Chapter 6
Inductance, Capacitance, and Mutual Inductance

Overview

- In addition to voltage sources, current sources, resistors, here we will discuss the remaining 2 types of basic elements: inductors, capacitors.
- Inductors and capacitors cannot generate nor dissipate but store energy.
- Their current-voltage (i-v) relations involve with integral and derivative of time, thus more complicated than resistors.

Key points

- Why the i-v relation of an inductor is $v = L\dfrac{di}{dt}$?
- Why the i-v relation of a capacitor is $i = C\dfrac{dv}{dt}$?
- Why the energies stored in an inductor and a capacitor are:

$$w = \frac{1}{2}Li^2,\ \frac{1}{2}Cv^2,\ \text{respectively?}$$

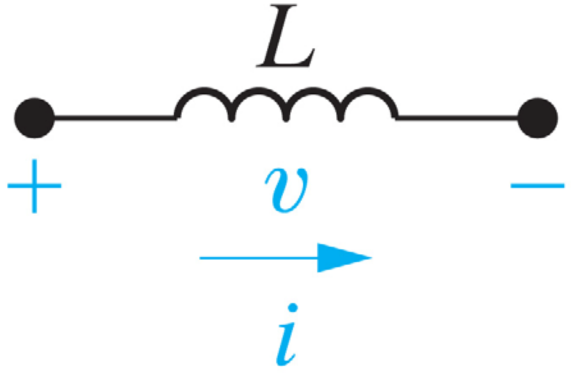
Section 6.1
The Inductor

1. Physics
2. i-v relation and behaviors
3. Power and energy

Fundamentals

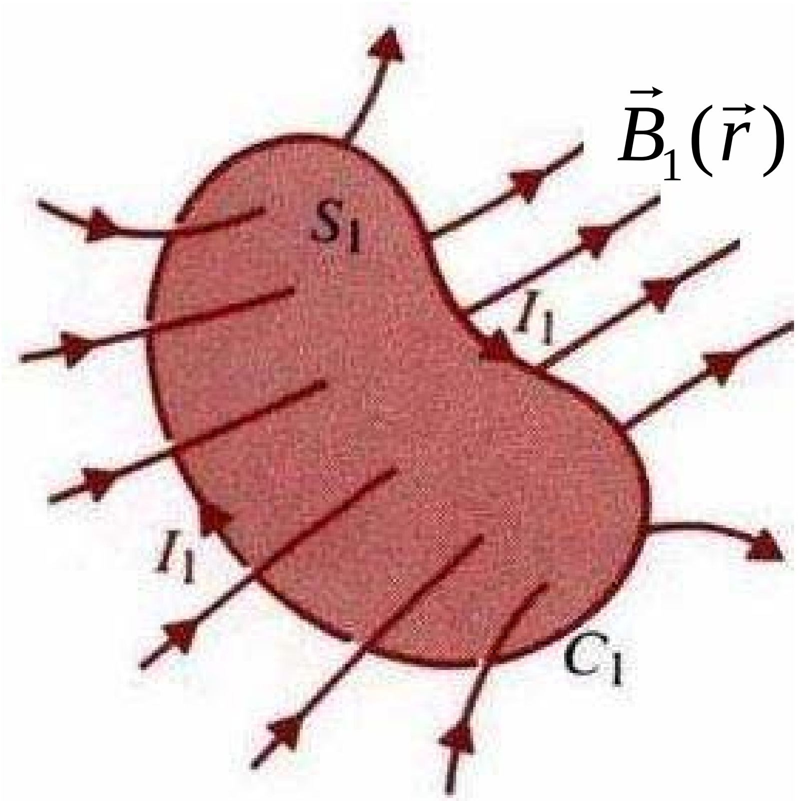
- An inductor of inductance L is symbolized by a solenoidal coil.
- Typical inductance L ranges from 10 μH to 10 mH.
- The i-v relation of an inductor (under the passive sign convention) is:

$$\boxed{v = L\frac{di}{dt},}$$



Physics of self-inductance (1)

- Consider an N_1-turn coil C_1 carrying current I_1. The resulting magnetic field $\vec{B}_1(\vec{r}) \propto N_1 I_1$ (Biot-Savart law) will pass through C_1 itself, causing a flux linkage λ_1, where



$$\lambda_1 = N_1 \phi_1,$$

$$\phi_1 = \int_{S_1} \vec{B}_1(\vec{r}) \cdot d\vec{s} = P_1 N_1 I_1,$$

P_1 is the permeance.

$$\Rightarrow \lambda_1 = P_1 N_1^2 I_1.$$

Physics of self-inductance (2)

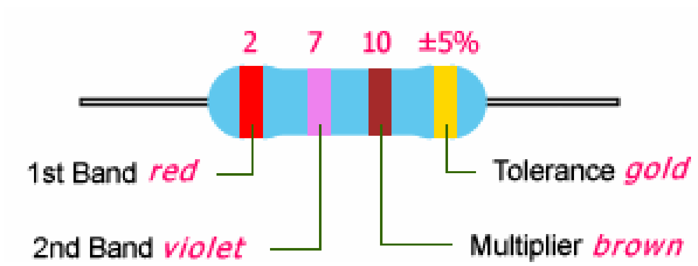
- The ratio of flux linkage to the driving current is defined as the self inductance of the loop:

$$\boxed{L_1 \equiv \frac{\lambda_1}{I_1}} = N_1^2 P_1,$$

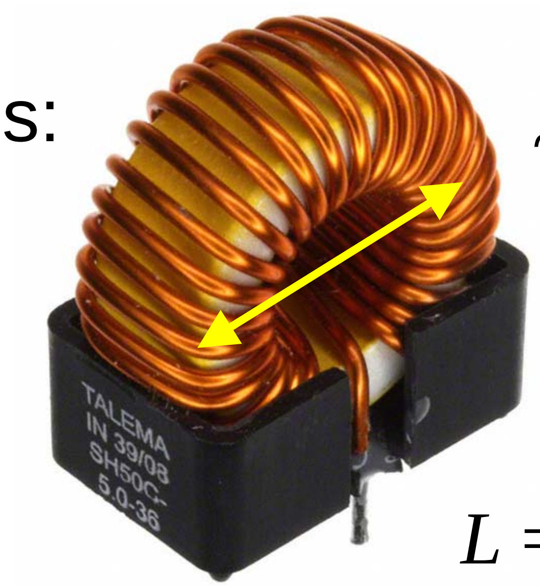
which describes how easy a coil current can introduce magnetic flux over the coil itself.

Examples

- Solenoidal & toroidal coils:



L = 270 μH



~2 cm

L = 36 μH

- RG59/U coaxial cable:

L = 351 nH/m.

The *i*-*v* relation

- Faraday's law states that the electromotive force v (emf, in units of volt) induced on a loop equals the time derivative of the magnetic flux linkage λ:

$$v \equiv \frac{d}{dt}\lambda, \;\Rightarrow v \equiv \frac{d}{dt}Li = L\frac{d}{dt}i.$$

- Note: The emf of a loop is a non-conservative force that can drive current flowing along the loop. In contrast, the current-driving force due to electric charges is conservative.

Behaviors of inductors

$$v = L\frac{di}{dt}$$

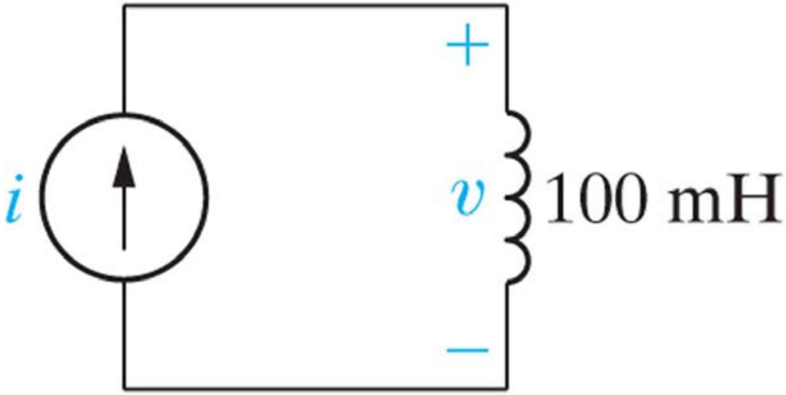
- DC-current: inductor behaves as a short circuit.
- Current cannot change instantaneously in an inductor, otherwise, infinite voltage will arise.
- Change of inductor current is the integral of voltage during the same time interval:

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau.$$

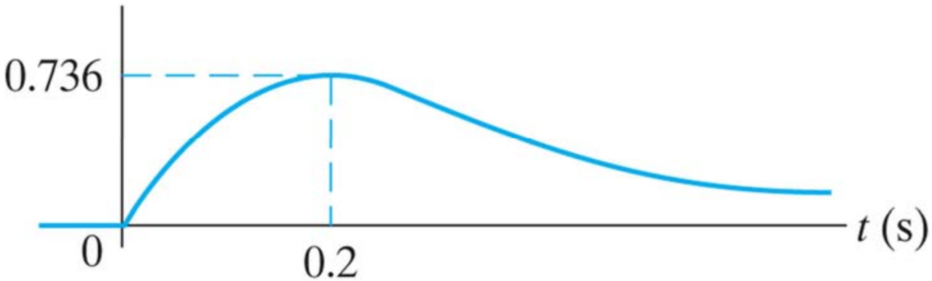
Inductive effect is everywhere!

- Nearly all electric circuits have currents flowing through conducting wires. Since it's difficult to shield magnetic fields, inductive effect occurs even we do not purposely add an inductor into the circuit.

Example 6.1: Inductor driven by a current pulse

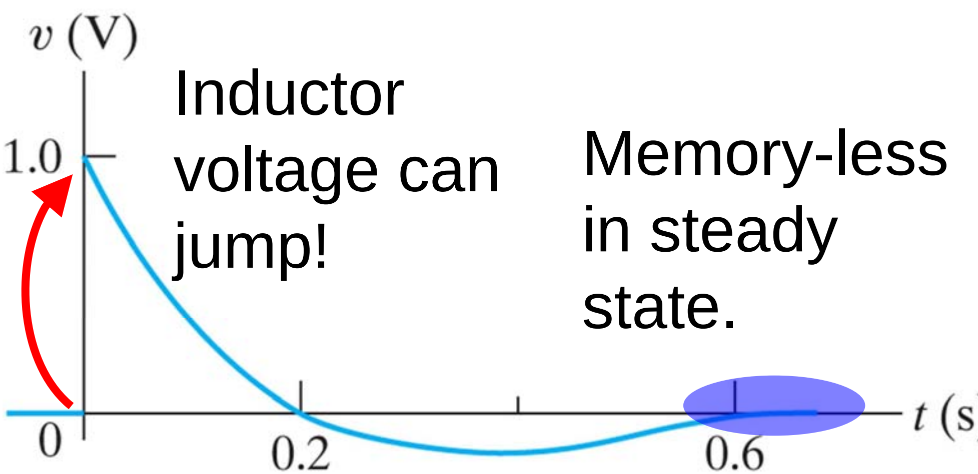


$$i(t)=\begin{cases}0, & t<0\\ 10te^{-5t}, & t>0\end{cases}$$



The inductor voltage is:

$$v(t)=L\frac{di}{dt}=\begin{cases}0, & t<0\\ e^{-5t}(1-5t), & t>0\end{cases}$$



Inductor voltage can jump!

Memory-less in steady state.

Power & energy (1)

- Consider an inductor of inductance L. The instantaneous power in the inductor is:

$$p = vi = Li\frac{di}{dt}.$$

- Assume there is no initial current (i.e. no initial energy), $\Rightarrow i(t=0)=0$, $w(t=0)=0$. We are interested in the energy W when the current increases from zero to I with arbitrary $i(t)$.

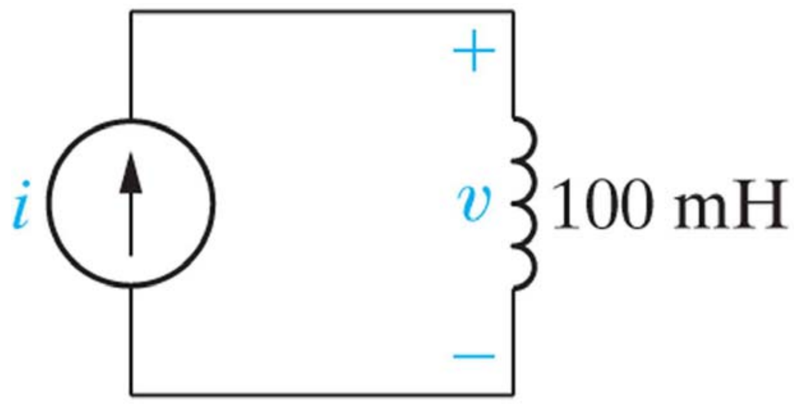
Power & energy (2)

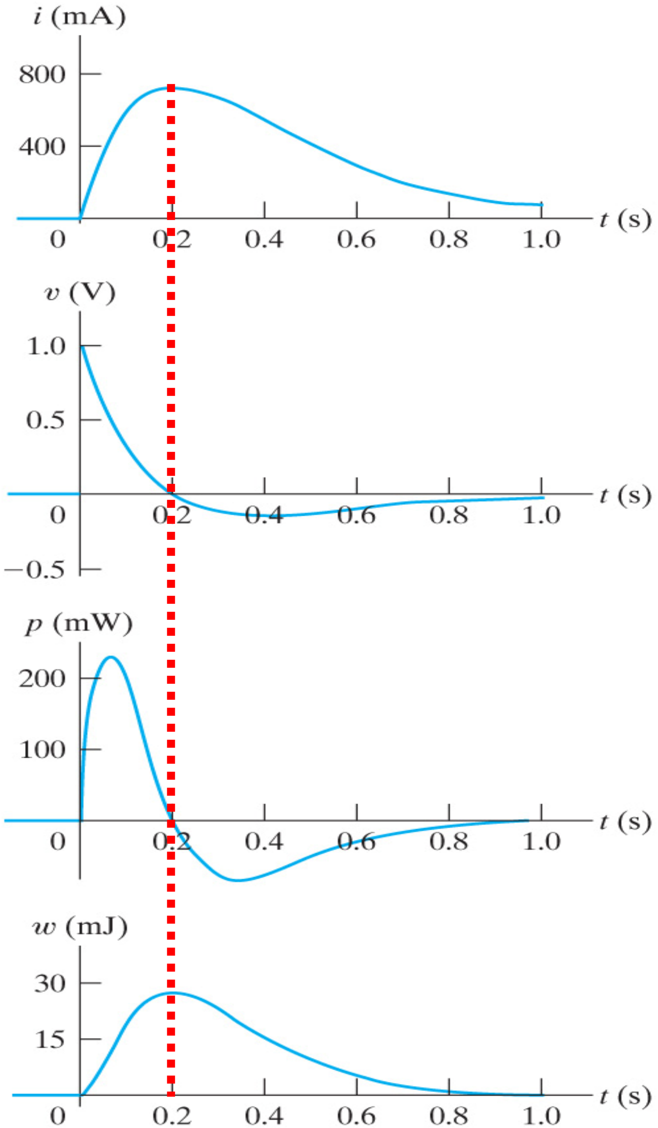
$$p = \frac{dw}{\cancel{dt}} = Li\frac{di}{\cancel{dt}}, \Rightarrow dw = Li \cdot di, \Rightarrow \int_0^W dw = L\int_0^I i di,$$

$$\Rightarrow W = L\frac{i^2}{2}\Big|_0^I = \frac{1}{2}LI^2, \quad \text{i.e.} \quad \boxed{w = \frac{1}{2}Li^2}$$

- How the current changes with time doesn't matter. It's the final current I determining the final energy.
- Inductor stores magnetic energy when there is nonzero current.

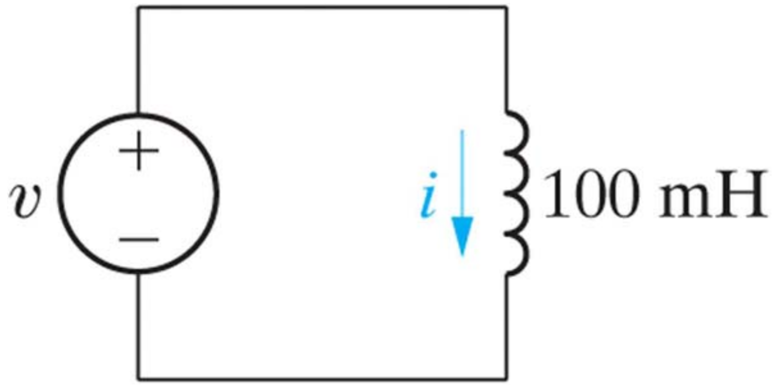
Example 6.3: Inductor driven by a current pulse

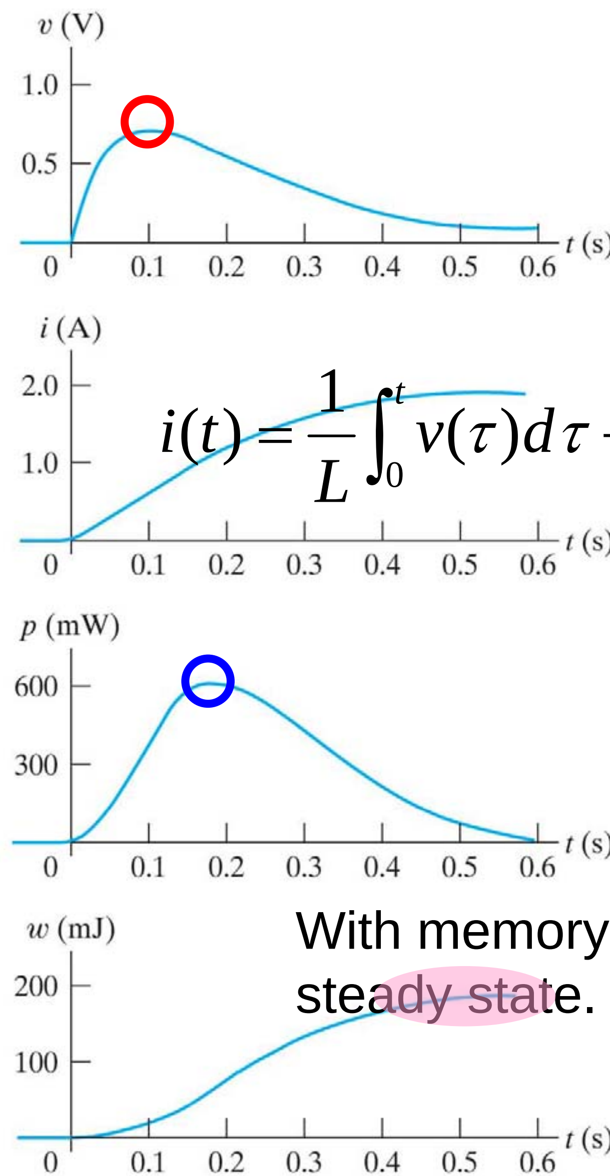




- $t < 0.2, \Rightarrow p > 0, w\uparrow$, charging.
- $t > 0.2, \Rightarrow p < 0, w\downarrow$, discharging.
- In steady state $(t \to \infty)$, $i \to 0$, $v \to 0, p \to 0, w \to 0$ (no energy).

Example 6.3: Inductor driven by a voltage pulse





$$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau + i(0)$$

With memory in steady state.

- $p > 0$, $w\uparrow$, always charging.
- In steady state $(t \to \infty)$, $i \to 2$ A, $v \to 0$, $p \to 0$, $w \to 200$ mJ (sustained current and constant energy).

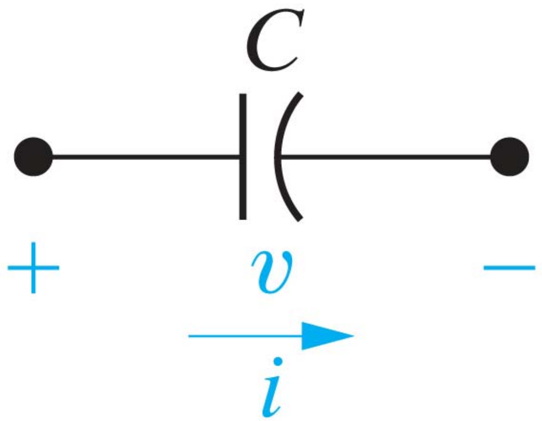
Section 6.2
The Capacitor

1. Physics
2. i-v relation and behaviors
3. Power and energy

Fundamentals

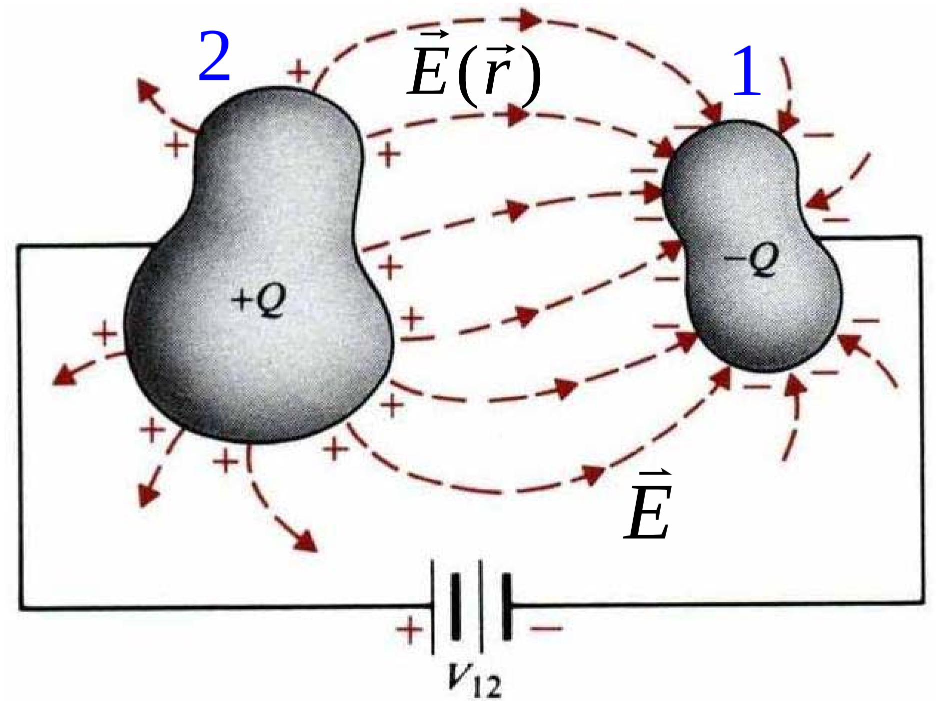
- A capacitor of capacitance C is symbolized by a parallel-plate.
- Typical capacitance C ranges from 10 pF to 470 μF.
- The i-v relation of an capacitor (under the passive sign convention) is:

$$i = C\frac{dv}{dt}.$$



Physics of capacitance (1)

- If we apply a voltage V_{12} between two isolated conductors, charge $\pm Q$ will be properly distributed over the conducting surfaces such that the resulting electric field $\vec{E}(\vec{r})$ satisfies:



$$\int_2^1 \vec{E}(\vec{r}) \cdot d\vec{l} = V_{12},$$

which is valid for any integral path linking the two conducting surfaces.

Physics of capacitance (2)

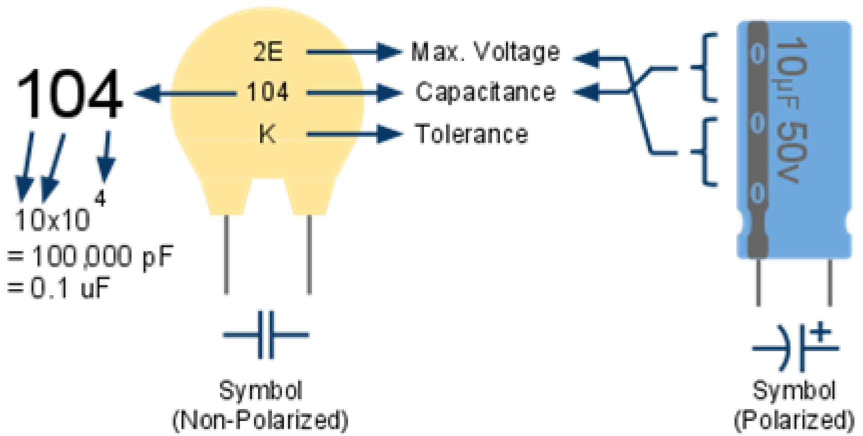
- If $V'_{12} = \alpha V_{12}, \Rightarrow Q' = \alpha Q$ while the spatial distribution of charge remains such that $\vec{E}'(\vec{r}) = \alpha\vec{E}(\vec{r}),\ V'_{12} = \int_2^1 \alpha\vec{E}(\vec{r})\cdot d\vec{l} = \alpha V_{12}.$
- The ratio of the deposited charge to the bias voltage is defined as the capacitance of the conducting pair:

$$\boxed{C \equiv \frac{Q}{V},}$$

describing how easy a bias voltage can deposit charge on the conducting pair.

Examples

- Ceramic disc & electrolytic:



Max. Operating Voltage

Code	Max. Voltage
1H	50V
2A	100V
2T	150V
2D	200V
2E	250V
2G	400V
2J	630V

Tolerance

Code	Percentage
B	± 0.1 pF
C	±0.25 pF
D	±0.5 pF
F	±1%
G	±2%
H	±3%
J	±5%
K	±10%
M	±20%
Z	+80%, -20%

- RG59/U coaxial cable:

C = 53 pF/m.

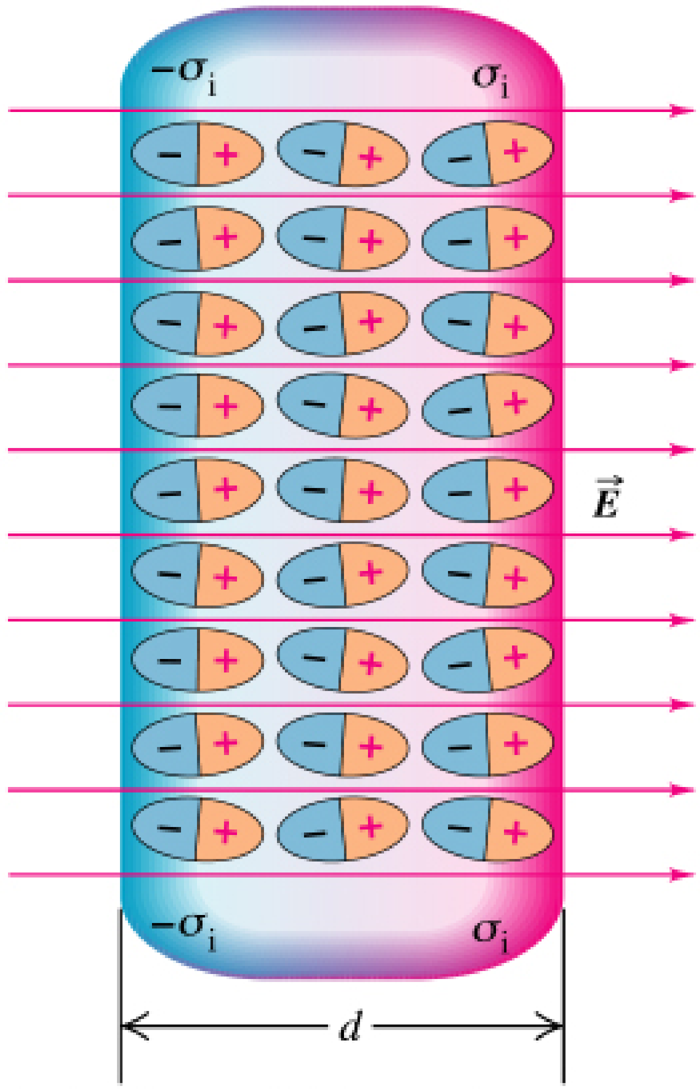
The *i*-*v* relation

- From the definition of capacitance:

$$C \equiv \frac{Q}{V}, \;\Rightarrow q(t) = Cv(t),\; \frac{d}{dt}q = C\frac{d}{dt}v,\; i = C\frac{d}{dt}v.$$

- Note: Charge cannot flow through the dielectric between the conductors. However, a time-varying voltage causes a time-varying electric field that can slightly displace the dielectric bound charge. It is the time-varying bound charge contributing to the "displacement current".

Polarization charge



Behaviors of capacitors

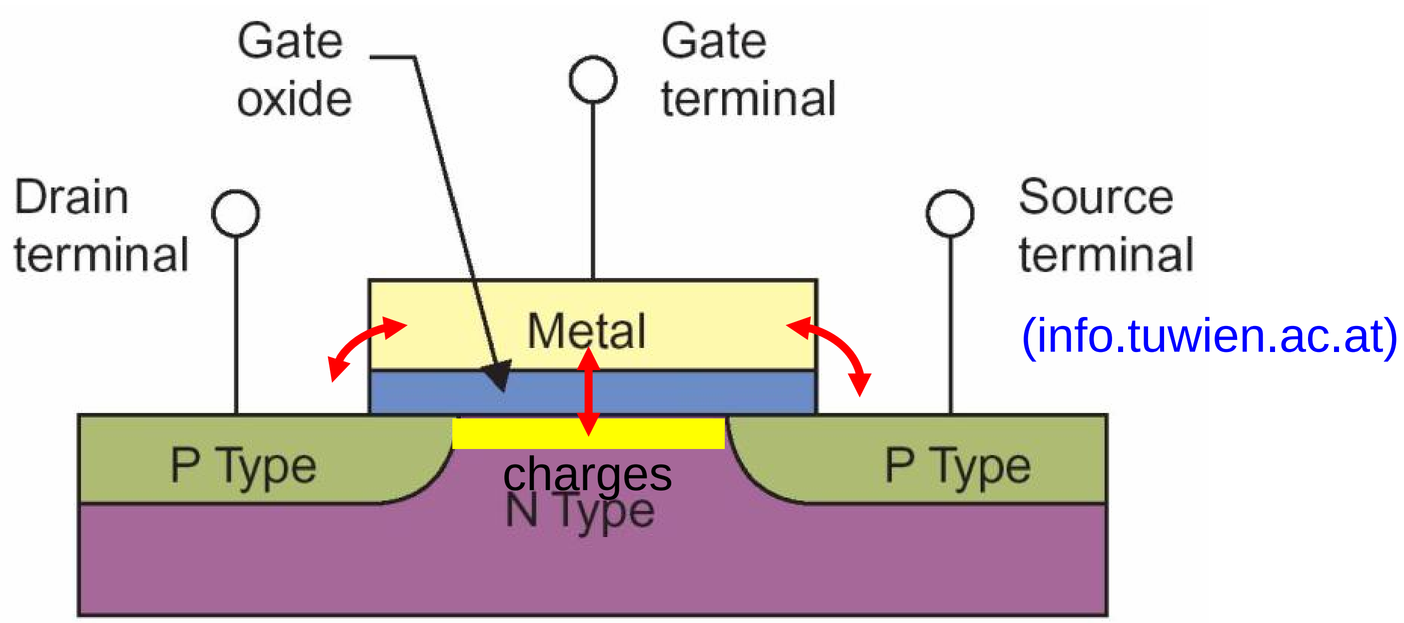
$$i = C\frac{dv}{dt}$$

- DC-voltage: capacitor behaves as an open circuit.
- Voltage cannot change instantaneously in an capacitor, otherwise, infinite current will arise.
- Change of capacitor voltage is the integral of current during the same time interval:

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau.$$

Capacitive effect is everywhere!

- A Metal-Oxide-Semiconductor (MOS) transistor has three conducting terminals (Gate, Source, Drain) separated by a dielectric layer with one another. Capacitive effect occurs even we do not purposely add a capacitor into the circuit.



Power & energy (1)

- Consider a capacitor of capacitance C. The instantaneous power in the capacitor is:

$$p = vi = Cv\frac{dv}{dt}.$$

- Assume there is no initial voltage (i.e. no initial energy), $\Rightarrow v(t=0)=0$, $w(t=0)=0$. We are interested in the energy W when the voltage increases from zero to V with arbitrary $v(t)$.

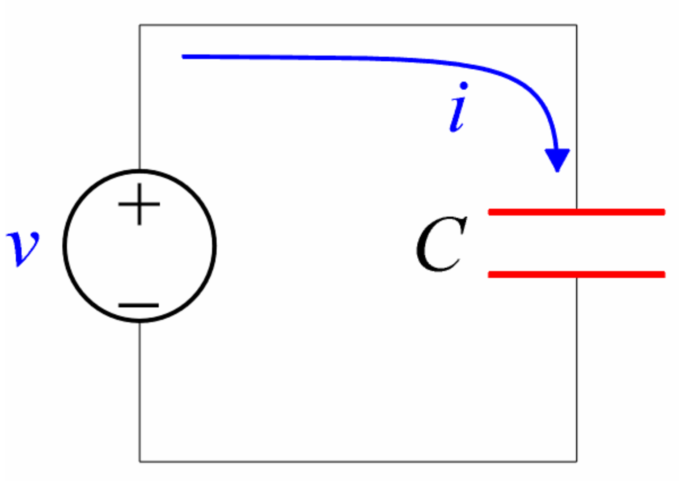
Power & energy (2)

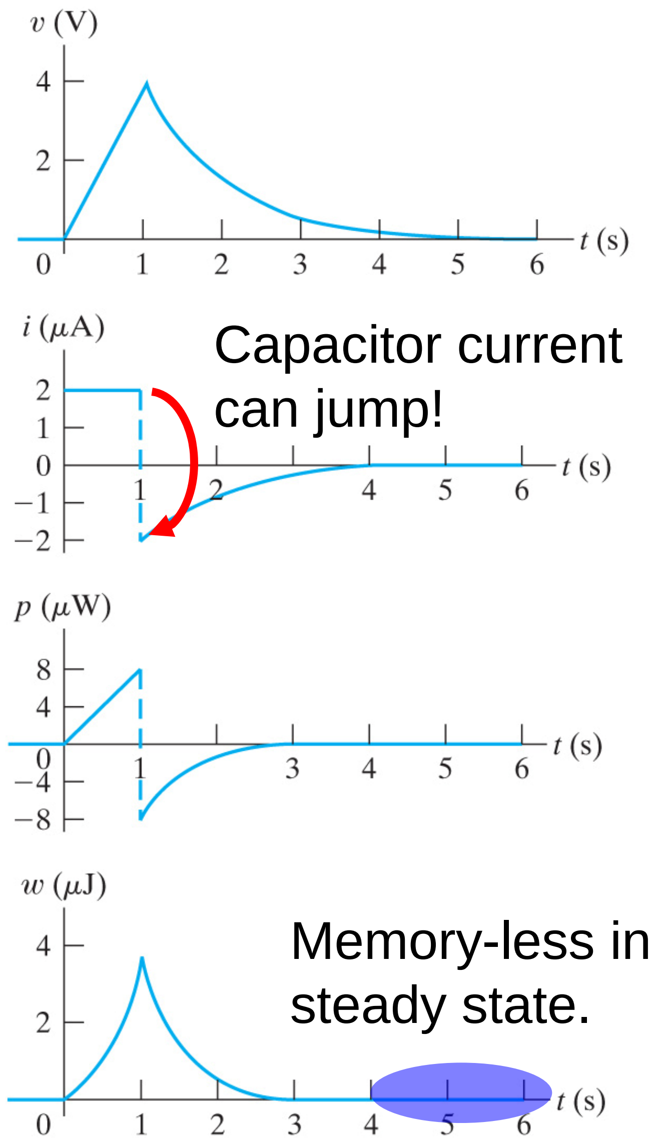
$$p = \frac{dw}{dt} = Cv\frac{dv}{dt}, \Rightarrow dw = Cv \cdot dv, \Rightarrow \int_0^W dw = C\int_0^V v dv,$$

$$\Rightarrow W = C\frac{v^2}{2}\bigg|_0^V = \frac{1}{2}CV^2, \text{ i.e. } \boxed{w = \frac{1}{2}Cv^2}$$

- How the voltage increases with time doesn't matter. It's the final voltage V determining the final energy.
- Capacitor stores electric energy when there is nonzero voltage.

Example 6.4: Capacitor driven by a voltage pulse



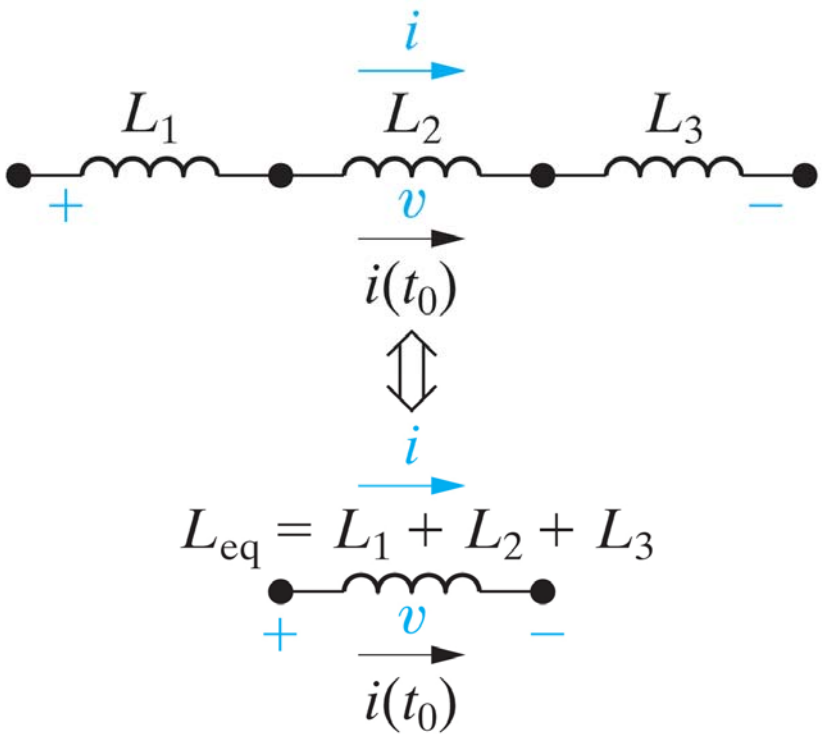


- $t<1, \Rightarrow p>0, w\uparrow$, charging.
- $t>1, \Rightarrow p<0, w\downarrow$, discharging.
- In steady state $(t\to\infty)$, $i\to 0$, $v\to 0$, $p\to 0$, $w\to 0$ (no energy).

Section 6.3 Series-Parallel Combinations

1. Inductors in series-parallel
2. Capacitors in series-parallel

Inductors in series



$$i = i_1 = i_2 = i_3,$$

$$v = v_1 + v_2 + v_3,$$

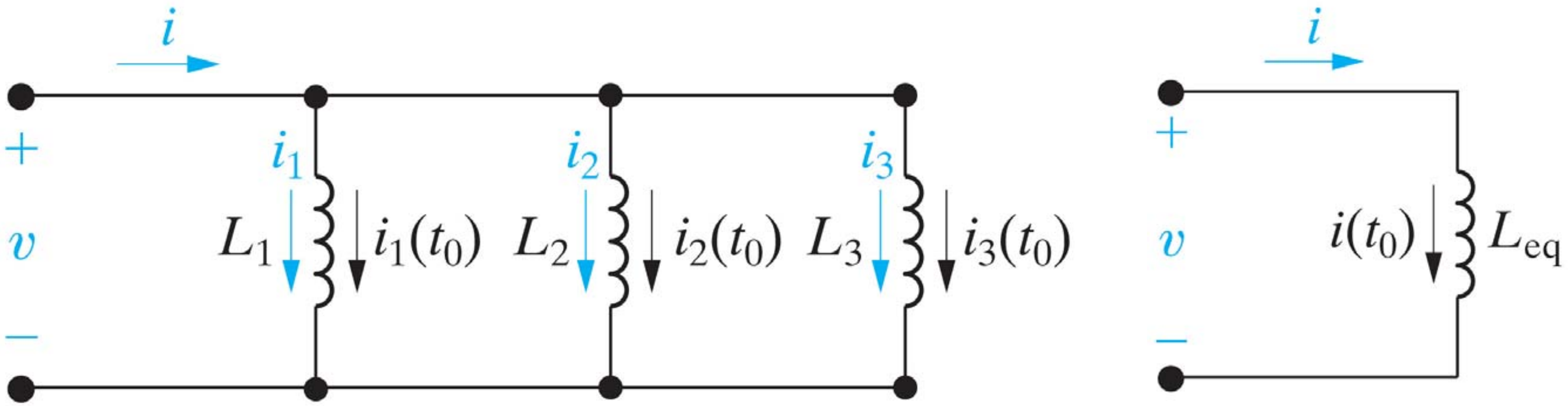
$$v_j = L_j \frac{di_j}{dt},$$

$$\Rightarrow v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$

$$= L_{eq} \frac{di}{dt},$$

$$\Rightarrow \boxed{L_{eq} = \sum_{j=1}^{n} L_j}$$

Inductors in parallel

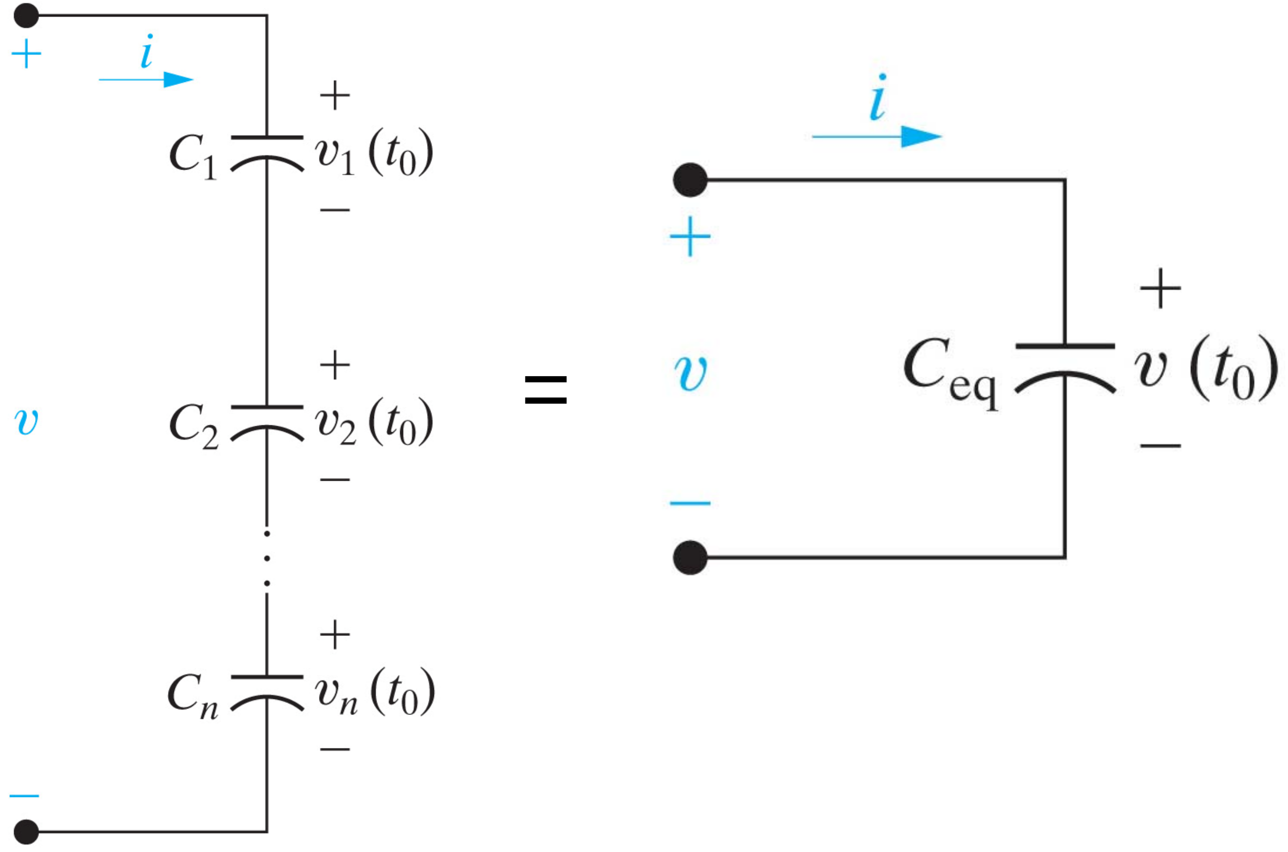


$$v = v_1 = v_2 = v_3, \quad i = i_1 + i_2 + i_3, \quad i_j = \frac{1}{L_j}\int_{t_0}^{t} v(\tau)d\tau + i_j(t_0),$$

$$\Rightarrow i = \sum_{j=1}^{3}\left(\frac{1}{L_j}\int_{t_0}^{t} v(\tau)d\tau + i_j(t_0)\right) = \left(\sum_{j=1}^{3}\frac{1}{L_j}\right)\left(\int_{t_0}^{t} v(\tau)d\tau\right) + \left[\sum_{j=1}^{3} i_j(t_0)\right]$$

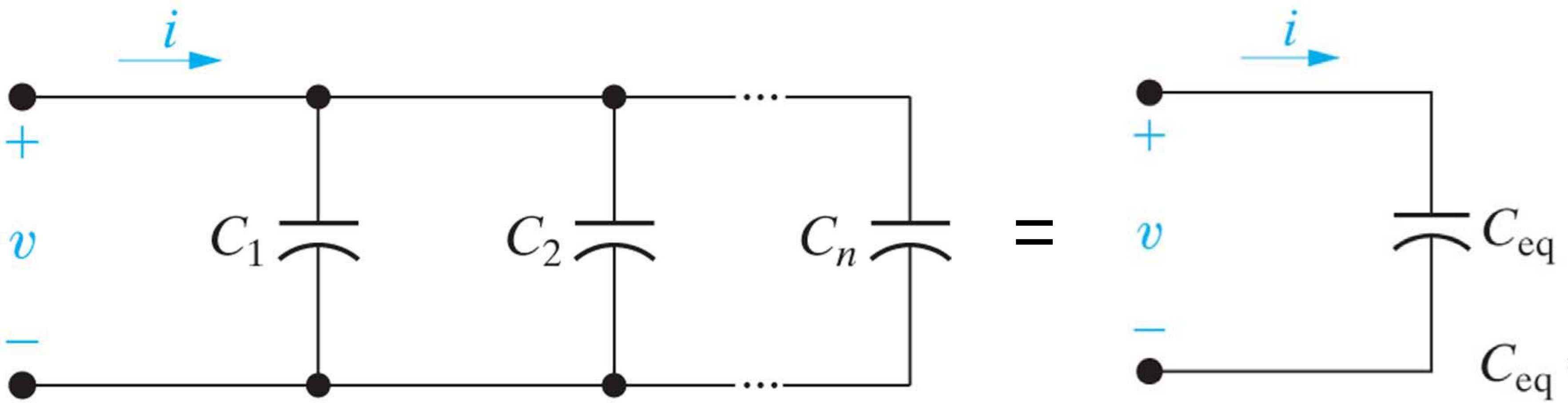
$$= \frac{1}{L_{eq}}\left(\int_{t_0}^{t} v(\tau)d\tau\right) + i(t_0), \Rightarrow \boxed{\frac{1}{L_{eq}} = \sum_{j=1}^{n}\frac{1}{L_j}}$$

Capacitors in series



$$\begin{cases} v = v_1 + v_2 + v_3, \ i = i_1 = i_2 = i_3, \\ v_j = \dfrac{1}{C_j}\displaystyle\int_{t_0}^{t} i\,d\tau + v_j(t_0), \ v = \dfrac{1}{C_{eq}}\displaystyle\int_{t_0}^{t} i\,d\tau + v(t_0), \end{cases} \Rightarrow \boxed{\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j}}$$

Capacitors in parallel



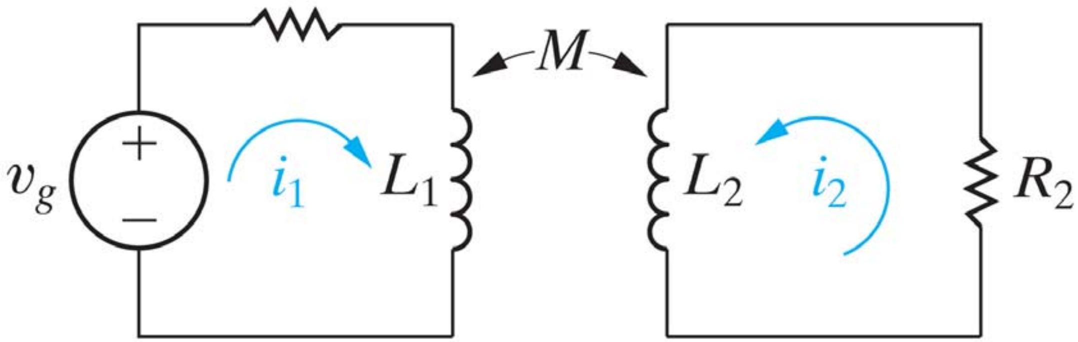
$$\begin{cases} v = v_1 = v_2 = v_3, \;\; i = i_1 + i_2 + i_3, \\ i_j = C_j \dfrac{dv}{dt}, \;\; i = C_{eq} \dfrac{dv}{dt}, \end{cases} \Rightarrow \boxed{C_{eq} = \sum_{j=1}^{n} C_j}$$

Section 6.4, 6.5
Mutual Inductance

1. Physics
2. i-v relation and dot convention
3. Energy

Fundamentals

- Mutual inductance M is a circuit parameter between two magnetically coupled coils.



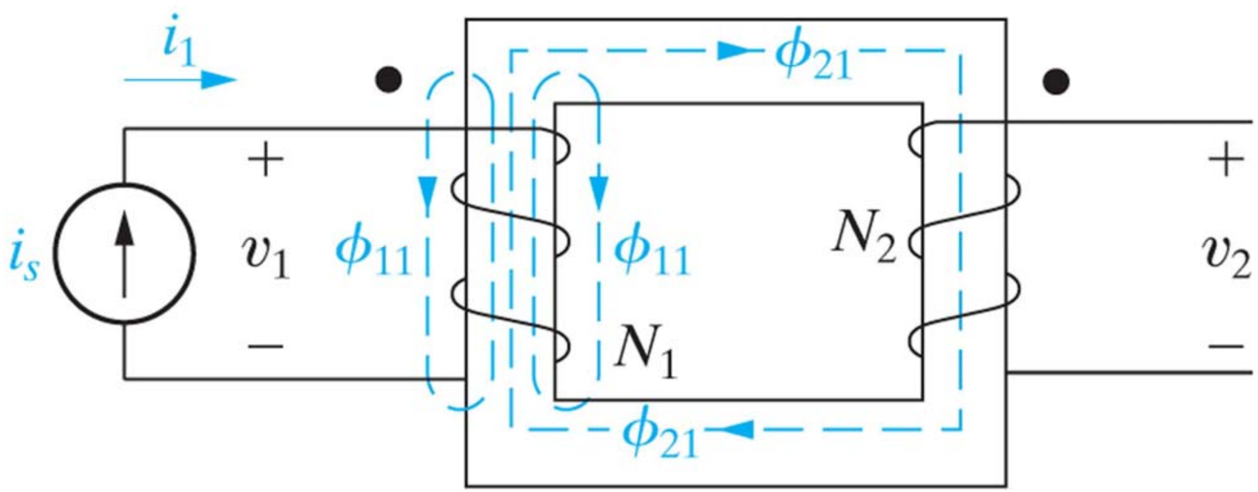
- The value of M satisfies $M = k\sqrt{L_1 L_2}$, where $0 \le k \le 1$ is the magnetic coupling coefficient.
- The emf induced in Coil 2 due to time-varying current in Coil 1 is proportional to $M\left(di_1/dt\right)$.

The i-v relation (1)

- Coil 1 of N_1 turns is driven by a time-varying current i_1, while Coil 2 of N_2 turns is open.
- The flux components linking (1) only Coil 1, (2) both coils, and (3) total flux linking Coil 1 are:

$$\phi_{11} = \int_{S_1} \vec{B}_1(\vec{r}) \cdot d\vec{s} = P_{11} N_1 I_1, \ \phi_{21} = \int_{S_2} \vec{B}_1(\vec{r}) \cdot d\vec{s} = P_{21} N_1 I_1.$$

$$\phi_1 = \phi_{11} + \phi_{21}, \ \Rightarrow P_1 N_1 I_1 = \left(P_{11} + P_{21}\right) N_1 I_1, \ P_1 = P_{11} + P_{21}.$$



The *i*-*v* relation (2)

- Faraday's law states that the emf induced on Coil 2 (when i_2 remains constant) is:

$$\boxed{v_2} = \frac{d}{dt} N_2 \phi_{21} = \frac{d}{dt} N_2 (P_{21} N_1 i_1) = N_2 N_1 P_{21} \frac{d}{dt} i_1 \boxed{= M_{21} \frac{d}{dt} i_1}.$$

- One can show that the emf induced on Coil 1 (when i_1 remains constant) is:

$$\boxed{v_1} = N_1 N_2 P_{12} \frac{d}{dt} i_2 \boxed{= M_{12} \frac{d}{dt} i_2}.$$

- For nonmagnetic media (e.g. air, silicon, plastic), $P_{21} = P_{12}$, $\Rightarrow M_{21} = M_{12} = M = N_1 N_2 P_{21}$.

Mutual inductance in terms of self-inductance

- The two self inductances and their product are:

$$L_1 = N_1^2 P_1,\ \ L_2 = N_2^2 P_2,$$

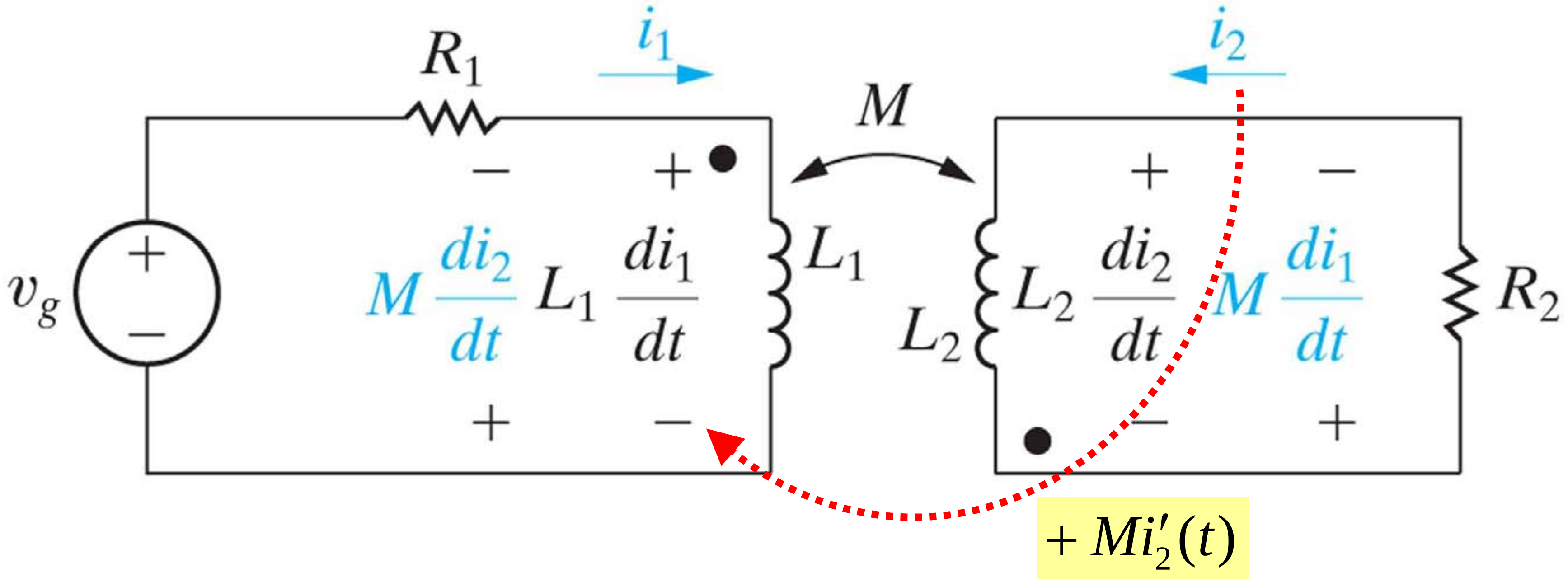
$$L_1 L_2 = N_1^2 N_2^2 P_1 P_2 = N_1^2 N_2^2 \left(P_{11} + P_{21}\right)\left(P_{22} + P_{12}\right)$$

$$= \left(N_1 N_2 P_{21}\right)^2 \left(\frac{P_{11}}{P_{21}} + 1\right)\left(\frac{P_{22}}{P_{21}} + 1\right),\ \text{if}\ P_{21} = P_{12}.$$

$$\text{Let}\ \frac{1}{k^2} = \left(\frac{P_{11}}{P_{21}} + 1\right)\left(\frac{P_{22}}{P_{21}} + 1\right),\ \Rightarrow L_1 L_2 = \frac{M^2}{k^2},\ M = k\sqrt{L_1 L_2}.$$

- The coupling coefficient (1) $k=0$, if $P_{21} \propto \phi_{21} = 0$ (i.e. no mutual flux), (2) $k=1$, if P_{11}=P_{22}=0 (i.e. $\phi_{11} = \phi_{22} = 0,\ \phi_1 = \phi_2 = \phi_{21}$, no flux leakage).

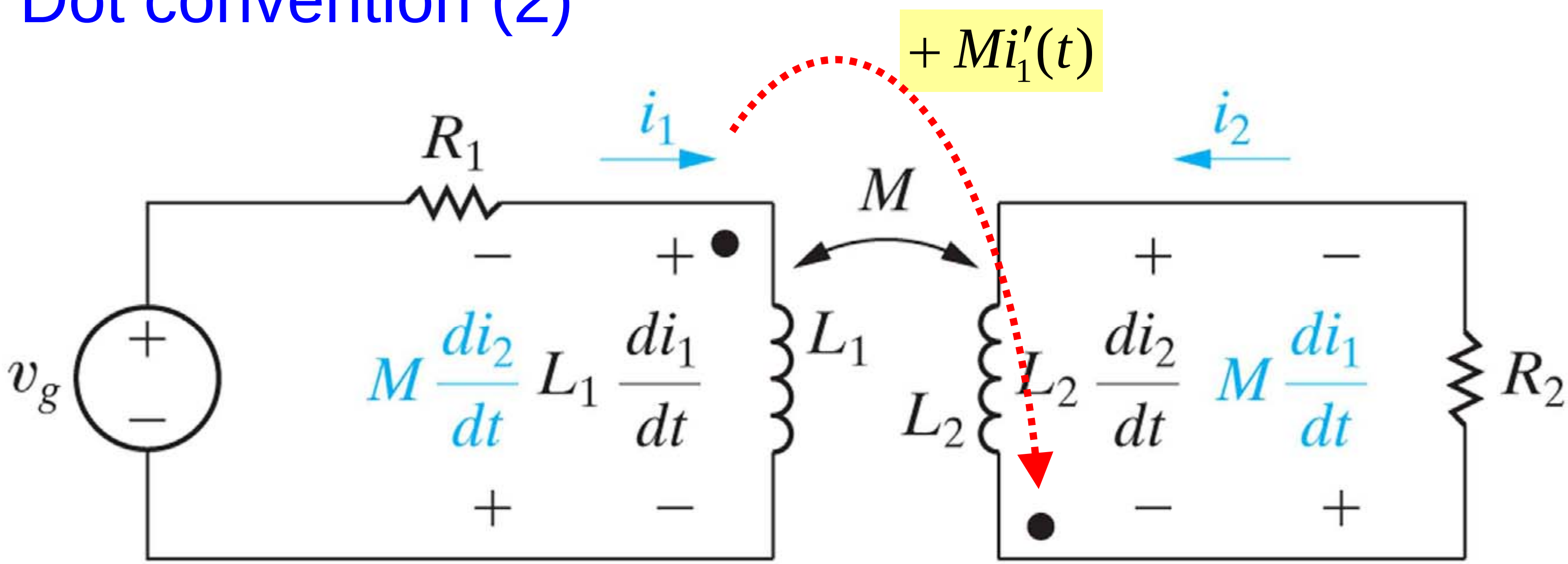
Dot convention (1)



- i_2 leaves the dot of L_2, $\Rightarrow$ the "+" polarity of $Mi_2'(t)$ is referred the terminal of L_1 without a dot.
- The total voltage across L_1 is:

$$v_1 = L_1 \frac{d}{dt} i_1 - M \frac{d}{dt} i_2.$$

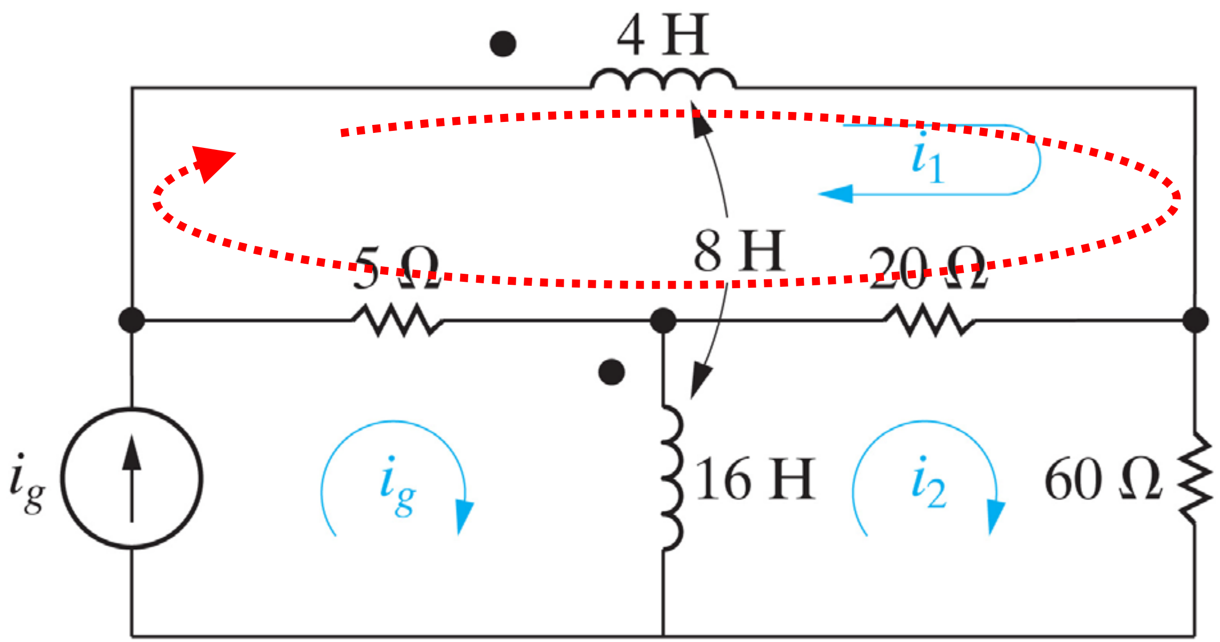
Dot convention (2)



- i_1 enters the dot of L_1, $\Rightarrow$ the "+" polarity of $Mi_1'(t)$ is referred the terminal of L_2 with a dot.
- The total voltage across L_2 is:

$$v_2 = L_2 \frac{d}{dt} i_2 - M \frac{d}{dt} i_1.$$

Example 6.6: Write a mesh current equation



$$(4\,\mathrm{H})\frac{d}{dt}i_1 + (20\,\Omega)(i_1 - i_2) + (5\,\Omega)(i_1 - i_g) + (8\,\mathrm{H})\frac{d}{dt}(i_g - i_2) = 0$$

Self-inductance,

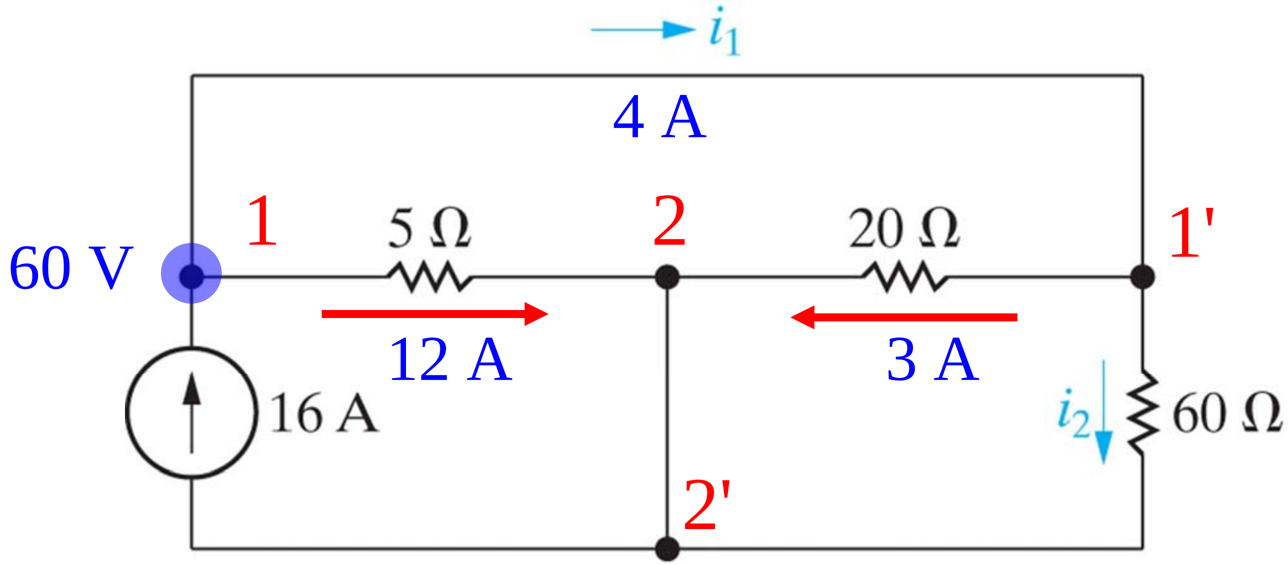
passive sign convention

Mutual-inductance,

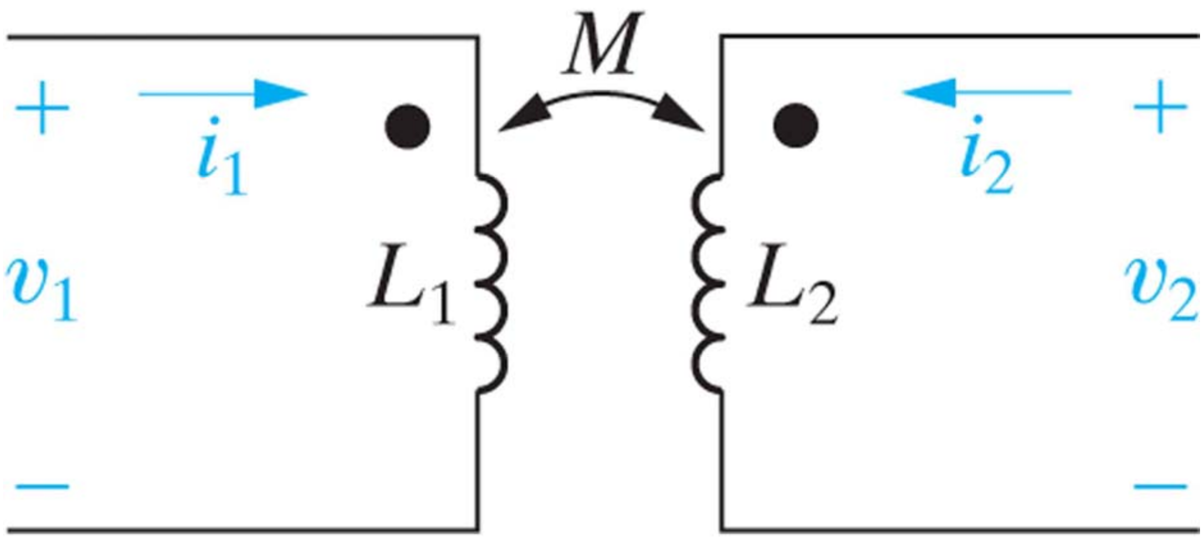
i_g–i_2 enters the dot of 16-H inductor

Example 6.6: Steady-state analysis

- In steady state ($t \to \infty$), inductors are short, $\Rightarrow$ the 3 resistors are in parallel ($R_{eq} = 3.75\ \Omega$).
- Let $v_2 = 0$. $\Rightarrow$ (1) $v_1 = (16\text{A})(3.75\ \Omega) = 60\ \text{V}$. (2) $i_{12} = (60\text{V})/(5\Omega) = 12\text{A}$, $\Rightarrow i_1 = (16\text{-}12) = 4\ \text{A}$ (not zero!). (3) $i_{1'2} = (60\text{V})/(20\Omega) = 3\ \text{A}$, $\Rightarrow i_{22'} = (12+3) = 15\ \text{A}$.



Energy of mutual inductance (1)



- Assume i_1, $i_2 = 0$ initially. Fix $i_2 = 0$ while increasing i_1 from 0 to some constant I_1. The energy stored in L_1 becomes:

$$W_1 = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2.$$

Energy of mutual inductance (2)

- Now fix $i_1 = I_1$, while increasing i_2 from 0 to I_2. During this period, emf's will be induced in loops 1 and 2 due to the time-varying i_2. The total power of the two inductors is:

$$p(t) = I_1 \cdot \left(M \frac{di_2}{dt} \right) + i_2(t) \cdot \left(L_2 \frac{di_2}{dt} \right).$$

- An extra energy of $W_{12}+W_2$ is stored in the pair:

$$W_{12} + W_2 = I_1 M \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2 = M I_1 I_2 + \frac{1}{2} L_2 I_2^2.$$

Energy of mutual inductance (3)

- The entire process contributes to a total energy

$$W_{tot} = \frac{1}{2} L_1 I_1^2 + M I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

for the two-inductor system.

- W_{tot} only depends on the final currents I_1, I_2 [independent of the time evolution of $i_1(t)$, $i_2(t)$].

Key points

- Why the i-v relation of an inductor is $v = L\dfrac{di}{dt}$?
- Why the i-v relation of a capacitor is $i = C\dfrac{dv}{dt}$?
- Why the energies stored in an inductor and a capacitor are:

$$w = \frac{1}{2}Li^2,\ \frac{1}{2}Cv^2,\ \text{respectively?}$$