Chapter 7

LTI Discrete-Time Systems in the Transform Domain

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Original PowerPoint slides prepared by S. K. Mitra

Types of Transfer Functions

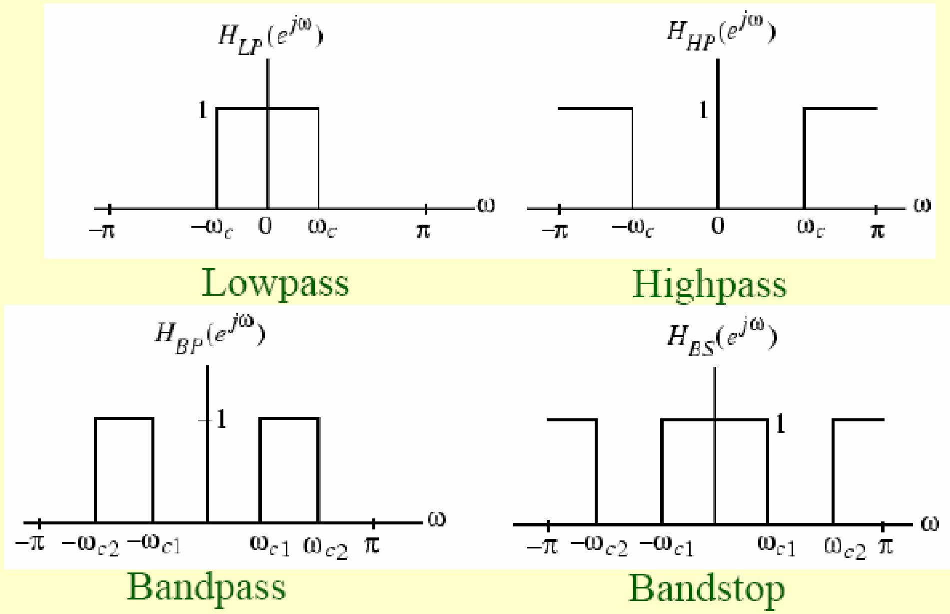
- The time-domain classification of an LTI digital transfer function sequence is based on the length of its impulse response:
  - **Finite impulse response (FIR)** transfer function
  - **Infinite impulse response (IIR)** transfer function
- In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications
  - Classification based on the shape of the magnitude function $|H(e^{j\omega})|$
  - Classification based on the form of the phase function $\theta(\omega)$

Classification Based on Magnitude Characteristics

- One common classification is based on ideal magnitude response
- A digital filter designed to pass signal components of certain frequencies without distortion should have a magnitude response equal to one at these frequencies, and zero at all other frequencies
- The range of frequencies where the frequency response takes the value of one is called the **passband**
- The range of frequencies where the frequency response takes the value of zero is called the **stopband**

Ideal Filters

- Frequency responses of the four popular types of ideal digital filters with real coefficients are shown below:



- The freq. ω_c, ω_{c1},and ω_{c2} are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a zero phase everywhere

Ideal Filters

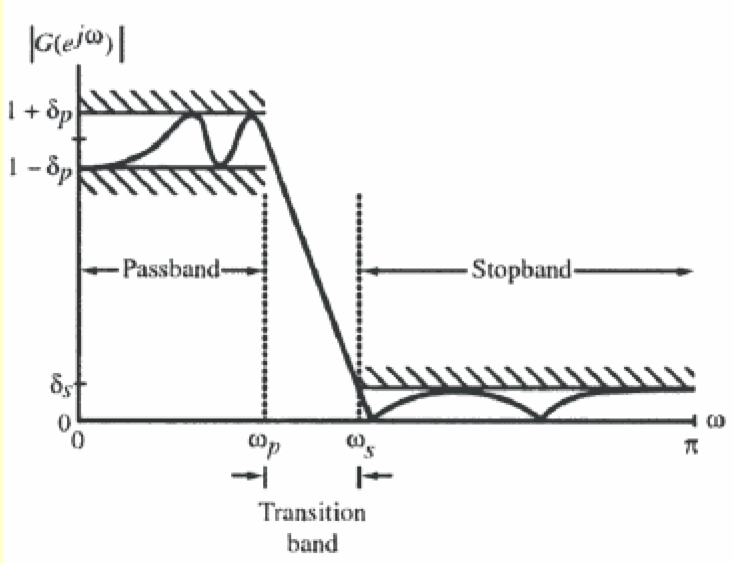
- The impulse response of the ideal lowpass filter:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- The above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable
- Also, $h_{LP}[n]$ is not causal and is of doubly infinite length
- The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and are not absolutely summable
- Thus, the ideal filters with the ideal “brick wall” frequency responses cannot be realized with finite dimensional LTI filter

Ideal Filters

- To develop stable and realizable transfer functions, the ideal freq. response specs. are relaxed by including a **transition band** between the passband and the stopband
- This allows the magnitude response to decay slowly from its max. value in the passband to the zero value in the stopband
- Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband



Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ is defined as a **bounded real (BR) transfer function** if

$$|H(e^{j\omega})| \leq 1 \text{ for all values of } \omega$$

- Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs
- Then the condition $|H(e^{j\omega})| \leq 1$ implies that

$$|Y(e^{j\omega})|^2 \leq |X(e^{j\omega})|^2$$

- Integrating the above from − π to π, and applying Parseval's relation we get

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Bounded Real Transfer Functions

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a **passive structure**
- If $|H(e^{j\omega})| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a **lossless system**
- A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})| = 1$ is thus called a **lossless bounded real (LBR) transfer function**
- The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity (see Sec. 12.9)

Bounded Real Transfer Functions

- Example – Consider the causal stable IIR transfer function

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

where K is a real constant

- Its square-magnitude function is given by

$$\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

- The maximum value of $|H(e^{j\omega})|^2$ is obtained when $2\alpha\cos\omega$ in the denominator is a maximum and the minimum value is obtained when $2\alpha\cos\omega$ is a minimum
- For $\alpha > 0$, maximum value of $2\alpha\cos\omega$ is equal to 2α at $\omega = 0$, and minimum value is -2α at $\omega = \pi$

Bounded Real Transfer Functions

- Thus, for $\alpha > 0$, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = \pi$
- On the other hand, for $\alpha < 0$, the maximum value of $2\alpha\cos\omega$ is equal to -2α at $\omega = \pi$, and the minimum value is equal to 2α at $\omega = 0$
- Here, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1+\alpha)^2$ at $\omega = \pi$ and the minimum value is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$
- Hence, the maximum value can be made equal to 1 by choosing $K = \pm(1-\alpha)$, in which case the minimum value becomes $(1-\alpha)^2/(1+\alpha)^2$

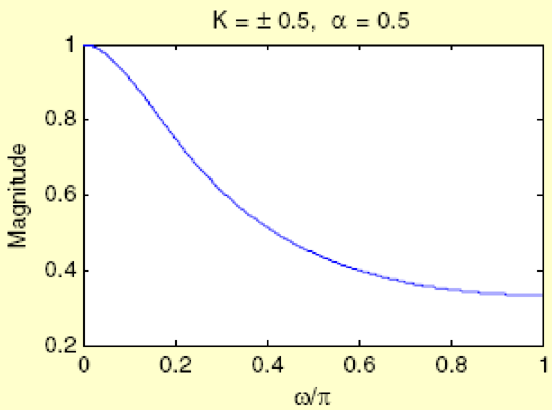
Bounded Real Transfer Functions

- Hence,

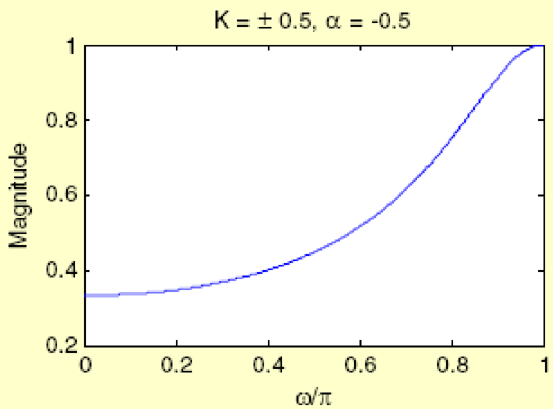
$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

  is a BR function for $K = \pm(1 - \alpha)$

- Plots of the magnitude function for with values of *K* chosen to make *H*(*z*) a BR function are shown below



Lowpass Filter



Highpass Filter

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All-Pass Transfer Functions

- An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$|H(e^{j\omega})|^2 = 1, \text{ for all } \omega$$

is called an **allpass transfer function**

- An M-th order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

- Denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:

$$D_M(z) = 1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

then it follows that $A_M(z)$ can be written as

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

All-Pass Transfer Functions

- Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = (1/r)e^{j\phi}$
- The numerator of a real-coefficient allpass transfer function is said to be the **mirror image polynomial** of the denominator, and vice versa
- We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-M polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z^{-1})$$

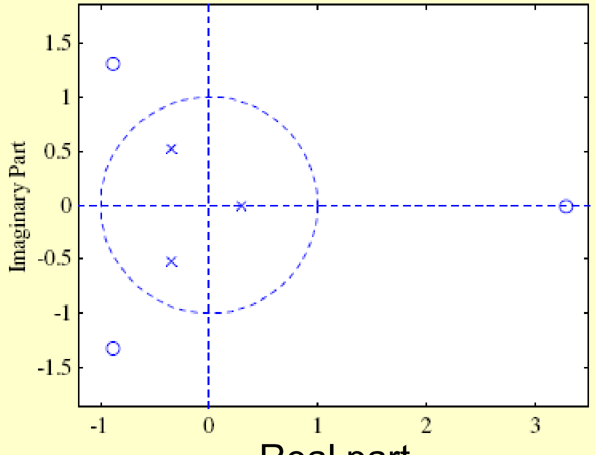
- The expression

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror image symmetry** in the z-plane

All-Pass Transfer Functions

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



- To show that $|A_M(e^{j\omega})| = 1$ we observe that

$$A_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Therefore

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})}$$

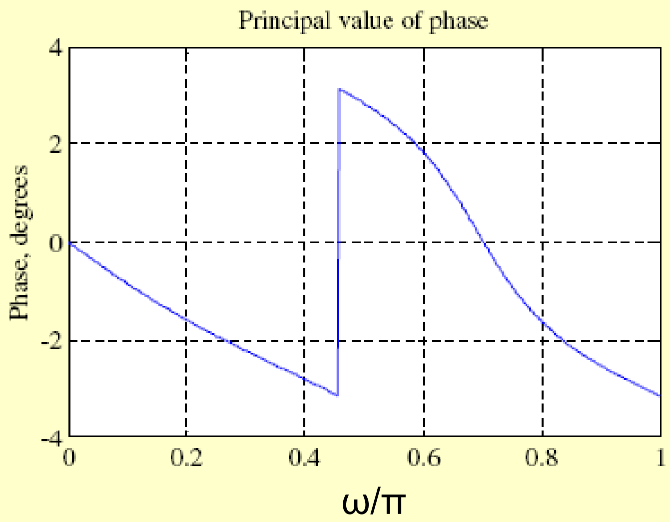
- Hence

$$|A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

All-Pass Transfer Functions

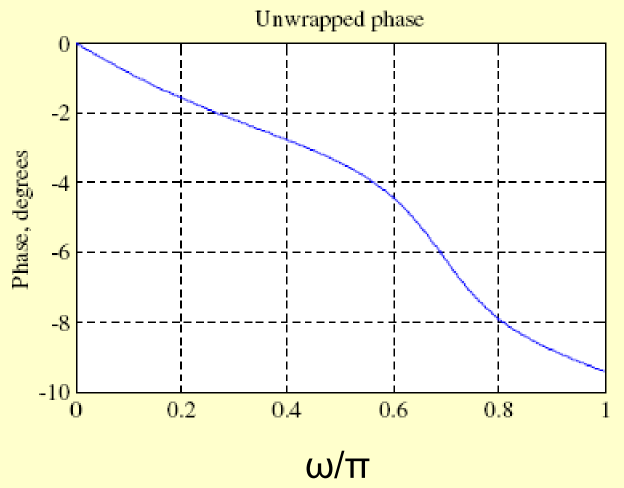
- Now, the poles of a causal stable transfer function must lie inside the unit circle in the *z*-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle
- Figure below shows the principal value of the phase of the 3rd-order allpass function

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



All-Pass Transfer Functions

- If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ as follows



- The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of ω

All-Pass Transfer Functions

Properties:

1. A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure
2. The magnitude function of a stable allpass function *A*(*z*) satisfies:

$$|A(z)| \begin{cases} <1, & \text{for } |z|>1 \\ =1, & \text{for } |z|=1 \\ >1, & \text{for } |z|<1 \end{cases}$$

3. Let τ(ω) denote the group delay function of an allpass filter *A*(*z*), i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

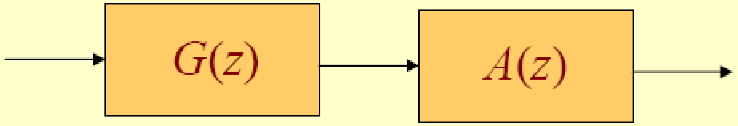
All-Pass Transfer Functions

- The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of ω so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$
- The group delay of an *M*-th order stable real-coefficient allpass transfer function satisfies $\int_0^{\pi} \tau(\omega) d\omega = M\pi$

A Simple Application:

- A simple but often used application of an allpass filter is as a **delay equalizer**
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade *has a* constant group delay in the band of interest

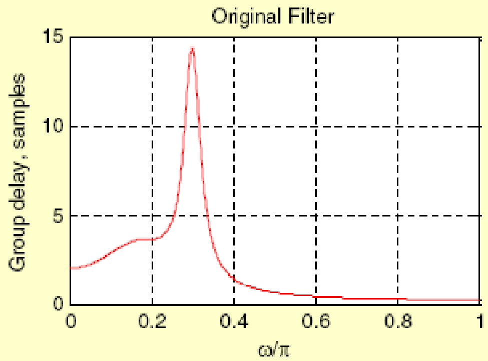
All-Pass Transfer Functions

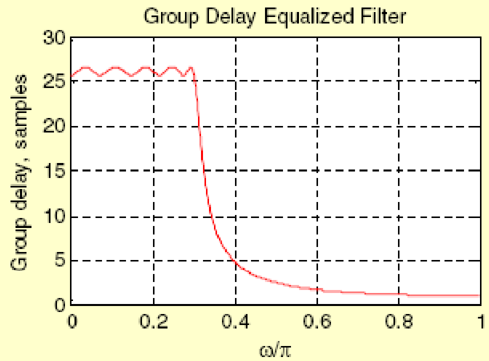


- Since $|A(e^{j\omega})| = 1$, we have

$$|A(e^{j\omega})\, G(e^{j\omega})| = |G(e^{j\omega})|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $A(z)$
- Example – Figure below shows the group delay of a 4th order elliptic filter with the following specifications: $\omega_p = 0.3\pi$, $\delta_p = 1$ dB, $\delta_s = 35$ dB
- The group delay of the original filter cascaded with an 8th order allpass filter is also shown



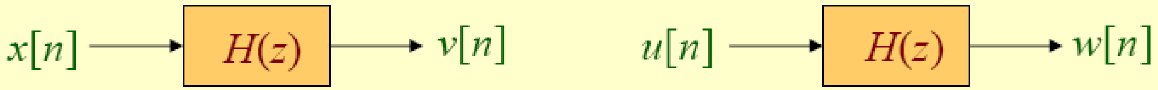


Classification Based on Phase Characteristics

- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband
- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase characteristic**
- However, it is not possible to design a causal digital filter with a zero phase
- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement

Zero-Phase Transfer Functions

- One zero-phase filtering scheme is sketched below



$$u[n] = v[-n], \qquad y[n] = w[-n]$$

- Let $X(e^{j\omega})$, $V(e^{j\omega})$, $U(e^{j\omega})$, $W(e^{j\omega})$, and $Y(e^{j\omega})$denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively
- Making use of the symmetry relations we arrive at the relations between various DTFTs as follows:

$$V(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega}),\ W(e^{j\omega}) = H(e^{j\omega})\, U(e^{j\omega})$$

$$U(e^{j\omega}) = V^*(\mathrm{e}^{j\omega}),\ Y(e^{j\omega}) = W^*(\mathrm{e}^{j\omega})$$

- Combining the above equations we get

$$Y(e^{j\omega}) = W^*(\mathrm{e}^{j\omega}) = H^*(e^{j\omega})\, U^*(e^{j\omega}) = H^*(e^{j\omega})\, V(e^{j\omega})$$

$$= H^*(e^{j\omega})\, H(e^{j\omega})\, X(e^{j\omega}) = |H(e^{j\omega})|^2\, X(e^{j\omega})$$

Linear-Phase Transfer Functions

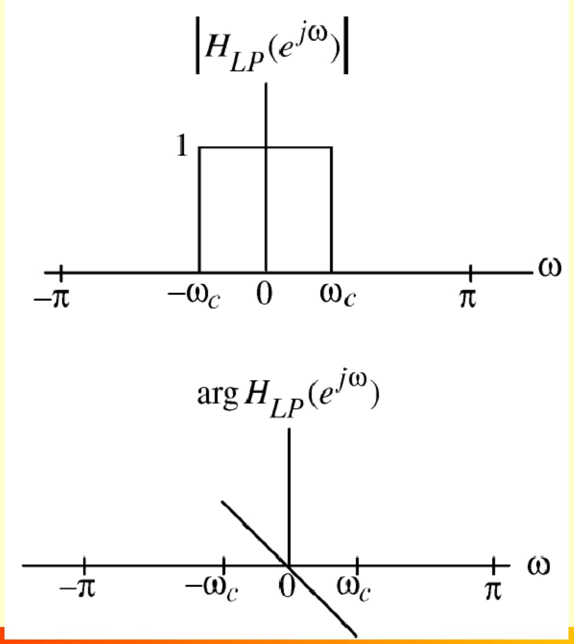
- The output $y[n]$ of a linear-phase filter to an input $x[n] = Ae^{j\omega n}$ is then given by

$$y[n] = Ae^{-j\omega D} e^{j\omega n} = Ae^{j\omega(n-D)}$$

- If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t = nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount D
- If D is an integer, then $y[n]$ is identical to $x[n]$, but delayed by D samples
- If D is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$
  - The waveform of the underlying continuous-time output is identical to the waveform of the continuous-time input and delayed D units of time

Linear-Phase Transfer Functions

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest
- Figure below shows the frequency response if a lowpass filter with a linear-phase characteristic in the passband
- Since the signal components in the stopban are blocked, the phase response in the stopband can be of any shape



Linear-Phase Transfer Functions

- Example – Determine the impulse response of an ideal lowpass filter with a linear phase response

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_o}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_o)}{\pi (n - n_o)}, \quad -\infty < n < \infty$$

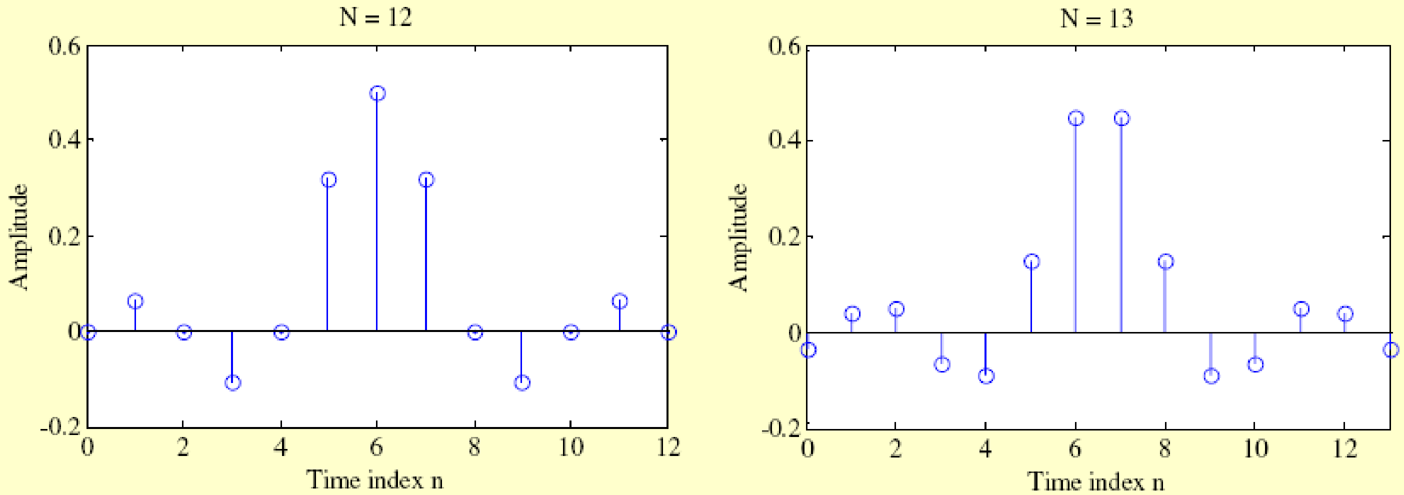
- As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable
- By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed

Linear-Phase Transfer Functions

- The truncated approximation may or may not exhibit linear phase, depending on the value of n_o chosen
- If we choose $n_o = N/2$ with N a positive integer, the truncated and shifted approximation

$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi (n - N/2)}, \quad 0 \le n \le N$$

will be a length N+1 causal linear-phase FIR filter



Zero-Phase Transfer Functions

- Because of the symmetry of the impulse response coefficients as indicated in the two figures, the frequency response of the truncated approximation can be expressed as:

$$\hat{H}_{LP}(e^{j\omega}) = \sum_{n=0}^{N} \hat{h}_{LP}[n] e^{-j\omega n} = e^{-j\omega N/2}\,\tilde{H}_{LP}(\omega)$$

- Where $\tilde{H}_{LP}(\omega)$, called the **zero-phase response** or **amplitude response**, is a real function of ω will be a length *N*+1 causal linear-phase FIR filter

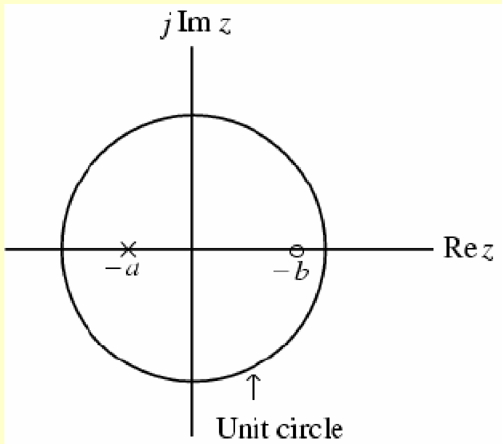
Minimum-Phase and Maximum-Phase Transfer Functions

- Consider the two 1st-order transfer functions:

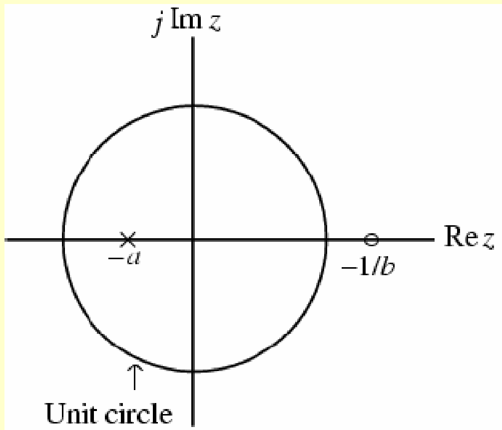
$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

- Both transfer functions have a pole inside the unit circle at the same location and are stable
- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$, whereas, the zero of $H_2(z)$ is at $z = -1/b$ situated in a mirror-image symmetry

$H_1(z)$



$H_2(z)$



Minimum-Phase and Maximum-Phase Transfer Functions

- However, both transfer functions have an identical magnitude function as

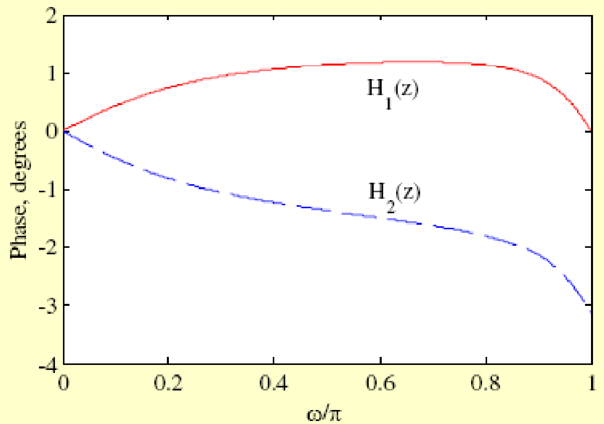
$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

- Figure below shows the unwrapped phase responses of the two transfer functions for a = 0.8 and b = − 0.5



Minimum-Phase and Maximum-Phase Transfer Functions

- As shown in the figure, $H_2(z)$ has an excess phase lag with respect to $H_1(z)$
- The excess phase lag property of $H_2(z)$ with respect to $H_1(z)$ can also be explained by observing that we can write

$$H_2(z) = \frac{bz+1}{z+a} = \underbrace{\left(\frac{z+b}{z+a}\right)}_{H_1(z)}\underbrace{\left(\frac{bz+1}{z+b}\right)}_{A(z)}$$

where $A(z) = (bz+1)/(z+b)$ is a stable allpass function

- The phase functions of $H_1(z)$ and $H_2(z)$ are related through

$$\arg[H_2(e^{j\omega})] = \arg[H_1(e^{j\omega})] + \arg[A(e^{j\omega})]$$

- As the unwrapped phase function of a stable first-order allpass function is a negative function of ω, it follows from the above that $H_2(z)$ has an excess phase lag with $H_1(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- Generalizing the above result, let $H_m(z)$ be a causal stable transfer function with all zeros inside the unit circle and let $H(z)$ be another causal stable transfer function satisfying $|H(e^{j\omega})| = |H_m(e^{j\omega})|$
- These two transfer functions are then related through $H(z) = H_m(z)\, A(z)$ where $A(z)$ is a causal stable allpass function
- The unwrapped phase functions of $H_m(z)$ and $H(z)$ are thus related through

$$\arg[H(e^{j\omega})] = \arg[H_{\rm m}(e^{j\omega})] + \arg[A(e^{j\omega})]$$

- $H(z)$ has an excess phase lag with $H_m(z)$
- A causal stable transfer function with all zeros inside the unit circle is called a **minimum-phase transfer function**

Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros outside the unit circle is called a **maximum-phase transfer function**
- A causal stable transfer function with zeros inside and outside the unit circle is called a **mixed-phase transfer function**
- Example – Consider the mixed-phase transfer function

$$H(z) = \frac{2(1+0.3z^{-1})(0.4-z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}$$

- We can rewrite $H(z)$ as

$$H(z) = \underbrace{\left[\frac{2(1+0.3z^{-1})(1-0.4z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}\right]}_{\text{Minimum-phase function}} \underbrace{\left(\frac{0.4-z^{-1}}{1-0.4z^{-1}}\right)}_{\text{Allpass function}}$$

Linear-Phase FIR Transfer Functions

- It is impossible to design an IIR transfer function with an exact linear-phase
- It is always possible to design an FIR transfer function with an exact linear-phase response
- We now develop the forms of the linear-phase FIR transfer function $H(z)$ with real impulse response $h[n]$
- Let $H(z) = \sum_{n=0}^{N} h[n] z^{-n}$
- If $H(z)$ is to have a linear-phase, its frequency response must be of the form

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega)$$

where c and β are constants, and $\breve{H}(\omega)$, called the amplitude response (zero-phase response), is a real function of ω

Linear-Phase FIR Transfer Functions

- For a real impulse response, the magnitude response $|H(e^{j\omega})|$ is an even function of ω, i.e., $|H(e^{j\omega})| = |H(e^{-j\omega})|$
- Since $|H(e^{j\omega})| = |\breve{H}(\omega)|$, the amplitude response is then either an even function or an odd function of ω, i.e., $\breve{H}(-\omega) = \pm\breve{H}(\omega)$
- The frequency response satisfies the relation

$$|H(e^{j\omega})| = |H^{*}(e^{-j\omega})|$$

or, equivalently, the relation

$$e^{j(c\omega+\beta)}\,\breve{H}(\omega) = e^{-j(-c\omega+\beta)}\,\breve{H}(-\omega)$$

- If $\breve{H}(\omega)$ is an even function, then the above relation leads to

$$e^{j\beta} = e^{-j\beta}$$

implying that either β = 0 or β = π

Linear-Phase FIR Transfer Functions

- From $H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega)$, we have $\breve{H}(\omega) = e^{-j(c\omega+\beta)}H(e^{j\omega})$
- Substituting the value of β in the above we get

$$\breve{H}(\omega) = \pm e^{-jc\omega}H(e^{j\omega}) = \pm\sum_n h[n]e^{-j\omega(c+n)}$$

- Replacing ω with in the previous equation we get

$$\breve{H}(-\omega) = \pm\sum_l h[l]e^{j\omega(c+l)}$$

- Let $l = N - n$, we rewrite the above equation as

$$\breve{H}(-\omega) = \pm\sum_n h[N-n]e^{j\omega(c+\mathrm{N}-n)}$$

- As $\breve{H}(\omega) = \breve{H}(-\omega)$, we have $h[n]e^{-j\omega(c+n)} = h[N-n]e^{j\omega(c+\mathrm{N}-n)}$
- The above leads to the condition:

$$h[n] = h[N-n],\ 0 \le n \le N, \text{ where } c = -N/2$$

- Thus, the FIR filter with an even amplitude response will have a inear phase if it has asymmetric impulse response

Linear-Phase FIR Transfer Functions

- If $\check{H}(\omega)$ is an odd function of ω, then from

$$e^{j(c\omega+\beta)}\,\check{H}(\omega) = e^{-j(c-\omega+\beta)}\,\check{H}(-\omega)$$

  we get $e^{j\beta} = -e^{-j\beta}$ as $\check{H}(-\omega) = -\check{H}(\omega)$

- The above is satisfied if $\beta = \pi/2$ or $\beta = -\pi/2$
- Then $H(e^{j\omega}) = e^{j(c\omega+\beta)}\,\check{H}(\omega)$ reduces to $H(e^{j\omega}) = je^{jc\omega}\,\check{H}(\omega)$
- The last equation can be rewritten as

$$\check{H}(\omega) = -je^{-jc\omega}\,H(e^{j\omega}) = -j\sum_n h[n]e^{-j\omega(c+n)}$$

- As $\check{H}(-\omega) = -\check{H}(\omega)$, from the above we get

$$\check{H}(-\omega) = j\sum_l h[l]e^{j\omega(c+l)}$$

- Making a change of variable $l = N - n$ we have

$$\check{H}(-\omega) = j\sum_n h[N-n]e^{j\omega(c+N-n)}$$

Linear-Phase FIR Transfer Functions

- Equating the above with

$$\check{H}(\omega) = -j \sum_l h[l] e^{-j\omega(c+l)}$$

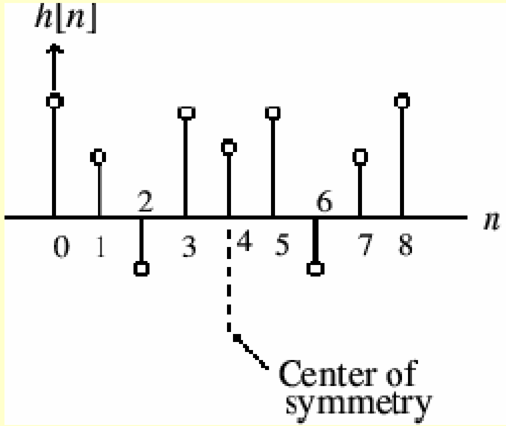
we arrive at the condition for linear phase as

$$h[n] = h[N - n],\ 0 \le n \le N, \text{ with } c = -N/2$$

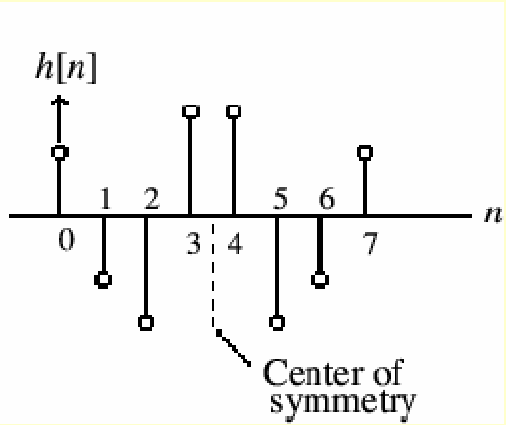
- Therefore, an FIR filter with an odd amplitude response will have linear-phase response if it has an antisymmetric impulse response
- Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions
- For an antisymmetric FIR filter of odd length, i.e., N even

$$h[N/2] = 0$$

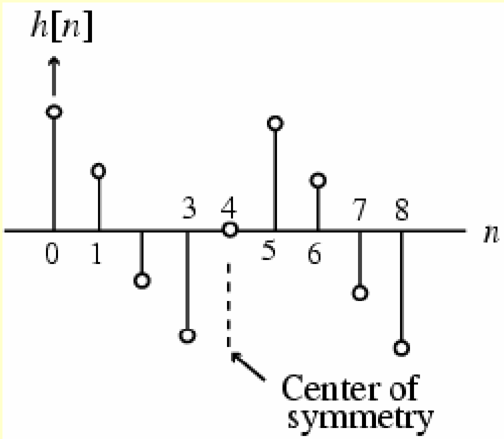
Linear-Phase FIR Transfer Functions



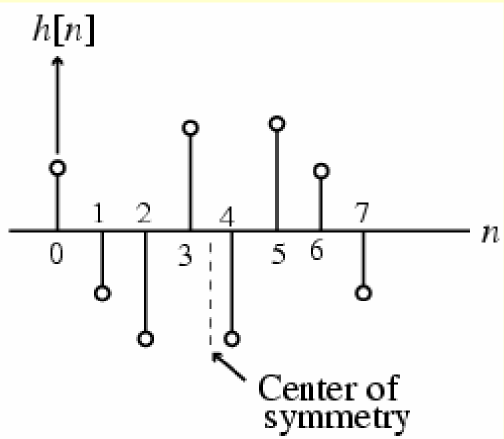
Type 1: $N = 8$



Type 2: $N = 7$



Type 3: $N = 8$



Type 4: $N = 7$

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Type-1 FIR Transfer Functions

Type 1: Symmetric Impulse Response with Odd Length

- In this case, the degree N is even (Assume $N = 8$)
- The transfer function $H(z)$ is given by

$$H(z) = h[0] + h[1]z^{-1} + h[2]\,z^{-2} + h[3]\,z^{-3} + h[4]\,z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8}$$

- Because of symmetry, we have $h[0] = h[8]$, $h[1] = h[7]$, $h[2] = h[6]$, and $h[3] = h[5]$
- Thus, we can write

$$H(z) = h[0](1+ z^{-8}) + h[1](z^{-1} + z^{-7}) + h[2](z^{-2} + z^{-6}) + h[3](z^{-3} + z^{-5}) + h[4]\,z^{-4}$$

$$= z^{-4}\{h[0](z^{4} + z^{-4}) + h[1](z^{3} + z^{-3}) + h[2](z^{2} + z^{-2}) + h[3](z + z^{-1}) + h[4]\}$$

Type-1 FIR Transfer Functions

- The corresponding frequency response is then given by

  $H(e^{j\omega}) = e^{-j4\omega}\{2h[0]\cos(4\omega) + 2h[1]\cos(3\omega) + 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]\}$

- The quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$
- The phase function here is given by

$$\theta(\omega) = -4\omega + \beta$$

  where β is either 0 or π, and hence it is a linear function of ω
- The group delay is given by

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4$$

  indicating a constant group delay of 4 samples

Type-1 FIR Transfer Functions

- In the general case for Type 1 FIR filters, the frequency response is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2}\tilde{H}(\omega)$$

where the **amplitude response** $\tilde{H}(\omega)$, also called the **zero-phase response**, is of the form

$$\tilde{H}(\omega) = h[\tfrac{N}{2}] + 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\cos(\omega n)$$

- Example – Consider

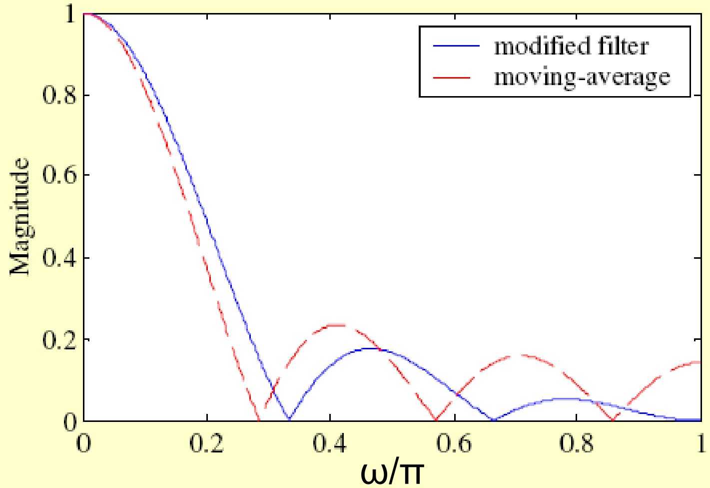
$$H_0(z) = \tfrac{1}{6}[\tfrac{1}{2} + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \tfrac{1}{2}z^{-6}]$$

which is seen to be a slightly modified version of a length-7 moving-average FIR filter

- The above transfer function has a symmetric impulse response and therefore a linear phase response

Type-1 FIR Transfer Functions

- The magnitude response of $H_0(z)$:



- Improved magnitude response obtained by changing the first and the last impulse response coefficients of MA filter
- This filter can be expressed as a cascade of a 2-point MA filter with a 6-point MA filter

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1}) \cdot \tfrac{1}{6}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

- Thus, $H_0(z)$ has a double zero at $z = -1$, i.e., ($\omega = \pi$)

Type-2 FIR Transfer Functions

Type 2: Symmetric Impulse Response with Even Length

- In this case, the degree N is odd (Assume $N = 7$)
- The transfer function $H(z)$ is of the form

$$H(z) = h[0] + h[1]z^{-1} + \ldots + h[7]z^{-7}$$

- Because of symmetry, we have $h[0] = h[7]$, $h[1] = h[6]$, $h[2] = h[5]$, and $h[3] = h[4]$
- Thus, we can write

$$H(z) = h[0](1 + z^{-7}) + h[1](z^{-1} + z^{-6}) + h[2](z^{-2} + z^{-5}) + h[3](z^{-3} + z^{-4})$$

$$= z^{-7/2}\{h[0](z^{7/2} + z^{-7/2}) + h[1](z^{5/2} + z^{-5/2}) + h[2](z^{3/2} + z^{-3/2}) + h[3](z^{1/2} + z^{-1/2})\}$$

Type-2 FIR Transfer Functions

- The corresponding frequency response is then given by

$$H(e^{j\omega}) = e^{-j7\omega/2}\{2h[0]\cos(7\omega/2) + 2h[1]\cos(5\omega/2) + 2h[2]\cos(3\omega/2) + 2h[3]\cos(\omega/2)\}$$

- The quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$
- The phase function here is given by

$$\theta(\omega) = -7\omega/2 + \beta$$

  where β is either 0 or π, and hence it is a linear function of ω
- The group delay is given by

$$\tau(\omega) = \frac{7}{2}$$

  indicating a constant group delay of 7/2 samples

Type-2 FIR Transfer Functions

- The expression for the frequency response in the general case for Type 2 FIR filters is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2}\tilde{H}(\omega)$$

- where the amplitude response is given by

$$\tilde{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2} - n]\cos(\omega(n - \tfrac{1}{2}))$$

Type-3 FIR Transfer Functions

Type 3: Antiymmetric Impulse Response with Odd Length

- In this case, the degree N is even (Assume $N = 8$)
- The transfer function $H(z)$ is of the form

$$H(z) = h[0] + h[1]z^{-1} + \ldots + h[8]z^{-8}$$

- Antisymmetric filter coefficients: $h[0] = -h[8]$, $h[1] = -h[7]$, $h[2] = -h[6]$, $h[3] = -h[5]$, and $h[4] = 0$
- Applying the symmetry condition we get

$$H(z) = z^{-4}\{h[0](z^4 - z^{-4}) + h[1](z^3 - z^{-3}) + h[2](z^2 - z^{-2}) + h[3](z - z^{-1})\}$$

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j4\omega}e^{j\pi/2}\{2h[0]\sin(4\omega) + 2h[1]\sin(3\omega) + 2h[2]\sin(2\omega) + 2h[3]\sin(\omega)\}$$

Type-3 FIR Transfer Functions

- It also exhibits a linear phase response given by

$$\theta(\omega) = -4\omega + \pi/2 + \beta$$

where β is either 0 or π

- The group delay here is

$$\tau(\omega) = 4$$

indicating a constant group delay of 4 samples

- In the general case

$$H(e^{j\omega}) = je^{-jN\omega/2}\tilde{H}(\omega)$$

where the amplitude response is of the form

$$\tilde{H}(\omega) = 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\sin(\omega n)$$

Type-4 FIR Transfer Functions

Type 4: Antiymmetric Impulse Response with Even Length

- In this case, the degree N is Odd (Assume $N = 7$)
- The transfer function $H(z)$ is of the form

$$H(z) = h[0] + h[1]z^{-1} + \ldots + h[7]z^{-7}$$

- Antisymmetric filter coefficients: $h[0] = -h[7]$, $h[1] = -h[6]$, $h[2] = -h[5]$, and $h[3] = -h[4]$
- Applying the symmetry condition we get

$$H(z) = z^{-7/2}\{h[0](z^{7/2} - z^{-7/2}) + h[1](z^{5/2} - z^{-5/2}) + h[2](z^{3/2} - z^{-3/2}) + h[3](z^{1/2} - z^{-1/2})\}$$

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j7\omega/2}e^{j\pi/2}\{2h[0]\sin(7\omega/2) + 2h[1]\sin(5\omega/2) + 2h[2]\sin(3\omega/2) + 2h[3]\sin(\omega/2)\}$$

General Form of Frequency Response

- In each of the four types of linear-phase FIR filters, the frequency response is of the form
$$H(e^{j\omega}) = e^{-jN\omega/2} e^{j\beta} \tilde{H}(\omega)$$
- The amplitude response $\tilde{H}(\omega)$ for each of the four types of linear-phase FIR filters can become negative over certain frequency ranges, typically in the stopband
- The magnitude and phase responses of the linear-phase FIR are given by
$$|H(e^{j\omega})| = |\tilde{H}(\omega)|$$
$$\theta(\omega) = \begin{cases} -\frac{N\omega}{2} + \beta, & \text{for } \tilde{H}(\omega) \geq 0 \\ -\frac{N\omega}{2} + \beta - \pi, & \text{for } \tilde{H}(\omega) < 0 \end{cases}$$
- The group delay in each case is $\tau(\omega) = N/2$

General Form of Frequency Response

- Note that, even though the group delay is constant, since in general $|H(e^{j\omega})|$ is not a constant, the output waveform is not a replica of the input waveform
- An FIR filter with a frequency response that is a real function of ω is often called a **zero-phase filter**
- Such a filter must have a noncausal impulse response

Zero Locations of Linear-Phase FIR Transfer Functions

- Consider first an FIR filter with a symmetric impulse response: $h[n] = h[N - n]$
- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} = \sum_{n=0}^{N} h[N-n]z^{-n}$$

- By making a change of variable $m = N - n$, we can write

$$\sum_{n=0}^{N} h[N-n]z^{-n} = \sum_{m=0}^{N} h[m]z^{-N+m} = z^{-N} \sum_{m=0}^{N} h[m]z^{m}$$

- But,

$$\sum_{m=0}^{N} h[m]z^{m} = H(z^{-1})$$

- Hence for an FIR filter with a symmetric impulse response of length N+1 we have $H(z) = z^{-N}H(z^{-1})$
- Such kind of $H(z)$ is called a **mirror-image polynomial (MIP)**

Zero Locations of Linear-Phase FIR Transfer Functions

- Now consider an FIR filter with a antisymmetric impulse response: $h[n] = -h[N-n]$
- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n} = -\sum_{n=0}^{N} h[N-n] z^{-n}$$

- By making a change of variable $m = N - n$, we get

$$-\sum_{n=0}^{N} h[N-n] z^{-n} = -\sum_{m=0}^{N} h[m] z^{-N+m} = -z^{-N} H(z^{-1})$$

- Hence, the transfer function $H(z)$ of an FIR filter with an antisymmetric impulse response satisfies the condition

$$H(z) = -z^{-N} H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called an **antimirror-image polynomial (AIP)**

Zero Locations of Linear-Phase FIR Transfer Functions

- It follows from the relation $H(z) = \pm z^{-N} H(z^{-1})$ that if $z = \xi_o$ is a zero of $H(z)$, so is $z = 1/\xi_o$
- Moreover, for an FIR filter with a real impulse response, the zeros of $H(z)$ occur in complex conjugate pairs
- Hence, a zero at $z = \xi_o$ is associated with a zero at $z = \xi_o^*$
- Thus, a complex zero that is not on the unit circle is associated with a set of 4 zeros given by

$$z = re^{\pm j\phi}, \; z = (1/r)e^{\pm j\phi}$$

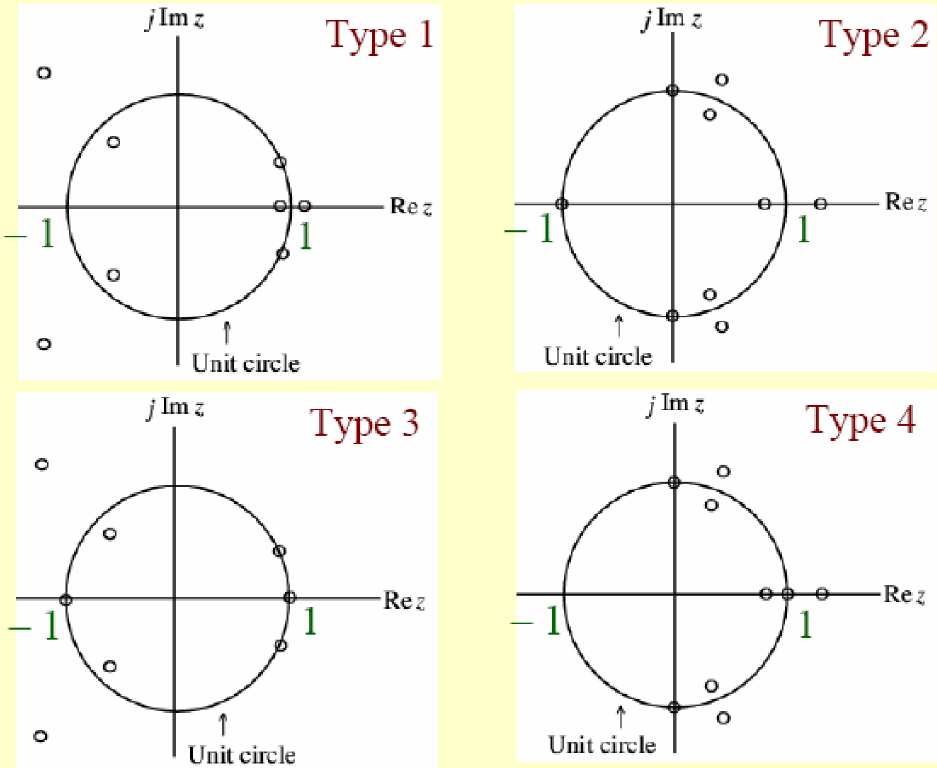
- A zero on the unit circle appear as a pair $z = e^{\pm j\phi}$, as its reciprocal is also its complex conjugate
- Since a zero at $z = \pm 1$ is its own reciprocal, it can appear only singly

Zero Locations of Linear-Phase FIR Transfer Functions

- Now a Type 2 FIR filter satisfies
$$H(z) = z^{-N}H(z^{-1})$$
with degree N odd
- Hence, $H(-1) = (-1)^{-N}\, H(-1) = -H(-1)$ implying that $H(-1) = 0$, i.e., $H(z)$ must have a zero at $z = -1$
- a Type 3 or 4 FIR filter satisfies
$$H(z) = -z^{-N}H(z^{-1})$$
- Thus, $H(1) = -(1)^{-N}\, H(1) = -H(1)$ implying that $H(z)$ must have a zero at $z = 1$
- On the other hand, only the Type 3 FIR filter is restricted to have a zero at $z = -1$ since here the degree N is even and hence, $H(-1) = -(-1)^{-N}\, H(-1) = -H(-1)$

Zero Locations of Linear-Phase FIR Transfer Functions

- Typical zero locations shown below



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Zero Locations of Linear-Phase FIR Transfer Functions

Summary

1. Type 1 FIR filter: Either an even number or no zeros at $z = 1$ and $z = -1$
2. Type 2 FIR filter: Either an even number or no zeros at $z = 1$, and an odd number of zeros at $z = -1$
3. Type 3 FIR filter: An odd number of zeros at $z = 1$ and $z = -1$
4. Type 4 FIR filter: An odd number of zeros at $z = 1$, and either an even number or no zeros at $z = -1$

Zero Locations of Linear-Phase FIR Transfer Functions

- The presence of zeros at $z = \pm 1$ leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters
- A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero $z = -1$
- A Type 3 FIR filter has zeros at both at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter
- A Type 4 FIR filter is not appropriate to design lowpass and bandstop filters due to the presence of a zero at $z = 1$
- Type 1 FIR filter has no such restrictions and can be used to design almost any type of filters

Simple Lowpass FIR Digital Filters

- The simplest lowpass FIR digital filter is the 2-point moving-average filter given by

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z}$$

- The above transfer function has a zero at $z = -1$ and a pole at $z = 0$
- Note that here the pole vector has a unity magnitude for all values of ω
- On the other hand, as ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2, the diameter of the unit circle, to 0
- Hence, the magnitude response $|H_0(e^{j\omega})|$ is a monotonically decreasing function of ω from ω = 0 to ω = π

Lowpass FIR Digital Filters

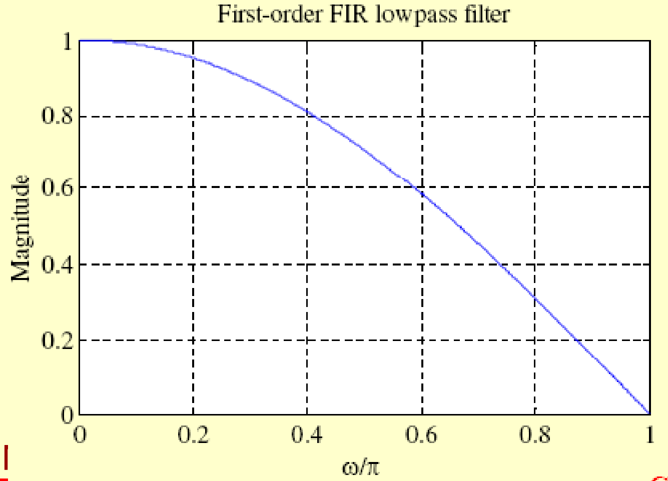
- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$, i.e.,

$$|H_0(e^{j0})| = 1, \qquad |H_0(e^{j\pi})| = 0$$

- The frequency response of the above filter is given by

$$|H_0(e^{j\omega})| = e^{-j\omega/2}\cos(\omega/2)$$

- The magnitude response $|H_0(e^{j\omega})| = e^{-j\omega/2}\cos(\omega/2)$ is a monotonically decreasing function of ω



Lowpass FIR Digital Filters

- The frequency $\omega = \omega_c$ at which

$$\left|H_0(e^{j\omega_c})\right| = \frac{1}{\sqrt{2}}\left|H_0(e^{j0})\right|$$

is of practical interest since here the gain $G(\omega_c)$ in dB is given by

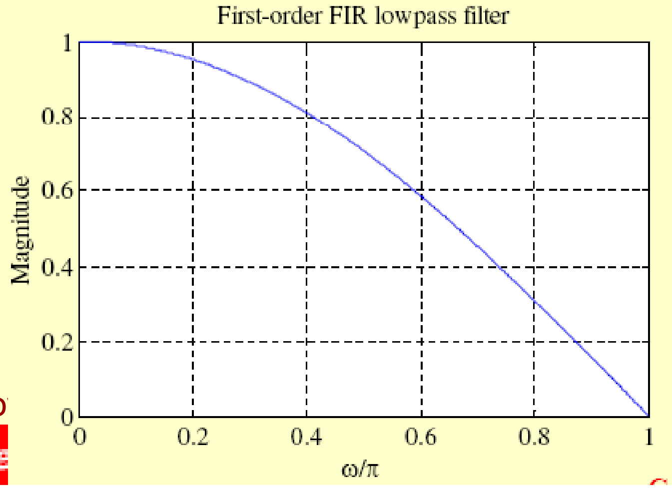
$$\begin{aligned} \mathcal{G}(\omega_c) &= 20\log_{10}\left|H(e^{j\omega_c})\right| \\ &= 20\log_{10}\left|H(e^{j0})\right| - 20\log_{10}\sqrt{2} \cong -3 \text{ dB} \end{aligned}$$

since the dc gain $G(0) = 20\log_{10}|H_0(e^{j0})| = 0$

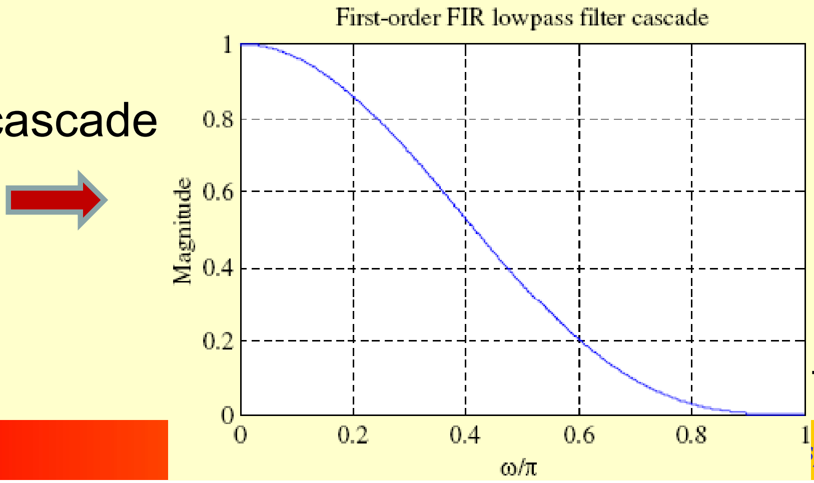
- Thus, the gain $G(\omega)$ at $\omega = \omega_c$ is approximately 3 dB less than the gain at $\omega = 0$
- As a result, ω_c is called the **3-dB cutoff frequency**
- To determine the value of ω_c we set $|H_0(e^{j\omega_c})|^2 = \cos^2(\omega_c/2) = \frac{1}{2}$ which yields $\omega_c = \pi / 2$

Lowpass FIR Digital Filters

- The 3-dB cutoff frequency ω_c can be considered as the passband edge frequency
- As a result, for the filter $H_0(z)$ the passband width is approximately π/2, and the stopband is from π/2 to π
- Note: $H_0(z)$ has a zero at $z = -1$ or $\omega = \pi$, which is in the stopband of the filter
- A cascade of 3 sections of the FIR filter $H_0(z) = ½(1 + z^{-1})$ results in an improved lowpass frequency



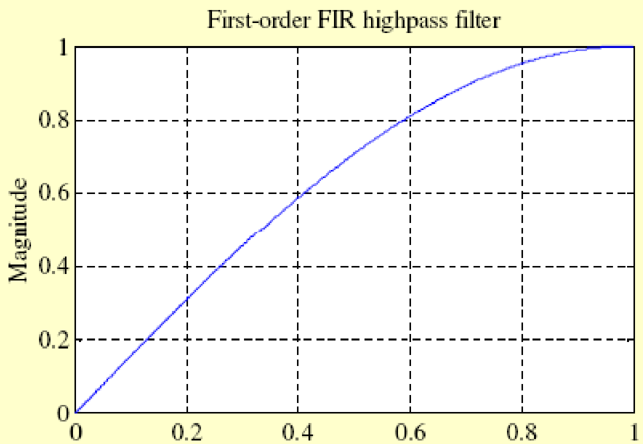
cascade



Highpass FIR Digital Filters

- The simplest highpass FIR filter is obtained from the simplest lowpass FIR filter by replacing z with $-z$

  For example: $H_1(z) = \frac{1}{2}(1 - z^{-1})$
- Corresponding frequency response is given by
- $H_1(e^{j\omega}) = je^{-j\omega/2}\sin(\omega/2)$



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Highpass FIR Digital Filters

- The monotonically increasing behavior of the magnitude function can again be demonstrated by examining the pole-zero pattern of the transfer function $H_1(z)$

  For example: $H_1(z) = \frac{1}{2}(1 - z^{-1})$
- The highpass transfer function $H_1(z)$ has a zero at $z = 1$ or $\omega = 0$ which is in the stopband of the filter
- Improved highpass magnitude response can be obtained by cascading several sections of the first-order highpass filter
- Alternately, a higher-order highpass filter of the form

$$H_1(z) = \frac{1}{M} \sum_{n=0}^{M-1} (-1)^n z^{-n}$$

  is obtained by replacing z with $-z$ in the transfer function of a moving average filter

Highpass FIR Digital Filters

- An application of the FIR highpass filters is in moving-target-indicator (MTI) radars
- In these radars, interfering signals, called **clutters**, are generated from fixed objects in the path of the radar beam
- The clutter, generated mainly from ground echoes and weather returns, has frequency components near zero frequency (dc)
- The clutter can be removed by filtering the radar return signal through a **two-pulse canceler**, which is the first-order FIR highpass filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$
- For a more effective removal it may be necessary to use a **three-pulse canceler** obtained by cascading two two-pulse cancelers

Simple IIR Digital Filters

- We have shown earlier that the first-order causal IIR transfer function

$$H(z) = \frac{K}{1-\alpha z^{-1}}, \quad 0 < \alpha < 1$$

has a lowpass magnitude response for α > 0

- On the other hand, the first-order causal IIR transfer function

$$H(z) = \frac{K}{1-\alpha z^{-1}}, \quad -1 < \alpha < 0$$

has a highpass magnitude response for α < 0

- However, the modified transfer function obtained with the addition of a factor $(1 + z^{-1})$ to the numerator

$$H(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}}, \quad -1 < \alpha < 0$$

exhibits a lowpass magnitude response

Simple IIR Digital Filters

- The modified first-order lowpass transfer function for both positive and negative values of α is then given by

$$H_{LP}(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2 to 0
- The maximum values of the magnitude function is $2K/(1-\alpha)$ at $\omega = 0$ and the minimum value is 0 at $\omega = \pi$, i.e.,

$$|H_{LP}(e^{j0})| = 2K/(1-\alpha), \quad |H_{LP}(e^{j\pi})| = 0$$

- Therefore, $|H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$

Simple IIR Digital Filters

- For most applications, it is usual to have a dc gain of 0 dB, that is to have $|H_{LP}(e^{j0})| = 1$
- To this end, we choose $K = (1-\alpha)/2$ resulting in the first-order IIR lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right), \quad 0 < |\alpha| < 1$$

- The above transfer function has a zero at i.e., at $\omega = \pi$ which is in the stopband

Lowpass IIR Digital Filters

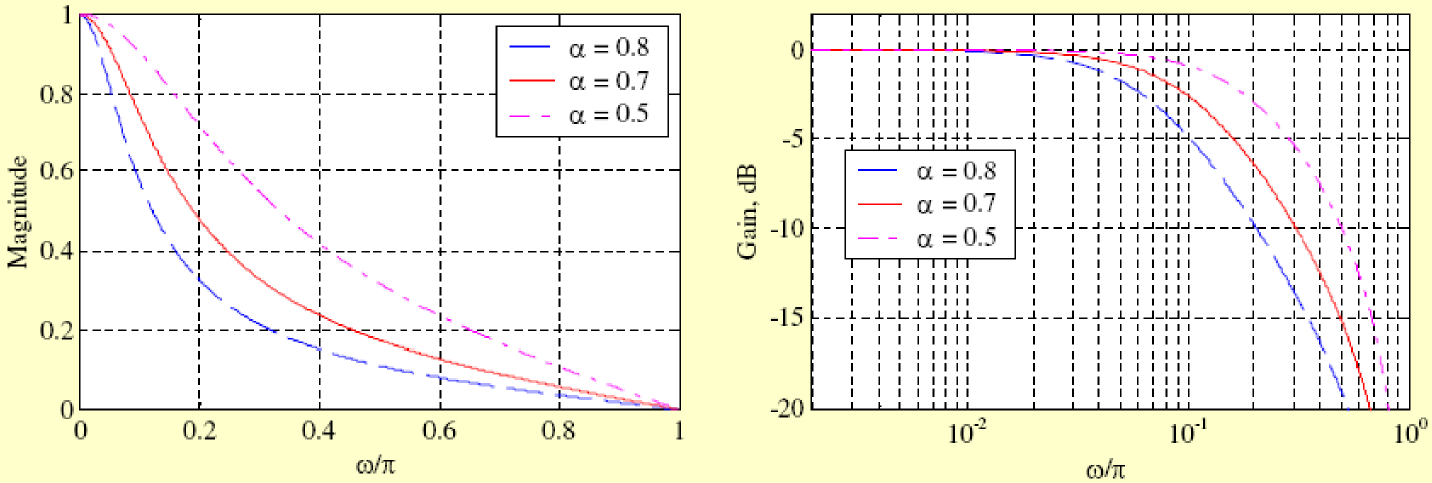
- A first-order causal lowpass IIR digital filter has a transfer function given by

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

  where $|\alpha| < 1$ for stability
- The above transfer function has a zero at $z = -1$ i.e., at $\omega = \pi$ which is in the stopband
- $H_{LP}(z)$ has a real pole at $z = \alpha$
- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2 to 0, whereas, for a positive value of α, the magnitude of the pole vector increases from a value of $1-\alpha$ to $1+\alpha$ function is 1 at $\omega = 0$, and the minimum
- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$

Lowpass IIR Digital Filters

- That is $|H_{LP}(e^{j0})| = 1$, $|H_{LP}(e^{j\pi})| = 0$
- Therefore, $|H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$ as indicated below



- The squared magnitude function is given by

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

Lowpass IIR Digital Filters

- The derivative of $|H_{LP}(e^{j\omega})|^2$ with respect to ω is given by

$$\frac{d\,|H_{LP}(e^{j\omega})|^2}{d\omega} = \frac{-(1-\alpha)^2(1+2\alpha+\alpha^2)\sin\omega}{2(1-2\alpha\cos\omega+\alpha^2)^2}$$

$d|H_{LP}(e^{j\omega})|^2 / d\omega \leq 0$ in the range $0 \leq \omega \leq \pi$ verifying again the monotonically decreasing behavior of the magnitude function

- To determine the 3-dB cutoff frequency we set

$$|H_{LP}(e^{j\omega_c})|^2 = \frac{1}{2}$$

- in the expression for the square magnitude function resulting in

$$\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)} = \frac{1}{2}$$

or $(1-\alpha)^2(1+\cos\omega_c) = 1+\alpha^2-2\alpha\cos\omega_c \Rightarrow \cos\omega_c = 2\alpha/(1+\alpha^2)$

Lowpass IIR Digital Filters

- The above quadratic equation can be solved for α yielding two solutions
- The solution resulting in a stable transfer function $H_{LP}(z)$ is given by

$$\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}$$

- It follows from

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2 - 2\alpha\cos\omega)}$$

that $H_{LP}(z)$ is a BR function for $|\alpha| < 1$

Highpass IIR Digital Filters

- A first-order causal highpass IIR digital filter has a transfer function given by

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

where $|\alpha| < 1$ for stability

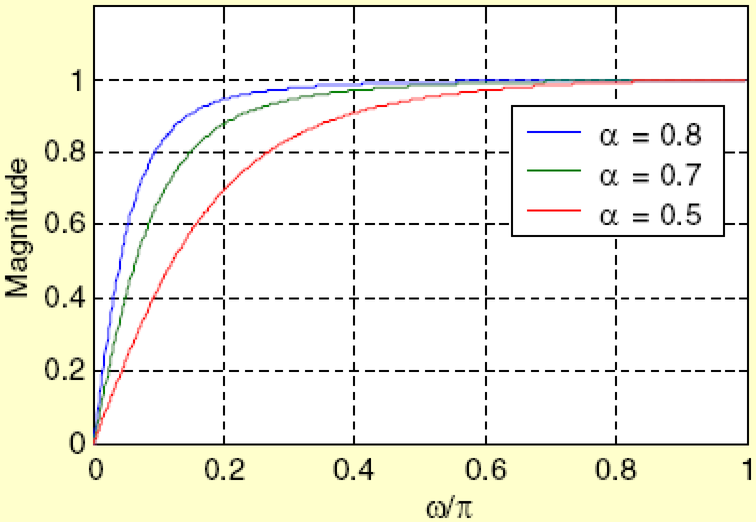
- The above transfer function has a zero at $z = 1$ i.e., at $\omega = 0$ which is in the stopband
- Its 3-dB cutoff frequency is given by

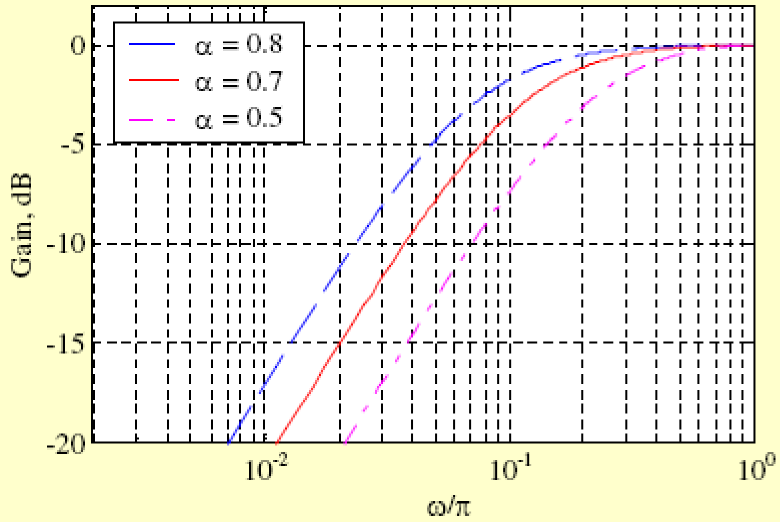
$$\alpha = (1 - \sin\omega_c)/\cos\omega_c$$

which is the same as that of $H_{LP}(z)$

Highpass IIR Digital Filters

- Magnitude and gain responses of $H_{HP}(z)$ are shown below





- $H_{HP}(z)$ is a BR function for $|\alpha| < 1$

Highpass IIR Digital Filters

- Magnitude and gain responses of $H_{HP}(z)$ are shown below
- Example - Design a first-order highpass digital filter with a 3-dB cutoff frequency of 0.8π
- Now, $\sin(\omega_c) = \sin(0.8\pi) = 0.587785$ and $\cos(0.8\pi) = -0.80902$
- Therefore $\alpha = (1 - \sin\omega_c)/\cos\omega_c = -0.5095245$
- Therefore,

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

$$= 0.245238\left(\frac{1-z^{-1}}{1+0.5095245\,z^{-1}}\right)$$

Bandpass IIR Digital Filters

- A 2nd-order bandpass digital transfer function is given by

$$H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- Its squared magnitude function is

$$\left|H_{BP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1+\beta^2(1+\alpha)^2+\alpha^2-2\beta(1+\alpha)^2\cos\omega+2\alpha\cos 2\omega]}$$

- $|H_{BP}(e^{j\omega})|^2$ goes to zero at $\omega = 0$ and $\omega = \pi$
- It assumes a maximum value of 1 at $\omega = \omega_o$ called the **center frequency** of the bandpass filter, where

$$\omega_o = \cos^{-1}(\beta)$$

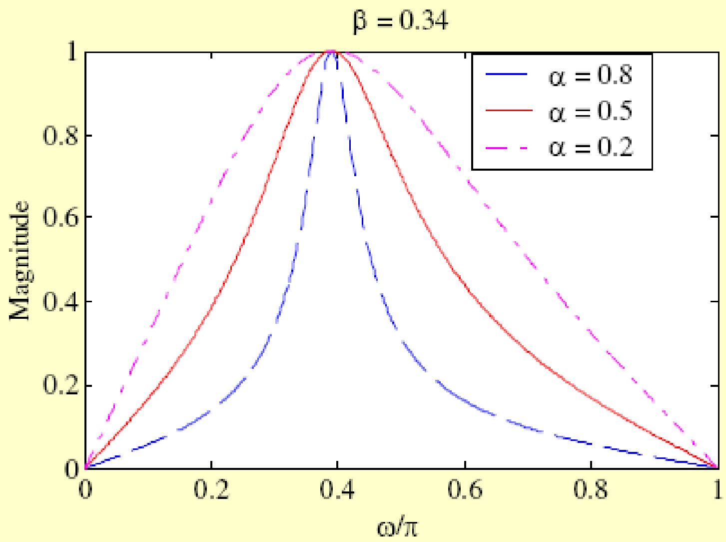
- The frequencies ω_{c1} and ω_{c2} where $|H_{BP}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**

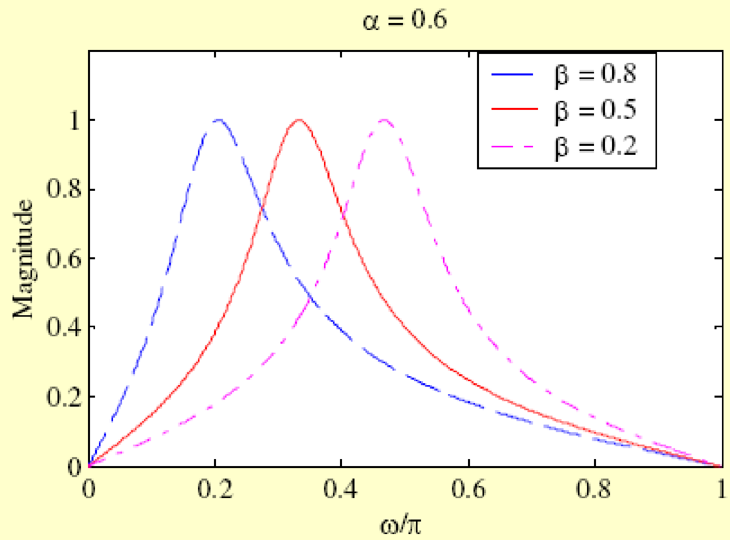
Bandpass IIR Digital Filters

- The difference between the two cutoff frequencies, assuming $\omega_{c1} > \omega_{c2}$ is called the **3-dB bandwidth** and is given by

$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

- The transfer function $H_{BP}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$





Original PowerPoint slides prepared by S. K. Mitra

Bandpass IIR Digital Filters

- Example - Design a 2nd order bandpass digital filter with central frequency at 0.4π and a 3-dB bandwidth of 0.1π
- Here $\beta = \cos(\omega_o) = \cos(0.4\pi) = 0.309017$ and

$$2\alpha/(1+\alpha)^2 = \cos(B_w) = \cos(0.1\pi) = 0.9510565$$

- The solution of the above equation yields: $\alpha = 1.376382$ and $\alpha = 0.72654253$
- The corresponding transfer functions are

$$H'_{BP}(z) = -0.18819 \frac{1 - z^{-2}}{1 - 0.7343424 z^{-1} + 1.37638 z^{-2}}$$

and

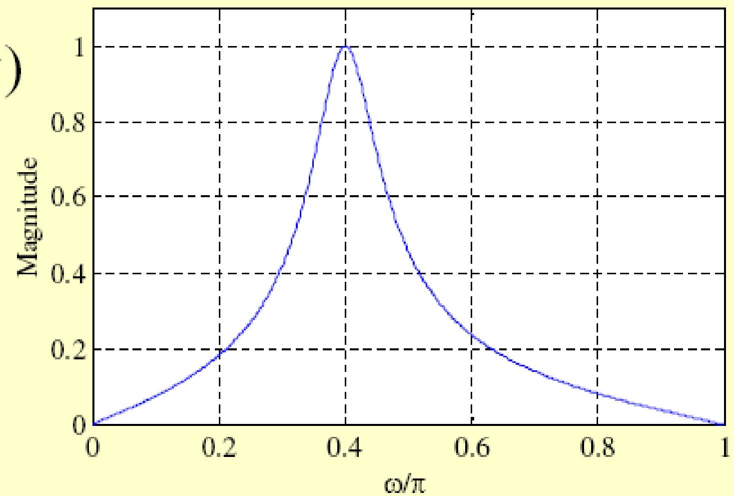
$$H''_{BP}(z) = 0.13673 \frac{1 - z^{-2}}{1 - 0.533531 z^{-1} + 0.72654253 z^{-2}}$$

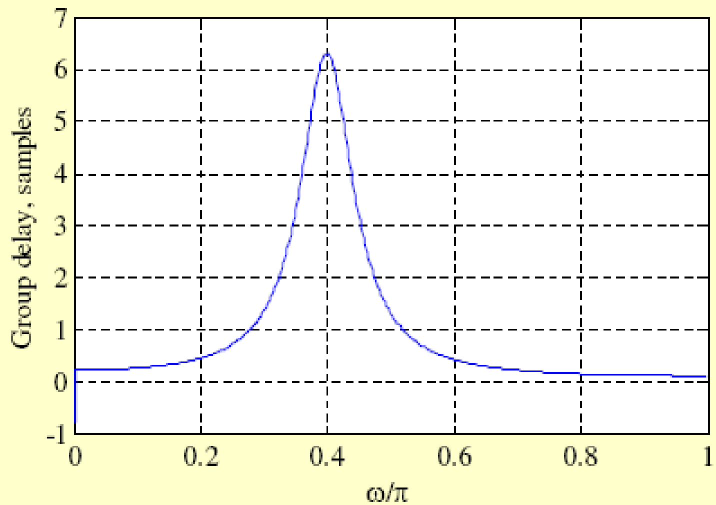
- The poles of $H'_{BP}(z)$ are at $z = 0.3671712 \pm j1.11425636$ and have a magnitude > 1

Bandpass IIR Digital Filters

- Thus, the poles of $H'_{BP}(z)$ are outside the unit circle making the transfer function unstable
- On the other hand, the poles of $H''_{BP}(z)$ are at *z* = 0.2667655 ± *j*0.8095546 and have a
- The solution of the above equation yields: α = 1.376382 and α = 0.72654253 and have a magnitude of 0.8523746
- Hence $H''_{BP}(z)$ is BIBO stable

$H''_{BP}(z)$



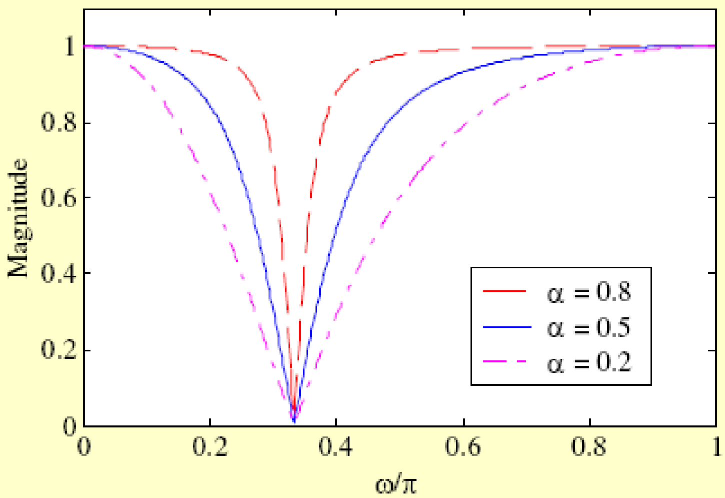


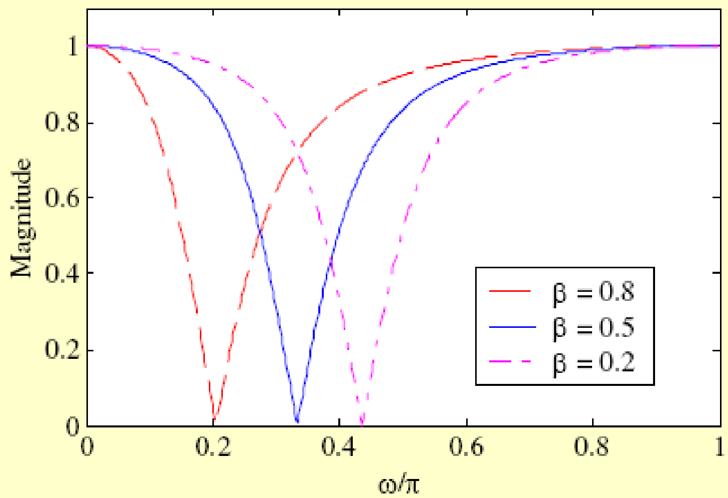
Bandstop IIR Digital Filters

- A 2nd-order bandstop digital filter has a transfer function given by

$$H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- The transfer function $H_{BS}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$





Bandstop IIR Digital Filters

- Here, the magnitude function takes the maximum value of 1 at $\omega = 0$ and $\omega = \pi$
- It goes to 0 at $\omega = \omega_o$, where ω_o, called the **notch frequency**, is given by

$$\omega_o = \cos^{-1}(\beta)$$

- The digital transfer function $H_{BS}(z)$ is more commonly called a **notch filter**
- The frequencies ω_{c1} and ω_{c2} where $|H_{BS}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**
- The difference between the two cutoff frequencies, assuming $\omega_{c1} > \omega_{c2}$ is called the **3-dB notch bandwidth** and is given by

$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

High-Order IIR Digital Filters

- By cascading the simple digital filters discussed so far, we can implement digital filters with sharper magnitude responses
- Consider a cascade of K first-order lowpass sections characterized by the transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

- The corresponding squared-magnitude function is given by

$$|G_{LP}(e^{j\omega})|^2 = \left[\frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}\right]^K$$

High-Order IIR Digital Filters

- To determine the relation between its 3-dB cutoff frequency ω_c and the parameter α, we set

$$\left[\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}\right]^K = \frac{1}{2}$$

which when solved for α, yields for a stable $G_{LP}(z)$

$$\alpha = \frac{1+(1-C)\cos\omega_c - \sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

where $C = 2^{(K-1)/K}$

- It should be noted that the expression for α given earlier reduces to

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

for $K = 1$

High-Order IIR Digital Filters

- Example - Design a lowpass filter with a 3- dB cutoff frequency at $\omega_c = 0.4\pi$ using a single first-order section and a cascade of 4 first-order sections, and compare their gain responses
- For the single first-order lowpass filter we have

$$\alpha = \frac{1+\sin\omega_c}{\cos\omega_c} = \frac{1+\sin(0.4\pi)}{\cos(0.4\pi)} = 0.1584$$

- For the cascade of 4 first-order sections, we substitute $K = 4$ and get

$C = 2^{(K-1)/K} = 2^{(4-1)/4} = 1.6818$

- Next we compute

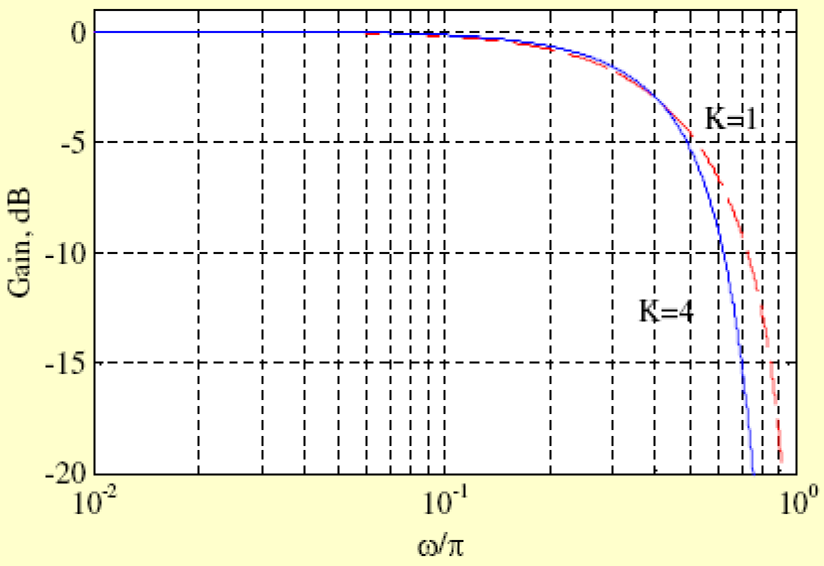
$$\alpha = \frac{1+(1-C)\cos\omega_c - \sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

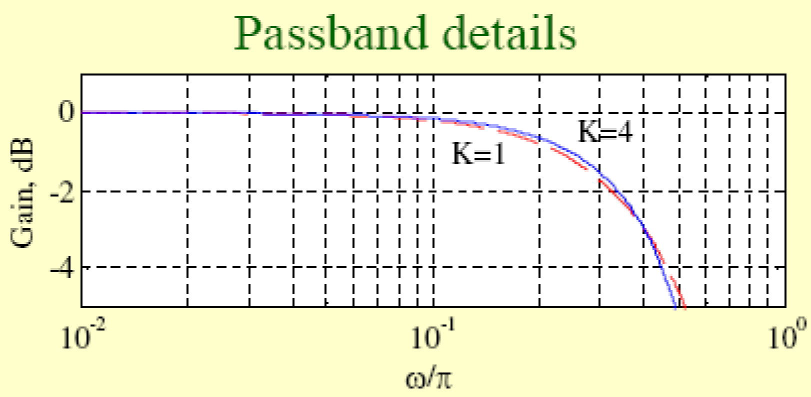
$$= \frac{1+(1-1.6818)\cos(0.4\pi)-\sin(0.4\pi)\sqrt{2(1.6818)-(1.6818)^2}}{1-1.6818+\cos(0.4\pi)}$$

$$= -0.251$$

High-Order IIR Digital Filters

- The gain responses of the two filters are shown below
- As can be seen, cascading has resulted in a sharper roll-off in the gain response





Comb Filters

- The simple filters discussed so far are characterized either by a single passband and/or a single stopband
- There are applications where filters with multiple passbands and stopbands are required
- The **comb filter** is an example of such filters
- In its most general form, a comb filter has a frequency response that is a periodic function of ω with a period $2\pi/L$, where L is a positive integer
- If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $G(z) = H(z^L)$

Comb Filters

- If $|H(e^{j\omega})|$ exhibits a peak at ω_p, then $|G(e^{j\omega})|$ will exhibit L peaks at $\omega_p k/L$, $0 \le k \le L - 1$ in the frequency range $0 \le \omega < 2\pi$
- Likewise, if $|H(e^{j\omega})|$ has a notch at ω_o, then then $|G(e^{j\omega})|$ have L notches at $\omega_o k/L$, $0 \le k \le L - 1$ in the frequency range $0 \le \omega < 2\pi$
- A comb filter can be generated from either an FIR or an IIR prototype filter
- For example, the comb filter generated from the prototype lowpass FIR filter $H_0(z) = 1/2\,(1 + z^{-1})$ has a transfer function

$$G_0(z) = H_0(z^L) = 1/2\,(1 + z^{-L})$$

- $|G_0(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi/L$ and L peaks at $\omega = 2\pi k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$

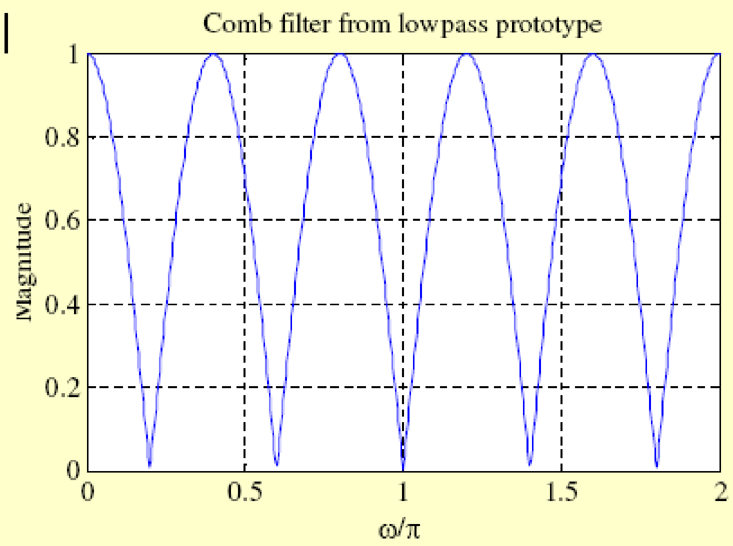
Comb Filters

- On the other hand, the comb filter generated from the prototype lowpass FIR filter $H_1(z) = 1/2\ (1 - z^{-1})$ has a transfer function

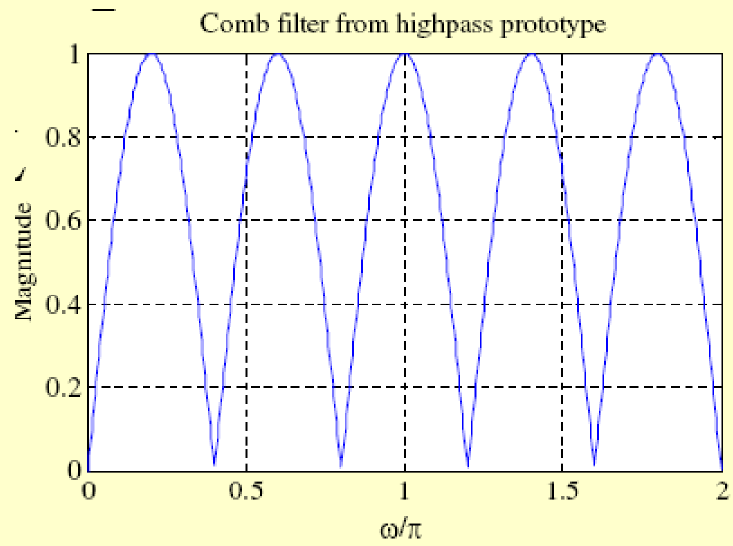
$$G_1(z) = H_1(z^L) = 1/2\ (1 - z^{-L})$$

- $|G_1(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi/L$ and L peaks at $\omega = 2\pi k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$

$|G_0(e^{j\omega})|$



$|G_1(e^{j\omega})|$



Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter
- For example, the M-point moving average filter

$$H(z) = \frac{1-z^{-M}}{M(1-z^{-1})}$$

has been used as a prototype
- This filter has a peak magnitude at $\omega = 0$, and $M - 1$ notches at $\omega = 2\pi l/M$, $1 \le l \le M - 1$
- The corresponding comb filter has a transfer function

$$G(z) = \frac{1-z^{-LM}}{M(1-z^{-L})}$$

whose magnitude has L peaks at $2\pi k/L$, $1 \le k \le L - 1$ and $L(M - 1)$ notches at $2\pi k/LM$, $1 \le k \le L(M - 1)$

Delay Complementary Transfer Functions

- A set of L transfer functions, $\{H_i(z)\}$, $0 \le i \le L-1$, is defined to be **delay-complementary** of each other if the sum of their transfer functions is equal to some integer multiple of unit delays, i.e.,

$$\sum_{i=0}^{L-1} H_i(z) = \beta z^{-n_o}, \qquad \beta \neq 0$$

  where n_o is a nonnegative integer
- A delay-complementary pair $\{H_0(z), H_1(z)\}$ can be readily designed if one of the pairs is a known Type 1 FIR transfer function of odd length
- Let $H_0(z)$ be a Type 1 FIR transfer function of length $M = 2K+1$, its delay-complementary transfer function is given by $H_1(z) = z^{-K} - H_0(z)$

Delay Complementary Transfer Functions

- Let the magnitude response of $H_0(z)$ be equal to $1 \pm \delta_p$ in the passband and less than or equal to δ_s in the stopband where δ_p and δ_s are very small numbers
- Now the frequency response of $H_0(z)$ can be expressed as

$$H_0(e^{j\omega}) = e^{-jK\omega}\breve{H}_0(\omega)$$

where $\breve{H}_0(\omega)$ is the **amplitude response**

- Its delay-complementary transfer function $H_1(z)$ has a frequency response given by

$$H_1(e^{j\omega}) = e^{-jK\omega}\breve{H}_1(\omega) = e^{-jK\omega}[1 - \breve{H}_0(\omega)]$$

- Now, in the passband, $1 - \delta_p \leq \breve{H}_0(\omega) \leq 1 + \delta_p$, and in the stopband, $-\delta_s \leq \breve{H}_0(\omega) \leq \delta_s$

Delay Complementary Transfer Functions

- It follows from the above equation that $-\delta_p \le \breve{H}_1(\omega) \le \delta_p$, and in the stopband, $1-\delta_s \le \breve{H}_1(\omega) \le 1+\delta_s$
- As a result, $H_1(z)$ has a complementary magnitude response characteristic to that of $H_0(z)$ with a stopband exactly identical to the passband of $H_0(z)$, and a passband that is exactly identical to the stopband of $H_0(z)$
- Thus, if $H_0(z)$ is a lowpass filter, $H_1(z)$ will be a highpass filter, and vice versa
- The frequency ω_o at which $\breve{H}_0(\omega_o) = \breve{H}_1(\omega_o) = 0.5$ the gain responses of both filters are 6 dB below their maximum values
- The frequency is thus called the **6-dB crossover frequency**

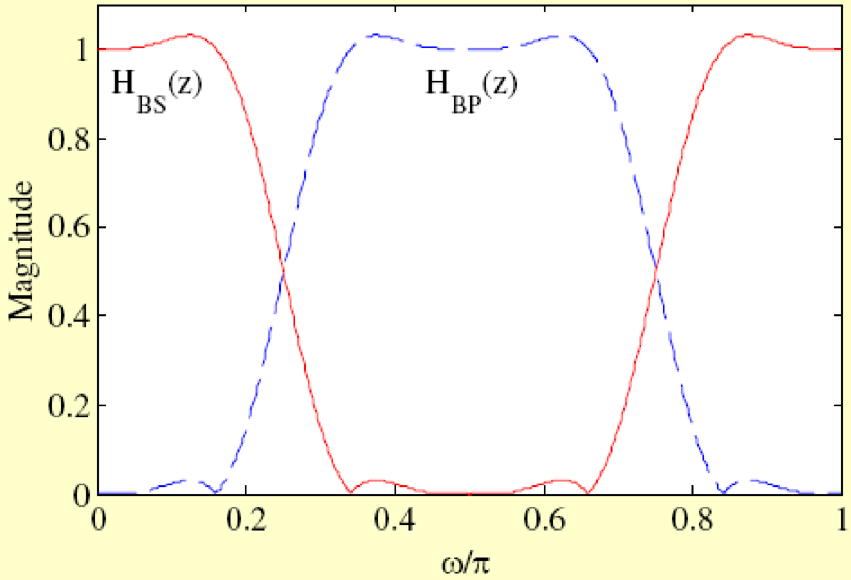
Delay Complementary Transfer Functions

- Example - Consider the Type 1 bandstop transfer function

$$H_{BS}(z)=\tfrac{1}{64}(1+z^{-2})^4(1-4z^{-2}+5z^{-4}+5z^{-8}-4z^{-10}+z^{-12})$$

- Its delay-complementary Type 1 bandpass transfer function is given by $H_{BP}(z)=z^{-10}-H_{BS}(z)$

$$=\tfrac{1}{64}(1-z^{-2})^4(1+4z^{-2}+5z^{-4}+5z^{-8}+4z^{-10}+z^{-12})$$



Allpass Complementary Filters

- A set of M transfer functions, $\{H_i(z)\}$, $0 \le i \le M - 1$, is defined to be **allpass-complementary** of each other, if the sum of their transfer functions is equal to an allpass function, i.e.,

$$\sum_{i=0}^{M-1} H_i(z) = A(z)$$

- Example - Consider the two transfer functions $H_0(z)$ and $H_1(z)$ given by

$$H_0(z) = \tfrac{1}{2}\,[A_0(z) + A_1(z)]$$
$$H_1(z) = \tfrac{1}{2}\,[A_0(z) - A_1(z)]$$

where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions

- Note that $H_0(z) + H_1(z) = A_0(z)$
- Hence, $H_0(z)$ and $H_1(z)$ are allpass complementary

Power Complementary Filters

- A set of M transfer functions, $\{H_i(z)\}$, $0 \leq i \leq M-1$, is defined to be **power-complementary** of each other, if the sum of their square-magnitude responses is equal to a constant K for all values of ω, i.e.,
$$\sum_{i=0}^{M-1} \left|H_i(e^{j\omega})\right|^2 = K, \qquad \text{for all } \omega$$
- By analytic continuation, the above property is equal to
$$\sum_{i=0}^{M-1} H_i(z)H_i(z^{-1}) = K, \qquad \text{for all } \omega$$
for real coefficient $H_0(z)$
- Usually, by scaling the transfer functions, the power-complementary property is defined for $K = 1$

Power Complementary Filters

- For a pair of power-complementary transfer functions, $H_0(z)$ and $H_1(z)$, the frequency ω_o where $|H_0(e^{j\omega o})|^2 = |H_1(e^{j\omega o})|^2 = 0.5$, is called the **cross-over frequency**
- At this frequency the gain responses of both filters are 3-dB below their maximum values
- As a result, is called the **3-dB cross-over frequency**
- Example - Consider the two transfer functions $H_0(z)$ and $H_1(z)$ given by

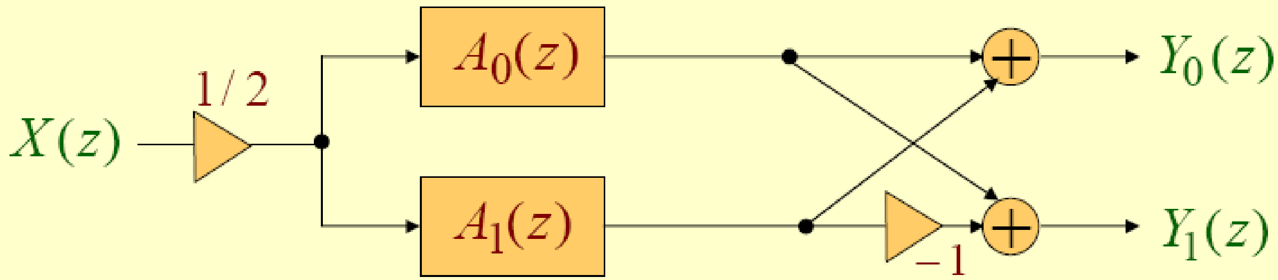
$$H_0(z) = ½ [A_0(z) + A_1(z)]$$

$$H_1(z) = ½ [A_0(z) - A_1(z)]$$

  where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions
- $H_0(z)$ and $H_1(z)$ are allpass and power complementary

Doubly Complementary Filters

- A set of M transfer functions satisfying both the allpass complementary and the power complementary properties is known as a **doubly-complementary set**
- A pair of doubly-complementary IIR transfer functions, $H_0(z)$ and $H_1(z)$, with a sum of allpass decomposition can be simply realized as indicated below



$$H_0(z) = \frac{Y_0(z)}{X(z)} \qquad H_1(z) = \frac{Y_1(z)}{X(z)}$$

Doubly Complementary Filters

- Example - The first-order lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

can be expressed as

$$H_{LP}(z) = \frac{1}{2}\left(1+\frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1}{2}[A_0(z)+A_1(z)]$$

where

$$A_0(z) = 1, \qquad A_1(z) = \frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}$$

- Its power-complementary highpass transfer function is thus given by

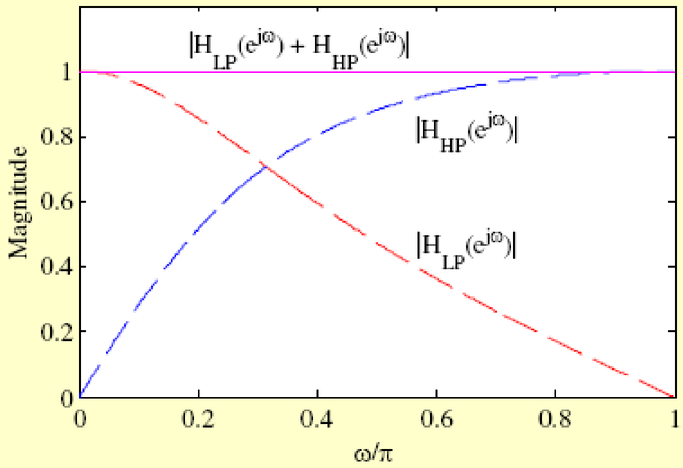
$$H_{HP}(z) = \frac{1}{2}[A_0(z)-A_1(z)] = \frac{1}{2}\left(1-\frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}\right)$$

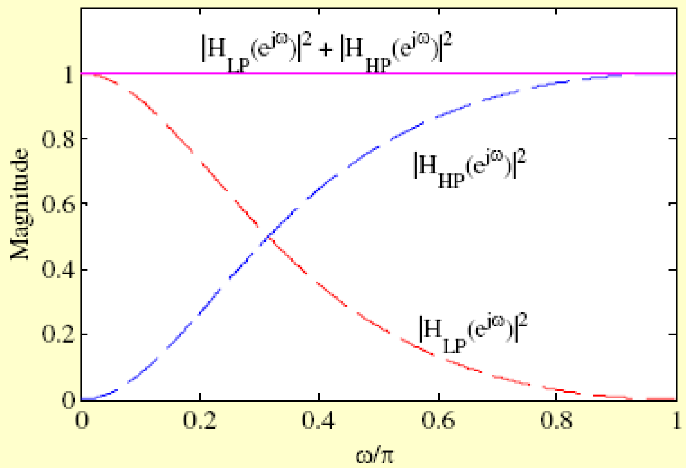
$$= \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

Doubly Complementary Filters

- The above expression is precisely the firstorder highpass transfer function described earlier
- Figure below demonstrates the allpass complementary property and the power complementary property of

  can be expressed as $H_{LP}(z)$ and $H_{HP}(z)$





Power Symmetric Filters

- A real-coefficient causal digital filter with a transfer function $H(z)$ is said to be a **power-symmetric filter** if it satisfies the condition

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K$$

where $K > 0$ is a constant

- It can be shown that the gain function $G(\omega)$ of a power-symmetric transfer function at $\omega = \pi$ is given by
- $10\log_{10}K - 3$ dB
- If we define $G(z) = H(-z)$, then it follows from the definition of the power-symmetric filter that $H(z)$ and $G(z)$ are power-complementary as

$$H(z)H(z^{-1}) + G(z)G(z^{-1}) = \text{a constant}$$

Conjugate Quadratic Filters

- If a power-symmetric filter has an FIR transfer function $H(z)$ of order N, then the FIR digital filter with a transfer function
$$G(z) = z^{-N}H(-z^{-1})$$
is called a **conjugate quadratic filter** of $H(z)$ and vice-versa
- It follows from the definition that $G(z)$ is also a power-symmetric causal filter quadratic filters
- It also can be seen that a pair of conjugate quadratic filters $H(z)$ and $G(z)$ are also power-complementary
- Example - Let $H(z) = 1 - 2z^{-1} + 6z^{-2} + 3z^{-3}$
$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 100$$
- $H(z)$ is a power-symmetric transfer function