

On Constructions of Optical Priority Queues Under a Priority-Based Routing Policy

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Abstract

In this paper, we consider Switched-Delay-Lines (SDL) constructions of optical priority queues by using optical (bufferless) crossbar switches and optical fiber delay lines. Priority queues is one of the most general and versatile buffering schemes, and includes the most commonly used first-in first-out (FIFO) queues and last-in first-out (LIFO) queues as special cases. The best results in the literature on SDL constructions of optical priority queues were obtained by Cheng *et al.* In this paper, we give constructions of optical priority queues by using a feedback system consisting of an optical (bufferless) crossbar switch and multiple groups of optical FIFO multiplexers with delay one (FM1's), and show that our constructions can be implemented by using an optical (bufferless) crossbar switch and multiple optical fiber delay lines. Our constructions extend and generalize those in an important class of constructions that contains the optimal constructions in the work of Cheng *et al.* As a result, we achieve larger buffer sizes and less construction complexities/costs than those in previous works. We obtain a closed-form expression for the maximum buffer size that is achieved by the optimal construction for the scenario that each group of FM1's in our constructions has the same number of FM1's. We also show that we can achieve a buffer size $2^{O(\sqrt{\alpha M})}$ by using an optical $(M+2) \times (M+2)$ crossbar switch and M fiber delay lines, where α is a constant that depends on the parameters used in the constructions and is better, i.e., larger, than that in the work of Cheng *et al.* in a very broad regime. One salient feature of our constructions is their fault-tolerant capability: we show that our constructions can tolerate the malfunctioning of some FM1's by using the generalization in this paper.

Index Terms

Fault tolerance, FIFO multiplexers, optical buffers, optical queues, optical switches, priority queues.

This work was supported in part by the Ministry of Science and Technology, Taiwan, under Contract MOST-111-2221-E-007-041-MY2. The authors are with the Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan (e-mails: jcheng@ee.nthu.edu.tw; d9564808@oz.nthu.edu.tw; dorayaki851211@gmail.com; whu910116hung1009@gmail.com; wantseng2004@gmail.com; morris.ych@gmail.com).

I. INTRODUCTION

Due to the lack of *optical* random-access memory, an important and challenging issue in all-optical packet-switched networks is the design and implementation of optical buffers for conflict resolution among packets competing for the same resources. One of the feasible approaches for the implementation of optical buffers is to use optical fiber delay lines to store optical packets and use optical (bufferless) crossbar switches to route optical packets through the optical fiber delay lines [1].

Apparently, the optical buffers in such Switched-Delay-Lines (SDL) constructions do not have random-access capability since optical packets can only enter the fibers from the inputs of the fibers and cannot be retrieved until they reach the outputs of the fibers. As such, there are two main issues in such an SDL approach: (i) The choice of the delays of the optical fiber delay lines. (ii) The design of the routing policy performed by the optical (bufferless) crossbar switches. If this is done properly, then optical packets can be routed to the right places at the right times, and exact emulations of the desired optical buffers can be achieved.

In the last three decades, there have been extensive studies on SDL constructions of optical buffers. These works include: (i) the early feasibility studies in [2]–[5], (ii) output-buffered switches in [6]–[11], (iii) first-in first-out (FIFO) multiplexers in [6] and [11]–[21], (iv) FIFO queues in [21]–[26], (v) last-in first-out (LIFO) queues in [23], [24], and [27], (vi) priority queues in [28]–[38], (vii) time slot interchanges in [21] and [39], (viii) linear compressors/decompressors, non-overtaking delay lines, and flexible delay lines in [21] and [40]–[45], and (ix) FIFO/LIFO/absolute contractors in [46]. Moreover, results on the fundamental complexity of SDL constructions of optical queues can be found in [47] and performance analysis for optical queues has been addressed in [48] and [49].

As in most works on SDL constructions of optical buffers, in this paper we focus on the theoretical aspect of the constructions of optical buffers. We are aware of many important practical feasibility issues such as: (i) router buffer sizing problem, (ii) fault-tolerant capability, and (iii) limitation on the number of times that an optical packet can recirculate through optical switches and fiber delay lines. For those interested in such issues, we refer to Sections V-A and V-C in [37] and the references therein for details. For review articles on SDL constructions of optical buffers as well as related implementation and feasibility issues, we refer to [50]–[55] and the references therein.

In this paper, we consider the constructions of optical priority queues. Priority queues is one of the most general and versatile buffering schemes, and includes the most commonly used FIFO queues and LIFO queues as special cases. In a priority queue, each packet is associated with a priority upon its arrival, the packet with the *highest* priority is sent out from the queue whenever there is a departure request and there are packets in the queue, and the packet with the *lowest* priority is dropped from the queue whenever there is a buffer overflow.

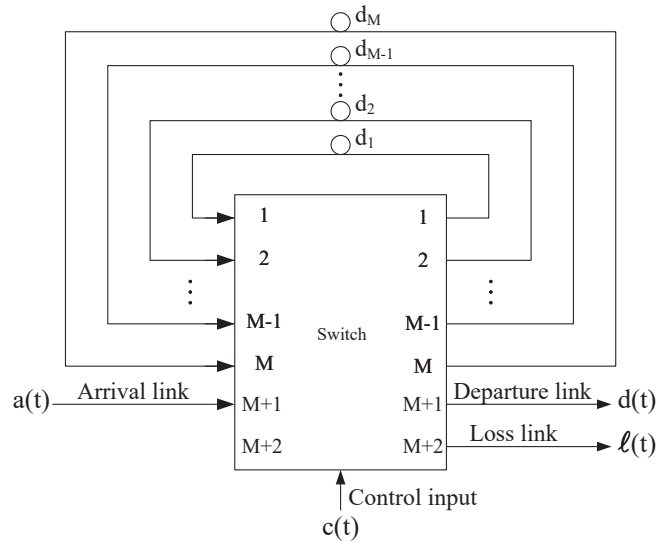


Fig. 1. A construction of an optical priority queue by using a feedback system consisting of an optical $(M + 2) \times (M + 2)$ (bufferless) crossbar switch and M fiber delay lines with delays d_1, d_2, \dots, d_M .

The first construction of optical priority queues was published in [28] by Sarwate and Anantharam of UC Berkeley. In [28], they showed that an optical priority queue with buffer size $O(M^2)$ can be implemented by using a feedback system consisting of an optical $(M + 2) \times (M + 2)$ (bufferless) crossbar switch and M fiber delay lines with appropriately chosen delays d_1, d_2, \dots, d_M (see Figure 1). Several improvements on the $O(M^2)$ buffer size have been made in [29]–[37] by using better choices of the delays d_1, d_2, \dots, d_M and better designs of the routing policy performed by the crossbar switch in Figure 1. The best result currently available in the literature was obtained in [37], and the buffer size achieved in [37] is $2^{O(\sqrt{\alpha M})}$, where α is a constant that depends on the parameters used in the constructions. The buffer size $2^{O(\sqrt{\alpha M})}$ is exponential in \sqrt{M} and significantly outperforms all previous results in [28]–[36] that are only polynomial in M .

In this paper, we adopt a generalization of the feedback system in [37] for the constructions of optical priority queues (see Figure 2). The feedback system in Figure 2 consists of an optical (bufferless) crossbar switch and multiple groups of optical FIFO multiplexers with delay one (FM1's). The FM1's in the feedback path are used for buffering packets and feeding packets from the outputs of the crossbar switch back to the inputs of the crossbar switch. By a detailed *counting* argument, we are able to obtain a better lower bound than that in [37] on the number of arrival links of the FM1's in each group for the accommodation of packets routed to that group of FM1's. As a result, we achieve less construction complexities/costs than those in [37]. Moreover, by using better choices of the parameters in our constructions, we achieve larger buffer sizes than those in [37].

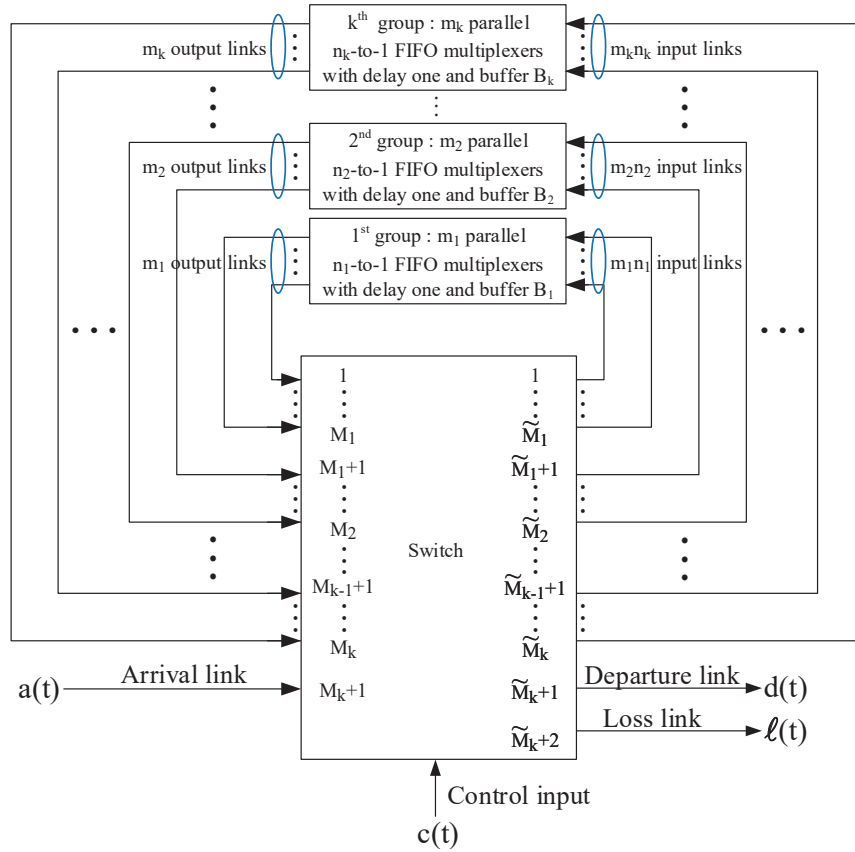


Fig. 2. A construction of an optical priority queue by using a feedback system consisting of an optical $(\sum_{i=1}^k m_i n_i + 2) \times (\sum_{i=1}^k m_i n_i + 2)$ (bufferless) crossbar switch and k groups of optical FIFO multiplexers with delay one (FM1's), where the i^{th} group has m_i parallel optical n_i -to-1 FM1's (n_i FM1's) with the same buffer size B_i ($B_i \geq 1$) for $i = 1, 2, \dots, k$. In the figure, we let $M_i = \sum_{j=1}^i m_j$ and $\tilde{M}_i = \sum_{j=1}^i m_j n_j$ for $i = 1, 2, \dots, k$.

The contributions of this paper are as follows: (i) Our constructions extend and subsume as special cases those in an important class of constructions that contains the optimal constructions in [37]. As a result, we achieve larger buffer sizes and less construction complexities/costs than those in [37]. This is confirmed by our numerical results, which show that we achieve a buffer size that is 1.27 to 1.33 times of that in [37] but with 10% to 30% less construction cost in most cases. (ii) Our constructions generalize those in [37] from the scenario that each group of FM1's has the same number of FM1's and each FM1 has the same number of arrival links to the scenario that each group of FM1's may have a different number of FM1's and the FM1's in different groups may have different numbers of arrival links. (iii) We obtain a closed-form expression for the maximum buffer size that is achieved by the optimal construction for the scenario that each group of FM1's has the same number of FM1's. (iv) The generalization in (ii) makes it possible for us to show that our constructions possess fault-tolerant capability. Fault-tolerance is an important practical issue in the design of any network element, and needs to be

taken into consideration during the design phase of a network element. We show that if some of the FM1's in the feedback system fail to function properly, then the feedback system in our constructions can still be operated as an optical priority queue with a *smaller* buffer size by using the remaining functioning FM1's. (v) We show that we can achieve a buffer size $2^{O(\sqrt{\alpha M})}$ by using an optical $(M + 2) \times (M + 2)$ crossbar switch and M fiber delay lines, where α is a constant that depends on the parameters used in the constructions. The achieved buffer size is larger than that in [37] in a very broad regime.

This paper is organized as follows. In Section II, we give a characterization of a priority queue, describe the feedback system used in our constructions, and describe the priority-based routing policy performed by the (bufferless) crossbar switch in the feedback system. In Section III, we show that the feedback system can be operated as an optical priority queue under the priority-based routing policy if the parameters in our constructions are properly chosen, and show the extension and generalization of our constructions over those in an important class of constructions that contains the optimal constructions in [37]. In Section IV, we give the optimal construction that achieves minimum construction complexity and maximum buffer size, obtain a closed-form expression for the maximum buffer size, and show that our constructions possess fault-tolerant capability. In Section V, we show that we can achieve a buffer size $2^{O(\sqrt{\alpha M})}$ by using an optical $(M + 2) \times (M + 2)$ crossbar switch and M fiber delay lines, where α is larger than that in [37] in a very broad regime. Finally, a brief conclusion is made in Section VI.

II. THE FEEDBACK SYSTEM AND ITS OPERATIONS IN OUR CONSTRUCTIONS

A. Characterization of a Priority Queue

As in most works on SDL constructions of optical queues in the literature, in this paper we consider the following discrete-time settings: (i) Time is slotted and synchronized. (ii) Packets are of the same size so that a packet can be transmitted through a link within a time slot. (iii) An optical $M \times M$ (bufferless) crossbar switch is a network element with M input links and M output links that can realize all of the $M!$ permutations between its inputs and its outputs. (iv) A fiber delay line with delay d is a network element with one input link and one output link that requires d time slots for a packet to traverse through. We note that variable-size packets can be taken care of with ease by implementing packet segmentation at the source and packet reassembly at the destination.

To be concise, in the rest of this paper we simply refer to time slot t as “slot t .” Since a packet can be transmitted through a link within a slot, there can be at most one packet from any link at any slot. Therefore, we can characterize a link by its link state, and we say that a link is in state 1 (resp., state 0) at slot t if there is a packet (resp., there is no packet) from that link at slot t . Furthermore, we assume that every network element is initially empty at slot $t = 0$.

A priority queue with buffer size B is a network element with one arrival link, one control input, one departure link, and one loss link (see Figure 3). We denote $a(t)$, $d(t)$, and $\ell(t)$ as

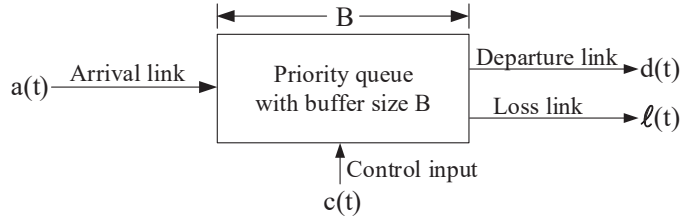


Fig. 3. A priority queue with buffer size B .

the link states of the arrival link, the departure link, and the loss link, respectively, at slot t . We denote $c(t) = 1$ (resp., $c(t) = 0$) if there is a departure request (resp., there is no departure request) from the controller at slot t . We also denote $q(t)$ as the number of packets stored in the buffer of the priority queue at slot t .

At each slot, every packet in the priority queue is assigned a *distinct* priority subject to the constraint that the *relative* priority order between any two packets remains unchanged as long as they are in the queue. For this, at each slot we assign a distinct *tag* to each packet in the queue according to its priority level. Specifically, the tag $\tau_p(t)$ of a packet p in the queue at slot t is assigned as $\tau_p(t) = i$ if packet p is the i^{th} -highest-priority packet among the $q(t-1) + a(t)$ packets in the queue at slot t (including the $q(t-1)$ packets buffered in the queue at slot $t-1$ and the $a(t)$ arrival packets at slot t). Therefore, the $q(t-1) + a(t)$ packets in the queue at slot t are assigned tags from 1 to $q(t-1) + a(t)$ in the order of decreasing priority.

We characterize a priority queue with buffer size B by the following three properties.

(P1) *Nonidling and priority departure*: If there is a departure request from the controller and there are packets in the queue at slot t , i.e., $c(t) = 1$ and $q(t-1) + a(t) \geq 1$, then there is a departure packet at slot t , i.e., $d(t) = 1$, and the departure packet at slot t is the packet in the queue at slot t with the *highest* priority, i.e., with tag assigned as 1 at slot t . Otherwise, there is no departure packet at slot t , i.e., $d(t) = 0$.

(P2) *Maximum buffer usage and priority loss*: If there is a buffer overflow at slot t , i.e., $c(t) = 0$, $q(t-1) = B$, and $a(t) = 1$, then there is a loss packet at slot t , i.e., $\ell(t) = 1$, and the loss packet at slot t is the packet in the queue at slot t with the *lowest* priority, i.e., with tag assigned as $B + 1$ at slot t . Otherwise, there is no loss packet at slot t , i.e., $\ell(t) = 0$.

(P3) *Flow conservation*: Packets arriving from the arrival link are either buffered in the queue or transmitted through the departure link or the loss link. Thus, we have $q(t) = q(t-1) + a(t) - d(t) - \ell(t)$.

B. The Feedback System in Our Constructions

In our constructions, we use the feedback system in Figure 2 that consists of an optical $(\sum_{i=1}^k m_i n_i + 2) \times (\sum_{i=1}^k m_i n_i + 2)$ (bufferless) crossbar switch and k groups of optical FM1's. The i^{th} group of FM1's has m_i parallel optical n_i -to-1 FM1's (n_i FM1's) with the same buffer size

B_i ($B_i \geq 1$) for $i = 1, 2, \dots, k$. We note that the feedback system in Figure 2 is a generalization of that used in [37] (as the feedback system in [37] is for the scenario that each group of FM1's has the same number of FM1's and each FM1 has the same number of arrival links, say $m_i = m$ and $n_i = n$ for all $1 \leq i \leq k$).

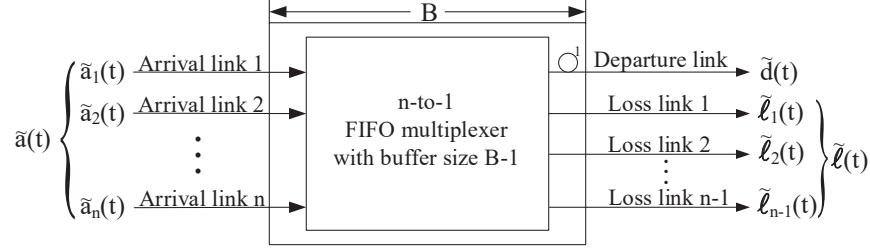


Fig. 4. An n -to-1 FIFO multiplexer with delay one (nFM1) and buffer size B .

In Figure 2, we use FM1's in the feedback path for buffering packets and feeding packets from the outputs of the crossbar switch back to the inputs of the crossbar switch. An nFM1 with buffer size B is a network element with n arrival links, one departure link, and $n - 1$ loss links, and is the concatenation of an n -to-1 FIFO multiplexer (nFM) with buffer size $B - 1$ and a delay line with delay equal to one (see Figure 4). We denote $\tilde{a}_i(t)$ as the link state of arrival link i at slot t for $i = 1, 2, \dots, n$, denote $\tilde{d}(t)$ as the link state of the departure link at slot t , and denote $\tilde{\ell}_i(t)$ as the link state of loss link i at slot t for $i = 1, 2, \dots, n - 1$. Let $\tilde{a}(t) = \sum_{i=1}^n \tilde{a}_i(t)$ be the number of arrival packets at slot t , let $\tilde{\ell}(t) = \sum_{i=1}^{n-1} \tilde{\ell}_i(t)$ be the number of loss packets at slot t , and let $\tilde{q}(t)$ be the number of packets buffered in the nFM1 at slot t .

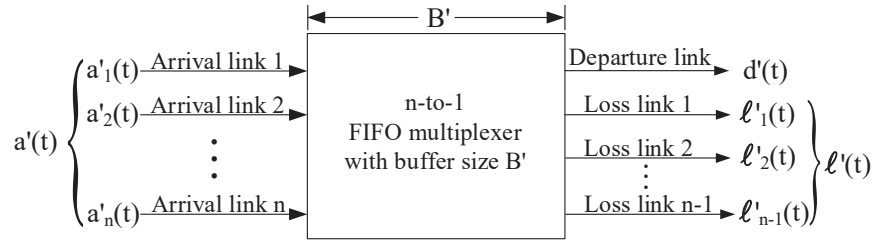


Fig. 5. An n -to-1 FIFO multiplexer (nFM) with buffer size B' .

In the following, we describe an nFM with buffer size B' . An nFM with buffer size B' is a network element with n arrival links, one departure link, and $n - 1$ loss links (see Figure 5). We denote $a'_i(t)$ as the link state of arrival link i at slot t for $i = 1, 2, \dots, n$, denote $d'(t)$ as the link state of the departure link at slot t , and denote $\ell'_i(t)$ as the link state of loss link i at slot t for $i = 1, 2, \dots, n - 1$. Let $a'(t) = \sum_{i=1}^n a'_i(t)$ be the number of arrival packets at slot t , let $\ell'(t) = \sum_{i=1}^{n-1} \ell'_i(t)$ be the number of loss packets at slot t , and let $q'(t)$ be the number of packets buffered in the nFM at slot t .

We characterize an nFM with buffer size B' by the following three properties.

(M1) *Nonidling and FIFO departure with prioritized arrival links*: If there are packets in the nFM at slot t , i.e., $q'(t-1) + a'(t) \geq 1$, then there is a departure packet at slot t , i.e., $d'(t) = 1$, and the departure packet at slot t is the packet in the nFM at slot t with the *earliest* arrival time. Otherwise, there is no departure packet at slot t , i.e., $d'(t) = 0$. To break the tie among packets arriving at the same time, we assume that the arrival links are prioritized so that a packet from an arrival link with a smaller link index is regarded as arriving earlier than that from an arrival link with a larger link index.

(M2) *Maximum buffer usage and FIFO loss with prioritized loss links*: If there is a buffer overflow at slot t , i.e., $q'(t-1) + a'(t) - d'(t) \geq B' + 1$, or, equivalently, $q'(t-1) + a'(t) - 1 \geq B' + 1$ (since in both cases we have $q'(t-1) + a'(t) \geq 1$ and hence it follows from the nonidling property in (M1) that $d'(t) = 1$), then there are $\ell'(t) = q'(t-1) + a'(t) - 1 - B'$ loss packets at slot t , and the loss packets at slot t are the $\ell'(t)$ packets in the nFM at slot t with the *latest* arrival times and they are dropped from loss links $1, 2, \dots, \ell'(t)$ in the order of increasing arrival times. Otherwise, there are no loss packets at slot t , i.e., $\ell'(t) = 0$.

(M3) *Flow conservation*: Packets arriving from the n arrival links are either buffered in the nFM or transmitted through the departure link or the $n - 1$ loss links. Thus, we have $q'(t) = q'(t-1) + a'(t) - d'(t) - \ell'(t)$.

Note that we have from the maximum buffer usage property in (M2) that $\ell'(t) = \max\{q'(t-1) + a'(t) - d'(t) - B', 0\}$, and hence it follows from the flow conservation property (M3) that $q'(t) = q'(t-1) + a'(t) - d'(t) - \ell'(t) = \min\{B', q'(t-1) + a'(t) - d'(t)\} \leq B'$. Therefore, there are at most B' packets buffered in an nFM with buffer size B' at any slot. As such, a packet admitted into an nFM with buffer size B' sees at most B' packets ahead of it, and it then follows from the nonidling and FIFO departure property in (M1) that the admitted packet is buffered in that nFM for at most B' slots.

Since an nFM1 with buffer size B is the concatenation of an nFM with buffer size $B - 1$ and a delay line with delay equal to one as shown Figure 4. we immediately see that there are at most B packets buffered in an nFM1 with buffer size B at any slot (as we have just seen that an nFM with buffer size $B - 1$ can buffer at most $B - 1$ packets and a delay line with delay equal to one can buffer at most one packet at any slot), and a packet admitted into an nFM1 with buffer size B is buffered in that nFM1 for at least one slot and at most B slots (as we have just seen that a packet admitted into an nFM with buffer size $B - 1$ is buffered in that nFM for at most $B - 1$ slots and it takes exactly one slot for a packet to traverse through a delay line with delay equal to one).

In Figure 6, we give a simple example to illustrate how an nFM1 with buffer size B works at a slot t , where $n = 4$ and $B = 3$: (i) We use a square box with a number inside it at a link to indicate that there is a packet from that link. (ii) We use three square boxes concatenated contiguously to represent the buffers of the nFM1 (note that $B = 3$). Each square box is capable

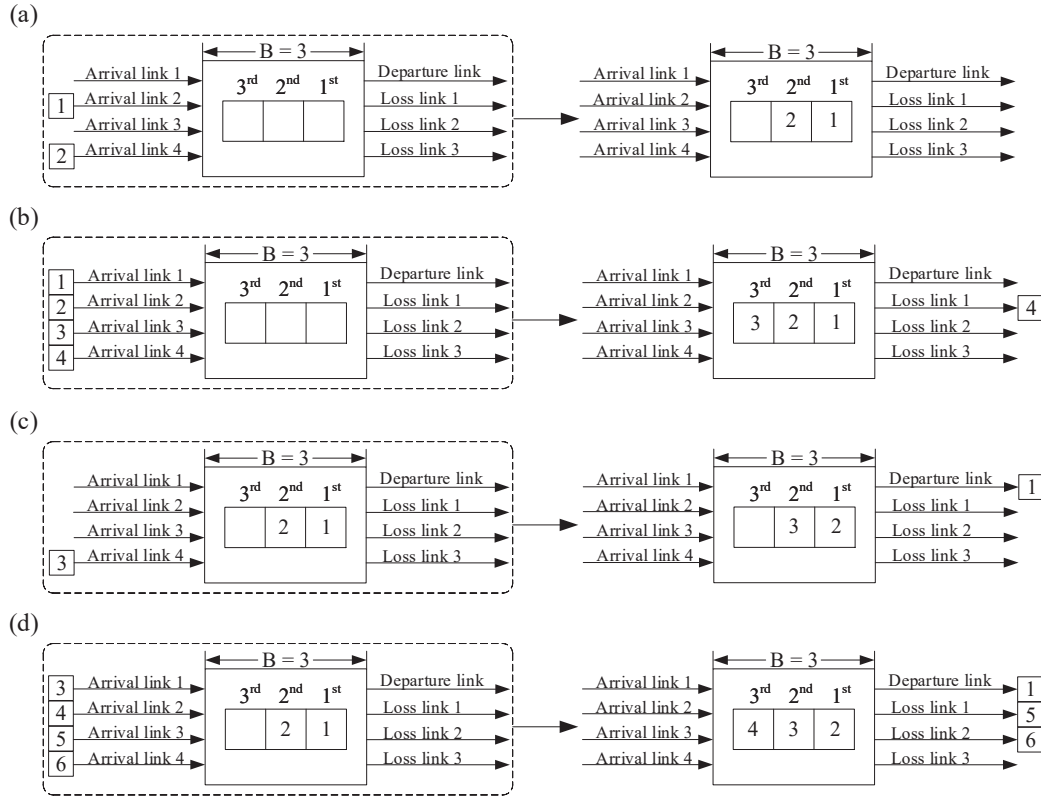


Fig. 6. An illustration of how an nFM1 with buffer size B works at a slot t , where $n = 4$ and $B = 3$: (a) $\tilde{q}(t-1) = 0$, $\tilde{a}_1(t) = \tilde{a}_3(t) = 0$, and $\tilde{a}_2(t) = \tilde{a}_4(t) = 1$. (b) $\tilde{q}(t-1) = 0$ and $\tilde{a}_1(t) = \tilde{a}_2(t) = \tilde{a}_3(t) = \tilde{a}_4(t) = 1$. (c) $\tilde{q}(t-1) = 2$, $\tilde{a}_1(t) = \tilde{a}_2(t) = \tilde{a}_3(t) = 0$, and $\tilde{a}_4(t) = 1$. (d) $\tilde{q}(t-1) = 2$ and $\tilde{a}_1(t) = \tilde{a}_2(t) = \tilde{a}_3(t) = \tilde{a}_4(t) = 1$. On the left-hand sides of (a)–(d), we show the system seen by an arrival packet at slot t . On the right-hand sides of (a)–(d), we show the system after it is operated as an nFM1 with buffer size B at slot t . Note that a packet in a square box with a smaller number inside it has an earlier arrival time than that in a square box with a larger number inside it.

of buffering one packet. A square box with (resp., without) a number inside it means that there is a packet (resp., there is no packet) buffered in that square box. (iii) Packets admitted into the nFM1 are stored in empty square boxes (starting from the rightmost and towards the leftmost empty box) according to their arrival times (from the earliest to the latest arrival time). (iv) A packet in a square box with a smaller number inside it has an earlier arrival time than that in a square box with a larger number inside it.

On the left-hand sides of Figure 6(a)–(d), we show the system seen by an arrival packet at slot t , i.e., we show the packets buffered in the nFM1 at slot “ $t-1$ ” and show the packets from the arrival links of the nFM1 at slot “ t .” On the right-hand sides of Figure 6(a)–(d), we show the system after it is operated as an nFM1 with buffer size B at slot t , i.e., we show the packets from the departure link and the loss links of the nFM1 at slot “ t ” and show the packets buffered in the nFM1 at slot “ t .” The details are described as follows:

- In Figure 6(a), two packets arrive from arrival links 2 and 4 at slot t and see an empty system, i.e., there are no packets buffered in the nFM1 at slot $t-1$ (see the left-hand side of Figure 6(a)). Thus, the packets from arrival links 2 and 4 are stored in the first square box and the second square box, respectively, at slot t (see the right-hand side of Figure 6(a)).
- In Figure 6(b), four packets arrive from the four arrival links at slot t and see an empty system (see the left-hand side of Figure 6(b)). Thus, the packets from arrival links 1, 2, and 3 are stored in the first square box, the second square box, and the third square box, respectively, at slot t , and the packet from arrival link 4 is immediately dropped from loss link 1 at slot t (see the right-hand side of Figure 6(b)).
- In Figure 6(c) and Figure 6(d), we show two cases that the arrival packets see a nonempty system, i.e., there are packets buffered in the nFM1 at slot $t-1$. In these cases, the packet buffered in the first square box at slot $t-1$ is sent out from the departure link of the nFM1 at slot t , each of the other packets buffered in the nFM1 at slot $t-1$ moves one step, i.e., one box, forward to the right and is stored there at slot t , and the arrival packets admitted into the nFM1 are stored in the remaining empty square boxes at slot t according to their arrival times (in the case that there is a buffer overflow, i.e., there are not enough empty square boxes to store all of the arrival packets, the latest arrival packets are immediately dropped from the loss links of the nFM1 at slot t).

C. The Operations of the Feedback System in Our Constructions

To describe the routing policy performed by the crossbar switch in the feedback system in Figure 2, at each slot we assign a distinct *buffering tag* to each of the packets that have to be buffered in the system (i.e., the packets that are not routed to the departure link or the loss link) according to its priority level. Specifically, the buffering tag $\tilde{\tau}_p(t)$ of a packet p that has to be buffered in the system at slot t is assigned as $\tilde{\tau}_p(t) = i$ if packet p is the i^{th} -highest-priority packet among the $q(t-1) + a(t) - d(t) - \ell(t)$ packets that have to be buffered in the system at slot t . Therefore, the $q(t-1) + a(t) - d(t) - \ell(t)$ packets that have to be buffered in the system at slot t are assigned buffering tags from 1 to $q(t-1) + a(t) - d(t) - \ell(t)$ in the order of decreasing priority.

Moreover, each group of FM1's in Figure 2 is associated with a *unique* set of buffering tags. Specifically, let U_k be the targeted buffer size of the optical priority queue in our constructions and let $U_0 = 0 < U_1 < U_2 < \dots < U_k$. Then we associate the i^{th} group of FM1's with the set $\Psi_i = \{U_{i-1} + 1, U_{i-1} + 2, \dots, U_i\}$ of buffering tags for $i = 1, 2, \dots, k$. Let $L_i = U_{i-1} + 1$ for $i = 1, 2, \dots, k$. Then we have $L_i \leq U_i$ and $\Psi_i = \{L_i, L_i + 1, \dots, U_i\}$ for $i = 1, 2, \dots, k$. Clearly, we have $U_i = \sum_{j=1}^i |\Psi_j|$ for $i = 1, 2, \dots, k$.

In our constructions, the crossbar switch in Figure 2 is operated according to the following routing policy at all slots $t \geq 1$.

(R1) *Routing to the departure link:* If there is a departure request from the controller and there are packets in the system at slot t , i.e., $c(t) = 1$ and $q(t-1) + a(t) \geq 1$, then the highest-priority packet (if any) among all of the packets from the arrival link or the output links of the *first* “ $s + 1$ ” groups of FM1’s is routed to the departure link at slot t . Otherwise, no packet is routed to the departure link at slot t .

(R2) *Routing to the loss link:* If there is a buffer overflow at slot t , i.e., $c(t) = 0$, $q(t-1) = U_k$, and $a(t) = 1$, then the lowest-priority packet (if any) among all of the packets from the arrival link or the output links of the *last* “ $s + 1$ ” groups of FM1’s is routed to the loss link at slot t . Otherwise, no packet is routed to the loss link at slot t .

(R3) *Priority-based routing to the k groups of FM1’s and round-robin routing to the FM1’s in each group and to the input links of each FM1:* (i) For a packet at an input link of the crossbar switch that has to be buffered in the system at slot t , it is routed to the group of FM1’s whose associated set of buffering tags contains the buffering tag of that packet. (ii) For packets routed to a group of FM1’s at slot t , they are *evenly* distributed to the FM1’s in that group in a *round-robin* manner, one by one, starting from the FM1 next to the one that was lastly used before slot t (or starting from the first FM1 in the case that no FM1 in that group has been used before slot t). (iii) Furthermore, for packets routed to an FM1 at slot t , they are also *evenly* distributed to the input links of that FM1 in a *round-robin* manner, one by one, starting from the input link next to the one that was lastly used before slot t (or starting from the first input link in the case that no input link of that FM1 has been used before slot t). As such, we can achieve load balancing among the FM1’s and fully utilize the buffering capacity of the FM1’s.

We will show in Theorem 1 (in Section III-A) that for $1 \leq s \leq k-1$ and $m_1, m_2, \dots, m_k \geq 1$, if we properly choose the parameters n_1, n_2, \dots, n_k and B_1, B_2, \dots, B_k in the feedback system in Figure 2 and the parameters $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ in the routing policy (R1)–(R3), then the feedback system in Figure 2 can be operated as an optical priority queue with buffer size U_k under the routing policy (R1)–(R3).

In the following, we give an example to illustrate how our construction of an optical priority queue works under the routing policy (R1)–(R3). In our example, we have $s = 1$, $k = 4$, and $m_1 = m_2 = m_3 = m_4 = 2$, and we choose $n_1 = n_2 = n_3 = n_4 = 4$, $B_1 = 1, B_2 = 2, B_3 = 2, B_4 = 1$, $|\Psi_1| = 1, |\Psi_2| = 3, |\Psi_3| = 3$, and $|\Psi_4| = 1$ (note that these parameters satisfy the conditions (A1)–(A3) in Theorem 1) so that we have $\Psi_1 = \{1\}$, $\Psi_2 = \{2, 3, 4\}$, $\Psi_3 = \{5, 6, 7\}$, and $\Psi_4 = \{8\}$ (see Figure 7(a)). Thus, the buffer size of the priority queue is $U_k = \sum_{i=1}^k |\Psi_i| = 8$. In our example, we only show the four groups of FM1’s (note that $k = 4$) in the feedback path of the feedback system in Figure 2: the four groups are arranged from the first group at the bottom to the fourth group at the top, and the two FM1’s in each group (note that $m_1 = m_2 = m_3 = m_4 = 2$) are so arranged that the first FM1 is on top of the second FM1 (see Figure 7(b)). For simplicity, we do not show the links of the FM1’s in Figure 7(b). Since the FM1’s are in the feedback path of the feedback system in Figure 2, we note that in Figure 7(b)

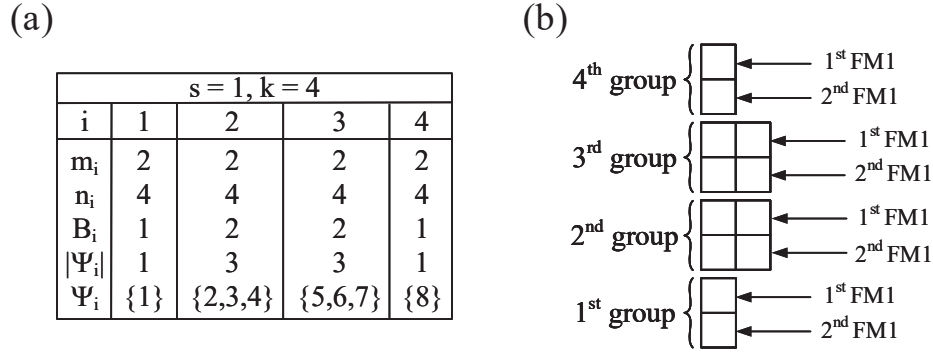


Fig. 7. (a) The parameters in our example for the illustration of how our construction of an optical priority queue works under the routing policy (R1)–(R3). (b) A schematic representation of the four groups of FM1’s in our example (for simplicity, we do not show the links of the FM1’s). Since the FM1’s are in the feedback path of the feedback system in Figure 2, we note that packets arrive at the FM1’s from the right and depart from the FM1’s from the left (note that Theorem 1 guarantees that there are no loss packets at the FM1’s).

packets arrive at the FM1’s from the right and depart from the FM1’s from the left (note that Theorem 1 guarantees that there are no loss packets at the FM1’s).

In Figure 8, we give a sample path for our example to illustrate how our construction of an optical priority queue works under the routing policy (R1)–(R3). In Figure 8(a), we show the states $a(t)$ and $c(t)$ of the arrival link and the control input, respectively, at slot t for $1 \leq t \leq 15$, and we also show the resulting states $d(t)$ and $\ell(t)$ of the departure link and the loss link, respectively, at slot t under the routing policy (R1)–(R3) for $1 \leq t \leq 15$. On the left-hand side for each slot t in Figure 8(b), we show the system seen by an arrival packet at slot t , i.e., we show the packets buffered in the system at slot “ $t - 1$ ” and show the packet from the arrival link of the system at slot “ t ,” and we also indicate which packet buffered in the system at slot $t - 1$ is routed to the departure link or the loss link of the system at slot “ t ” if the crossbar switch in Figure 2 is operated under the routing policy (R1)–(R3) at slot t . On the right-hand side for each slot t in Figure 8(b), we show the packets buffered in the system at slot “ t ” after the crossbar switch in Figure 2 is operated under the routing policy (R1)–(R3) at slot t .

We note that for each slot t in Figure 8(b), the number inside a square box is the *buffering tag* of the corresponding packet at slot “ t .” Furthermore, on the *left*-hand side for each slot t in Figure 8(b), the *dot* on the left of an FM1 in a group indicates that the corresponding FM1 is the FM1 in that group that was lastly used before slot “ t ” (in the case that no FM1 in a group has been used before slot “ t ,” there is no dot on the left of any FM1 in that group). Therefore, on the *right*-hand side for each slot t in Figure 8(b), the dot on the left of an FM1 in a group indicates that the corresponding FM1 is the FM1 in that group that was lastly used before or at slot t , i.e., before slot “ $t + 1$ ” (in the case that no FM1 in a group has been used before or at slot t , i.e., before slot “ $t + 1$,” there is no dot on the left of any FM1 in that group).

We describe how the routing policy (R1)–(R3) works for a few slots as follows.

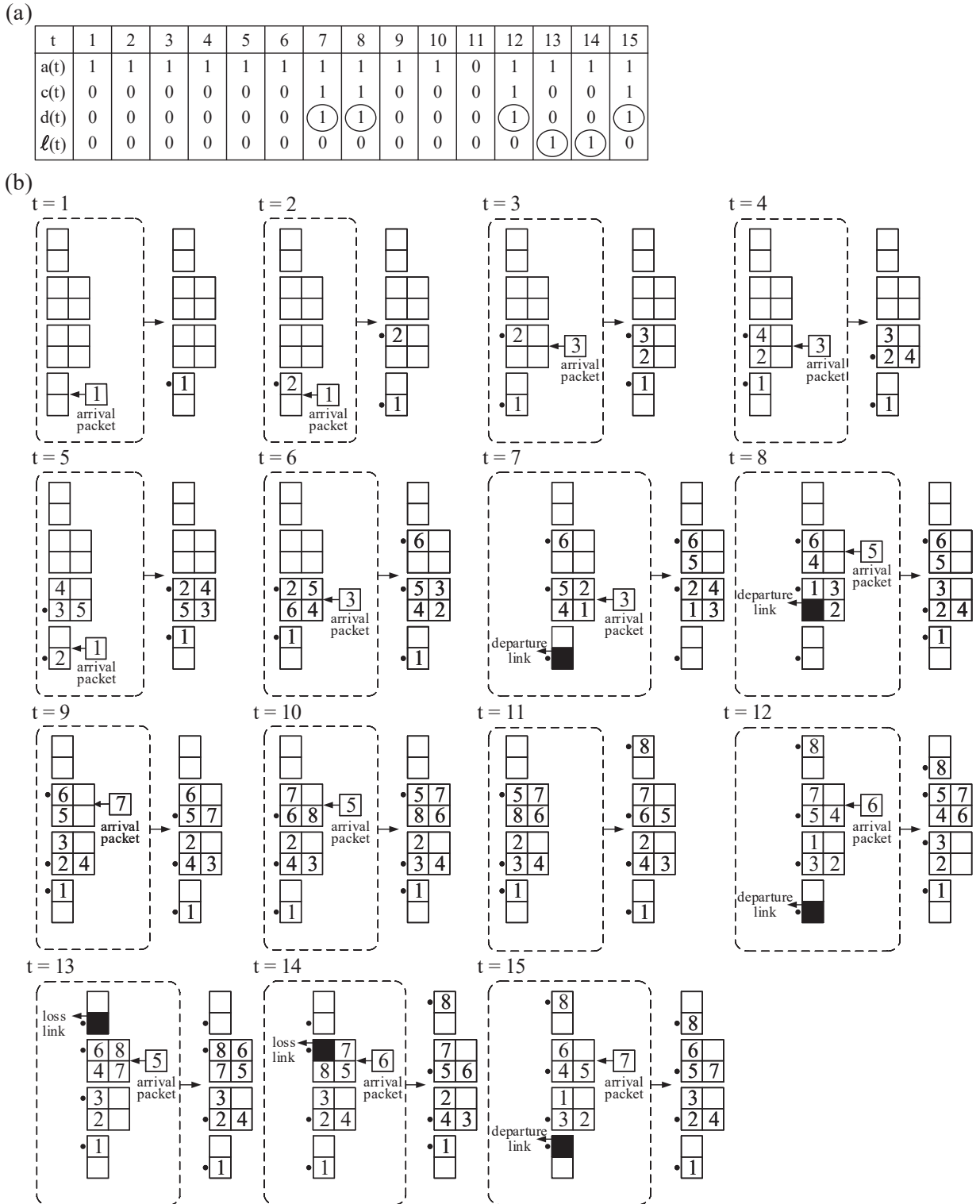


Fig. 8. A sample path for our example to illustrate how our construction of an optical priority queue works under the routing policy (R1)–(R3). (a) The states of the links in the system at slot t for $1 \leq t \leq 15$. (b) On the left-hand side for each slot t , we show the system seen by an arrival packet at slot t , and we also indicate which packet buffered in the system at slot $t - 1$ is routed to the departure link or the loss link of the system at slot “ t ” if the crossbar switch in Figure 2 is operated under the routing policy (R1)–(R3) at slot t . On the right-hand side for each slot t , we show the packets buffered in the system at slot “ t ” after the crossbar switch in Figure 2 is operated under the routing policy (R1)–(R3) at slot t . Note that for each slot t , the number inside a square box is the *buffering tag* of the corresponding packet at slot “ t .” Also note that on the *left*-hand side (resp., *right*-hand side) for each slot t , the dot on the left of an FMI in a group indicates that the corresponding FMI is the FMI in that group that was lastly used before slot “ t ” (resp., “ $t + 1$ ”).

- At slot $t = 1$, a packet arrives and sees an empty system, i.e., there are no packets buffered in the system at slot $t - 1 = 0$ (note that we have assumed that the system is initially empty at slot $t - 1 = 0$), and there is no departure request (see the left-hand side for slot $t = 1$ and recall that the number inside a square box is the *buffering tag* of the corresponding packet at slot $t = 1$). Thus, the arrival packet at slot $t = 1$ has to be buffered in the system and its buffering tag is assigned as 1 at slot $t = 1$. As such, as shown on the right-hand side for slot $t = 1$, the arrival packet is routed to the first group of FM1's (as $1 \in \Psi_1$) and stored in the first FM1 in the first group at slot $t = 1$ (as no FM1 in the first group has been used before slot $t = 1$).
- At slot $t = 2$, a packet arrives and sees a system with one packet, i.e., the packet buffered in the system at slot $t - 1 = 1$, where the packet buffered in the system at slot $t - 1 = 1$ has lower priority than the arrival packet, and there is no departure request (see the left-hand side for slot $t = 2$). Thus, the two packets in the system at slot $t = 2$ have to be buffered in the system and their buffering tags are assigned as 1 and 2 at slot $t = 2$. As such, as shown on the right-hand side for slot $t = 2$, the packet with buffering tag assigned as 1 is routed to the first group of FM1's (as $1 \in \Psi_1$) and stored in the second FM1 in the first group at slot $t = 2$ (as the first FM1 is the FM1 in the first group that was lastly used before slot $t = 2$), and the packet with buffering tag assigned as 2 is routed to the second group of FM1's (as $2 \in \Psi_2$) and stored in the first FM1 in the second group at slot $t = 2$ (as no FM1 in the second group has been used before slot $t = 2$).
- At slot $t = 7$, a packet arrives and sees a system with six packets, i.e., the six packets buffered in the system at slot $t - 1 = 6$, where three of the six packets buffered in the system at slot $t - 1 = 6$ have lower priorities than the arrival packet, and there is a departure request (see the left-hand side for slot $t = 7$). Thus, there is a departure packet at slot $t = 7$, and the packet with *tag* assigned as 1 at slot $t = 7$, i.e., the packet buffered in the second FM1 in the first group at slot $t - 1 = 6$, is routed to the departure link of the priority queue at slot $t = 7$ (see the left-hand side for slot $t = 7$). Moreover, as shown on the right-hand side for slot $t = 7$, the packets with buffering tags assigned as 1 and 2 at slot $t = 7$ move one step forward to the left at slot $t = 7$, the packets with buffering tags assigned as 3 and 4 at slot $t = 7$ are routed to the second group of FM1's (as $3, 4 \in \Psi_2$) and stored in the second FM1 and the first FM1, respectively, in the second group at slot $t = 7$ (as the first FM1 is the FM1 in the second group that was lastly used before slot $t = 7$), and the packets with buffering tags assigned as 5 and 6 at slot $t = 7$ are routed to the third group of FM1's (as $5, 6 \in \Psi_3$) and stored in the second FM1 and the first FM1, respectively, in the third group at slot $t = 7$ (as the first FM1 is the FM1 in the third group that was lastly used before slot $t = 7$).
- At slot $t = 13$, a packet arrives and sees a full system, i.e., there are $U_k = 8$ packets buffered in the system at slot $t - 1 = 12$, where four of the eight packets buffered in the

system at slot $t - 1 = 12$ have lower priorities than the arrival packet, and there is no departure request (see the left-hand side for slot $t = 13$). Thus, there is a buffer overflow at slot $t = 13$, and the packet with *tag* assigned as $U_k + 1 = 9$ at slot $t = 13$, i.e., the packet buffered in the second FM1 in the fourth group at slot $t - 1 = 12$, is routed to the loss link of the priority queue at slot $t = 13$ (see the left-hand side for slot $t = 13$). Moreover, as shown on the right-hand side for slot $t = 13$, the packet with buffering tag assigned as 1 at slot $t = 13$ is routed to the first group of FM1's (as $1 \in \Psi_1$) and stored in the second FM1 in the first group at slot $t = 13$ (as the first FM1 is the FM1 in the first group that was lastly used before slot $t = 13$), the packets with buffering tags assigned as 2, 3, and 4 at slot $t = 13$ are routed to the second group of FM1's (as $2, 3, 4 \in \Psi_2$) and stored in the second FM1, the first FM1, and the second FM1, respectively, in the second group at slot $t = 13$ (as the first FM1 is the FM1 in the second group that was lastly used before slot $t = 13$), the packets with buffering tags assigned as 7 and 8 at slot $t = 13$ move one step forward to the left at slot $t = 13$, and the packets with buffering tags assigned as 5 and 6 at slot $t = 13$ are routed to the third group of FM1's (as $5, 6 \in \Psi_3$) and stored in the second FM1 and the first FM1, respectively, in the third group at slot $t = 13$ (as the first FM1 is the FM1 in the third group that was lastly used before slot $t = 13$).

III. THE CONSTRUCTIONS AND THE COMPARISON WITH THE CONSTRUCTIONS IN [37]

A. The Constructions

For given $1 \leq s \leq k - 1$ and $m_1, m_2, \dots, m_k \geq 1$, we show in the following theorem how to choose the parameters n_1, n_2, \dots, n_k , B_1, B_2, \dots, B_k , and $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ so that the feedback system in Figure 2 can be operated as an optical priority queue under the routing policy (R1)–(R3).

Theorem 1 *Assume that the feedback system in Figure 2 is operated under the routing policy (R1)–(R3) at all slots. Suppose that $1 \leq s \leq k - 1$ and $m_1, m_2, \dots, m_k \geq 1$. Also suppose that the parameters n_1, n_2, \dots, n_k , B_1, B_2, \dots, B_k , and $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ satisfy the following conditions (A1)–(A3):*

(A1) *The condition for n_1, n_2, \dots, n_k :*

$$n_i \geq \lceil (\sum_{j=j_1}^{j_2} m_j + 1) / m_i \rceil, \quad (1)$$

where j_1 and j_2 are given as follows (note that j_1 and j_2 depend on i): If k is even, say $k = 2\ell$, then

$$\begin{cases} j_1 = \max\{i - 1, 1\} \text{ and } j_2 = \min\{i + s, \ell + 1\}, & \text{if } 1 \leq i \leq \ell, \\ j_1 = \max\{i - s, \ell\} \text{ and } j_2 = \min\{i + 1, k\}, & \text{if } \ell + 1 \leq i \leq k. \end{cases} \quad (2)$$

On the other hand, if k is odd, say $k = 2\ell - 1$, then

$$\begin{cases} j_1 = \max\{i - 1, 1\} \text{ and } j_2 = \min\{i + s, \ell\}, & \text{if } 1 \leq i \leq \ell - 1, \\ j_1 = i - 1 \text{ and } j_2 = i + 1, & \text{if } i = \ell, \\ j_1 = \max\{i - s, \ell\} \text{ and } j_2 = \min\{i + 1, k\}, & \text{if } \ell + 1 \leq i \leq k. \end{cases} \quad (3)$$

(A2) The condition for B_1, B_2, \dots, B_k :

$$1 \leq B_i \leq \begin{cases} U_{i-1} + 1, & \text{if } 1 \leq i \leq s + 1, \\ U_{i-1} - U_{i-s-1}, & \text{if } s + 2 \leq i \leq k, \end{cases} \quad (4)$$

and

$$1 \leq B_i \leq \begin{cases} U_{i+s} - U_i, & \text{if } 1 \leq i \leq k - s - 1, \\ U_k - U_i + 1, & \text{if } k - s \leq i \leq k. \end{cases} \quad (5)$$

Note that we have $B_1 = B_k = 1$.

(A3) The condition for $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$:

$$1 \leq |\Psi_i| \leq (m_i - 1)B_i + 1 \text{ for } 1 \leq i \leq k, \quad (6)$$

and

$$|\Psi_i| \geq \begin{cases} B_{i-1}, & \text{if } 2 \leq i \leq \lceil k/2 \rceil, \\ B_{i+1}, & \text{if } \lfloor k/2 \rfloor + 1 \leq i \leq k - 1. \end{cases} \quad (7)$$

Then the feedback system in Figure 2 can be operated as an optical priority queue with buffer size $U_k = \sum_{i=1}^k |\Psi_i|$ at all slots.

We need the following results in Lemma 2 from [37] for the proof of Theorem 1. Roughly speaking, the upper (resp., lower) bound in Lemma 2(i) holds because there is at most one arrival (resp., departure) packet with higher priority than a given packet at any slot, and Lemma 2(ii)(iii) hold because a packet admitted into an FM1 with buffer size B_i can be buffered in that FM1 for at most B_i slots for all $1 \leq i \leq k$.

Lemma 2 Assume that the feedback system in Figure 2 is operated under the routing policy (R1)–(R3) at all slots. Suppose that the properties (P1) and (P2) are satisfied up to slot t , and the property (P3) is satisfied up to slot $t - 1$.

(i) [37, Theorem 3] The buffering tag of a packet can either increase by at most one or decrease by at most one at slot t . Precisely, if a packet p is buffered in the system at slot $t - 1$ and has to be buffered in the system at slot t , then we have

$$-1 \leq \tilde{\tau}_p(t) - \tilde{\tau}_p(t - 1) \leq 1. \quad (8)$$

(ii) [37, Theorem 5] The buffering tag of a packet p buffered in the i^{th} group of FM1's at slot t is limited to the range $[L_i - B_i + 1, U_i + B_i - 1]$, i.e.,

$$L_i - B_i + 1 \leq \tilde{\tau}_p(t) \leq U_i + B_i - 1, \text{ for all } 1 \leq i \leq k. \quad (9)$$

(iii) [37, Theorem 6] There are at most $|\Psi_i| + B_i - 1$ packets buffered in or routed to the i^{th} group of FM1's at slot t for all $1 \leq i \leq k$.

Remark 3 We note that all of the bounds in Lemma 2 are tight as can be seen from our illustration sample path in Figure 8(b):

(i) The inequality for the upper (resp., lower) bound in (8) holds with equality at slots $t = 2, 4, 5, 6, 10, 13,$ and 14 (resp., slots $t = 7, 8, 12,$ and 15) in Figure 8(b).

(ii) For $i = 1$, we have $U_i + B_i - 1 = 1$ and $L_i - B_i + 1 = 1$, and hence both the inequality for the upper bound and the inequality for the lower bound in (9) hold with equalities at slots $t \in \{1, 2, \dots, 15\} \setminus \{7\}$. For $i = 2$, we have $U_i + B_i - 1 = 5$ and $L_i - B_i + 1 = 1$, and hence the inequality for the upper (resp., lower) bound in (9) holds with equality at slots $t = 5$ and 6 (resp. slot $t = 7$). For $i = 3$, we have $U_i + B_i - 1 = 8$ and $L_i - B_i + 1 = 4$, and hence the inequality for the upper (resp., lower) bound in (9) holds with equality at slots $t = 10$ and 13 (resp. slot $t = 12$). Finally, for $i = 4$, we have $U_i + B_i - 1 = 8$ and $L_i - B_i + 1 = 8$, and hence both the inequality for the upper bound and the inequality for the lower bound in (9) hold with equalities at slots $t = 11, 12, 14,$ and 15 .

(iii) For $i = 1$, we have $|\Psi_i| + B_i - 1 = 1$, and hence the upper bound in Lemma 2(iii) holds with equality at slots $t \in \{1, 2, \dots, 15\} \setminus \{7\}$. For $i = 2$, we have $|\Psi_i| + B_i - 1 = 4$, and hence the upper bound in Lemma 2(iii) holds with equality at slots $t = 5, 6,$ and 7 . For $i = 3$, we have $|\Psi_i| + B_i - 1 = 4$, and hence the upper bound in Lemma 2(iii) holds with equality at slots $t = 10, 12,$ and 13 . Finally, for $i = 4$, we have $|\Psi_i| + B_i - 1 = 1$, and hence the upper bound in Lemma 2(iii) holds with equality at slots $t = 11, 12, 14,$ and 15 .

Proof. (Proof of Theorem 1) We prove Theorem 1 by showing that the properties (P1)–(P3) are satisfied at all slots $t \geq 1$. We do so by induction on t . First consider slot $t = 1$. (i) If there is no departure request from the controller or there are no packets in the system at slot $t = 1$, then no packet is routed to the departure link at slot $t = 1$ (according to the routing policy (R1)), and hence the nonidling and priority departure property (P1) is satisfied at slot $t = 1$. So assume that there is a departure request from the controller and there are packets in the system at slot $t = 1$. Since we have assumed that the system is initially empty at slot $t - 1 = 0$, i.e., $q(t - 1) = 0$, it is clear that there is a packet from the arrival link at slot $t = 1$ and that arrival packet is the only packet in the system at slot $t = 1$. It follows that the arrival packet at slot $t = 1$ is the highest-priority packet in the system and is routed to the departure link immediately at slot $t = 1$ (according to the routing policy (R1)). Thus, the nonidling and priority departure property (P1) is satisfied at slot $t = 1$. (ii) As it is clear from $q(t - 1) = 0 < U_k$ that there is no buffer overflow at slot $t = 1$, no packet is routed to the loss link at slot $t = 1$ (according to the routing policy (R2)). Thus, the maximum buffer usage and priority loss property (P2) is satisfied at slot $t = 1$. (iii) Since the FM1's are initially empty at slot $t - 1 = 0$, it is clear that there is no buffer overflow at any FM1 at slot $t = 1$. Moreover, since there are no packets from the

outputs of the FM1's at slot $t = 1$ (as the FM1's are initially empty at slot $t - 1 = 0$) and there is at most one arrival packet at slot $t = 1$ (as there is only one arrival link), we see that there is at most one packet at the input links of the crossbar switch at slot $t = 1$ and hence there is at most one packet routed to any input link of any FM1 at slot $t = 1$ (according to the routing policy (R3)). As such, there is no collision at any input link of any FM1 at slot $t = 1$. Thus, there is no internal packet loss in the system at slot $t = 1$, and hence the flow conservation property (P3) is satisfied at slot $t = 1$.

Now assume as the induction hypothesis that the properties (P1)–(P3) are satisfied up to slot $t - 1$ for some $t - 1 \geq 1$. In the following, we complete the induction by showing that the properties (P1)–(P3) are satisfied at slot t .

(i) *The nonidling and priority departure property (P1) is satisfied at slot t .* If there is no departure request from the controller or there are no packets in the system at slot t , then no packet is routed to the departure link at slot t (according to the routing policy (R1)), and hence the property (P1) is satisfied at slot t . So assume that there is a departure request from the controller and there are packets in the system at slot t , i.e., $c(t) = 1$ and $q(t-1) + a(t) \geq 1$. Let packet p be the highest-priority packet in the system at slot t , i.e., $\tau_p(t) = 1$. In the following, we show that packet p is either from the arrival link or from the output links of the *first* " $s + 1$ " groups of FM1's at slot t . It then follows from the routing policy (R1) that packet p is sent out from the departure link at slot t . Thus, the property (P1) is satisfied at slot t .

If packet p is an arrival packet at slot t , then we are done. So assume that packet p is not an arrival packet at slot t . Then packet p must be buffered in the system at slot $t - 1$. Let $a_p(t)$ be the number of arrival packets at slot t with higher priorities than packet p . Since the flow conservation property (P3) is satisfied at slot $t - 1$ (by the induction hypothesis), we know that there is no internal packet loss at slot $t - 1$ and hence we have

$$\tau_p(t) = \tilde{\tau}_p(t-1) + a_p(t). \quad (10)$$

From (10) and $a_p(t) \geq 0$, we have $\tilde{\tau}_p(t-1) = \tau_p(t) - a_p(t) \leq \tau_p(t) = 1$. As we also have $\tilde{\tau}_p(t-1) \geq 1$, it follows that $\tilde{\tau}_p(t-1) = 1$.

We claim that packet p is buffered in the first $s + 1$ groups of FM1's at slot $t - 1$. Assume on the contrary that packet p is buffered in the last $k - s - 1$ groups of FM1's at slot $t - 1$, say packet p is buffered in the i^{th} group of FM1's at slot $t - 1$ for some $s + 2 \leq i \leq k$. Then we have

$$L_i - B_i + 1 = (U_{i-1} + 1) - B_i + 1 \geq U_{i-s-1} + 2 > U_0 + 2 = 2 > 1 = \tilde{\tau}_p(t-1),$$

where the first inequality follows from (4) in (A2) and the second inequality follows from $U_0 < U_1 < \dots < U_k$ and $i - s - 1 > 0$. Thus, we have reached a contradiction to $L_i - B_i + 1 \leq \tilde{\tau}_p(t-1)$ in Lemma 2(ii).

We have just shown that packet p is buffered in the first $s + 1$ groups of FM1's at slot $t - 1$, say packet p is buffered in the i^{th} group of FM1's at slot $t - 1$ for some $1 \leq i \leq s + 1$. Let t'

be the slot that packet p is routed to the i^{th} group of FM1's for the last time before or at slot $t - 1$. Since we know that a packet can be buffered in an FM1 with buffer size B_i for at most B_i slots, we have $t' \leq t - 1 \leq t' + B_i - 1$. If $t' \leq t - 1 \leq t' + B_i - 2$, then we have

$$\begin{aligned} \tilde{\tau}_p(t - 1) &= \tilde{\tau}_p(t') + \sum_{\ell=1}^{t-t'-1} (\tilde{\tau}_p(t' + \ell) - \tilde{\tau}_p(t' + \ell - 1)) \geq L_i - (t - t' - 1) \cdot 1 \\ &\geq L_i - (B_i - 2) = (U_{i-1} + 1) - (B_i - 2) \geq 2 > 1 = \tilde{\tau}_p(t - 1), \end{aligned}$$

where the first inequality follows from $L_i \leq \tilde{\tau}_p(t') \leq U_i$ (according to the routing policy (R3)), Lemma 2(i), and $t - t' - 1 \geq 0$, the second inequality follows from $t - t' - 1 \leq B_i - 2$, and the third inequality follows from (4) in (A2). Thus, we have reached an apparent contradiction. As a result, it must be the case that $t - 1 = t' + B_i - 1$ and packet p is buffered in one of the FM1's in the i^{th} group at slots $t', t' + 1, \dots, t - 1$ so that it is buffered there for $t - t' = B_i$ slots. As we know that a packet can be buffered in an FM1 with buffer size B_i for at most B_i slots, it is clear that packet p must leave from the i^{th} group of FM1's at slot t , and we are done. (In our illustration sample path in Figure 8(b), we have $s = 1$: for the case that $i = 1$, see slots $t = 7, 12$, and 15 for such a departure packet; and for the case that $i = 2 = s + 1$, see slot $t = 8$ for such a departure packet.)

(ii) *The maximum buffer usage and priority loss property (P2) is satisfied at slot t .* If there is no buffer overflow at slot t , then no packet is routed to the loss link at slot t (according to the routing policy (R2)), and hence the property (P2) is satisfied at slot t . So assume that there is a buffer overflow at slot t , i.e., $c(t) = 0$, $q(t - 1) = U_k$, and $a(t) = 1$. Let packet p be the lowest-priority packet in the system at slot t , i.e., $\tau_p(t) = U_k + 1$. In the following, we show that packet p is either from the arrival link or from the output links of the *last* " $s + 1$ " groups of FM1's at slot t . It then follows from the routing policy (R2) that packet p is dropped from the loss link at slot t . Thus, the property (P2) is satisfied at slot t .

If packet p is an arrival packet at slot t , then we are done. So assume that packet p is not an arrival packet at slot t . Then packet p must be buffered in the system at slot $t - 1$. Clearly, (10) in the proof of (i) above still holds. From (10) and $a_p(t) \leq 1$, we have $\tilde{\tau}_p(t - 1) = \tau_p(t) - a_p(t) \geq \tau_p(t) - 1 = U_k$. As we also have $\tilde{\tau}_p(t - 1) \leq q(t - 1) = U_k$, it follows that $\tilde{\tau}_p(t - 1) = U_k$.

We claim that packet p is buffered in the last $s + 1$ groups of FM1's at slot $t - 1$. Assume on the contrary that packet p is buffered in the first $k - s - 1$ groups of FM1's at slot $t - 1$, say packet p is buffered in the i^{th} group of FM1's at slot $t - 1$ for some $1 \leq i \leq k - s - 1$. Then we have

$$U_i + B_i - 1 \leq U_{i+s} - 1 < U_k - 1 < U_k = \tilde{\tau}_p(t - 1),$$

where the first inequality follows from (5) in (A2) and the second inequality follows from $U_0 < U_1 < \dots < U_k$ and $i + s < k$. Thus, we have reached a contradiction to $\tilde{\tau}_p(t - 1) \leq U_i + B_i - 1$ in Lemma 2(ii).

We have just shown that packet p is buffered in the last $s + 1$ groups of FM1's at slot $t - 1$, say packet p is buffered in the i^{th} group of FM1's at slot $t - 1$ for some $k - s \leq i \leq k$. Let t'

be the slot that packet p is routed to the i^{th} group of FM1's for the last time before or at slot $t - 1$. As in the proof of (i) above, we have $t' \leq t - 1 \leq t' + B_i - 1$. If $t' \leq t - 1 \leq t' + B_i - 2$, then we have

$$\begin{aligned} \tilde{\tau}_p(t - 1) &= \tilde{\tau}_p(t') + \sum_{\ell=1}^{t-t'-1} (\tilde{\tau}_p(t' + \ell) - \tilde{\tau}_p(t' + \ell - 1)) \leq U_i + (t - t' - 1) \cdot 1 \\ &\leq U_i + (B_i - 2) \leq U_k - 1 < U_k = \tilde{\tau}_p(t - 1), \end{aligned}$$

where the first inequality follows from $L_i \leq \tilde{\tau}_p(t') \leq U_i$ (according to the routing policy (R3)), Lemma 2(i), and $t - t' - 1 \geq 0$, the second inequality follows from $t - t' - 1 \leq B_i - 2$, and the third inequality follows from (5) in (A2). Thus, we have reached an apparent contradiction. As in the proof of (i) above, it must be the case that $t - 1 = t' + B_i - 1$ and packet p must leave from the i^{th} group of FM1's at slot t , and we are done. (In our illustration sample path in Figure 8(b), we have $s = 1$ and $k = 4$: for the case that $i = k - s = 3$, see slot $t = 14$ for such a loss packet; and for the case that $i = 4 = k$, see slot $t = 13$ for such a loss packet.)

(iii) *The flow conservation property (P3) is satisfied at slot t .* From the induction hypothesis and the results in (i) and (ii) above, we see that the properties (P1) and (P2) are satisfied up to slot t and the property (P3) is satisfied up to slot $t - 1$. This will be used to prove the following results: (a) There is no buffer overflow at any FM1 at slot t . (b) There is no collision at any input link of any FM1 at slot t . If this can be done, then there is no internal packet loss in the system at slot t , and hence the flow conservation property (P3) is satisfied at slot t .

(a) *There is no buffer overflow at any FM1 at slot t .* Consider the i^{th} group of FM1's, where $1 \leq i \leq k$. From Lemma 2(iii) and (6) in (A3), we know that the number of packets buffered in or routed to the i^{th} group of FM1's at slot t is at most $|\Psi_i| + B_i - 1 \leq m_i B_i$. As packets routed to the i^{th} group of FM1's are evenly distributed to the m_i FM1's in the i^{th} group (according to the round-robin routing policy (R3)), it is easy to see that the number of packets buffered in or routed to each FM1 in the i^{th} group at slot t is at most $\lceil (m_i B_i) / m_i \rceil = B_i$. Thus, there is no buffer overflow at any FM1 in the i^{th} group at slot t .

(b) *There is no collision at any input link of any FM1 at slot t .* The proof consists of two steps. First, for each $1 \leq j \leq k$, we will identify the groups of FM1's to which packets that are from the output links of the j^{th} group of FM1's and have to be buffered in the system at slot t can possibly be routed. Second, for each $1 \leq i \leq k$, we will use the results obtained in the first step to show that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $j_1 \leq j \leq j_2$, where j_1 and j_2 are given by (2) and (3). The proof involves a detailed counting argument and will be given in Appendix A.

If this can be done, then there are at most $\sum_{j=j_1}^{j_2} m_j + 1$ packets routed to the i^{th} group of FM1's at slot t for all $1 \leq i \leq k$. As there are $m_i n_i$ input links at the i^{th} group of FM1's and we have from (A1) that $m_i n_i \geq \sum_{j=j_1}^{j_2} m_j + 1$, it follows from the round-robin routing policy (R3) that there is at most one packet routed to any input link of any FM1 in the i^{th} group of

FM1's at slot t for all $1 \leq i \leq k$. Therefore, there is no collision at any input link of any FM1 in the i^{th} group of FM1's at slot t for all $1 \leq i \leq k$, and the proof is completed. ■

For the scenario that each group of FM1's in Figure 2 has the same number of FM1's, say $m_i = m$ for all $1 \leq i \leq k$, the condition (A1) can be simplified as follows. If k is even, say $k = 2\ell$, then $\ell = k/2 \geq \lceil (s+1)/2 \rceil \geq 1$ and the condition (A1) can be simplified as follows:

$$n_i \geq \begin{cases} \min\{i+s, \ell+1\} - \max\{i-1, 1\} + 2, & \text{if } 1 \leq i \leq \ell, \\ \min\{i+1, k\} - \max\{i-s, \ell\} + 2, & \text{if } \ell+1 \leq i \leq k. \end{cases} \quad (11)$$

When $1 \leq \ell \leq s$, (11) can be expressed as

$$n_i \geq \begin{cases} \ell+2, & \text{if } i=1, \\ \ell-i+4, & \text{if } 2 \leq i \leq \ell, \\ i-\ell+3, & \text{if } \ell+1 \leq i \leq k-1, \\ \ell+2, & \text{if } i=k. \end{cases} \quad (12)$$

When $\ell \geq s+1$, (11) can be expressed as

$$n_i \geq \begin{cases} s+2, & \text{if } i=1, \\ s+3, & \text{if } 2 \leq i \leq \ell-s, \\ \ell-i+4, & \text{if } \ell-s+1 \leq i \leq \ell, \\ i-\ell+3, & \text{if } \ell+1 \leq i \leq \ell+s, \\ s+3, & \text{if } \ell+s+1 \leq i \leq k-1, \\ s+2, & \text{if } i=k. \end{cases} \quad (13)$$

On the other hand, if k is odd, say $k = 2\ell - 1$, then $\ell = (k+1)/2 \geq \lceil ((s+1)+1)/2 \rceil \geq 2$ and the condition (A1) can be simplified as follows:

$$n_i \geq \begin{cases} \min\{i+s, \ell\} - \max\{i-1, 1\} + 2, & \text{if } 1 \leq i \leq \ell-1, \\ 4, & \text{if } i=\ell, \\ \min\{i+1, k\} - \max\{i-s, \ell\} + 2, & \text{if } \ell+1 \leq i \leq k. \end{cases} \quad (14)$$

When $2 \leq \ell \leq s+1$, (14) can be expressed as

$$n_i \geq \begin{cases} \ell+1, & \text{if } i=1, \\ \ell-i+3, & \text{if } 2 \leq i \leq \ell-1, \\ 4, & \text{if } i=\ell, \\ i-\ell+3, & \text{if } \ell+1 \leq i \leq k-1, \\ \ell+1, & \text{if } i=k. \end{cases} \quad (15)$$

When $\ell \geq s + 2$, (14) can be expressed as

$$n_i \geq \begin{cases} s + 2, & \text{if } i = 1, \\ s + 3, & \text{if } 2 \leq i \leq \ell - s - 1, \\ \ell - i + 3, & \text{if } \ell - s \leq i \leq \ell - 1, \\ 4, & \text{if } i = \ell, \\ i - \ell + 3, & \text{if } \ell + 1 \leq i \leq \ell + s, \\ s + 3, & \text{if } \ell + s + 1 \leq i \leq k - 1, \\ s + 2, & \text{if } i = k. \end{cases} \quad (16)$$

Furthermore, for the scenario that each group of FM1's in Figure 2 has the same number of FM1's and each FM1 in Figure 2 has the same number of arrival links, say $m_i = m$ and $n_i = n$ for all $1 \leq i \leq k$, the condition (A1) can be simplified as follows. If k is even, say $k = 2\ell$, then we see from (12) and (13) that

$$n \geq \min\{\ell + 2, s + 3\} = \begin{cases} \ell + 2, & \text{if } 1 \leq \ell \leq s, \\ s + 3, & \text{if } \ell \geq s + 1. \end{cases} \quad (17)$$

On the other hand, if k is odd, say $k = 2\ell - 1$, then we see from (15) and (16) that

$$n \geq \min\{\max\{\ell + 1, 4\}, s + 3\} = \begin{cases} 4, & \text{if } \ell = 2, \\ \ell + 1, & \text{if } 3 \leq \ell \leq s + 1, \\ s + 3, & \text{if } \ell \geq s + 2. \end{cases} \quad (18)$$

It is easy to see that we can combine (17) and (18) into the following condition for n :

$$n \geq \begin{cases} k + 1, & \text{if } 2 \leq k \leq 3, \\ \min\{\lfloor k/2 \rfloor + 2, s + 3\}, & \text{if } k \geq 4. \end{cases} \quad (19)$$

B. Comparison With the Constructions in [37]

The constructions in [37] are for the scenario that each group of FM1's in Figure 2 has the same number of FM1's and each FM1 in Figure 2 has the same number of arrival links, say $m_i = m$ and $n_i = n$ for all $1 \leq i \leq k$. In the constructions in [37, Theorem 7], the parameters n , B_1, B_2, \dots, B_k , and $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ have to satisfy the following conditions (A1')–(A3'): (A1') The condition for n :

$$n \geq \min\{k + 1, 2s + 2\}. \quad (20)$$

If k is even, say $k = 2\ell$, then $\ell = k/2 \geq \lceil (s + 1)/2 \rceil \geq 1$ and (20) can be expressed as

$$n \geq \begin{cases} 2\ell + 1, & \text{if } 1 \leq \ell \leq s, \\ 2s + 2, & \text{if } \ell \geq s + 1. \end{cases} \quad (21)$$

On the other hand, if k is odd, say $k = 2\ell - 1$, then $\ell = (k+1)/2 \geq \lceil ((s+1)+1)/2 \rceil \geq 2$ and (20) can be expressed as

$$n \geq \begin{cases} 2\ell, & \text{if } 2 \leq \ell \leq s+1, \\ 2s+2, & \text{if } \ell \geq s+2. \end{cases} \quad (22)$$

(A2') The condition for B_1, B_2, \dots, B_k : $B_1 = B_k = 1$,

$$1 \leq B_i \leq \begin{cases} U_{i-1}, & \text{if } 2 \leq i \leq s+1, \\ U_{i-1} - U_{i-s-1}, & \text{if } s+2 \leq i \leq k, \end{cases} \quad (23)$$

and

$$1 \leq B_i \leq \begin{cases} U_{i+s} - U_i, & \text{if } 1 \leq i \leq k-s-1, \\ U_k - U_i, & \text{if } k-s \leq i \leq k-1. \end{cases} \quad (24)$$

(A3') The condition for $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$:

$$1 \leq |\Psi_i| \leq (m-1)B_i + 1 \text{ for all } 1 \leq i \leq k, \quad (25)$$

We claim that the condition for n in (20) is more restricted than that in (19). This can be proved as follows. If $2 \leq k \leq 3$, then both (19) and (20) give the same lower bound on n , i.e., $n \geq k+1$. If $k \geq 4$ and $s = 1$, then both (19) and (20) give the same lower bound on n , i.e., $n \geq 4$. Finally, if $k \geq 4$ and $2 \leq s \leq k-1$, then the lower bound in (19) is *smaller* than that in (20) (this can be seen from (17), (18), (21), and (22) as follows: if k is even, say $k = 2\ell$, then we have from $k \geq 4$ that $\ell \geq 2$, and hence the lower bound $\ell + 2$ in (17) is smaller than the lower bound $2\ell + 1$ in (21) when $2 \leq \ell \leq s$, and the lower bound $s + 3$ in (17) is smaller than the lower bound $2s + 2$ in (21) when $\ell \geq s + 1$; on the other hand, if k is odd, say $k = 2\ell - 1$, then we have from $k \geq 4$ that $\ell \geq 3$, and hence the lower bound $\ell + 1$ in (18) is smaller than the lower bound 2ℓ in (22) when $3 \leq \ell \leq s + 1$, and the lower bound $s + 3$ in (18) is smaller than the lower bound $2s + 2$ in (22) when $\ell \geq s + 2$).

Also, it is apparent that the condition for B_1, B_2, \dots, B_k in (23) and (24) is more restricted than that in (4) and (5), and condition for $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ in (25) is the same as that in (6) (with $m_i = m$ in (6) for $1 \leq i \leq k$).

From the above arguments, we immediately see that our constructions subsume those in [37] that satisfy (7) as special cases. In particular, our constructions subsume the optimal constructions in [37] as special cases (since the optimal constructions in [37] satisfy (7)). Therefore, our constructions can be regarded as an extension of those in [37] that satisfy (7).

Moreover, our constructions generalize those in [37] from the scenario that each group of FM1's has the same number of FM1's and each FM1 has the same number of arrival links to the scenario that each group of FM1's may have a different number of FM1's and the FM1's in different groups may have different numbers of arrival links, i.e., m_1, m_2, \dots, m_k can be different and n_1, n_2, \dots, n_k can also be different. The generalization will be used to show that our constructions possess fault-tolerant capability in Section IV-C.

IV. OPTIMAL CONSTRUCTIONS WITH MINIMUM CONSTRUCTION COMPLEXITIES AND MAXIMUM BUFFER SIZES

In this section, we give an analysis on the optimal constructions that achieve minimum construction complexities and maximum buffer sizes among the constructions in Theorem 1. Due to the symmetry, i.e., i versus $k - i + 1$ for all $1 \leq i \leq \lceil k/2 \rceil$, that can be observed from the conditions (A1)–(A3) in Theorem 1, in our analysis we focus on the scenario that the i^{th} group and the $(k - i + 1)^{\text{th}}$ group of FM1's in Figure 2 have the same number of FM1's, i.e., $m_i = m_{k-i+1}$, for all $1 \leq i \leq \lceil k/2 \rceil$. Such a scenario is general enough and includes the scenario considered in [37] as a special case.

A. The Optimal Constructions

In the following theorem, we give the parameters $n_1, n_2, \dots, n_k, B_1, B_2, \dots, B_k$, and $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ for the optimal construction that achieves minimum construction complexity and maximum buffer size among the constructions in Theorem 1 for the scenario that $m_i = m_{k-i+1} \geq 2$ for all $1 \leq i \leq \lceil k/2 \rceil$.

Theorem 4 *Suppose that $1 \leq s \leq k - 1$ and $m_i = m_{k-i+1} \geq 2$ for all $1 \leq i \leq \lceil k/2 \rceil$. Then the parameters $n_1, n_2, \dots, n_k, B_1, B_2, \dots, B_k$, and $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ for the optimal construction that achieves minimum construction complexity and maximum buffer size among the constructions in Theorem 1 are given as follows:*

(A1*) *The optimal choice for n_1, n_2, \dots, n_k :*

$$n_i = \lceil (\sum_{j=j_1}^{j_2} m_j + 1) / m_i \rceil, \quad (26)$$

where j_1 and j_2 are given by (2) and (3) (note that j_1 and j_2 depend on i), for $1 \leq i \leq k$.

(A2*) *The optimal choice for B_1, B_2, \dots, B_k : If $s + 1 \leq k \leq 2s + 2$, then B_1, B_2, \dots, B_k are recursively given by*

$$B_i = B_{k-i+1} = \sum_{j=1}^{i-1} ((m_j - 1)B_j + 1) + 1 \text{ for } 1 \leq i \leq \lceil k/2 \rceil, \quad (27)$$

where we have adopted the convention that the sum is zero if the upper limit is smaller than the lower limit of a summation. On the other hand, if $k \geq 2s + 3$, then B_1, B_2, \dots, B_k are recursively given by

$$B_i = B_{k-i+1} = \begin{cases} \sum_{j=1}^{i-1} ((m_j - 1)B_j + 1) + 1, & \text{if } 1 \leq i \leq s + 1, \\ \sum_{j=i-s}^{i-1} ((m_j - 1)B_j + 1), & \text{if } s + 2 \leq i \leq \lceil k/2 \rceil. \end{cases} \quad (28)$$

Note that we have $B_1 = B_k = 1$.

(A3*) *The optimal choice for $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$:*

$$|\Psi_i| = (m_i - 1)B_i + 1 \text{ for } 1 \leq i \leq k. \quad (29)$$

Furthermore, the maximum buffer size U_k achieved by the optimal construction is given by

$$U_k = \sum_{i=1}^k ((m_i - 1)B_i + 1). \quad (30)$$

Proof. Since an optical FM1 can be operated as an optical FM1 with a smaller number of arrival links, it is clear that the construction complexity/cost of an optical FM1 increases with the number of its arrival links. Thus, to minimize the construction complexity of an optical FM1, we should choose the number of its arrival links as small as possible. As such, it is clear from (1) in the condition (A1) that we should choose n_1, n_2, \dots, n_k as given by (26) in (A1*).

To maximize the buffer size $U_k = \sum_{i=1}^k |\Psi_i|$ among the constructions in Theorem 1, it is clear that we should choose $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ as large as possible. Thus, if (7) in the condition (A3) is satisfied, then it is clear from (6) in the condition (A3) that we should choose $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ as given by (29) in (A3*).

So assume that we choose $|\Psi_i| = (m_i - 1)B_i + 1$ for $1 \leq i \leq k$ as given by (29). Then the buffer size is $U_k = \sum_{i=1}^k ((m_i - 1)B_i + 1)$ as given by (30). As such, it is clear from (30) that we should choose B_1, B_2, \dots, B_k as large as possible in order to maximize the buffer size U_k . As we have assumed that $m_i = m_{k-i+1}$ for $1 \leq i \leq \lceil k/2 \rceil$, it is easy to deduce from (4) and (5) in the condition (A2) that we should choose B_1, B_2, \dots, B_k as given by (27) and (28) in (A2*).

It remains to show that the parameters B_1, B_2, \dots, B_k as given by (27) and (28) and the parameters $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ as given by (29) satisfy (7) in the condition (A3). If $2 \leq i \leq \lceil k/2 \rceil$, then we have

$$|\Psi_i| = (m_i - 1)B_i + 1 > B_i \geq (m_{i-1} - 1)B_{i-1} + 1 > B_{i-1}, \quad (31)$$

where the equality follows from (29), the first inequality follows from the assumption that $m_i \geq 2$, the second inequality follows from (27) and (28) (note that $i \geq 2$ and $s \geq 1$), and the last inequality follows from the assumption that $m_{i-1} \geq 2$. On the other hand, if $\lfloor k/2 \rfloor + 1 \leq i \leq k-1$, then we have

$$|\Psi_i| = (m_i - 1)B_i + 1 = (m_{k-i+1} - 1)B_{k-i+1} + 1 = |\Psi_{k-i+1}| > B_{k-i} = B_{k-(k-i)+1} = B_{i+1}, \quad (32)$$

where the first equality follows from (29), the second equality follows from $m_i = m_{k-i+1}$ in the assumption and $B_i = B_{k-i+1}$ in (27) and (28), the third equality follows from (29), the inequality follows from $2 \leq k - i + 1 \leq k - \lfloor k/2 \rfloor = \lceil k/2 \rceil$ and (31), and the fourth equality follows from (27) and (28). By combining (31) and (32), we obtain (7). ■

Remark 5 (i) For the scenario that each group of FMI's in Figure 2 has the same number of FMI's, say $m_i = m \geq 2$ for all $1 \leq i \leq k$, we see from Theorem 4 that the optimal choice for n_1, n_2, \dots, n_k is given by (11)–(16) (with the inequalities in (11)–(16) replaced by equalities), the optimal choice for B_1, B_2, \dots, B_k is given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$), the optimal choice for $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ is given by (29) (with $m_i = m$ in (29)

for $1 \leq i \leq k$), and the maximum buffer size U_k achieved by the optimal construction is given by (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$).

(ii) Furthermore, for the scenario that each group of FMI's in Figure 2 has the same number of FMI's and each FMI in Figure 2 has the same number of arrival links, say $m_i = m \geq 2$ and $n_i = n$ for all $1 \leq i \leq k$, we also see from Theorem 4 that the optimal choice for n is given by (19) (with the inequality in (19) replaced by equality), the optimal choice for B_1, B_2, \dots, B_k is given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$), the optimal choice for $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ is given by (29) (with $m_i = m$ in (29) for $1 \leq i \leq k$), and the maximum buffer size U_k achieved by the optimal construction is given by (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$).

B. The Maximum Buffer Size for the Scenario that Each Group Has the Same Number of FMI's

In this subsection, we consider the scenario that each group has the same number of FMI's, say $m_i = m \geq 2$ for all $1 \leq i \leq k$. We show that the parameters B_1, B_2, \dots, B_k given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$) can be obtained in closed form as follows.

Theorem 6 Suppose that $1 \leq s \leq k - 1$ and $m_i = m \geq 2$ for $1 \leq i \leq k$. Also suppose that B_1, B_2, \dots, B_k are given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$). Let q_i be the unique nonnegative integer such that $q_i(s + 1) + 1 \leq i \leq (q_i + 1)(s + 1)$, i.e., $q_i = \lceil i/(s + 1) \rceil - 1$, for $1 \leq i \leq \lceil k/2 \rceil$.

(i) If $2 \leq i \leq \lceil k/2 \rceil$, then B_i and B_{k-i+1} can be recursively given as follows:

$$B_i = B_{k-i+1} = \begin{cases} mB_{i-1} + 1, & \text{if } 2 \leq i \leq s + 1, \\ mB_{i-1} - (m - 1)B_{i-s-1} - 1, & \text{if } i = s + 2, \\ mB_{i-1} - (m - 1)B_{i-s-1}, & \text{if } i \geq s + 3. \end{cases} \quad (33)$$

(ii) If $1 \leq i \leq \lceil k/2 \rceil$, then B_i and B_{k-i+1} can be expressed in closed form as follows:

$$B_i = B_{k-i+1} = \sum_{j=0}^{q_i} (-1)^j (1/j!) [j(i - j(s + 1))_{j-1} + m(i - j(s + 1))_j] (m - 1)^{j-1} m^{i-j(s+1)-1} - 1/(m - 1), \quad (34)$$

where $(a)_j$ is the Pochhammer symbol given by $(a)_{-1} = (a)_0 = 1$ and $(a)_j = a(a + 1)(a + 2) \cdots (a + j - 1)$ for every positive integer j .

Proof. See Appendix B. ■

Remark 7 It is easy to see from (34) and $q_i = 0$ for $1 \leq i \leq s + 1$ that

$$B_i = B_{k-i+1} = (m^i - 1)/(m - 1) \text{ for } 1 \leq i \leq \min\{\lceil k/2 \rceil, s + 1\}. \quad (35)$$

Thus, we have from (35) and $m \geq 2$ the following upper bound and lower bound for B_i :

$$m^{i-1} \leq B_i \leq 2m^{i-1} \text{ for } 1 \leq i \leq \min\{\lceil k/2 \rceil, s+1\}. \quad (36)$$

For the special case that $s = 1$, a simpler closed-form expression for B_1, B_2, \dots, B_k given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$) can be obtained as follows.

Theorem 8 *Suppose that $s = 1$, $k \geq 2$, and $m_i = m \geq 2$ for $1 \leq i \leq k$. Also suppose that B_1, B_2, \dots, B_k are given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$). Then we have*

$$B_i = B_{k-i+1} = \begin{cases} 1, & \text{if } i = 1, \\ i + 1, & \text{if } m = 2 \text{ and } 2 \leq i \leq \lceil k/2 \rceil, \\ ((m^2 - m - 1)(m - 1)^{i-2} - 1)/(m - 2), & \text{if } m \geq 3 \text{ and } 2 \leq i \leq \lceil k/2 \rceil. \end{cases} \quad (37)$$

Proof. Since we have $B_i = B_{k-i+1}$ for $1 \leq i \leq \lceil k/2 \rceil$ in (27) and (28), it suffices to show that B_i is given by (37) for $1 \leq i \leq \lceil k/2 \rceil$. We will show that B_i is given by (38) below (as (37) follows trivially from (38)):

$$B_i = \begin{cases} 1, & \text{if } i = 1, \\ \sum_{j=0}^{i-1} (m-1)^j + (m-1)^{i-2}, & \text{if } 2 \leq i \leq \lceil k/2 \rceil. \end{cases} \quad (38)$$

As we have from (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$) that $B_1 = 1$, (38) holds for $i = 1$. If $k = 2$, then we are done. So assume that $k \geq 3$. In the following, we prove by induction on i that (38) holds for $2 \leq i \leq \lceil k/2 \rceil$.

It is clear from (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$) that $B_2 = ((m-1)B_1 + 1) + 1 = m + 1$, and hence (38) holds for $i = 2$. Assume as the induction hypothesis that (38) holds for $i-1$ for some $2 \leq i-1 \leq \lceil k/2 \rceil - 1$. Then we have from (28) (note that $s = 1$ and $s+2 = 3 \leq i \leq \lceil k/2 \rceil$) and the induction hypothesis that

$$\begin{aligned} B_i &= (m-1)B_{i-1} + 1 = (m-1) \left(\sum_{j=0}^{i-2} (m-1)^j + (m-1)^{i-3} \right) + 1 \\ &= \sum_{j=0}^{i-1} (m-1)^j + (m-1)^{i-2}. \end{aligned}$$

The proof is completed. ■

In the following theorem, we show that the maximum buffer size U_k given by (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) can be obtained in closed form by using the results in Theorem 6 and Theorem 8.

Theorem 9 *Suppose that $1 \leq s \leq k-1$ and $m_i = m \geq 2$ for $1 \leq i \leq k$. Also suppose that B_1, B_2, \dots, B_k are given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$), and U_k is given by (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$),*

(i) *Suppose that $s = 1$, $k \geq 3$, and $m \geq 3$. If k is even, say $k = 2\ell$ for some $\ell \geq 2$ (note that $k \geq 3$), then we have*

$$U_k = [2(m^2 - m - 1)(m - 1)^\ell - 4(m - 1)^2 - 2(\ell - 1)(m - 1) + 2\ell]/(m - 2)^2. \quad (39)$$

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$ (note that $k \geq 3$), then we have

$$U_k = [m(m^2 - m - 1)(m - 1)^{\ell-1} - 4(m - 1)^2 - (2\ell - 3)(m - 1) + 2\ell - 1]/(m - 2)^2. \quad (40)$$

(ii) Suppose that $s \geq 2$ and $s + 1 \leq k \leq 2s + 2$, If k is even, say $k = 2\ell$ for some $\lceil (s + 1)/2 \rceil \leq \ell \leq \lfloor (2s + 2)/2 \rfloor = s + 1$, then we have

$$U_k = 2m(m^\ell - 1)/(m - 1). \quad (41)$$

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\lceil (s + 2)/2 \rceil \leq \ell \leq \lfloor (2s + 3)/2 \rfloor = s + 1$, then we have

$$U_k = m(m^\ell + m^{\ell-1} - 2)/(m - 1). \quad (42)$$

(iii) Suppose that $s \geq 2$ and $k \geq 2s + 3$. Then B_1, B_2, \dots, B_k are given in closed form in (34). If k is even, say $k = 2\ell$ for some $\ell \geq \lceil (2s + 3)/2 \rceil = s + 2$, then we have

$$U_k = 2m \sum_{r=0}^{q_\ell} B_{\ell-r(s+1)} + 2q_\ell. \quad (43)$$

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\ell \geq \lceil (2s + 4)/2 \rceil = s + 2$, then we have

$$U_k = 2m \sum_{r=1}^{q_\ell} B_{\ell-r(s+1)} + (m + 1)B_\ell + 2q_\ell - 1. \quad (44)$$

Proof. (i) As $s = 1$, $k \geq 3$, and $m \geq 3$, we have from (37) that $B_1 = B_k = 1$ and $B_i = B_{k-i+1} = ((m^2 - m - 1)(m - 1)^{i-2} - 1)/(m - 2)$ for $2 \leq i \leq \lceil k/2 \rceil$. If k is even, say $k = 2\ell$ for some $\ell \geq 2$. then we have from (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) that

$$\begin{aligned} U_k &= 2m + 2 \sum_{i=2}^{\ell} [(m - 1)((m^2 - m - 1)(m - 1)^{i-2} - 1)/(m - 2) + 1] \\ &= [2(m^2 - m - 1)(m - 1)^\ell - 4(m - 1)^2 - 2(\ell - 1)(m - 1) + 2\ell]/(m - 2)^2. \end{aligned}$$

Thus, we have obtained (39).

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$, then we have from (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) that

$$\begin{aligned} U_k &= 2m + 2 \sum_{i=2}^{\ell-1} [(m - 1)((m^2 - m - 1)(m - 1)^{i-2} - 1)/(m - 2) + 1] \\ &\quad + (m - 1)((m^2 - m - 1)(m - 1)^{\ell-2} - 1)/(m - 2) + 1 \\ &= [m(m^2 - m - 1)(m - 1)^{\ell-1} - 4(m - 1)^2 - (2\ell - 3)(m - 1) + 2\ell - 1]/(m - 2)^2. \end{aligned}$$

Thus, we have obtained (40).

(ii) As $k \leq 2s + 2$, we have from (35) that $B_i = B_{k-i+1} = (m^i - 1)/(m - 1)$ for $1 \leq i \leq \min\{\lceil k/2 \rceil, s + 1\} = \lceil k/2 \rceil$. If k is even, say $k = 2\ell$ for some $\lceil (s + 1)/2 \rceil \leq \ell \leq \lfloor (2s + 2)/2 \rfloor = s + 1$, then we have from (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) and $\lceil k/2 \rceil = \ell$ that

$$U_k = 2 \sum_{i=1}^{\ell} ((m^i - 1) + 1) = 2m(m^\ell - 1)/(m - 1).$$

Thus, we have obtained (41).

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\lceil (s+2)/2 \rceil \leq \ell \leq \lfloor (2s+3)/2 \rfloor = s+1$, then we have from (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) and $\lceil k/2 \rceil = \ell$ that

$$U_k = 2\sum_{i=1}^{\ell-1}((m^i - 1) + 1) + (m^\ell - 1) + 1 = m(m^\ell + m^{\ell-1} - 2)/(m - 1).$$

Thus, we have obtained (42).

(iii) From Theorem 6, we know that B_1, B_2, \dots, B_k are given in closed form in (34). First consider the case that k is even, say $k = 2\ell$ for some $\ell \geq \lceil (2s+3)/2 \rceil = s+2$. Then we have from (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) that

$$\begin{aligned} U_k &= 2\sum_{j=1}^{\ell}((m-1)B_j + 1) \\ &= 2[\sum_{j=1}^{\ell-q_\ell(s+1)}((m-1)B_j + 1) + \sum_{r=0}^{q_\ell-1}\sum_{j=\ell-(r+1)(s+1)+1}^{\ell-r(s+1)}((m-1)B_j + 1)]. \end{aligned} \quad (45)$$

It is easy to see from (28) (note that $k \geq 2s+3$ and $1 \leq \ell - q_\ell(s+1) \leq s+1$) that

$$\begin{aligned} \sum_{j=1}^{\ell-q_\ell(s+1)}((m-1)B_j + 1) &= \sum_{j=1}^{\ell-q_\ell(s+1)-1}((m-1)B_j + 1) + (m-1)B_{\ell-q_\ell(s+1)} + 1 \\ &= (B_{\ell-q_\ell(s+1)} - 1) + (m-1)B_{\ell-q_\ell(s+1)} + 1 = mB_{\ell-q_\ell(s+1)}. \end{aligned} \quad (46)$$

For $0 \leq r \leq q_\ell - 1$, we also have from (28) (note that $k \geq 2s+3$ and $\ell - r(s+1) \geq (q_\ell(s+1) + 1) - (q_\ell - 1)(s+1) = s+2$) that

$$\begin{aligned} &\sum_{j=\ell-(r+1)(s+1)+1}^{\ell-r(s+1)}((m-1)B_j + 1) \\ &= \sum_{j=\ell-(r+1)(s+1)+1}^{\ell-r(s+1)-1}((m-1)B_j + 1) + (m-1)B_{\ell-r(s+1)} + 1 \\ &= B_{\ell-r(s+1)} + (m-1)B_{\ell-r(s+1)} + 1 = mB_{\ell-r(s+1)} + 1. \end{aligned} \quad (47)$$

By substituting (46) and (47) into (45), we obtain (43).

Now consider the case that k is odd, say $k = 2\ell - 1$ for some $\ell \geq \lceil (2s+4)/2 \rceil = s+2$. Then we have from (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) and (43) that

$$\begin{aligned} U_k &= 2\sum_{j=1}^{\ell-1}((m-1)B_j + 1) + ((m-1)B_\ell + 1) \\ &= 2\sum_{j=1}^{\ell}((m-1)B_j + 1) - ((m-1)B_\ell + 1) \\ &= 2m\sum_{r=0}^{q_\ell}B_{\ell-r(s+1)} + 2q_\ell - ((m-1)B_\ell + 1) \\ &= 2m\sum_{r=1}^{q_\ell}B_{\ell-r(s+1)} + (m+1)B_\ell + 2q_\ell - 1. \end{aligned}$$

Thus, we have obtained (44). ■

C. Fault-Tolerant Capability of Our Constructions

Fault-tolerance is an important practical issue in the design of any network element, and it deals with the situation that some of the components of a network element fail to function properly. Without taking the fault-tolerant capability into consideration during the design phase, a network element consisting of hundreds or thousands of components may be in a total breakdown even when only a single component fails to function properly.

In this subsection, we consider the scenario that each group of FM1's in Figure 2 has the same number of FM1's and each FM1 in Figure 2 has the same number of arrival links, and show that the optimal constructions given in Remark 5(ii) possess fault-tolerant capability.

Theorem 10 *Suppose that $1 \leq s \leq k - 1$, and $m_i = m \geq 2$ and $n_i = n$ for $1 \leq i \leq k$. Then the optimal construction given in Remark 5(ii) possess fault-tolerant capability and can tolerate up to f broken FM1's, i.e., the feedback system in Figure 2 can still be operated as an optical priority queue with a smaller buffer size after up to f FM1's fail to function properly, where $f = \lfloor (m - 1)/(n - 1) \rfloor$.*

Proof. Recall that for the optimal construction given in Remark 5(ii), n is given by (19) (with the inequality in (19) replaced by equality), B_1, B_2, \dots, B_k are given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$), $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ are given by (29) (with $m_i = m$ in (29) for $1 \leq i \leq k$), and the maximum buffer size U_k achieved by the optimal construction is given by (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$).

Assume that there are f_i broken FM1's in the i^{th} group of FM1's in Figure 2 for $1 \leq i \leq k$, where $\sum_{i=1}^k f_i \leq f$. Let $f'_i = \max\{f_i, f_{k-i+1}\}$ and $m'_i = m - f'_i$ for $1 \leq i \leq k$. Then it is clear that $f'_i = f'_{k-i+1} \leq f$ and $m'_i = m'_{k-i+1}$ for $1 \leq i \leq \lceil k/2 \rceil$. For $1 \leq i \leq k$, we have

$$\begin{aligned} m'_i &= m - f'_i \geq m - f = m - \lfloor (m - 1)/(n - 1) \rfloor = \lceil m - (m - 1)/(n - 1) \rceil \\ &\geq \lceil m - (m - 1)/2 \rceil = \lceil (m + 1)/2 \rceil \geq 2, \end{aligned}$$

where the first inequality follows from $f'_i \leq f$, the second inequality follows from $n \geq 3$ in (19) (with the inequality in (19) replaced by equality), and the third inequality follows from $m \geq 2$.

We claim that n_1, n_2, \dots, n_k and m'_1, m'_2, \dots, m'_k satisfy the condition (A1). To see this, suppose that $1 \leq i \leq k$. Let j_1 and j_2 be given by (2) and (3) (note that j_1 and j_2 depend on i). Then we have

$$\begin{aligned} \sum_{j=j_1}^{j_2} m'_j + 1 &= \sum_{j=j_1}^{j_2} (m - f'_j) + 1 = (j_2 - j_1 + 1)m - \sum_{j=j_1}^{j_2} f'_j + 1 \\ &= (j_2 - j_1 + 1)(m'_i + f'_i) - \sum_{j=j_1}^{j_2} f'_j + 1 \\ &\leq (n - 1)(m'_i + f) - f'_i + 1 \leq (n - 1)m'_i + (m - 1) - f'_i + 1 \\ &= (n - 1)m'_i + m'_i = n \cdot m'_i, \end{aligned} \tag{48}$$

where the first inequality follows from $n = n_i \geq j_2 - j_1 + 2$ in (1) (with $m_j = m$ in (1) for $1 \leq j \leq k$), $f'_i \leq f$, and $\sum_{j=j_1}^{j_2} f'_j \geq f'_i$ (as $j_1 \leq i \leq j_2$), and the second inequality follows from $f = \lfloor (m-1)/(n-1) \rfloor \leq (m-1)/(n-1)$. Thus, we have $n_i = n \geq \lceil (\sum_{j=j_1}^{j_2} m'_j + 1)/m'_i \rceil$.

Let B'_1, B'_2, \dots, B'_k be given by (27) and (28) (with m_i in (27) and (28) replaced by m'_i for $1 \leq i \leq k$), and let $|\Psi'_1|, |\Psi'_2|, \dots, |\Psi'_k|$ be given by (29) (with m_i in (29) replaced by m'_i for $1 \leq i \leq k$). Then (4) and (5) in the condition (A2) and (6) in the condition (A3) are satisfied. By using the same argument as that in the last paragraph in the proof of Theorem 4, we see that (7) in the condition (A3) is also satisfied.

Since it is clear from $m'_i = m - f'_i \leq m$ that $B'_i \leq B_i$, we see that the nFM1's with buffer size B_i can be used as nFM1's with buffer size B'_i for $1 \leq i \leq k$. Therefore, we have from Theorem 1 that the feedback system in Figure 2 can still be operated as an optical priority queue with buffer size $U'_k = \sum_{i=1}^k ((m'_i - 1)B'_i + 1)$. Note that we have from $m'_i \leq m$ and $B'_i \leq B_i$ for $1 \leq i \leq k$ that

$$U'_k = \sum_{i=1}^k ((m'_i - 1)B'_i + 1) \leq \sum_{i=1}^k ((m - 1)B_i + 1) = U_k.$$

The proof is completed. ■

V. COMPLEXITY ANALYSIS AND NUMERICAL RESULTS

A. Constructions by Using a Single Optical Crossbar Switch and Multiple Fiber Delay Lines

In this subsection, we show that the constructions in Theorem 1 by using a single optical (bufferless) crossbar switch and multiple groups of FM1's as in Figure 2 can be implemented by using a single optical (bufferless) crossbar switch and multiple fiber delay lines as in Figure 1.

Theorem 11 *Suppose that $1 \leq s \leq k - 1$ and $m_1, m_2, \dots, m_k \geq 1$. Also suppose that the parameters n_1, n_2, \dots, n_k , B_1, B_2, \dots, B_k , and $|\Psi_1|, |\Psi_2|, \dots, |\Psi_k|$ satisfy the conditions (A1)–(A3). Then an optical priority queue with buffer size $U_k = \sum_{i=1}^k |\Psi_i|$ can be constructed by using a feedback system consisting of an optical $(M + 2) \times (M + 2)$ (bufferless) crossbar switch and M fiber delay lines as in Figure 1, where*

$$M = \sum_{i=1}^k m_i ((n_i - 1) \lceil \log_{n_i} B_i \rceil + n_i + 1). \quad (49)$$

Proof. To prove the theorem, we use the best constructions currently available in the literature, i.e., the constructions in [13], to implement the optical FM1's in Figure 2. From [13, Figure 3], we know that an optical nFM with buffer size $B - 1$ can be implemented by using a feedback system consisting of an optical $((n-1) \lceil \log_n B \rceil + n) \times ((n-1) \lceil \log_n B \rceil + n)$ (bufferless) crossbar switch and $(n-1) \lceil \log_n B \rceil$ fiber delay lines.

Since an nFM1 with buffer size B is the concatenation of an nFM with buffer size $B - 1$ and a delay line with delay equal to one as in Figure 4, it is easy to see that an optical nFM1 with buffer size B can be implemented by using an optical $((n-1) \lceil \log_n B \rceil + n + 1) \times ((n-1) \lceil \log_n B \rceil + n + 1)$

(bufferless) crossbar switch and $(n-1)\lceil \log_n B \rceil + 1$ fiber delay lines. As such, we immediately see that the constructions in Theorem 1 can be implemented by using a feedback system consisting of an optical $(M+2) \times (M+2)$ (bufferless) crossbar switch and M fiber delay lines as in Figure 1, where M is given by (49). \blacksquare

For the scenario that each group of FM1's in Figure 2 has the same number of FM1's and each FM1 in Figure 2 has the same number of arrival links, say $m_i = m \geq 2$ and $n_i = n$ for $1 \leq i \leq k$, we show that the maximum buffer size U_k (which is given by (30) with $m_i = m$ for $1 \leq i \leq k$) achieved by the optimal construction can be expressed in terms of the switch size M (which is given by (49) with $m_i = m$ and $n_i = n$ for $1 \leq i \leq k$) as $2^{O(\sqrt{\alpha M})}$, where α is a constant that depends on the parameters s , k , and m used in the constructions.

Theorem 12 *Suppose that $1 \leq s \leq k-1$, and $m_i = m \geq 2$ and $n_i = n$ for $1 \leq i \leq k$. Also suppose that n is given by (19) (with the inequality in (19) replaced by equality), B_1, B_2, \dots, B_k are given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$), U_k is given by (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$), and M is given by (49) (with $m_i = m$ and $n_i = n$ in (49) for $1 \leq i \leq k$).*

(i) *Suppose that $s = 1$, $k \geq 3$, and $m \geq 3$. If k is even, say $k = 2\ell$ for some $\ell \geq 2$ (note that $k \geq 3$), then we have*

$$2^{\sqrt{2M \log_2(m-1)/(3m) - 7 \log_2(m-1) + 1}} \leq U_k \leq 2^{\sqrt{2M \log_2(m-1)/(3m) + \log_2(10(m-1))}}. \quad (50)$$

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$ (note that $k \geq 3$), then we have

$$2^{\sqrt{2M \log_2(m-1)/(3m) - 7 \log_2(m-1)}} \leq U_k \leq 2^{\sqrt{2M \log_2(m-1)/(3m) + \log_2(9(m-1))}}. \quad (51)$$

Therefore, we have

$$U_k = 2^{O(\sqrt{2M \log_2(m-1)/(3m)})}. \quad (52)$$

(ii) *Suppose that $s \geq 2$ and $s+1 \leq k \leq 2s+2$. Then we have*

$$n = \begin{cases} 4, & \text{if } k = 3, \\ \lfloor k/2 \rfloor + 2, & \text{if } \max\{s+1, 4\} \leq k \leq 2s+2. \end{cases} \quad (53)$$

If k is even, say $k = 2\ell$ for some $\lceil (s+1)/2 \rceil \leq \ell \leq \lfloor (2s+2)/2 \rfloor = s+1$, then we have

$$2^{\sqrt{M \log_2 n \log_2 m / ((n-1)m) - 4 \log_2 n + 1}} \leq U_k \leq 2^{\sqrt{M \log_2 n \log_2 m / ((n-1)m) + \log_2 m + 2}}. \quad (54)$$

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\lceil (s+2)/2 \rceil \leq \ell \leq \lfloor (2s+3)/2 \rfloor = s+1$, then we have

$$2^{\sqrt{M \log_2 n \log_2 m / ((n-1)m) - 4 \log_2 n}} \leq U_k \leq 2^{\sqrt{M \log_2 n \log_2 m / ((n-1)m) + \log_2(3m)}}. \quad (55)$$

Therefore, we have

$$U_k = 2^{O(\sqrt{M \log_2 n \log_2 m / ((n-1)m)})}. \quad (56)$$

(iii) Suppose that $s \geq 2$ and $k \geq 2s + 3$. Then we have

$$m^{i-s-1} \leq B_i \leq 2m^{i-1} \text{ for } s+2 \leq i \leq (m^{s+1}-1)/(m-1)+s. \quad (57)$$

If k is even, say $k = 2\ell$ for some $\ell \geq \lceil (2s+3)/2 \rceil = s+2$, and $\ell \leq (m^{s+1}-1)/(m-1)+s$, then we have

$$\begin{aligned} & 2\sqrt{M \log_2 (s+3) \log_2 m / ((s+2)m) - 4 \log_2 (s+3) - s \log_2 m + 1} \\ & \leq U_k \leq 2\sqrt{M \log_2 (s+3) \log_2 m / ((s+2)m) + (s+1) \log_2 m + \log_2 6}. \end{aligned} \quad (58)$$

On the other hand, if k is odd, say $k = 2\ell - 1$ for some $\ell \geq \lceil (2s+4)/2 \rceil = s+2$, and $\ell \leq (m^{s+1}-1)/(m-1)+s$, then we have

$$\begin{aligned} & 2\sqrt{M \log_2 (s+3) \log_2 m / ((s+2)m) - 4 \log_2 (s+3) - s \log_2 m} \\ & \leq U_k \leq 2\sqrt{M \log_2 (s+3) \log_2 m / ((s+2)m) + (s+1) \log_2 m + \log_2 6}. \end{aligned} \quad (59)$$

Therefore, we have

$$U_k = 2^{O(\sqrt{M \log_2 (s+3) \log_2 m / ((s+2)m)})}. \quad (60)$$

Proof. (i) Since $s = 1$ and $k \geq 3$, it is clear from (19) (with the inequality in (19) replaced by equality) that $n = 4$. As $m \geq 3$, we have from (37) that $B_1 = B_k = 1$ and $B_i = B_{k-i+1} = ((m^2 - m - 1)(m - 1)^{i-2} - 1)/(m - 2)$ for $2 \leq i \leq \lceil k/2 \rceil$.

First consider the case that k is even, say $k = 2\ell$ for some $\ell \geq 2$. From (39), $\ell \geq 2$, and $m \geq 3$, we can see that

$$\begin{aligned} U_k & \geq [2(m^2 - m - 1)(m - 1)^\ell - 4(m - 1)^\ell - 2\ell(m - 1)]/(m - 2)^2 \\ & = 2(m - 1)^\ell + [2(3m - 7)(m - 1)^\ell - 2\ell(m - 1)]/(m - 2)^2 \\ & \geq 2(m - 1)^\ell + [4\ell(m - 1) - 2\ell(m - 1)]/(m - 2)^2 \geq 2(m - 1)^\ell, \end{aligned} \quad (61)$$

$$\begin{aligned} U_k & \leq [2(m^2 - m - 1)(m - 1)^\ell - 16 - 4(\ell - 1) + 2\ell]/(m - 2)^2 \\ & \leq 2(m^2 - m - 1)(m - 1)^\ell/(m - 2)^2 \leq 10(m - 1)^\ell. \end{aligned} \quad (62)$$

For $2 \leq i \leq \lceil k/2 \rceil$, we can see from $m \geq 3$ that $B_i \leq (m^2 - m)(m - 1)^{i-2}/(m - 2) \leq 3(m - 1)^{i-1}$ and $B_i \geq ((m^2 - m - 1)(m - 1)^{i-2} - (m - 1)^{i-2})/(m - 2) = (m + 1)(m - 1)^{i-2} \geq (m - 1)^{i-1}$. Note that $(m - 1)^{i-1} \leq B_i \leq 3(m - 1)^{i-1}$ also holds for $i = 1$ (as $B_1 = 1$). Thus, we have $\lceil \log_4 B_i \rceil < \log_4 B_i + 1 \leq (i - 1) \log_4 (m - 1) + \log_4 3 + 1 \leq (i - 1) \log_4 (m - 1) + 2$ and $\lceil \log_4 B_i \rceil \geq \log_4 B_i \geq (i - 1) \log_4 (m - 1)$ for $1 \leq i \leq \lceil k/2 \rceil$. As such, it follows from (49) (with $m_i = m$ and $n_i = n$ in (49) for $1 \leq i \leq k$) and $m \geq 3$ that

$$\begin{aligned} \frac{M}{m} & \leq 2 \sum_{i=1}^{\ell} [3((i - 1) \log_4 (m - 1) + 2) + 5] \\ & = 3\ell(\ell - 1) \log_4 (m - 1) + 22\ell \leq 3(\ell + 7)^2 \log_4 (m - 1), \end{aligned} \quad (63)$$

$$\begin{aligned} \frac{M}{m} & \geq 2 \sum_{i=1}^{\ell} [3(i - 1) \log_4 (m - 1) + 5] \\ & = 3\ell(\ell - 1) \log_4 (m - 1) + 10\ell \geq 3(\ell - 1)^2 \log_4 (m - 1). \end{aligned} \quad (64)$$

By combining $U_k \geq 2^{\ell \log_2(m-1)+1}$ in (61) and $\ell \geq \sqrt{2M/(3m \log_2(m-1))} - 7$ in (63), we obtain the lower bound for U_k in (50). Similarly, by combining $U_k \leq 2^{\ell \log_2(m-1)+\log_2 10}$ in (62) and $\ell \leq \sqrt{2M/(3m \log_2(m-1))} + 1$ in (64), we obtain the upper bound for U_k in (50).

Now consider the case that k is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$. From (40), $\ell \geq 2$, and $m \geq 3$, we can see that

$$\begin{aligned} U_k &\geq [m(m^2 - m - 1)(m - 1)^{\ell-1} - 4(m - 1)^\ell - (2\ell - 3)(m - 2)]/(m - 2)^2 \\ &= (m - 1)^\ell + [(4m^2 - 13m + 8)(m - 1)^{\ell-1} - (2\ell - 3)(m - 2)]/(m - 2)^2 \\ &\geq (m - 1)^\ell + [5(\ell - 1)(m - 1) - (2\ell - 3)(m - 2)]/(m - 2)^2 \geq (m - 1)^\ell, \end{aligned} \quad (65)$$

$$\begin{aligned} U_k &\leq [m(m^2 - m)(m - 1)^{\ell-1} - 16 - 2(2\ell - 3) + 2\ell - 1]/(m - 2)^2 \\ &\leq m^2(m - 1)^\ell/(m - 2)^2 \leq 9(m - 1)^\ell. \end{aligned} \quad (66)$$

As in the proof for the case that k is even, we have

$$\begin{aligned} \frac{M}{m} &\leq 2\sum_{i=1}^{\ell-1} [3((i - 1) \log_4(m - 1) + 2) + 5] + 3((\ell - 1) \log_4(m - 1) + 2) + 5 \\ &= 3(\ell - 1)^2 \log_4(m - 1) + 22\ell - 11 \leq 3(\ell + 7)^2 \log_4(m - 1), \end{aligned} \quad (67)$$

$$\begin{aligned} \frac{M}{m} &\geq 2\sum_{i=1}^{\ell-1} [3(i - 1) \log_4(m - 1) + 5] + 3(\ell - 1) \log_4(m - 1) + 5 \\ &= 3(\ell - 1)^2 \log_4(m - 1) + 10\ell - 5 \geq 3(\ell - 1)^2 \log_4(m - 1). \end{aligned} \quad (68)$$

By combining $U_k \geq 2^{\ell \log_2(m-1)}$ in (65) and $\ell \geq \sqrt{2M/(3m \log_2(m-1))} - 7$ in (67), we obtain the lower bound for U_k in (51). Similarly, by combining $U_k \leq 2^{\ell \log_2(m-1)+\log_2 9}$ in (66) and $\ell \leq \sqrt{2M/(3m \log_2(m-1))} + 1$ in (68), we obtain the upper bound for U_k in (51).

(ii) Since $k \geq s + 1 \geq 3$, it is clear that (53) follows from (19) (with the inequality in (19) replaced by equality) and $\lfloor k/2 \rfloor + 2 \leq \lfloor (2s + 2)/2 \rfloor + 2 = s + 3$. As $k \leq 2s + 2$, we have from (35) that $B_i = B_{k-i+1} = (m^i - 1)/(m - 1)$ for $1 \leq i \leq \min\{\lfloor k/2 \rfloor, s + 1\} = \lfloor k/2 \rfloor$.

First consider the case that k is even, say $k = 2\ell$ for some $\lfloor (s + 1)/2 \rfloor \leq \ell \leq \lfloor (2s + 2)/2 \rfloor = s + 1$. Then we see from $s \geq 2$ that $\ell \geq 2$. From (41), $\ell \geq 2$, and $m \geq 2$, we can see that

$$2m^\ell \leq U_k \leq 4m^\ell. \quad (69)$$

From (36) and $n \geq 4$ (by (53)), we can see that $\lceil \log_n B_i \rceil < \log_n B_i + 1 \leq (i - 1) \log_n m + \log_n 2 + 1 \leq (i - 1) \log_n m + 2$ and $\lceil \log_n B_i \rceil \geq \log_n B_i \geq (i - 1) \log_n m$ for $1 \leq i \leq \lfloor k/2 \rfloor$. It then follows from (49) (with $m_i = m$ and $n_i = n$ in (49) for $1 \leq i \leq k$) that

$$\begin{aligned} \frac{M}{m} &\leq 2\sum_{i=1}^{\ell} [(n - 1)((i - 1) \log_n m + 2) + (n + 1)] \\ &= (n - 1)\ell(\ell - 1) \log_n m + (3n - 1)(2\ell) \leq (n - 1)(\ell + 4/\log_n m)^2 \log_n m, \end{aligned} \quad (70)$$

$$\begin{aligned} \frac{M}{m} &\geq 2\sum_{i=1}^{\ell} [(n - 1)(i - 1) \log_n m + (n + 1)] \\ &= (n - 1)\ell(\ell - 1) \log_n m + (n + 1)(2\ell) \geq (n - 1)(\ell - 1)^2 \log_n m. \end{aligned} \quad (71)$$

By combining $U_k \geq 2^{\ell \log_2 m+1}$ in (69) and $\ell \geq \sqrt{M \log_2 n / ((n-1)m \log_2 m)} - 4 \log_2 n / \log_2 m$ in (70), we obtain the lower bound for U_k in (54). Similarly, by combining $U_k \leq 2^{\ell \log_2 m+2}$ in (69) and $\ell \leq \sqrt{M \log_2 n / ((n-1)m \log_2 m)} + 1$ in (71), we obtain the upper bound for U_k in (54).

Now consider the case that k is odd, say $k = 2\ell - 1$ for some $\lceil (s+2)/2 \rceil \leq \ell \leq \lfloor (2s+3)/2 \rfloor = s+1$. Then we see from $s \geq 2$ that $\ell \geq 2$. From (42), $\ell \geq 2$, and $m \geq 2$, we can see that

$$m^\ell \leq U_k \leq 3m^\ell. \quad (72)$$

As in the proof for the case that k is even, we have

$$\begin{aligned} \frac{M}{m} &\leq 2 \sum_{i=1}^{\ell-1} [(n-1)((i-1) \log_n m + 2) + (n+1)] + (n-1)((\ell-1) \log_n m + 2) + (n+1) \\ &= (n-1)(\ell-1)^2 \log_n m + (3n-1)(2\ell-1) \leq (n-1)(\ell+4/\log_n m)^2 \log_n m, \end{aligned} \quad (73)$$

$$\begin{aligned} \frac{M}{m} &\geq 2 \sum_{i=1}^{\ell-1} [(n-1)(i-1) \log_n m + (n+1)] + (n-1)(\ell-1) \log_n m + (n+1) \\ &= (n-1)(\ell-1)^2 \log_n m + (n+1)(2\ell-1) \geq (n-1)(\ell-1)^2 \log_n m. \end{aligned} \quad (74)$$

By combining $U_k \geq 2^{\ell \log_2 m}$ in (72) and $\ell \geq \sqrt{M \log_2 n / ((n-1)m \log_2 m)} - 4 \log_2 n / \log_2 m$ in (73), we obtain the lower bound for U_k in (55). Similarly, by combining $U_k \leq 2^{\ell \log_2 m + \log_2 3}$ in (72) and $\ell \leq \sqrt{M \log_2 n / ((n-1)m \log_2 m)} + 1$ in (74), we obtain the upper bound for U_k in (55).

(iii) The proof of (57) is given in Appendix C. As $k \geq 2s+3 > 4$, it is clear from (19) (with the inequality in (19) replaced by equality that $n = \min\{\lfloor k/2 \rfloor + 2, s+3\} = s+3$).

First consider the case that k is even, say $k = 2\ell$ for some $\ell \geq \lceil (2s+3)/2 \rceil = s+2$, and $\ell \leq (m^{s+1} - 1)/(m-1) + s$. Then we have from $B_i \geq 1$, (36), and (57) that

$$\max\{m^{i-s-1}, 1\} \leq B_i \leq 2m^{i-1} \text{ for } 1 \leq i \leq \ell. \quad (75)$$

From (43), (75), $2q_\ell \leq q_\ell(s+1) \leq \ell \leq m^\ell$, and $m^{s+1} \geq 2^3 = 8$, we can see that

$$U_k = 2m \sum_{r=0}^{q_\ell} B_{\ell-r(s+1)} + 2q_\ell \geq 2m B_\ell \geq 2m^{\ell-s} \quad (76)$$

$$\begin{aligned} U_k &= 2m \sum_{r=0}^{q_\ell} B_{\ell-r(s+1)} + 2q_\ell \leq 2m \sum_{r=0}^{q_\ell} (2m^{\ell-r(s+1)-1}) + m^\ell \\ &\leq 4m^\ell / (1 - 1/m^{s+1}) + m^\ell \leq (39/7)m^\ell \leq 6m^\ell. \end{aligned} \quad (77)$$

From (75), we can see that $\lceil \log_{s+3} B_i \rceil < \log_{s+3} B_i + 1 \leq (i-1) \log_{s+3} m + \log_{s+3} 2 + 1 \leq (i-1) \log_{s+3} m + 2$ and $\lceil \log_{s+3} B_i \rceil \geq \log_{s+3} B_i \geq \max\{(i-s-1) \log_{s+3} m, 0\}$. As such, it follows from (49) (with $m_i = m$ and $n_i = n$ in (49) for $1 \leq i \leq k$) that

$$\begin{aligned} \frac{M}{m} &\leq 2 \sum_{i=1}^{\ell} [(s+2)((i-1) \log_{s+3} m + 2) + s+4] \\ &= (s+2)\ell(\ell-1) \log_{s+3} m + (3s+8)(2\ell) \leq (s+2)(\ell+4/\log_{s+3} m)^2 \log_{s+3} m \end{aligned} \quad (78)$$

$$\begin{aligned} \frac{M}{m} &\geq 2 \sum_{i=1}^s [(s+2) \cdot 0 + s+4] + 2 \sum_{i=s+1}^{\ell} [(s+2)(i-s-1) \log_{s+3} m + s+4] \\ &\geq (s+2)(\ell-s-1)^2 \log_{s+3} m. \end{aligned} \quad (79)$$

By combining $U_k \geq 2^{(\ell-s)\log_2 m+1}$ in (76) and $\ell \geq \sqrt{M \log_2 (s+3)/((s+2)m \log_2 m)} - 4 \log_2 (s+3)/\log_2 m$ in (78), we obtain the lower bound for U_k in (58). Similarly, by combining $U_k \leq 2^{\ell \log_2 m + \log_2 6}$ in (77) and $\ell \leq \sqrt{M \log_2 (s+3)/((s+2)m \log_2 m)} + s + 1$ in (79), we obtain the upper bound for U_k in (58).

Now consider the case that k is odd, say $k = 2\ell - 1$ for some $\ell \geq \lceil (2s+4)/2 \rceil = s+2$, and $\ell \leq (m^{s+1} - 1)/(m - 1) + s$. Then (75) still holds. From (44), (75), $2q_\ell \leq q_\ell(s+1) \leq \ell \leq m^\ell$, and $m^{s+1} \geq 2^3 = 8$, we can see that

$$U_k = 2m \sum_{r=1}^{q_\ell} B_{\ell-r(s+1)} + (m+1)B_\ell + 2q_\ell - 1 \geq mB_\ell \geq m^{\ell-s} \quad (80)$$

$$\begin{aligned} U_k &= 2m \sum_{r=1}^{q_\ell} B_{\ell-r(s+1)} + (m+1)B_\ell + 2q_\ell - 1 \\ &= 2m \sum_{r=0}^{q_\ell} B_{\ell-r(s+1)} - (m-1)B_\ell + 2q_\ell - 1 \leq 2m \sum_{r=0}^{q_\ell} (2m^{\ell-r(s+1)-1}) + m^\ell \\ &\leq 4m^\ell / (1 - 1/m^{s+1}) + m^\ell \leq (39/7)m^\ell \leq 6m^\ell. \end{aligned} \quad (81)$$

As in the proof for the case that k is even, we have

$$\begin{aligned} \frac{M}{m} &\leq 2 \sum_{i=1}^{\ell-1} [(s+2)((i-1) \log_{s+3} m + 2) + s + 4] + (s+2)((\ell-1) \log_{s+3} m + 2) + s + 4 \\ &= (s+2)(\ell-1)^2 \log_{s+3} m + (3s+8)(2\ell-1) \leq (s+2)(\ell+4/\log_{s+3} m)^2 \log_{s+3} m \quad (82) \\ \frac{M}{m} &\geq 2 \sum_{i=1}^s [(s+2) \cdot 0 + s + 4] + 2 \sum_{i=s+1}^{\ell-1} [(s+2)(i-s-1) \log_{s+3} m + s + 4] \\ &\quad + (s+2)(\ell-s-1) \log_{s+3} m + s + 4 \\ &\geq (s+2)(\ell-s-1)^2 \log_{s+3} m. \end{aligned} \quad (83)$$

By combining $U_k \geq 2^{(\ell-s)\log_2 m}$ in (80) and $\ell \geq \sqrt{M \log_2 (s+3)/((s+2)m \log_2 m)} - 4 \log_2 (s+3)/\log_2 m$ in (82), we obtain the lower bound for U_k in (59). Similarly, by combining $U_k \leq 2^{\ell \log_2 m + \log_2 6}$ in (81) and $\ell \leq \sqrt{M \log_2 (s+3)/((s+2)m \log_2 m)} + s + 1$ in (83), we obtain the upper bound for U_k in (59). \blacksquare

Remark 13 (i) We see from Theorem 12 that we can achieve a buffer size $U_k = 2^{O(\sqrt{\alpha M})}$, where α is a constant that depends on the parameters s , k , and m used in the constructions:

- If $s = 1$, $k \geq 3$, and $m \geq 3$, then we have from Theorem 12(i) that $\alpha = 2 \log_2 (m-1)/(3m)$, which is the same as that in [37, Theorem 11(iii)].
- If $s = 2$, $k = 3$, and $m \geq 2$, then we have from Theorem 12(ii) that $\alpha = 2 \log_2 m/(3m)$, which is the same as that in [37, Theorem 11(iv)].
- If $s \geq 2$, $\max\{s+1, 4\} \leq k \leq 2s$, and $m \geq 2$, then we have from Theorem 12(ii) that $\alpha = \log_2 n \log_2 m / ((n-1)m)$, where $n = \lfloor k/2 \rfloor + 2$, which is better, i.e., larger, than $\log_2 (k+1) \log_2 m / (km)$ in [37, Theorem 11(iv)] (this is because $\log_2 x / (x-1)$ is decreasing in x over $x \geq 3$ and $3 < \lfloor k/2 \rfloor + 2 < k+1$ for $k \geq 4$).
- If $s \geq 2$, $2s+1 \leq k \leq 2s+2$, and $m \geq 2$, then we have from Theorem 12(ii) that $\alpha = \log_2 n \log_2 m / ((n-1)m)$, where $n = \lfloor k/2 \rfloor + 2$, which is better, i.e., larger, than

$\log_2(2s+2)\log_2 m/((2s+1)m)$ in [37, Theorem 11(v)] (this is because we have from $s \geq 2$ and $k \geq 2s+1$ that $3 < \lfloor k/2 \rfloor + 2 < 2s+2$ for $2s+1 \leq k \leq 2s+2$).

- If $s \geq 2$, $k \geq 2s+3$, and $m \geq 2$, then we have from Theorem 12(iii) that $\alpha = \log_2(s+3)\log_2 m/((s+2)m)$, which is better, i.e., larger, than $\log_2(2s+2)\log_2 m/((2s+1)m)$ in [37, Theorem 11(v)] (this is because we have from $s \geq 2$ that $3 < s+3 < 2s+2$).

(ii) From (i), we see that our constructions achieve larger buffer sizes than those in [37] in the regime that $s \geq 2$, $k \geq \max\{s+1, 4\}$, and $m \geq 2$. This agrees with the analysis in Section III-B that our choice of n is better than that in [37] in this regime. Indeed, our numerical results in Section V-B below also show that we achieve larger buffer sizes than those in [37] with less construction cost in this regime.

B. Numerical Results

In this subsection, we present our numerical results for the scenario considered in [37], i.e., $m_i = m \geq 2$ and $n_i = n$ for all $1 \leq i \leq k$. We use the number of 2×2 (bufferless) crossbar switches required in the construction of an optical priority queue as a measure of its construction cost. Note that an $M \times M$ (bufferless) crossbar switch can be implemented by using $M \log_2 M - M/2$ 2×2 switches via the Benes network [56].

In our optimal constructions, the parameter n is given by (19) (with the inequality in (19) replaced by equality), and the parameters B_1, B_2, \dots, B_k are given by (27) and (28) (with $m_i = m$ in (27) and (28) for $1 \leq i \leq k$). For $m = 2$, $2 \leq s \leq 6$, and $s+1 \leq k \leq 50$, we calculate the maximum buffer size U_k achieved in our construction by using (30) (with $m_i = m$ in (30) for $1 \leq i \leq k$) and calculate the number $N_k = (M+2)\log_2(M+2) - (M+2)/2$ of 2×2 switches required in our construction, where M is calculated by using (49) (with $m_i = m$ and $n_i = n$ in (49) for $1 \leq i \leq k$); we also calculate the maximum buffer size U'_k achieved in the construction in [37] by using [37, Eq. (13)] and calculate the number $N'_k = (M'+2)\log_2(M'+2) - (M'+2)/2$ of 2×2 switches required in the construction in [37], where M' is calculated by using [37, Eq. (14)].

From Figure 9 and Figure 10, we can see that our constructions achieve larger buffer sizes than those in [37], i.e., $U_k/U'_k > 1$, but with less construction cost, i.e., $N_k/N'_k \leq 1$. For moderate values of k , say $k \geq 7$, we achieve a buffer size that is 1.27 to 1.33 times of those in [37], but with 10% to 30% less construction cost. Furthermore, we observe that the improvement of our constructions over those in [37] gets better as s gets larger, i.e., the ratio U_k/U'_k gets larger and the ratio N_k/N'_k get smaller as s gets larger.

Similar results can be observed from Figure 11 and Figure 12 for $m = 3$, $2 \leq s \leq 6$, and $s+1 \leq k \leq 50$.

VI. CONCLUSION

In this paper, we have given constructions of optical priority queues by using a feedback system consisting of an optical (bufferless) crossbar switch and multiple groups of optical FM1's, and

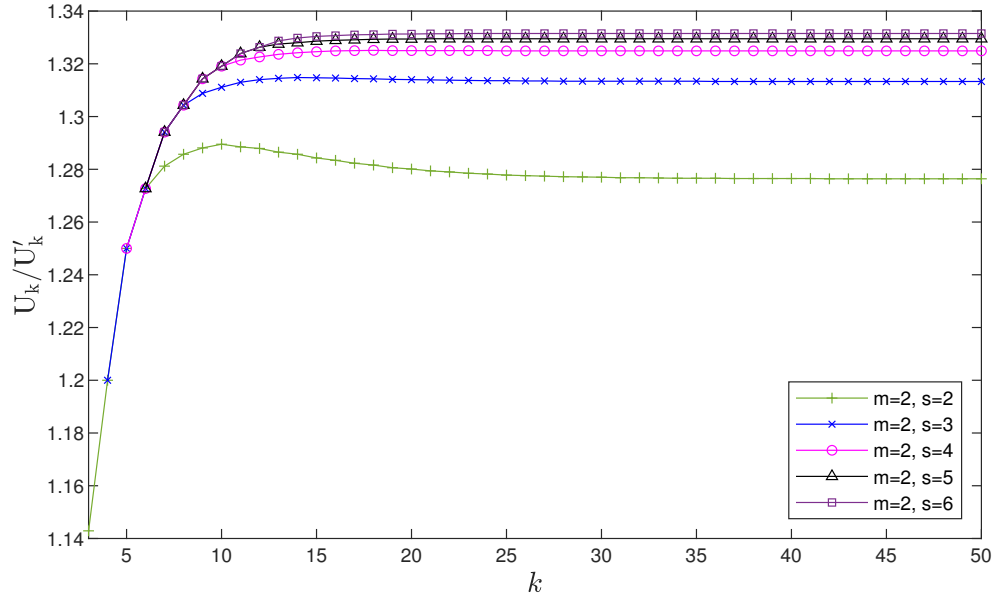


Fig. 9. The ratio U_k/U'_k between the maximum buffer sizes U_k and U'_k achieved in our construction and in the construction in [37], respectively, for $m = 2$, $2 \leq s \leq 6$, and $s + 1 \leq k \leq 50$.

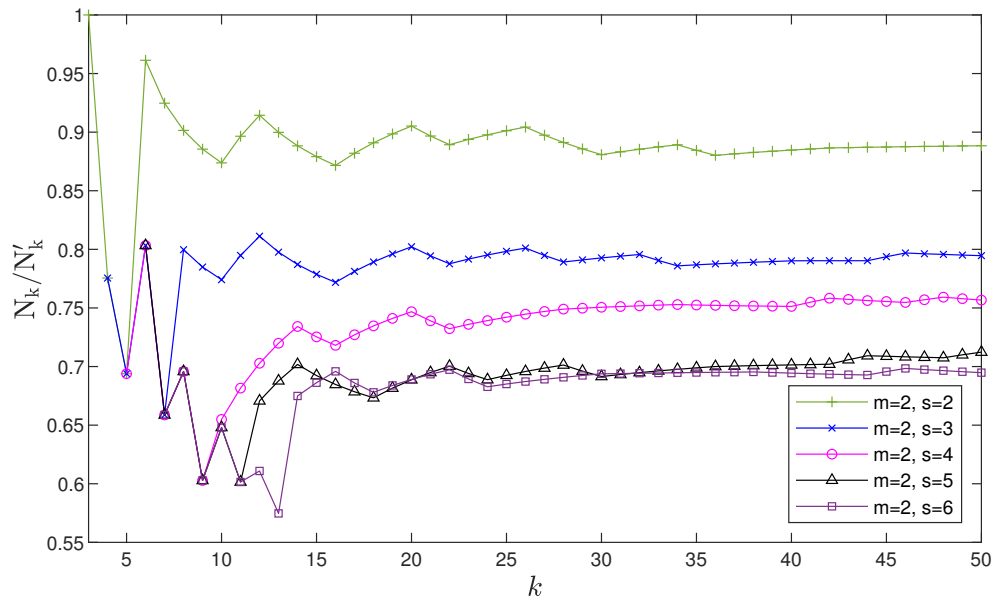


Fig. 10. The ratio N_k/N'_k between the numbers N_k and N'_k of 2×2 switches required in our construction and in the construction in [37], respectively, for $m = 2$, $2 \leq s \leq 6$, and $s + 1 \leq k \leq 50$.

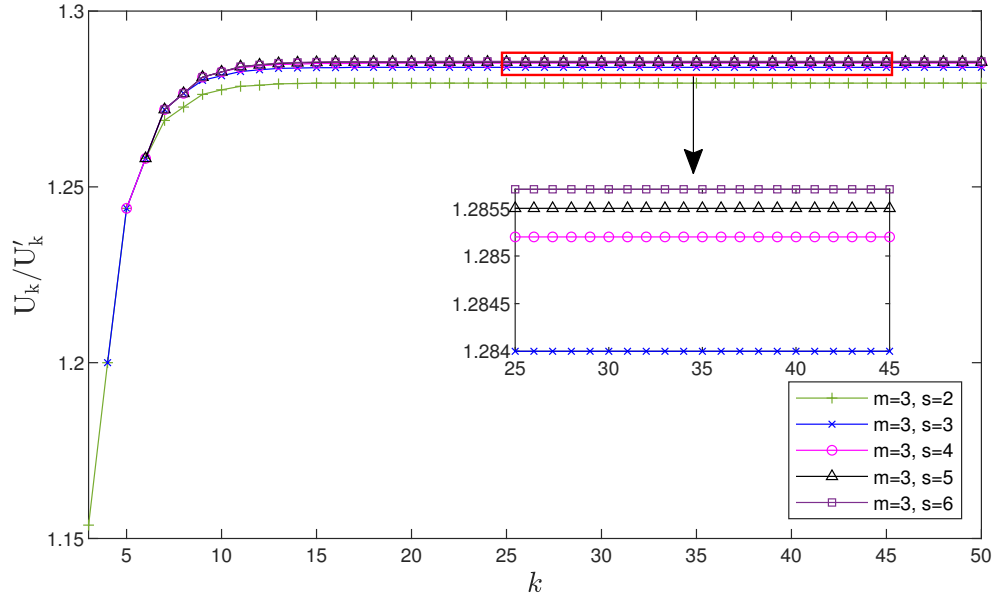


Fig. 11. The ratio U_k/U'_k between the maximum buffer sizes U_k and U'_k achieved in our construction and in the construction in [37], respectively, for $m = 3$, $2 \leq s \leq 6$, and $s + 1 \leq k \leq 50$.

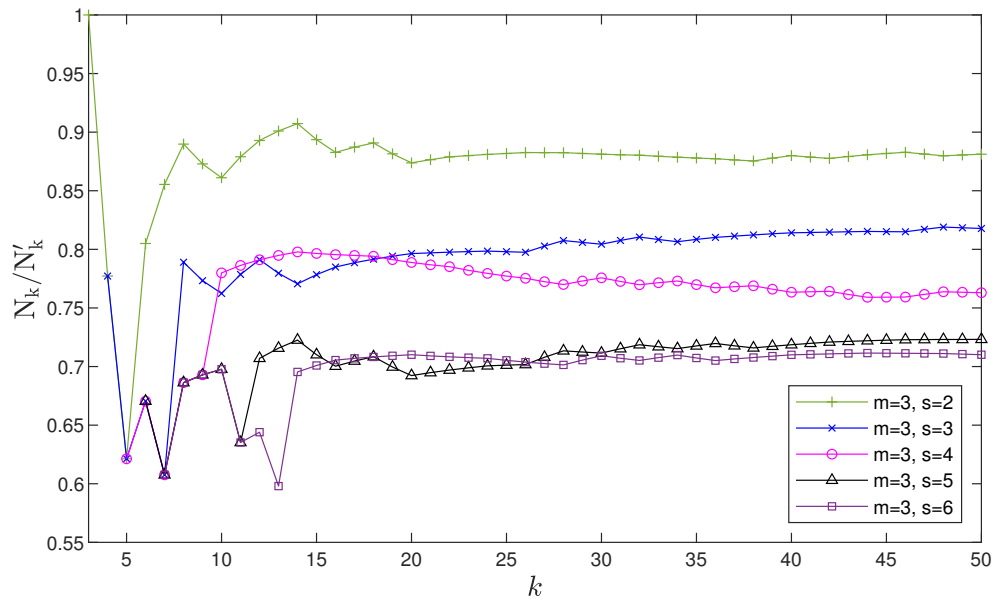


Fig. 12. The ratio N_k/N'_k between the numbers N_k and N'_k of 2×2 switches required in our construction and in the construction in [37], respectively, for $m = 3$, $2 \leq s \leq 6$, and $s + 1 \leq k \leq 50$.

have shown that our constructions can be implemented by using an optical (bufferless) crossbar switch and multiple optical fiber delay lines. We have shown that our constructions extend, improve, and generalize those in an important class of constructions that contains the optimal constructions in [37]. As a result of the generalization in this paper, we have also shown that our constructions possess fault-tolerant capability that can tolerate some malfunctioning FM1's in our constructions.

APPENDIX A

PROOF OF “(B) THERE IS NO COLLISION AT ANY INPUT LINK OF ANY FM1 AT SLOT t ” IN THE PROOF OF THEOREM 1

In this appendix, we complete the proof of the two steps mentioned at the end of the proof of Theorem 1: First, for each $1 \leq j \leq k$, we will identify the groups of FM1's to which packets that are from the output links of the j^{th} group of FM1's and have to be buffered in the system at slot t can possibly be routed. Second, for each $1 \leq i \leq k$, we will use the results obtained in the first step to show that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $j_1 \leq j \leq j_2$, where j_1 and j_2 are given by (2) and (3).

We divide the proof into two parts: (i) k is even and (ii) k is odd.

(i) k is even, say $k = 2\ell$. Note that $\ell = k/2 \geq \lceil (s+1)/2 \rceil \geq 1$. Also note that in this case (7) in (A3) can be written as follows:

$$|\Psi_j| \geq \begin{cases} B_{j-1}, & \text{if } 2 \leq j \leq \ell, \\ B_{j+1}, & \text{if } \ell + 1 \leq j \leq k - 1. \end{cases} \quad (84)$$

We discuss the two cases $1 \leq \ell \leq s+1$ and $\ell \geq s+2$ separately.

Case 1: $1 \leq \ell \leq s+1$. First, for each $1 \leq j \leq k$, we will identify the groups of FM1's to which packets that are from the output links of the j^{th} group of FM1's and have to be buffered in the system at slot t can possibly be routed. For this purpose, consider $1 \leq j \leq k$ and consider a packet p that is buffered in the j^{th} group of FM1's at slot $t-1$, leaves the j^{th} group of FM1's at slot t , and has to be buffered in the system at slot t . Then we have from Lemma 2(i) and Lemma 2(ii) that

$$\tilde{\tau}_p(t) \geq \tilde{\tau}_p(t-1) - 1 \geq L_j - B_j \quad (85)$$

and

$$\tilde{\tau}_p(t) \leq \tilde{\tau}_p(t-1) + 1 \leq U_j + B_j. \quad (86)$$

We consider the following subcases.

Subcase 1(a): $1 \leq j \leq \ell - 1$. In this subcase, we have $\tilde{\tau}_p(t) \geq 1 = L_1$, and we also have from (86) and (84) in (A3) (note that $2 \leq j+1 \leq \ell$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_j + |\Psi_{j+1}| = U_{j+1}$.

Thus, we have $\tilde{\tau}_p(t) \in \{L_1, L_1 + 1, \dots, U_{j+1}\} = \cup_{i=1}^{j+1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq j + 1$.

Subcase 1(b): $j = \ell$. If $1 \leq \ell \leq s$, then clearly packet p can only be routed to one of the k groups of FM1's at slot t , i.e., packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq k$. On the other hand, if $\ell = s + 1$, then we have $\tilde{\tau}_p(t) \geq 1 = L_1$, and we also have from (86) and (5) in (A2) (note that $j = \ell = 2\ell - \ell = k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s} = U_{\ell+(\ell-1)} = U_{k-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_1, L_1 + 1, \dots, U_{k-1}\} = \cup_{i=1}^{k-1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq k - 1$.

Subcase 1(c): $j = \ell + 1$. If $1 \leq \ell \leq s$, then clearly packet p can only be routed to one of the k groups of FM1's at slot t , i.e., packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq k$. On the other hand, if $\ell = s + 1$, then we have $\tilde{\tau}_p(t) \leq U_k$, and we also have from (85) and (4) in (A2) (note that $j = \ell + 1 = s + 2$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s} = L_{(\ell+1)-(\ell-1)} = L_2$. Thus, we have $\tilde{\tau}_p(t) \in \{L_2, L_2 + 1, \dots, U_k\} = \cup_{i=2}^k \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $2 \leq i \leq k$.

Subcase 1(d): $\ell + 2 \leq j \leq k$. In this subcase, we have $\tilde{\tau}_p(t) \leq U_k$, and we also have from (85) and (84) in (A3) (note that $\ell + 1 \leq j - 1 \leq k - 1$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq (U_{j-1} + 1) - |\Psi_{j-1}| = U_{j-2} + 1 = L_{j-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-1}, L_{j-1} + 1, \dots, U_k\} = \cup_{i=j-1}^k \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - 1 \leq i \leq k$.

We summarize the results in Subcases 1(a)–(d) in the following table.

$k = 2\ell$ and $1 \leq \ell \leq s + 1$				
j	$1 \leq j \leq \ell - 1$	$j = \ell$	$j = \ell + 1$	$\ell + 2 \leq j \leq k$
i	$1 \leq i \leq j + 1$	$\begin{cases} 1 \leq i \leq k, & \text{if } 1 \leq \ell \leq s \\ 1 \leq i \leq k - 1, & \text{if } \ell = s + 1 \end{cases}$	$\begin{cases} 1 \leq i \leq k, & \text{if } 1 \leq \ell \leq s \\ 2 \leq i \leq k, & \text{if } \ell = s + 1 \end{cases}$	$j - 1 \leq i \leq k$

TABLE I

FOR EACH $1 \leq j \leq k$, THE RANGE OF i GIVEN IN SUBCASES 1(A)–(D) SUCH THAT PACKETS FROM THE OUTPUT LINKS OF THE j^{TH} GROUP OF FM1'S CAN POSSIBLY BE ROUTED TO THE i^{TH} GROUP OF FM1'S AT SLOT t .

Second, for each $1 \leq i \leq k$, we will use the results in Subcases 1(a)–(d) above to identify the groups of FM1's such that packets that are from the output links of these groups (but not other groups) of FM1's and have to be buffered in the system at slot t can possibly be routed to the i^{th} group of FM1's at slot t , and show that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $j_1 \leq j \leq j_2$, where j_1 and j_2 are given by (2) (note that k is even). For this purpose, consider $1 \leq i \leq k$ and consider the following subcases.

Subcase 1(a'): $i = 1$. If $1 \leq \ell \leq s$, then we have from (2) that $j_1 = \max\{i - 1, 1\} = 1$ (as $i - 1 = 0 < 1$) and $j_2 = \min\{i + s, \ell + 1\} = \ell + 1$ (as $i + s = s + 1 \geq \ell + 1$). On the other hand, if

$\ell = s + 1$, then we have from (2) that $j_1 = \max\{i - 1, 1\} = 1$ and $j_2 = \min\{i + s, \ell + 1\} = s + 1$ (as $i + s = s + 1 = \ell < \ell + 1$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) holds (as $i = 1$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $i = 1$). Similarly, if $\ell = s + 1$, then the condition $1 \leq i \leq k - 1$ in Subcase 1(b) also holds (as $i = 1$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell + 1$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(c) holds (as $i = 1$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t . On the other hand, if $\ell = s + 1$, then the condition $2 \leq i \leq k$ in Subcase 1(c) does not hold (as $i = 1 < 2$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 2 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(d) does not hold (as $i = 1 < \ell + 1 \leq j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

As such, we deduce the following results:

- ▲ If $1 \leq \ell \leq s$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $1 \leq j \leq \ell + 1$, i.e., $j_1 \leq j \leq j_2$.
- ▲ On the other hand, if $\ell = s + 1$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $1 \leq j \leq \ell = s + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 1(b'): $2 \leq i \leq \ell$. Note that $\ell \geq 2$ in this subcase. In this subcase, we have from (2) that $j_1 = \max\{i - 1, 1\} = i - 1$ (as $i - 1 \geq 1$) and $j_2 = \min\{i + s, \ell + 1\} = \ell + 1$ (as $i + s \geq s + 2 \geq \ell + 1$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) holds if and only if $i - 1 = \max\{i - 1, 1\} \leq j \leq \ell - 1$. Thus, if $i - 1 \leq j \leq \ell - 1$ (resp., $1 \leq j \leq i - 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $1 < 2 \leq i \leq \ell < 2\ell = k$). Similarly, if $\ell = s + 1$, then the condition $1 \leq i \leq k - 1$ in Subcase 1(b) also holds (as $1 < 2 \leq i \leq \ell < 2\ell - 1 = k - 1$). Thus, packets from the

output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .

- Suppose that $j = \ell + 1$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(c) holds (as $1 < 2 \leq i \leq \ell < 2\ell = k$). Similarly, if $\ell = s + 1$, then the condition $2 \leq i \leq k$ in Subcase 1(c) also holds (as $2 \leq i \leq \ell < 2\ell = k$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 2 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(d) does not hold (as $i \leq \ell \leq j - 2 < j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $i - 1 \leq j \leq \ell + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 1(c'): $\ell + 1 \leq i \leq k - 1$. Note that $\ell \geq 2$ in this subcase. In this subcase, we have from (2) that $j_2 = \min\{i + 1, k\} = i + 1$ (as $i + 1 \leq k$) and $j_1 = \max\{i - s, \ell\} = \ell$ (as $i - s \leq k - 1 - s \leq k - \ell = \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) does not hold (as $i \geq \ell + 1 \geq j + 2 > j + 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $1 < \ell + 1 \leq i \leq k - 1 < k$). Similarly, if $\ell = s + 1$, then the condition $1 \leq i \leq k - 1$ in Subcase 1(b) also holds (as $1 < \ell + 1 \leq i \leq k - 1$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell + 1$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(c) holds (as $1 < \ell + 1 \leq i \leq k - 1 < k$). Similarly, if $\ell = s + 1$, then the condition $2 \leq i \leq k$ in Subcase 1(c) also holds (as $2 < \ell + 1 \leq i \leq k - 1 < k$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 2 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(d) holds if and only if $i + 1 = \min\{i + 1, k\} \geq j \geq \ell + 2$. Thus, if $i + 1 \geq j \geq \ell + 2$ (resp., $k \geq j \geq i + 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $i + 1 \geq j \geq \ell$, i.e., $j_2 \geq j \geq j_1$.

Subcase 1(d'): $i = k$. If $1 \leq \ell \leq s$, then we have from (2) that $j_2 = \min\{i + 1, k\} = k$ (as $i + 1 = k + 1 > k$) and $j_1 = \max\{i - s, \ell\} = \ell$ (as $i - s = k - s \leq k - \ell = \ell$). On the other hand, if $\ell = s + 1$, then we have from (2) that $j_2 = \min\{i + 1, k\} = k$ and $j_1 = \max\{i - s, \ell\} = k - s$ (as $i - s = k - s = k - (\ell - 1) = \ell + 1 > \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) does not hold (as $i = k > \ell \geq j + 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $i = k$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t . On the other hand, if $\ell = s + 1$, then the condition $1 \leq i \leq k - 1$ in Subcase 1(b) does not hold (as $i = k > k - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell + 1$. If $1 \leq \ell \leq s$, then the condition $1 \leq i \leq k$ in Subcase 1(c) holds (as $i = k$). Similarly, if $\ell = s + 1$, then the condition $2 \leq i \leq k$ in Subcase 1(c) also holds (as $i = k$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 2 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(d) holds (as $i = k$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .

As such, we deduce the following results:

- ▲ If $1 \leq \ell \leq s$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $k \geq j \geq \ell$, i.e., $j_2 \geq j \geq j_1$.
- ▲ On the other hand, if $\ell = s + 1$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $k \geq j \geq \ell + 1 = 2\ell - (\ell - 1) = k - s$, i.e., $j_2 \geq j \geq j_1$.

Case 2: $\ell \geq s + 2$. First, for each $1 \leq j \leq k$, we will identify the groups of FM1's to which packets that are from the output links of the j^{th} group of FM1's and have to be buffered in the system at slot t can possibly be routed. For this purpose, consider $1 \leq j \leq k$ and consider a packet p that is buffered in the j^{th} group of FM1's at slot $t - 1$, leaves the j^{th} group of FM1's at slot t , and has to be buffered in the system at slot t . Note that (85) and (86) still hold in this case.

We consider the following subcases.

Subcase 2(a): $1 \leq j \leq s + 1$. In this subcase, we have $\tilde{\tau}_p(t) \geq 1 = L_1$, and we also have from (86) and (84) in (A3) (note that $2 \leq j + 1 \leq s + 2 \leq \ell$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_j + |\Psi_{j+1}| = U_{j+1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_1, L_1 + 1, \dots, U_{j+1}\} = \cup_{i=1}^{j+1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq j + 1$.

Subcase 2(b): $s + 2 \leq j \leq \ell - 1$. Note that $\ell \geq s + 3$ in this subcase. In this subcase, we have from (85) and (4) in (A2) (note that $s + 2 \leq j \leq \ell - 1 < k$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s}$, and we also have from (86) and (84) in (A3) (note

that $2 < s + 3 \leq j + 1 \leq \ell$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_j + |\Psi_{j+1}| = U_{j+1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-s}, L_{j-s} + 1, \dots, U_{j+1}\} = \cup_{i=j-s}^{j+1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - s \leq i \leq j + 1$.

Subcase 2(c): $j = \ell$. In this subcase, we have from (85) and (4) in (A2) (note that $s + 2 \leq j = \ell < k$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s}$, and we also have from (86) and (5) in (A2) (note that $1 < s + 2 \leq j = \ell \leq k - s - 2 < k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-s}, L_{j-s} + 1, \dots, U_{j+s}\} = \cup_{i=j-s}^{j+s} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - s \leq i \leq j + s$.

Subcase 2(d): $j = \ell + 1$. In this subcase, we have from (86) and (5) in (A2) (note that $1 < s + 3 \leq j = \ell + 1 \leq k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s}$, and we also have from (85) and (4) in (A2) (note that $s + 2 < j = \ell + 1 < k$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-s}, L_{j-s} + 1, \dots, U_{j+s}\} = \cup_{i=j-s}^{j+s} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - s \leq i \leq j + s$.

Subcase 2(e): $\ell + 2 \leq j \leq k - s - 1$. Note that $\ell \geq s + 3$ in this subcase. In this subcase, we have from (86) and (5) in (A2) (note that $1 < \ell + 2 \leq j \leq k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s}$, and we also have from (85) and (84) in (A3) (note that $\ell + 1 \leq j - 1 \leq k - s - 2 < k - 1$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq (U_{j-1} + 1) - |\Psi_{j-1}| = U_{j-2} + 1 = L_{j-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-1}, L_{j-1} + 1, \dots, U_{j+s}\} = \cup_{i=j-1}^{j+s} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - 1 \leq i \leq j + s$.

Subcase 2(f): $k - s \leq j \leq k$. In this subcase, we have $\tilde{\tau}_p(t) \leq U_k$, and we also have from (85) and (84) in (A3) (note that $\ell + 1 \leq k - s - 1 \leq j - 1 \leq k - 1$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq (U_{j-1} + 1) - |\Psi_{j-1}| = U_{j-2} + 1 = L_{j-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-1}, L_{j-1} + 1, \dots, U_k\} = \cup_{i=j-1}^k \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - 1 \leq i \leq k$.

We summarize the results in Subcases 2(a)–(f) in the following table.

$k = 2\ell$ and $\ell \geq s + 2$						
j	$1 \leq j \leq s + 1$	$s + 2 \leq j \leq \ell - 1$	$j = \ell$	$j = \ell + 1$	$\ell + 2 \leq j \leq k - s - 1$	$k - s \leq j \leq k$
i	$1 \leq i \leq j + 1$	$j - s \leq i \leq j + 1$	$j - s \leq i \leq j + s$	$j - s \leq i \leq j + s$	$j - 1 \leq i \leq j + s$	$j - 1 \leq i \leq k$

TABLE II

FOR EACH $1 \leq j \leq k$, THE RANGE OF i GIVEN IN SUBCASES 2(A)–(F) SUCH THAT PACKETS FROM THE OUTPUT LINKS OF THE j^{TH} GROUP OF FM1'S CAN POSSIBLY BE ROUTED TO THE i^{TH} GROUP OF FM1'S AT SLOT t .

Second, for each $1 \leq i \leq k$, we will use the results in Subcases 2(a)–(f) above to identify the groups of FM1's such that packets that are from the output links of these groups (but not other groups) of FM1's and have to be buffered in the system at slot t can possibly be routed to the i^{th} group of FM1's at slot t , and show that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's

at slot t for some $j_1 \leq j \leq j_2$, where j_1 and j_2 are given by (2) (note that k is even). For this purpose, consider $1 \leq i \leq k$ and consider the following subcases.

Subcase 2(a'): $i = 1$. In this subcase, we have from (2) that $j_1 = \max\{i - 1, 1\} = 1$ (as $i - 1 = 0 < 1$) and $j_2 = \min\{i + s, \ell + 1\} = s + 1$ (as $i + s = s + 1 < (s + 2) + 1 \leq \ell + 1$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) holds (as $i = 1$) and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $s + 2 \leq j \leq \ell - 1$. Note that $\ell \geq s + 3$. Then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) does not hold (as $i = 1 < 2 \leq j - s$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. Then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) does not hold (as $i = 1 < 2 \leq \ell - s = j - s$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell + 1$. Then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) does not hold (as $i = 1 < 3 \leq (\ell + 1) - s = j - s$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 2 \leq j \leq k - s - 1$. Note that $\ell \geq s + 3$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) does not hold (as $i = 1 < \ell + 1 \leq j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(f) does not hold (as $i = 1 < s + 3 \leq 2\ell - s - 1 = k - s - 1 \leq j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $1 \leq j \leq s + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 2(b'): $2 \leq i \leq s + 2$. In this subcase, we have from (2) that $j_1 = \max\{i - 1, 1\} = i - 1$. If $s + 2 \leq \ell \leq 2s + 1$, then we have from (2) that

$$j_2 = \min\{i + s, \ell + 1\} = \begin{cases} i + s, & \text{if } 2 \leq i \leq \ell - s, \\ \ell + 1, & \text{if } \ell - s + 1 \leq i \leq s + 2. \end{cases}$$

On the other hand, if $\ell \geq 2s + 2$, then we have from (2) that $j_2 = \min\{i + s, \ell + 1\} = i + s$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) holds if and only if $i - 1 = \max\{i - 1, 1\} \leq j \leq s + 1$.

- Suppose that $s + 2 \leq j \leq \ell - 1$. Note that $\ell \geq s + 3$. If $s + 3 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if

$$s + 2 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \begin{cases} i + s, & \text{if } 2 \leq i \leq \ell - s - 1, \\ \ell - 1, & \text{if } \ell - s \leq i \leq s + 2. \end{cases}$$

On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if $s + 2 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = i + s$.

- Suppose that $j = \ell$. If $s + 2 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $\ell - s = \max\{\ell - s, 2\} \leq i \leq \min\{\ell + s, s + 2\} = s + 2$. On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) does not hold (as $i \leq s + 2 \leq \ell - s - 1 = j - s - 1 < j - s$).
- Suppose that $j = \ell + 1$. If $s + 2 \leq \ell \leq 2s + 1$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) holds if and only if $\ell - s + 1 = \max\{(\ell + 1) - s, 2\} \leq i \leq \min\{(\ell + 1) + s, s + 2\} = s + 2$. On the other hand, if $\ell \geq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) does not hold (as $i \leq s + 2 \leq \ell - s = j - s - 1 < j - s$).
- Suppose that $\ell + 2 \leq j \leq k - s - 1$. Note that $\ell \geq s + 3$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) does not hold (as $i \leq s + 2 \leq \ell - 1 \leq j - 3 < j - 1$).
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(f) does not hold (as $i \leq s + 2 \leq 2\ell - s - 2 = k - s - 2 \leq j - 2 < j - 1$).

As such, we deduce the following results:

- ▲ If $\ell = s + 2$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some (note that the contribution from the result in the first bullet point above is $i - 1 \leq j \leq s + 1 = \ell - 1$)

$$i - 1 \leq j \leq \begin{cases} \ell = s + 2 = i + s, & \text{if } i = 2, \\ \ell + 1, & \text{if } 3 \leq i \leq s + 2, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

- ▲ If $s + 3 \leq \ell \leq 2s + 1$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i - 1 \leq j \leq \begin{cases} i + s, & \text{if } 2 \leq i \leq \ell - s - 1, \\ \ell = i + s, & \text{if } i = \ell - s, \\ \ell + 1, & \text{if } \ell - s + 1 \leq i \leq s + 2, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

- ▲ If $\ell = 2s + 2$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i - 1 \leq j \leq \begin{cases} i + s, & \text{if } 2 \leq i \leq s + 1, \\ \ell = i + s, & \text{if } i = s + 2, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

- ▲ Finally, if $\ell \geq 2s + 3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i - 1 \leq j \leq i + s$, i.e., $j_1 \leq j \leq j_2$.

Subcase 2(c'): $s + 3 \leq i \leq \ell$. Note that $\ell \geq s + 3$ in this subcase. In this subcase, we have from (2) that $j_1 = \max\{i - 1, 1\} = i - 1$. If $s + 3 \leq \ell \leq 2s + 2$, then we have from (2) that $j_2 = \min\{i + s, \ell + 1\} = \ell + 1$. On the other hand, if $\ell \geq 2s + 3$, then we have from (2) that

$$j_2 = \min\{i + s, \ell + 1\} = \begin{cases} i + s, & \text{if } s + 3 \leq i \leq \ell - s, \\ \ell + 1, & \text{if } \ell - s + 1 \leq i \leq \ell. \end{cases}$$

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i \geq s + 3 \geq j + 2 > j + 1$).
- Suppose that $s + 2 \leq j \leq \ell - 1$. If $s + 3 \leq \ell \leq 2s + 3$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if $i - 1 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \ell - 1$. On the other hand, if $\ell \geq 2s + 4$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if

$$i - 1 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \begin{cases} i + s, & \text{if } s + 3 \leq i \leq \ell - s - 1, \\ \ell - 1, & \text{if } \ell - s \leq i \leq \ell. \end{cases}$$

- Suppose that $j = \ell$. If $s + 3 \leq \ell \leq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds (as $j - s = \ell - s \leq s + 3 \leq i \leq \ell < \ell + s = j + s$). On the other hand, if $\ell \geq 2s + 4$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $\ell - s = \max\{\ell - s, s + 3\} \leq i \leq \min\{\ell + s, \ell\} = \ell$.
- Suppose that $j = \ell + 1$. If $s + 3 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) holds (as $j - s = (\ell + 1) - s \leq s + 3 \leq i \leq \ell < (\ell + 1) + s = j + s$). On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) holds if and only if $\ell - s + 1 = \max\{(\ell + 1) - s, s + 3\} \leq i \leq \min\{(\ell + 1) + s, \ell\} = \ell$.
- Suppose that $\ell + 2 \leq j \leq k - s - 1$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) does not hold (as $i \leq \ell \leq j - 2 < j - 1$).
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(f) does not hold (as $i \leq \ell \leq 2\ell - s - 3 = k - s - 3 \leq j - 3 < j - 1$).

As such, we deduce the following results:

- ▲ If $s + 3 \leq \ell \leq 2s + 2$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i - 1 \leq j \leq \ell + 1$, i.e., $j_1 \leq j \leq j_2$.

- ▲ If $\ell = 2s + 3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i - 1 \leq j \leq \begin{cases} \ell = i + s, & \text{if } i = s + 3, \\ \ell + 1, & \text{if } s + 4 \leq i \leq \ell, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

- ▲ Finally, if $\ell \geq 2s + 4$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i - 1 \leq j \leq \begin{cases} i + s, & \text{if } s + 3 \leq i \leq \ell - s - 1, \\ \ell = i + s, & \text{if } i = \ell - s, \\ \ell + 1, & \text{if } \ell - s + 1 \leq i \leq \ell, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

Subcase 2(d'): $\ell + 1 \leq i \leq k - s - 2$. Note that $\ell \geq s + 3$ in this subcase. In this subcase, we have from (2) that $j_2 = \min\{i + 1, k\} = i + 1$. If $s + 3 \leq \ell \leq 2s + 2$, then we have from (2) that $j_1 = \max\{i - s, \ell\} = \ell$. On the other hand, if $\ell \geq 2s + 3$, then we have from (2) that

$$j_1 = \max\{i - s, \ell\} = \begin{cases} \ell, & \text{if } \ell + 1 \leq i \leq \ell + s, \\ i - s, & \text{if } \ell + s + 1 \leq i \leq k - s - 2. \end{cases}$$

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i \geq \ell + 1 \geq s + 4 \geq j + 3 > j + 1$).
- Suppose that $s + 2 \leq j \leq \ell - 1$. Then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) does not hold (as $i \geq \ell + 1 \geq j + 2 > j + 1$).
- Suppose that $j = \ell$. If $s + 3 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds (as $j - s = \ell - s < \ell + 1 \leq i \leq k - s - 2 \leq \ell + s = j + s$). On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $\ell + 1 = \max\{\ell - s, \ell + 1\} \leq i \leq \min\{\ell + s, k - s - 2\} = \ell + s$.
- Suppose that $j = \ell + 1$. If $s + 3 \leq \ell \leq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) holds (as $j - s = (\ell + 1) - s < \ell + 1 \leq i \leq k - s - 2 \leq (\ell + 1) + s = j + s$). On the other hand, if $\ell \geq 2s + 4$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) holds if and only if $\ell + 1 = \max\{(\ell + 1) - s, \ell + 1\} \leq i \leq \min\{(\ell + 1) + s, k - s - 2\} = \ell + s + 1$.
- Suppose that $\ell + 2 \leq j \leq k - s - 1$. If $s + 3 \leq \ell \leq 2s + 3$, then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) holds if and only if $i + 1 = \min\{i + 1, k - s - 1\} \geq j \geq \max\{i - s, \ell + 2\} = \ell + 2$. On the other hand, if $\ell \geq 2s + 4$, then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) holds if and only if

$$\begin{aligned} i + 1 = \min\{i + 1, k - s - 1\} \geq j \geq \max\{i - s, \ell + 2\} \\ = \begin{cases} \ell + 2, & \text{if } \ell + 1 \leq i \leq \ell + s + 1, \\ i - s, & \text{if } \ell + s + 2 \leq i \leq k - s - 2. \end{cases} \end{aligned}$$

- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(f) does not hold (as $i \leq k - s - 2 \leq j - 2 < j - 1$).

As such, we deduce the following results:

- ▲ If $s + 3 \leq \ell \leq 2s + 2$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i + 1 \geq j \geq \ell$, i.e., $j_2 \geq j \geq j_1$.
- ▲ If $\ell = 2s + 3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i + 1 \geq j \geq \begin{cases} \ell, & \text{if } 2s + 4 \leq i \leq 3s + 3, \\ \ell + 1 = i - s, & \text{if } i = 3s + 4, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

- ▲ Finally, if $\ell \geq 2s + 4$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i + 1 \geq j \geq \begin{cases} \ell, & \text{if } \ell + 1 \leq i \leq \ell + s, \\ \ell + 1 = i - s, & \text{if } i = \ell + s + 1, \\ i - s, & \text{if } \ell + s + 2 \leq i \leq k - s - 2, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

Subcase 2(e'): $k - s - 1 \leq i \leq k - 1$. In this subcase, we have from (2) that $j_2 = \min\{i + 1, k\} = i + 1$. If $s + 2 \leq \ell \leq 2s + 1$, then we have from (2) that

$$j_1 = \max\{i - s, \ell\} = \begin{cases} \ell, & \text{if } k - s - 1 \leq i \leq \ell + s, \\ i - s, & \text{if } \ell + s + 1 \leq i \leq k - 1. \end{cases}$$

On the other hand, if $\ell \geq 2s + 2$, then we have from (2) that $j_1 = \max\{i - s, \ell\} = i - s$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i \geq k - s - 1 \geq s + 3 \geq j + 2 > j + 1$).
- Suppose that $s + 2 \leq j \leq \ell - 1$. Note that $\ell \geq s + 3$. Then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) does not hold (as $i \geq k - s - 1 \geq \ell + 2 \geq j + 3 > j + 1$).
- Suppose that $j = \ell$. If $s + 2 \leq \ell \leq 2s + 1$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $k - s - 1 = \max\{\ell - s, k - s - 1\} \leq i \leq \min\{\ell + s, k - 1\} = \ell + s$. On the other hand, if $\ell \geq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) does not hold (as $i \geq k - s - 1 \geq \ell + s + 1 = j + s + 1 > j + s$).
- Suppose that $j = \ell + 1$. If $s + 2 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) holds if and only if $k - s - 1 = \max\{(\ell + 1) - s, k - s - 1\} \leq i \leq \min\{(\ell + 1) + s, k - 1\} = \ell + s + 1$. On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) does not hold (as $i \geq k - s - 1 \geq \ell + s + 2 = j + s + 1 > j + s$).

- Suppose that $\ell + 2 \leq j \leq k - s - 1$. Note that $\ell \geq s + 3$. If $s + 3 \leq \ell \leq 2s + 2$, then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) holds if and only if

$$\begin{aligned} k - s - 1 = \min\{i + 1, k - s - 1\} &\geq j \geq \max\{i - s, \ell + 2\} \\ &= \begin{cases} \ell + 2, & \text{if } k - s - 1 \leq i \leq \ell + s + 1, \\ i - s, & \text{if } \ell + s + 2 \leq i \leq k - 1. \end{cases} \end{aligned}$$

On the other hand, if $\ell \geq 2s + 3$, then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) holds if and only if $k - s - 1 = \min\{i + 1, k - s - 1\} \geq j \geq \max\{i - s, \ell + 2\} = i - s$.

- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(f) holds if and only if $i + 1 = \min\{i + 1, k\} \geq j \geq k - s$.

As such, we deduce the following results:

- ▲ If $\ell = s + 2$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some (note that the contribution from the result in the last bullet point above is $i + 1 \geq j \geq k - s = s + 4 = \ell + 2$)

$$i + 1 \geq j \geq \begin{cases} \ell, & \text{if } k - s - 1 \leq i \leq \ell + s = 2s + 2 = k - 2, \\ \ell + 1 = s + 3 = i - s, & \text{if } i = 2s + 3 = k - 1, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

- ▲ If $s + 3 \leq \ell \leq 2s + 1$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i + 1 \geq j \geq \begin{cases} \ell, & \text{if } k - s - 1 \leq i \leq \ell + s, \\ \ell + 1 = i - s, & \text{if } i = \ell + s + 1, \\ i - s, & \text{if } \ell + s + 2 \leq i \leq k - 1, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

- ▲ If $\ell = 2s + 2$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i + 1 \geq j \geq \begin{cases} \ell + 1 = i - s, & \text{if } i = 3s + 3, \\ i - s, & \text{if } 3s + 4 \leq i \leq k - 1, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

- ▲ Finally, if $\ell \geq 2s + 3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i + 1 \geq j \geq i - s$, i.e., $j_2 \geq j \geq j_1$.

Subcase 2(f'): $i = k$. In this subcase, we have from (2) that $j_2 = \min\{i + 1, k\} = k$ and $j_1 = \max\{i - s, \ell\} = k - s$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i = k > \ell \geq s + 2 \geq j + 1$).

- Suppose that $s + 2 \leq j \leq \ell - 1$. Note that $\ell \geq s + 3$. Then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) does not hold (as $i = k > \ell \geq j + 1$).
- Suppose that $j = \ell$. Then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) does not hold (as $i = k > \ell + s = j + s$).
- Suppose that $j = \ell + 1$. Then the condition $j - s \leq i \leq j + s$ in Subcase 2(d) does not hold (as $i = k > \ell + s + 1 = j + s$).
- Suppose that $\ell + 2 \leq j \leq k - s - 1$. Note that $\ell \geq s + 3$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(e) does not hold (as $i = k > k - 1 \geq j + s$).
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(f) holds (as $i = k$).

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $k \geq j \geq k - s$, i.e., $j_2 \geq j \geq j_1$.

(ii) k is odd, say $k = 2\ell - 1$. Note that $\ell = (k + 1)/2 \geq \lceil ((s + 1) + 1)/2 \rceil \geq 2$. Also note that in this case (7) in (A3) can be written as follows:

$$|\Psi_j| \geq \begin{cases} B_{j-1}, & \text{if } 2 \leq j \leq \ell, \\ B_{j+1}, & \text{if } \ell \leq j \leq k - 1. \end{cases} \quad (87)$$

We discuss the two cases $2 \leq \ell \leq s + 2$ and $\ell \geq s + 3$ separately.

Case 1: $2 \leq \ell \leq s + 2$. First, for each $1 \leq j \leq k$, we will identify the groups of FM1's to which packets that are from the output links of the j^{th} group of FM1's and have to be buffered in the system at slot t can possibly be routed. For this purpose, consider $1 \leq j \leq k$ and consider a packet p that is buffered in the j^{th} group of FM1's at slot $t - 1$, leaves the j^{th} group of FM1's at slot t , and has to be buffered in the system at slot t . Note that (85) and (86) still hold in this case.

We consider the following subcases.

Subcase 1(a): $1 \leq j \leq \ell - 1$. In this subcase, we have $\tilde{\tau}_p(t) \geq 1 = L_1$, and we also have from (86) and (87) in (A3) (note that $2 \leq j + 1 \leq \ell$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_j + |\Psi_{j+1}| = U_{j+1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_1, L_1 + 1, \dots, U_{j+1}\} = \cup_{i=1}^{j+1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq j + 1$.

Subcase 1(b): $j = \ell$. If $2 \leq \ell \leq s + 1$, then clearly packet p can only be routed to one of the k groups of FM1's at slot t , i.e., packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq k$. On the other hand, if $\ell = s + 2$, then we have from (85) and (4) in (A2) (note that $j = \ell = s + 2$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s} = L_{\ell-(\ell-2)} = L_2$, and we also have from (86) and (5) in (A2) (note that $j = \ell = (2\ell - 1) - (\ell - 1) = k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s} = U_{\ell+(\ell-2)} = U_{k-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_2, L_2 + 1, \dots, U_{k-1}\} = \cup_{i=2}^{k-1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $2 \leq i \leq k - 1$.

Subcase 1(c): $\ell + 1 \leq j \leq k$. In this subcase, we have $\tilde{\tau}_p(t) \leq U_k$, and we also have from (85) and (87) in (A3) (note that $\ell \leq j - 1 \leq k - 1$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq (U_{j-1} + 1) - |\Psi_{j-1}| = U_{j-2} + 1 = L_{j-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-1}, L_{j-1} + 1, \dots, U_k\} = \cup_{i=j-1}^k \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - 1 \leq i \leq k$.

We summarize the results in Subcases 1(a)–(c) in the following table.

$k = 2\ell - 1$ and $2 \leq \ell \leq s + 2$			
j	$1 \leq j \leq \ell - 1$	$j = \ell$	$\ell + 1 \leq j \leq k$
i	$1 \leq i \leq j + 1$	$\begin{cases} 1 \leq i \leq k, & \text{if } 2 \leq \ell \leq s + 1 \\ 2 \leq i \leq k - 1, & \text{if } \ell = s + 2 \end{cases}$	$j - 1 \leq i \leq k$

TABLE III

FOR EACH $1 \leq j \leq k$, THE RANGE OF i GIVEN IN SUBCASES 1(A)–(C) SUCH THAT PACKETS FROM THE OUTPUT LINKS OF THE j^{TH} GROUP OF FM1'S CAN POSSIBLY BE ROUTED TO THE i^{TH} GROUP OF FM1'S AT SLOT t .

Second, for each $1 \leq i \leq k$, we will use the results in Subcases 1(a)–(c) above to identify the groups of FM1's such that packets that are from the output links of these groups (but not other groups) of FM1's and have to be buffered in the system at slot t can possibly be routed to the i^{th} group of FM1's at slot t , and show that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $j_1 \leq j \leq j_2$, where j_1 and j_2 are given by (3) (note that k is odd). For this purpose, consider $1 \leq i \leq k$ and consider the following subcases.

Subcase 1(a'): $i = 1$. If $2 \leq \ell \leq s + 1$, then we have from (3) that $j_1 = \max\{i - 1, 1\} = 1$ (as $i - 1 = 0 < 1$) and $j_2 = \min\{i + s, \ell\} = \ell$ (as $i + s = s + 1 \geq \ell$). On the other hand, if $\ell = s + 2$, then we have from (3) that $j_1 = \max\{i - 1, 1\} = 1$ and $j_2 = \min\{i + s, \ell\} = s + 1$ (as $i + s = s + 1 < \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) holds (as $i = 1$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $2 \leq \ell \leq s + 1$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $i = 1$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t . On the other hand, if $\ell = s + 2$, then the condition $2 \leq i \leq k - 1$ in Subcase 1(b) does not hold (as $i = 1 < 2$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 1 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(c) does not hold (as $i = 1 < \ell \leq j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

As such, we deduce the following results:

- ▲ If $2 \leq \ell \leq s + 1$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $1 \leq j \leq \ell$, i.e., $j_1 \leq j \leq j_2$.
- ▲ On the other hand, if $\ell = s + 2$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $1 \leq j \leq \ell - 1 = s + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 1(b'): $2 \leq i \leq \ell - 1$. Note that $\ell \geq 3$ in this subcase. In this subcase, we have from (3) that $j_1 = \max\{i - 1, 1\} = i - 1$ (as $i - 1 \geq 1$) and $j_2 = \min\{i + s, \ell\} = \ell$ (as $i + s \geq s + 2 \geq \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) holds if and only if $i - 1 = \max\{i - 1, 1\} \leq j \leq \ell - 1$. Thus, if $i - 1 \leq j \leq \ell - 1$ (resp., $1 \leq j \leq i - 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $2 \leq \ell \leq s + 1$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $1 < 2 \leq i \leq \ell - 1 < 2\ell - 1 = k$). Similarly, if $\ell = s + 2$, then the condition $2 \leq i \leq k - 1$ in Subcase 1(b) also holds (as $2 \leq i \leq \ell - 1 < 2\ell - 2 = k - 1$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 1 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(c) does not hold (as $i \leq \ell - 1 \leq j - 2 < j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $i - 1 \leq j \leq \ell$, i.e., $j_1 \leq j \leq j_2$.

Subcase 1(c'): $i = \ell$. In this subcase, we have from (3) that $j_1 = i - 1 = \ell - 1$ and $j_2 = i + 1 = \ell + 1$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) holds if and only if $\ell - 1 = \max\{\ell - 1, 1\} = \max\{i - 1, 1\} \leq j \leq \ell - 1$, i.e., $j = \ell - 1$. Thus, if $j = \ell - 1$ (resp., $1 \leq j \leq \ell - 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $2 \leq \ell \leq s + 1$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $1 < i = \ell < 2\ell - 1 = k$). Similarly, if $\ell = s + 2$, then the condition $2 \leq i \leq k - 1$ in Subcase 1(b) also holds (as $2 \leq i = \ell \leq 2\ell - 2 = k - 1$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 1 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(c) holds if and only if $\ell + 1 = \min\{\ell + 1, k\} = \min\{i + 1, k\} \geq j \geq \ell + 1$, i.e., $j = \ell + 1$. Thus, if

$j = \ell + 1$ (resp., $k \geq j \geq \ell + 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $\ell - 1 \leq j \leq \ell + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 1(d'): $\ell + 1 \leq i \leq k - 1$. Note that $\ell \geq 3$ in this subcase. In this subcase, we have from (3) that $j_2 = \min\{i + 1, k\} = i + 1$ (as $i + 1 \leq k$) and $j_1 = \max\{i - s, \ell\} = \ell$ (as $i - s \leq k - 1 - s \leq k - 1 - (\ell - 2) = \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) does not hold (as $i \geq \ell + 1 \geq j + 2 > j + 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $2 \leq \ell \leq s + 1$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $1 < \ell + 1 \leq i \leq k - 1 < k$). Similarly, if $\ell = s + 2$, then the condition $2 \leq i \leq k - 1$ in Subcase 1(b) also holds (as $2 < \ell + 1 \leq i \leq k - 1$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 1 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(c) holds if and only if $i + 1 = \min\{i + 1, k\} \geq j \geq \ell + 1$. Thus, if $i + 1 \geq j \geq \ell + 1$ (resp., $k \geq j \geq i + 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $i + 1 \geq j \geq \ell$, i.e., $j_2 \geq j \geq j_1$.

Subcase 1(e'): $i = k$. If $2 \leq \ell \leq s + 1$, then we have from (3) that $j_2 = \min\{i + 1, k\} = k$ (as $i + 1 = k + 1 > k$) and $j_1 = \max\{i - s, \ell\} = \ell$ (as $i - s = k - s \leq k - (\ell - 1) = \ell$). On the other hand, if $\ell = s + 2$, then we have from (3) that $j_2 = \min\{i + 1, k\} = k$ and $j_1 = \max\{i - s, \ell\} = \ell + 1$ (as $i - s = k - s = k - (\ell - 2) = \ell + 1 > \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq \ell - 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 1(a) does not hold (as $i = k > \ell \geq j + 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. If $2 \leq \ell \leq s + 1$, then the condition $1 \leq i \leq k$ in Subcase 1(b) holds (as $i = k$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t . On the other hand, if $\ell = s + 2$, then the condition $1 \leq i \leq k - 1$ in Subcase 1(b) does not hold (as $i = k > k - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

- Suppose that $\ell + 1 \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 1(c) holds (as $i = k$), and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .

As such, we deduce the following results:

- ▲ If $2 \leq \ell \leq s + 1$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $k \geq j \geq \ell$, i.e., $j_2 \geq j \geq j_1$.
- ▲ On the other hand, if $\ell = s + 2$, then packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $k \geq j \geq \ell + 1$, i.e., $j_2 \geq j \geq j_1$.

Case 2: $\ell \geq s + 3$. First, for each $1 \leq j \leq k$, we will identify the groups of FM1's to which packets that are from the output links of the j^{th} group of FM1's and have to be buffered in the system at slot t can possibly be routed. For this purpose, consider $1 \leq j \leq k$ and consider a packet p that is buffered in the j^{th} group of FM1's at slot $t - 1$, leaves the j^{th} group of FM1's at slot t , and has to be buffered in the system at slot t . Note that (85) and (86) still hold in this case.

We consider the following subcases.

Subcase 2(a): $1 \leq j \leq s + 1$. In this subcase, we have $\tilde{\tau}_p(t) \geq 1 = L_1$, and we also have from (86) and (87) in (A3) (note that $2 \leq j + 1 \leq s + 2 \leq \ell$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_j + |\Psi_{j+1}| = U_{j+1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_1, L_1 + 1, \dots, U_{j+1}\} = \cup_{i=1}^{j+1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $1 \leq i \leq j + 1$.

Subcase 2(b): $s + 2 \leq j \leq \ell - 1$. In this subcase, we have from (85) and (4) in (A2) (note that $s + 2 \leq j \leq \ell - 1 < k$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s}$, and we also have from (86) and (87) in (A3) (note that $2 < s + 3 \leq j + 1 \leq \ell$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_j + |\Psi_{j+1}| = U_{j+1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-s}, L_{j-s} + 1, \dots, U_{j+1}\} = \cup_{i=j-s}^{j+1} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - s \leq i \leq j + 1$.

Subcase 2(c): $j = \ell$. In this subcase, we have from (85) and (4) in (A2) (note that $s + 2 < j = \ell < k$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq U_{j-s-1} + 1 = L_{j-s}$, and we also have from (86) and (5) in (A2) (note that $1 < s + 3 \leq j = \ell \leq k - s - 2 < k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-s}, L_{j-s} + 1, \dots, U_{j+s}\} = \cup_{i=j-s}^{j+s} \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j - s \leq i \leq j + s$.

Subcase 2(d): $\ell + 1 \leq j \leq k - s - 1$. In this subcase, we have from (86) and (5) in (A2) (note that $1 < \ell + 1 \leq j \leq k - s - 1$) that $\tilde{\tau}_p(t) \leq U_j + B_j \leq U_{j+s}$, and we also have from (85) and (87) in (A3) (note that $\ell \leq j - 1 \leq k - s - 2 < k - 1$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1} + 1) - B_j \geq (U_{j-1} + 1) - |\Psi_{j-1}| = U_{j-2} + 1 = L_{j-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-1}, L_{j-1} + 1, \dots, U_{j+s}\} = \cup_{i=j-1}^{j+s} \Psi_i$.

It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j-1 \leq i \leq j+s$.

Subcase 2(e): $k-s \leq j \leq k$. In this subcase, we have $\tilde{\tau}_p(t) \leq U_k$, and we also have from (85) and (87) in (A3) (note that $\ell < k-s-1 \leq j-1 \leq k-1$) that $\tilde{\tau}_p(t) \geq L_j - B_j = (U_{j-1}+1) - B_j \geq (U_{j-1}+1) - |\Psi_{j-1}| = U_{j-2} + 1 = L_{j-1}$. Thus, we have $\tilde{\tau}_p(t) \in \{L_{j-1}, L_{j-1} + 1, \dots, U_k\} = \cup_{i=j-1}^k \Psi_i$. It then follows from the routing policy (R3) that packet p can only be routed to the i^{th} group of FM1's at slot t for some $j-1 \leq i \leq k$.

We summarize the results in Subcases 2(a)–(e) in the following table.

$k = 2\ell - 1$ and $\ell \geq s + 3$					
j	$1 \leq j \leq s+1$	$s+2 \leq j \leq \ell-1$	$j = \ell$	$\ell+1 \leq j \leq k-s-1$	$k-s \leq j \leq k$
i	$1 \leq i \leq j+1$	$j-s \leq i \leq j+1$	$j-s \leq i \leq j+s$	$j-1 \leq i \leq j+s$	$j-1 \leq i \leq k$

TABLE IV

FOR EACH $1 \leq j \leq k$, THE RANGE OF i GIVEN IN SUBCASES 2(A)–(E) SUCH THAT PACKETS FROM THE OUTPUT LINKS OF THE j^{TH} GROUP OF FM1'S CAN POSSIBLY BE ROUTED TO THE i^{TH} GROUP OF FM1'S AT SLOT t .

Second, for each $1 \leq i \leq k$, we will use the results in Subcases 2(a)–(e) above to identify the groups of FM1's such that packets that are from the output links of these groups (but not other groups) of FM1's and have to be buffered in the system at slot t can possibly be routed to the i^{th} group of FM1's at slot t , and show that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $j_1 \leq j \leq j_2$, where j_1 and j_2 are given by (3) (note that k is odd). For this purpose, consider $1 \leq i \leq k$ and consider the following subcases.

Subcase 2(a'): $i = 1$. In this subcase, we have from (3) that $j_1 = \max\{i-1, 1\} = 1$ (as $i-1 = 0 < 1$) and $j_2 = \min\{i+s, \ell\} = s+1$ (as $i+s = s+1 < s+3 \leq \ell$).

It is easy to see the following results:

- Suppose that $1 \leq j \leq s+1$. Then the condition $1 \leq i \leq j+1$ in Subcase 2(a) holds (as $i = 1$) and hence packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $s+2 \leq j \leq \ell-1$. Then the condition $j-s \leq i \leq j+1$ in Subcase 2(b) does not hold (as $i = 1 < 2 \leq j-s$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. Then the condition $j-s \leq i \leq j+s$ in Subcase 2(c) does not hold (as $i = 1 < 2 < \ell-s = j-s$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell+1 \leq j \leq k-s-1$. Then the condition $j-1 \leq i \leq j+s$ in Subcase 2(d) does not hold (as $i = 1 < \ell \leq j-1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(e) does not hold (as $i = 1 < s + 4 \leq 2\ell - s - 2 = k - s - 1 \leq j - 1$), and hence packets from the output links of the j^{th} group of FM1's cannot be routed to the i^{th} group of FM1's at slot t .

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $1 \leq j \leq s + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 2(b'): $2 \leq i \leq s + 2$. In this subcase, we have from (3) that $j_1 = \max\{i - 1, 1\} = i - 1$. If $s + 3 \leq \ell \leq 2s + 2$, then we have from (3) that

$$j_2 = \min\{i + s, \ell\} = \begin{cases} i + s, & \text{if } 2 \leq i \leq \ell - s - 1, \\ \ell, & \text{if } \ell - s \leq i \leq s + 2. \end{cases}$$

On the other hand, if $\ell \geq 2s + 3$, then we have from (3) that $j_2 = \min\{i + s, \ell\} = i + s$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) holds if and only if $i - 1 = \max\{i - 1, 1\} \leq j \leq s + 1$.
- Suppose that $s + 2 \leq j \leq \ell - 1$. If $s + 3 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if

$$s + 2 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \begin{cases} i + s, & \text{if } 2 \leq i \leq \ell - s - 1, \\ \ell - 1, & \text{if } \ell - s \leq i \leq s + 2. \end{cases}$$

On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if $s + 2 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = i + s$.

- Suppose that $j = \ell$. If $s + 3 \leq \ell \leq 2s + 2$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $\ell - s = \max\{\ell - s, 2\} \leq i \leq \min\{\ell + s, s + 2\} = s + 2$. On the other hand, if $\ell \geq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) does not hold (as $i \leq s + 2 \leq \ell - s - 1 = j - s - 1 < j - s$).
- Suppose that $\ell + 1 \leq j \leq k - s - 1$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(d) does not hold (as $i \leq s + 2 \leq \ell - 1 \leq j - 2 < j - 1$).
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(e) does not hold (as $i \leq s + 2 \leq 2\ell - s - 4 = k - s - 3 \leq j - 3 < j - 1$).

As such, we deduce the following results:

- ▲ If $s + 3 \leq \ell \leq 2s + 2$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i - 1 \leq j \leq \begin{cases} i + s, & \text{if } 2 \leq i \leq \ell - s - 1, \\ \ell, & \text{if } \ell - s \leq i \leq s + 2, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

- ▲ On the other hand, if $\ell \geq 2s + 3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i - 1 \leq j \leq i + s$, i.e., $j_1 \leq j \leq j_2$.

Subcase 2(c'): $s + 3 \leq i \leq \ell - 1$. Note that $\ell \geq s + 4$ in this subcase. In this subcase, we have from (3) that $j_1 = \max\{i - 1, 1\} = i - 1$. If $s + 4 \leq \ell \leq 2s + 3$, then we have from (3) that $j_2 = \min\{i + s, \ell\} = \ell$. On the other hand, if $\ell \geq 2s + 4$, then we have from (3) that

$$j_2 = \min\{i + s, \ell\} = \begin{cases} i + s, & \text{if } s + 3 \leq i \leq \ell - s - 1, \\ \ell, & \text{if } \ell - s \leq i \leq \ell - 1. \end{cases}$$

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i \geq s + 3 \geq j + 2 > j + 1$).
- Suppose that $s + 2 \leq j \leq \ell - 1$. If $s + 4 \leq \ell \leq 2s + 3$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if $i - 1 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \ell - 1$. On the other hand, if $\ell \geq 2s + 4$, then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if

$$i - 1 = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \begin{cases} i + s, & \text{if } s + 3 \leq i \leq \ell - s - 1, \\ \ell - 1, & \text{if } \ell - s \leq i \leq \ell - 1. \end{cases}$$

- Suppose that $j = \ell$. If $s + 4 \leq \ell \leq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds (as $j - s = \ell - s \leq s + 3 \leq i \leq \ell - 1 < \ell + s = j + s$). On the other hand, if $\ell \geq 2s + 4$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $\ell - s = \max\{\ell - s, s + 3\} \leq i \leq \min\{\ell + s, \ell - 1\} = \ell - 1$.
- Suppose that $\ell + 1 \leq j \leq k - s - 1$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(d) does not hold (as $i \leq \ell - 1 \leq j - 2 < j - 1$).
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(e) does not hold (as $i \leq \ell - 1 \leq 2\ell - s - 5 = k - s - 4 \leq j - 4 < j - 1$).

As such, we deduce the following results:

- ▲ If $s + 4 \leq \ell \leq 2s + 3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i - 1 \leq j \leq \ell$, i.e., $j_1 \leq j \leq j_2$.
- ▲ On the other hand, if $\ell \geq 2s + 4$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i - 1 \leq j \leq \begin{cases} i + s, & \text{if } s + 3 \leq i \leq \ell - s - 1, \\ \ell, & \text{if } \ell - s \leq i \leq \ell - 1, \end{cases}$$

i.e., $j_1 \leq j \leq j_2$.

Subcase 2(d'): $i = \ell$. In this subcase, we have from (3) that $j_1 = i - 1 = \ell - 1$ and $j_2 = i + 1 = \ell + 1$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i = \ell \geq s + 3 \geq j + 2 > j + 1$).
- Suppose that $s + 2 \leq j \leq \ell - 1$. Then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) holds if and only if $\ell - 1 = \max\{\ell - 1, s + 2\} = \max\{i - 1, s + 2\} \leq j \leq \min\{i + s, \ell - 1\} = \min\{\ell + s, \ell - 1\} = \ell - 1$, i.e., $j = \ell - 1$. Thus, if $j = \ell - 1$ (resp., $s + 2 \leq j \leq \ell - 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .
- Suppose that $j = \ell$. Then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds (as $j - s = \ell - s < i = \ell < \ell + s = j + s$). Thus, packets from the output links of the j^{th} group of FM1's can possibly be routed to the i^{th} group of FM1's at slot t .
- Suppose that $\ell + 1 \leq j \leq k - s - 1$. Then the condition $j - 1 \leq i \leq j + s$ in Subcase 2(d) holds if and only if $\ell + 1 = \min\{\ell + 1, k - s - 1\} = \min\{i + 1, k - s - 1\} \geq j \geq \max\{i - s, \ell + 1\} = \max\{\ell - s, \ell + 1\} = \ell + 1$, i.e., $j = \ell + 1$. Thus, if $j = \ell + 1$ (resp., $k - s - 1 \geq j \geq \ell + 2$), then packets from the output links of the j^{th} group of FM1's can possibly (resp., cannot) be routed to the i^{th} group of FM1's at slot t .
- Suppose that $k - s \leq j \leq k$. Then the condition $j - 1 \leq i \leq k$ in Subcase 2(e) does not hold (as $i = \ell \leq 2\ell - s - 3 = k - s - 2 \leq j - 2 < j - 1$).

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $\ell - 1 \leq j \leq \ell + 1$, i.e., $j_1 \leq j \leq j_2$.

Subcase 2(e'): $\ell + 1 \leq i \leq k - s - 2$. Note that $\ell \geq s + 4$ in this subcase. In this subcase, we have from (3) that $j_2 = \min\{i + 1, k\} = i + 1$. If $s + 4 \leq \ell \leq 2s + 3$, then we have from (3) that $j_1 = \max\{i - s, \ell\} = \ell$. On the other hand, if $\ell \geq 2s + 4$, then we have from (3) that

$$j_1 = \max\{i - s, \ell\} = \begin{cases} \ell, & \text{if } \ell + 1 \leq i \leq \ell + s, \\ i - s, & \text{if } \ell + s + 1 \leq i \leq k - s - 2. \end{cases}$$

It is easy to see the following results:

- Suppose that $1 \leq j \leq s + 1$. Then the condition $1 \leq i \leq j + 1$ in Subcase 2(a) does not hold (as $i \geq \ell + 1 \geq s + 5 \geq j + 4 > j + 1$).
- Suppose that $s + 2 \leq j \leq \ell - 1$. Then the condition $j - s \leq i \leq j + 1$ in Subcase 2(b) does not hold (as $i \geq \ell + 1 \geq j + 2 > j + 1$).
- Suppose that $j = \ell$. If $s + 4 \leq \ell \leq 2s + 3$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds (as $j - s = \ell - s < \ell + 1 \leq i \leq k - s - 2 \leq \ell + s = j + s$). On the other hand, if $\ell \geq 2s + 4$, then the condition $j - s \leq i \leq j + s$ in Subcase 2(c) holds if and only if $\ell + 1 = \max\{\ell - s, \ell + 1\} \leq i \leq \min\{\ell + s, k - s - 2\} = \ell + s$.

- Suppose that $\ell+1 \leq j \leq k-s-1$. If $s+4 \leq \ell \leq 2s+3$, then the condition $j-1 \leq i \leq j+s$ in Subcase 2(d) holds if and only if $i+1 = \min\{i+1, k-s-1\} \geq j \geq \max\{i-s, \ell+1\} = \ell+1$. On the other hand, if $\ell \geq 2s+4$, then the condition $j-1 \leq i \leq j+s$ in Subcase 2(d) holds if and only if

$$\begin{aligned} i+1 = \min\{i+1, k-s-1\} \geq j &\geq \max\{i-s, \ell+1\} \\ &= \begin{cases} \ell+1, & \text{if } \ell+1 \leq i \leq \ell+s, \\ i-s, & \text{if } \ell+s+1 \leq i \leq k-s-2. \end{cases} \end{aligned}$$

- Suppose that $k-s \leq j \leq k$. Then the condition $j-1 \leq i \leq k$ in Subcase 2(e) does not hold (as $i \leq k-s-2 \leq j-2 < j-1$).

As such, we deduce the following results:

- ▲ If $s+4 \leq \ell \leq 2s+3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i+1 \geq j \geq \ell$, i.e., $j_2 \geq j \geq j_1$.
- ▲ On the other hand, if $\ell \geq 2s+4$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i+1 \geq j \geq \begin{cases} \ell, & \text{if } \ell+1 \leq i \leq \ell+s, \\ i-s, & \text{if } \ell+s+1 \leq i \leq k-s-2, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

Subcase 2(f'): $k-s-1 \leq i \leq k-1$. In this subcase, we have from (3) that $j_2 = \min\{i+1, k\} = i+1$. If $s+3 \leq \ell \leq 2s+2$, then we have from (3) that

$$j_1 = \max\{i-s, \ell\} = \begin{cases} \ell, & \text{if } k-s-1 \leq i \leq \ell+s, \\ i-s, & \text{if } \ell+s+1 \leq i \leq k-1. \end{cases}$$

On the other hand, if $\ell \geq 2s+3$, then we have from (3) that $j_1 = \max\{i-s, \ell\} = i-s$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s+1$. Then the condition $1 \leq i \leq j+1$ in Subcase 2(a) does not hold (as $i \geq k-s-1 \geq s+4 \geq j+3 > j+1$).
- Suppose that $s+2 \leq j \leq \ell-1$. Then the condition $j-s \leq i \leq j+1$ in Subcase 2(b) does not hold (as $i \geq k-s-1 \geq \ell+1 \geq j+2 > j+1$).
- Suppose that $j = \ell$. If $s+3 \leq \ell \leq 2s+2$, then the condition $j-s \leq i \leq j+s$ in Subcase 2(c) holds if and only if $k-s-1 = \max\{\ell-s, k-s-1\} \leq i \leq \min\{\ell+s, k-1\} = \ell+s$. On the other hand, if $\ell \geq 2s+3$, then the condition $j-s \leq i \leq j+s$ in Subcase 2(c) does not hold (as $i \geq k-s-1 \geq \ell+s+1 = j+s+1 > j+s$).

- Suppose that $\ell+1 \leq j \leq k-s-1$. If $s+3 \leq \ell \leq 2s+2$, then the condition $j-1 \leq i \leq j+s$ in Subcase 2(d) holds if and only if

$$\begin{aligned} k-s-1 = \min\{i+1, k-s-1\} \geq j &\geq \max\{i-s, \ell+1\} \\ &= \begin{cases} \ell+1, & \text{if } k-s-1 \leq i \leq \ell+s, \\ i-s, & \text{if } \ell+s+1 \leq i \leq k-1. \end{cases} \end{aligned}$$

On the other hand, if $\ell \geq 2s+3$, then the condition $j-1 \leq i \leq j+s$ in Subcase 2(d) holds if and only if $k-s-1 = \min\{i+1, k-s-1\} \geq j \geq \max\{i-s, \ell+1\} = i-s$.

- Suppose that $k-s \leq j \leq k$. Then the condition $j-1 \leq i \leq k$ in Subcase 2(e) holds if and only if $i+1 = \min\{i+1, k\} \geq j \geq k-s$.

As such, we deduce the following results:

- ▲ If $s+3 \leq \ell \leq 2s+2$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some

$$i+1 \geq j \geq \begin{cases} \ell, & \text{if } k-s-1 \leq i \leq \ell+s, \\ i-s, & \text{if } \ell+s+1 \leq i \leq k-1, \end{cases}$$

i.e., $j_2 \geq j \geq j_1$.

- ▲ On the other hand, if $\ell \geq 2s+3$, then packets routed to the i^{th} group of FM1's at slot t are either from the arrival link or from the output links of the j^{th} group of FM1's at slot t for some $i+1 \geq j \geq i-s$, i.e., $j_2 \geq j \geq j_1$.

Subcase 2(g'): $i = k$. In this subcase, we have from (3) that $j_2 = \min\{i+1, k\} = k$ and $j_1 = \max\{i-s, \ell\} = k-s$.

It is easy to see the following results:

- Suppose that $1 \leq j \leq s+1$. Then the condition $1 \leq i \leq j+1$ in Subcase 2(a) does not hold (as $i = k > \ell-1 \geq s+2 \geq j+1$).
- Suppose that $s+2 \leq j \leq \ell-1$. Then the condition $j-s \leq i \leq j+1$ in Subcase 2(b) does not hold (as $i = k > \ell \geq j+1$).
- Suppose that $j = \ell$. Then the condition $j-s \leq i \leq j+s$ in Subcase 2(c) does not hold (as $i = k > \ell+s = j+s$).
- Suppose that $\ell+1 \leq j \leq k-s-1$. Then the condition $j-1 \leq i \leq j+s$ in Subcase 2(d) does not hold (as $i = k > k-1 \geq j+s$).
- Suppose that $k-s \leq j \leq k$. Then the condition $j-1 \leq i \leq k$ in Subcase 2(e) holds (as $i = k$).

As such, we deduce that packets routed to the i^{th} group of FM1's at slot t can only come from either the arrival link or the output links of the j^{th} group of FM1's at slot t for some $k \geq j \geq k-s$, i.e., $j_2 \geq j \geq j_1$.

APPENDIX B
PROOF OF THEOREM 6

Note that we have $B_i = B_{k-i+1}$ for $1 \leq i \leq \lceil k/2 \rceil$ in (27) and (28).

(i) It suffices to show that B_i is recursively given by (33) for $2 \leq i \leq \lceil k/2 \rceil$. Suppose that $2 \leq i \leq \lceil k/2 \rceil$. We consider the following three cases.

Case 1: $2 \leq i \leq s+1$. For $s+1 \leq k \leq 2s+2$ (resp., $k \geq 2s+3$), we have from (27) (resp., (28)) and $1 \leq i-1 < i \leq \lceil k/2 \rceil$ (resp., $1 \leq i-1 < i \leq s+1$) that

$$\begin{aligned} B_i &= \sum_{j=1}^{i-1} ((m-1)B_j + 1) + 1 = \sum_{j=1}^{i-2} ((m-1)B_j + 1) + ((m-1)B_{i-1} + 1) + 1 \\ &= (B_{i-1} - 1) + ((m-1)B_{i-1} + 1) + 1 = mB_{i-1} + 1. \end{aligned}$$

Case 2: $i = s+2$. In this case, we have $k \geq 2s+3$ (as $\lceil k/2 \rceil \geq i = s+2$), and hence it follows from (28) and $i-1 = s+1$ that

$$\begin{aligned} B_i &= \sum_{j=i-s}^{i-1} ((m-1)B_j + 1) \\ &= \sum_{j=i-s-1}^{i-2} ((m-1)B_j + 1) + ((m-1)B_{i-1} + 1) - ((m-1)B_{i-s-1} + 1) \\ &= (B_{i-1} - 1) + (m-1)B_{i-1} - (m-1)B_{i-s-1} = mB_{i-1} - (m-1)B_{i-s-1} - 1. \end{aligned}$$

Case 3: $i \geq s+3$. In this case, we have $k > 2s+3$ (as $\lceil k/2 \rceil \geq i \geq s+3$), and hence it follows from (28) and $i > i-1 \geq s+2$ that

$$\begin{aligned} B_i &= \sum_{j=i-s}^{i-1} ((m-1)B_j + 1) \\ &= \sum_{j=i-s-1}^{i-2} ((m-1)B_j + 1) + ((m-1)B_{i-1} + 1) - ((m-1)B_{i-s-1} + 1) \\ &= B_{i-1} + (m-1)B_{i-1} - (m-1)B_{i-s-1} = mB_{i-1} - (m-1)B_{i-s-1}. \end{aligned}$$

(ii) It suffices to show that B_i is given by (34) for $1 \leq i \leq \lceil k/2 \rceil$. We prove this by induction on i . It is clear from (27) and (28) that $B_1 = 1$ and hence (34) holds for $i = 1$ (note that $q_1 = \lceil 1/(s+1) \rceil - 1 = 0$). Assume as the induction hypothesis that (34) holds up to $i-1$ for some $1 \leq i-1 \leq \lceil k/2 \rceil - 1$. Now consider the following three cases.

Case 1: $q_i = 0$. In this case, we have $2 \leq i \leq (q_i + 1)(s+1) = s+1$. Thus, we have $q_{i-1} = \lceil (i-1)/(s+1) \rceil - 1 = 0$. It then follows from (33) and the induction hypothesis, i.e., (34) holds for $i-1$ (note that $q_{i-1} = 0$), that

$$B_i = mB_{i-1} + 1 = m(m^{i-1}/(m-1) - 1/(m-1)) + 1 = m^i/(m-1) - 1/(m-1).$$

Thus, we have proved that (34) holds for i (note that $q_i = 0$).

Case 2: $q_i \geq 1$ and $i = q_i(s+1) + 1$. In this case, we have $q_{i-1} = \lceil (i-1)/(s+1) \rceil - 1 = q_i - 1$ and $q_{i-s-1} = \lceil (i-s-1)/(s+1) \rceil - 1 = q_i - 1$. We then consider the two subcases $q_i = 1$ and $q_i \geq 2$ separately.

Subcase 2(a): $q_i = 1$. In this subcase, we have $i = s + 2$ and it follows from (33) and the induction hypothesis, i.e., (34) holds for $i - 1$ (note that $q_{i-1} = q_i - 1 = 0$), that

$$\begin{aligned} B_i &= mB_{i-1} - (m-1)B_{i-s-1} - 1 = m(m^{i-1}/(m-1) - 1/(m-1)) - (m-1)B_1 - 1 \\ &= m^i/(m-1) - (1+m) - 1/(m-1). \end{aligned}$$

Thus, we have proved that (34) holds for i (note that $q_i = 1$).

Subcase 2(b): $q_i \geq 2$. In this subcase, we have from (33) (note that $i = q_i(s+1) + 1 \geq 2(s+1) + 1 > s+3$) and the induction hypothesis, i.e., (34) holds for $i - 1$ and $i - s - 1$ (note that $i - 1 > i - s - 1 = (q_i - 1)(s+1) + 1 \geq 1 \cdot (s+1) + 1 > 1$ and $q_{i-1} = q_{i-s-1} = q_i - 1$), that

$$\begin{aligned} B_i &= mB_{i-1} - (m-1)B_{i-s-1} \\ &= \sum_{j=0}^{q_i-1} (-1)^j (1/j!) [j(i-j(s+1)-1)_{j-1} + m(i-j(s+1)-1)_j] \\ &\quad \times (m-1)^{j-1} m^{i-j(s+1)-1} - m/(m-1) \\ &\quad - \sum_{j=0}^{q_i-1} (-1)^j (1/j!) [j(i-(j+1)(s+1))_{j-1} + m(i-(j+1)(s+1))_j] \\ &\quad \times (m-1)^j m^{i-(j+1)(s+1)-1} + 1 \\ &= m^i/(m-1) \\ &\quad + \sum_{j=1}^{q_i-1} (-1)^j (1/j!) \{ [j(i-j(s+1)-1)_{j-1} + j(j-1)(i-j(s+1))_{j-2}] \\ &\quad + m[(i-j(s+1)-1)_j + j(i-j(s+1))_{j-1}] \} (m-1)^{j-1} m^{i-j(s+1)-1} \\ &\quad - (-1)^{q_i-1} (1+m)(m-1)^{q_i-1} - 1/(m-1) \\ &= \sum_{j=0}^{q_i} (-1)^j (1/j!) [j(i-j(s+1))_{j-1} + m(i-j(s+1))_j] (m-1)^{j-1} m^{i-j(s+1)-1} \\ &\quad - 1/(m-1), \end{aligned}$$

where in the last two equalities we have used $i - q_i(s+1) = 1$, $(1)_j = j!$ for $q_i - 2 \leq j \leq q_i$, $j(i-j(s+1)-1)_{j-1} + j(j-1)(i-j(s+1))_{j-2} = j(i-j(s+1))_{j-1}$ for $1 \leq j \leq q_i - 1$, and $(i-j(s+1)-1)_j + j(i-j(s+1))_{j-1} = (i-j(s+1))_j$ for $1 \leq j \leq q_i - 1$. Thus, we have proved that (34) holds for i .

Case 3: $q_i \geq 1$ and $q_i(s+1) + 2 \leq i \leq (q_i+1)(s+1)$. In this case, we have $q_{i-1} = \lceil (i-1)/(s+1) \rceil - 1 = q_i$ and $q_{i-s-1} = \lceil (i-s-1)/(s+1) \rceil - 1 = q_i - 1$. Similar to the proof in Subcase 2(b) above, we have from (33) (note that $i \geq q_i(s+1) + 2 \geq 1 \cdot (s+1) + 2 = s+3$) and the induction hypothesis, i.e., (34) holds for $i - 1$ and $i - s - 1$ (note that $i - 1 > i - s - 1 \geq (q_i - 1)(s+1) + 2 > 1$, $q_{i-1} = q_i$, and $q_{i-s-1} = q_i - 1$), that

$$\begin{aligned} B_i &= mB_{i-1} - (m-1)B_{i-s-1} \\ &= \sum_{j=0}^{q_i} (-1)^j (1/j!) [j(i-j(s+1)-1)_{j-1} + m(i-j(s+1)-1)_j] \\ &\quad \times (m-1)^{j-1} m^{i-j(s+1)-1} - m/(m-1) \\ &\quad - \sum_{j=0}^{q_i-1} (-1)^j (1/j!) [j(i-(j+1)(s+1))_{j-1} + m(i-(j+1)(s+1))_j] \\ &\quad \times (m-1)^j m^{i-(j+1)(s+1)-1} + 1 \end{aligned}$$

$$\begin{aligned}
&= m^i/(m-1) \\
&\quad + \sum_{j=1}^{q_i} (-1)^j (1/j!) \{ [j(i-j(s+1)-1)_{j-1} + j(j-1)(i-j(s+1))_{j-2}] \\
&\quad\quad + m[(i-j(s+1)-1)_j + j(i-j(s+1))_{j-1}] \} (m-1)^{j-1} m^{i-j(s+1)-1} \\
&\quad - 1/(m-1) \\
&= \sum_{j=0}^{q_i} (-1)^j (1/j!) [j(i-j(s+1))_{j-1} + m(i-j(s+1))_j] (m-1)^{j-1} m^{i-j(s+1)-1} \\
&\quad - 1/(m-1),
\end{aligned}$$

where in the last equality we have used $j(i-j(s+1)-1)_{j-1} + j(j-1)(i-j(s+1))_{j-2} = j(i-j(s+1))_{j-1}$ and $(i-j(s+1)-1)_j + j(i-j(s+1))_{j-1} = (i-j(s+1))_j$ for $1 \leq j \leq q_i$. Thus, we have proved that (34) holds for i .

APPENDIX C PROOF OF (57)

Suppose that $s+2 \leq i \leq (m^{s+1}-1)/(m-1) + s$. Then we have $q_i = \lceil i/(s+1) \rceil - 1 \geq 1$ and

$$\begin{aligned}
m^{s+1} &\geq (i-s)(m-1) + 1 \geq (q_i(s+1) + 1 - s)(m-1) + 1 \\
&= (q_i - 1)(s+1)(m-1) + 2m - 1.
\end{aligned} \tag{88}$$

Write B_i in (34) as follows:

$$B_i = \sum_{j=0}^{q_i} (-1)^j \delta_j - 1/(m-1), \tag{89}$$

where

$$\delta_j = (1/j!) [j(i-j(s+1))_{j-1} + m(i-j(s+1))_j] (m-1)^{j-1} m^{i-j(s+1)-1} \tag{90}$$

for $0 \leq j \leq q_i$. Note that $\delta_j > 0$ (as $i-j(s+1) \geq i-q_i(s+1) \geq 1$ and $m \geq 2$) for $0 \leq j \leq q_i$.

We claim that the sequence $(\delta_j)_{j=1}^{q_i}$ is strictly decreasing. Suppose $1 \leq j \leq q_i - 1$. Note that

$$\begin{aligned}
\delta_{j+1} &= (1/(j+1)!) [(j+1)(i-(j+1)(s+1))_j + m(i-(j+1)(s+1))_{j+1}] \\
&\quad \times (m-1)^j m^{i-(j+1)(s+1)-1}.
\end{aligned} \tag{91}$$

To show that $\delta_j > \delta_{j+1}$, we first give upper bounds for the two terms $(i-(j+1)(s+1))_j$ and $(i-(j+1)(s+1))_{j+1}$ in (91) as follows: We have

$$\begin{aligned}
(i-(j+1)(s+1))_j &= (i-(j+1)(s+1))_{j-1} (i-(j+1)(s+1) + j - 1) \\
&\leq (i-j(s+1))_{j-1} (q_i - 1)(s+1),
\end{aligned} \tag{92}$$

where the inequality follows from $(i-j(s+1))_{j-1} \geq (i-(j+1)(s+1))_{j-1}$ (as $i-j(s+1) > i-(j+1)(s+1) \geq i-q_i(s+1) \geq 1$ and $j-1 \geq 0$) and $0 < i-(j+1)(s+1) + j - 1 \leq$

$(q_i + 1)(s + 1) - (j + 1)(s + 1) + j - 1 = (q_i - 1)(s + 1) - (j - 1)s \leq (q_i - 1)(s + 1)$. Similarly, we have

$$\begin{aligned} (i - (j + 1)(s + 1))_{j+1} &= (i - (j + 1)(s + 1))_j (i - (j + 1)(s + 1) + j) \\ &\leq (i - j(s + 1))_j ((q_i - 1)(s + 1) + 1). \end{aligned} \quad (93)$$

As such, it follows that

$$\begin{aligned} \delta_j - \delta_{j+1} &= (1/j!) [j(i - j(s + 1))_{j-1} + m(i - j(s + 1))_j] (m - 1)^{j-1} m^{i-j(s+1)-1} \\ &\quad - (1/(j + 1)!) [(j + 1)(i - (j + 1)(s + 1))_j + m(i - (j + 1)(s + 1))_{j+1}] \\ &\quad \times (m - 1)^j m^{i-(j+1)(s+1)-1} \\ &\geq (1/j!) [j m^{s+1} - (q_i - 1)(s + 1)(m - 1)] (i - j(s + 1))_{j-1} (m - 1)^{j-1} m^{i-(j+1)(s+1)-1} \\ &\quad + (1/(j + 1)!) [(j + 1) m^{s+1} - ((q_i - 1)(s + 1) + 1)(m - 1)] \\ &\quad \times (i - j(s + 1))_j (m - 1)^{j-1} m^{i-(j+1)(s+1)} \\ &\geq (1/j!) (2m - 1) (i - j(s + 1))_{j-1} (m - 1)^{j-1} m^{i-(j+1)(s+1)-1} \\ &\quad + (1/(j + 1)!) m (i - j(s + 1))_j (m - 1)^{j-1} m^{i-(j+1)(s+1)} \\ &> 0, \end{aligned}$$

where the equality follows from (90) and (91), the first inequality follows from (92), (93), and $m \geq 2$, and the second inequality follows from $j \geq 1$ and (88). Thus, we have proved that the sequence $(\delta_j)_{j=1}^{q_i}$ is strictly decreasing.

We also claim that

$$\delta_0 - \delta_1 - 1/(m - 1) \leq B_i \leq \delta_0 - 1/(m - 1). \quad (94)$$

If q_i is odd, say $q_i = 2\ell - 1$ for some $\ell \geq 1$, then it follows from (89), the strict monotonicity of the sequence $(\delta_j)_{j=1}^{q_i}$, and the positivity of the δ_j 's that

$$B_i = \delta_0 - \sum_{j=1}^{\ell-1} (\delta_{2j-1} - \delta_{2j}) - \delta_{2\ell-1} - 1/(m - 1) \leq \delta_0 - 1/(m - 1)$$

and

$$B_i = \delta_0 - \delta_1 + \sum_{j=1}^{\ell-1} (\delta_{2j} - \delta_{2j+1}) - 1/(m - 1) \geq \delta_0 - \delta_1 - 1/(m - 1).$$

On the other hand, if q_i is even, say $q_i = 2\ell$ for some $\ell \geq 1$, then we have

$$B_i = \delta_0 - \sum_{j=1}^{\ell} (\delta_{2j-1} - \delta_{2j}) - 1/(m - 1) \leq \delta_0 - 1/(m - 1)$$

and

$$B_i = \delta_0 - \delta_1 + \sum_{j=1}^{\ell-1} (\delta_{2j} - \delta_{2j+1}) + \delta_{2\ell} - 1/(m - 1) \geq \delta_0 - \delta_1 - 1/(m - 1).$$

Thus, we have proved that (94) holds.

We are now in a position to show that (57) holds. From (94), (90), and $m \geq 2$, we have

$$B_i \leq \delta_0 - 1/(m-1) < \delta_0 = m^i/(m-1) \leq 2m^{i-1}. \quad (95)$$

From (94), (90), $m^{s+1} \geq (i-s)(m-1) + 1$, and $i \geq s+2$, we have

$$\begin{aligned} B_i &\geq \delta_0 - \delta_1 - 1/(m-1) \\ &= m^i/(m-1) - [1 + m(i-s-1)]m^{i-s-2} - 1/(m-1) \\ &= [m^{s+1} - (i-s-1)(m-1)]m^{i-s-1}/(m-1) - m^{i-s-2} - 1/(m-1) \\ &\geq m^{i-s}/(m-1) - m^{i-s-2} - 1/(m-1) \\ &= m^{i-s-1} + (m^{i-s-2} - 1)/(m-1) \geq m^{i-s-1}. \end{aligned} \quad (96)$$

By combining (95) and (96), we obtain (57).

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