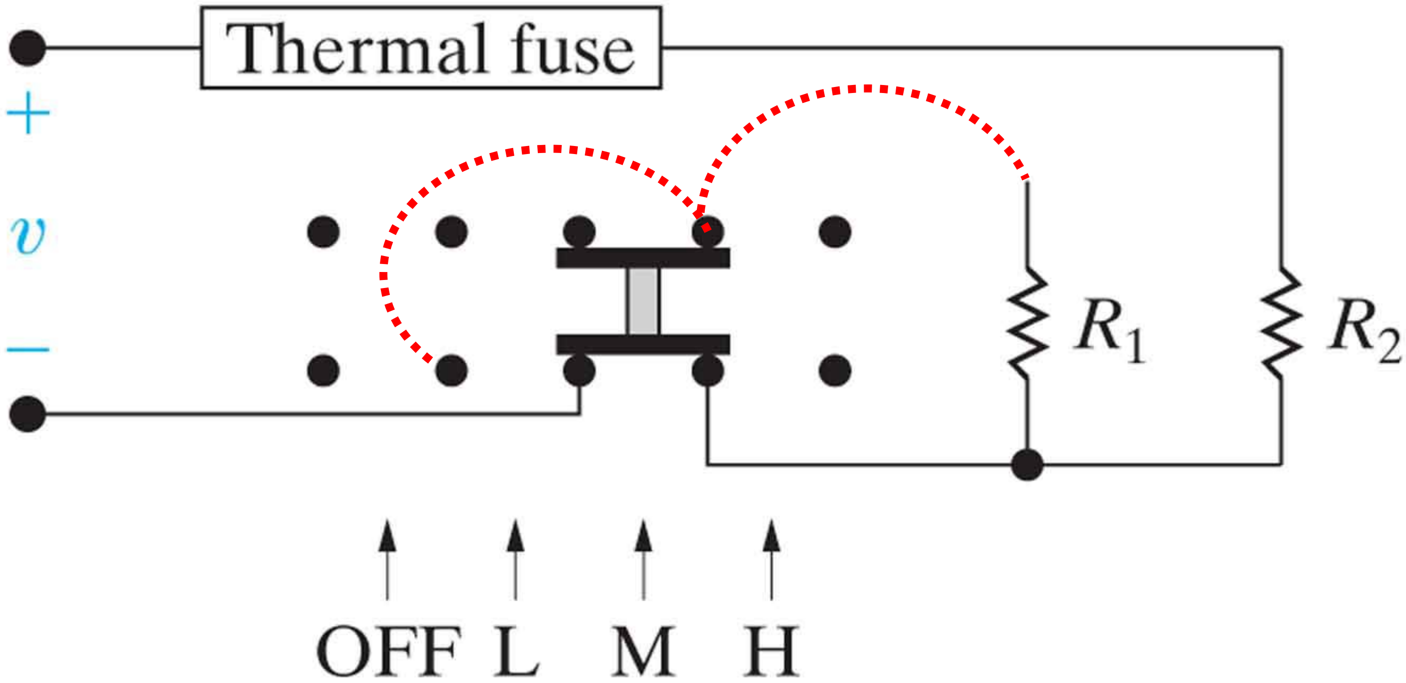
Low switch setting

- The four-position switch makes R_1 and R_2 in series, giving the lowest output power.



Medium switch setting

- The four-position switch makes R_1 in vain, giving the medium output power.



High switch setting

- The four-position switch makes R_1 and R_2 in parallel, giving the highest output power.