Chapter 2

Discrete-Time Signals & Systems

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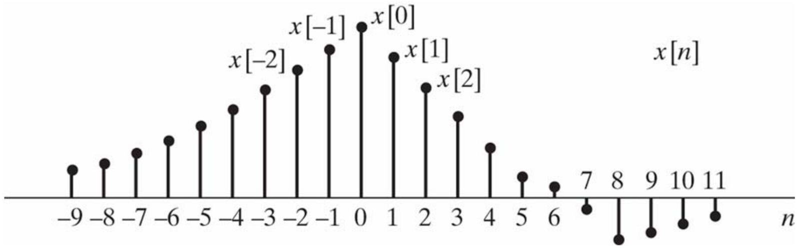
Discrete-Time Signals

- Signals are represented as sequences of numbers, called **samples**
- Sample value of a typical signal or sequence denoted as $x = \{x[n]\}$ with $-\infty \le n \le \infty$
- $x[n]$ is defined only for integer values of n and undefined for non-integer values of n
- Representation of discrete-time signals:
  - **Functional representation** $x[n] = \begin{cases} n-2, & n \ge 0 \\ -3, & n < 0 \end{cases}$
  - **Tabular representation**
  - **Sequence representation**

$x(n) = \{\ldots, -0.2, \underline{2.2}, 1.1, 0.2, -3.7, 2.9. \ldots\}$

Discrete-Time Signals

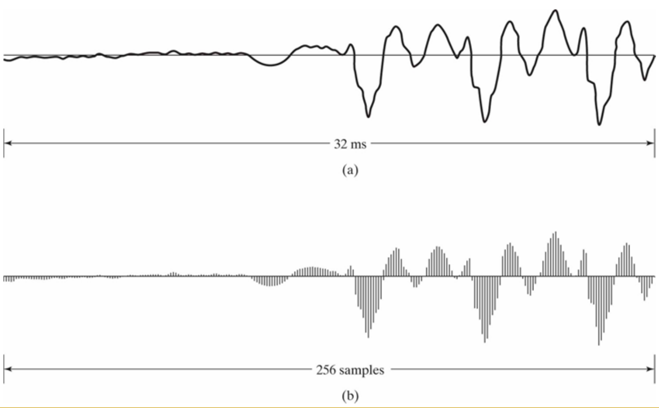
- **Graphical representation**



Discrete-Time Signals

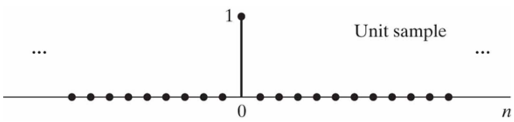
- Sampling a speech signal

$$x[n] = x_a(nT), \quad -\infty < n < \infty$$



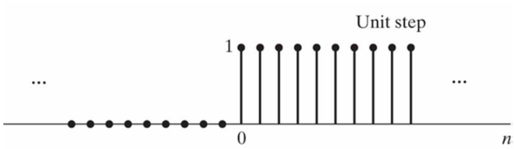
Basic Sequences

- **Unit sample sequence** - $\delta[n]=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$



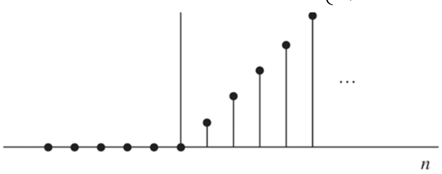
- **Unit step sequence** - $u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$

$$u[n]=\sum_{k=-\infty}^{n}\delta[k]$$
$$=\sum_{k=0}^{\infty}\delta[n-k]$$
$$\delta[n]=u[n]-u[n-1]$$



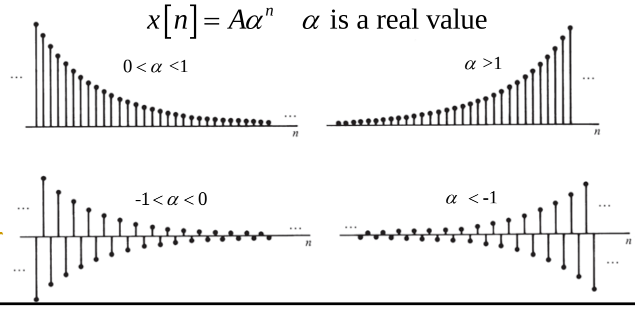
Basic Sequences

- **Unit ramp signal** - $u_r[n]=\begin{cases}n, & n\geq 0\\ 0, & n<0\end{cases}$



- **Real exponential signal** -

$x[n]=A\alpha^{n}$ α is a real value



Basic Sequences

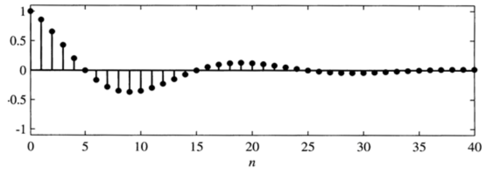
- **Complex exponential signal -**

$$x(n) = A\alpha^n = |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \qquad \alpha \equiv |\alpha| e^{j\omega_0}$$

$$= |A||\alpha|^n \left(\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)\right)$$

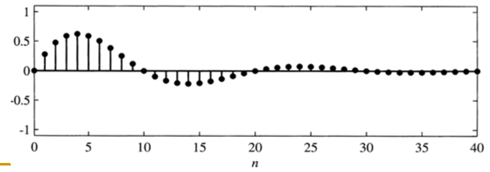
$$x[n] = x_R[n] + jx_I[n]$$

$$x_R[n] = |A||\alpha|^n \cos(\omega_0 n + \phi)$$



(a)

$$x_I[n] = |A||\alpha|^n \sin(\omega_0 n + \phi)$$



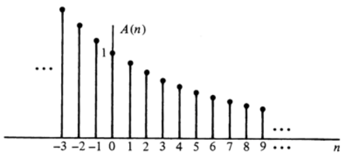
(b)

Basic Sequences

- **Complex exponential signal -**

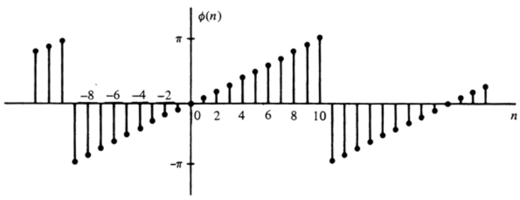
$$x[n] = x_R[n] + jx_I[n] = |x[n]| \angle x[n]$$

$$|x[n]| = A[n] \equiv r^n$$



(a) Graph of $A(n) = r^n$, $r = 0.9$

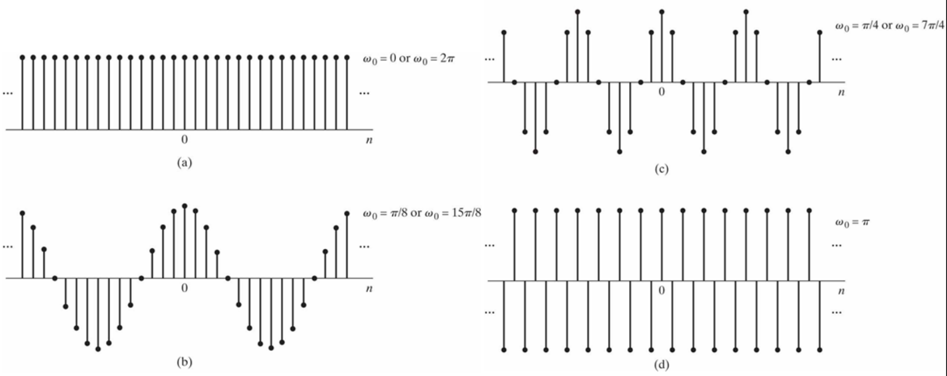
$$\angle x[n] = \phi[n] \equiv \theta n$$



(b) Graph of $\phi(n) = \frac{\pi}{10}n$, modulo 2π plotted in the range $(-\pi, \pi)$

Basic Sequences

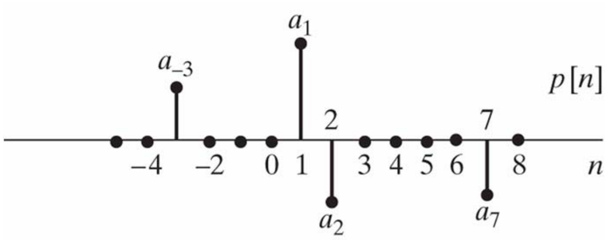
- **Sinusoidal signals with different frequencies**



Basic Sequences

- An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its delayed (advanced) versions

$$p[n] = \sum_{k=-\infty}^{\infty} p[k]\delta[n-k]$$



The Norm of a Discrete-Time Signal

- **Size of a Signal** - given by the norm of the signal

  L_p-norm :
  $$\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{\frac{1}{p}}$$

  where p is a positive integer

- The value of p is typically 1 or 2 or ∞

  L_2-norm $\|x\|_2$ is the root-mean-squared (rms) value of $\{x[n]\}$

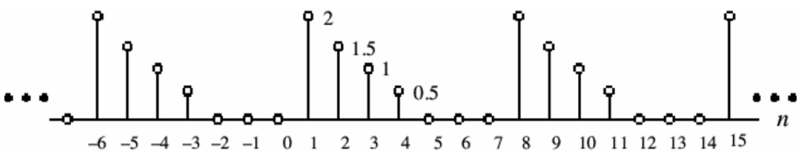
  L_1-norm $\|x\|_1$ is the mean absolute value of $\{x[n]\}$

  L_∞-norm $\|x\|_\infty$ is the peak absolute value of $\{x[n]\}$ (why?)

  $$\|x\|_\infty = |x|_{\max}$$

Classification of Discrete-Time Signals

- Periodic signals and aperiodic signals
  - A signal is periodic with period N ($N > 0$) if and only if
    $$x[n+N] = x[n] \quad \text{for all } n$$
  - The smallest value of N for which the above condition holds is called the (**fundamental**) **period**



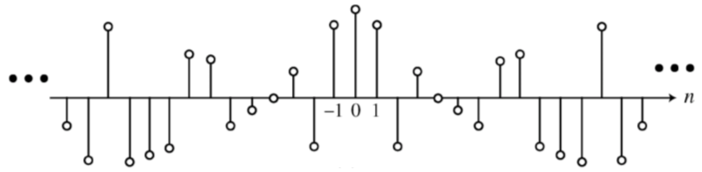
  - A signal not satisfying the periodicity condition is called **nonperiodic** or **aperiodic**

Classification of Discrete-Time Signals

- **Conjugate-symmetric** sequence:

$$x[n] = x^*[-n]$$

  - If $x[n]$ is real, then it is an **even sequence**
  - for a conjugate-symmetric sequence $\{x[n]\}$, $x[0]$ must be a real number

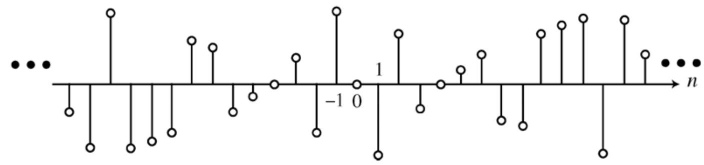


Classification of Discrete-Time Signals

- **Conjugate-antisymmetric** sequence:

$$x[n] = -x^*[-n]$$

  - If $x[n]$ is real, then it is an **odd sequence**
  - for a conjugate anti-symmetric sequence $\{y[n]\}$, $y[0]$ must be an imaginary number



Classification of Discrete-Time Signals

- Any complex sequence can be expressed as a sum of its conjugate-symmetric and conjugate-antisymmetric parts:

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$\begin{cases} x_{cs}[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right) \\ x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right) \end{cases}$$

- Any real sequence can be expressed as a sum of its even part and its odd part:

$$x[n] = x_{ev}[n] + x_{od}[n]$$

where

$$\begin{cases} x_{ev}[n] = \frac{1}{2}\left(x[n] + x[-n]\right) \\ x_{od}[n] = \frac{1}{2}\left(x[n] - x[-n]\right) \end{cases}$$

Classification of Discrete-Time Signals

- Periodic signals and aperiodic signals
  - A signal is periodic with period N ($N > 0$) if and only if

$$x[n+N] = x[n] \quad \text{for all } n$$

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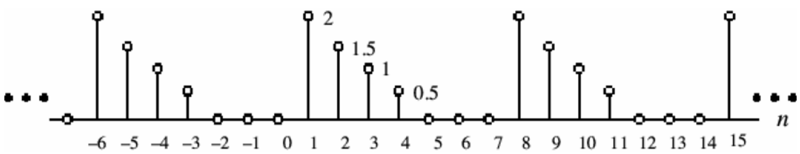


  - A signal not satisfying the periodicity condition is called **nonperiodic** or **aperiodic**

Classification of Discrete-Time Signals

- Energy signals and power signals
  - The total **energy** of a signal $x(n)$ is defined by

$$E \equiv \sum_{n=-\infty}^{\infty} |x[n]|^2$$

  - An infinite length sequence with finite sample values may or may not be an **energy signal** (with finite energy)
  - The **average power** of a discrete-time signal $x[n]$ is defined by

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

  - Define the signal **energy** of $x(n)$ over the finite interval $-N \le n \le N$ as

$$E_N = \sum_{n=-N}^{N} |x[n]|^2$$

Classification of Discrete-Time Signals

- **Energy signals and power signals**
  - The signal energy can then be expressed as

$$E \equiv \lim_{N\to\infty} E_N$$

  - The average power of $x(n)$ becomes

$$P = \lim_{N\to\infty} \frac{1}{2N+1} E_N$$

  - If E is finite, $P = 0$. On the other hand, if E is infinite, the average power P may be either finite or infinite
  - If P is finite (and nonzero), the signal is called a **power signal**

Classification of Discrete-Time Signals

- **Energy signals and power signals**
  - **Example** – Determine the power and energy of the unit step sequence

    The average power of the unit step signal is

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=0}^{N} 1 = \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}$$

    It's a power signal with infinite energy
  - **Example** - Consider the causal sequence defined by

$$x[n] = \begin{cases} 3(-1)^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

    Note: $x(n)$ has infinite energy, its average power is

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\left(9\sum_{n=0}^{N} 1\right) = 4.5$$

Classification of Discrete-Time Signals

- An infinite energy signal with finite average power is called a **power signal**
  - Example - A periodic sequence which has a finite average power but infinite energy
- A finite energy signal with zero average power is called an **energy signal**
  - Example - A finite-length sequence which has finite energy but zero average power

Classification of Discrete-Time Signals

- A sequence $x[n]$ is said to be **bounded** if

$$|x[n]| \leq B_x < \infty$$

  - Example - The sequence $x[n] = \cos 0.3\pi n$ is a bounded sequence as

$$|x[n]| = |\cos(0.3\pi n)| \leq 1$$

- A sequence $x[n]$ is said to be **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  - Example - The following sequence is absolutely summable

$$y[n] = \begin{cases} 0.3^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Classification of Discrete-Time Signals

- A sequence $x[n]$ is said to be **square summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

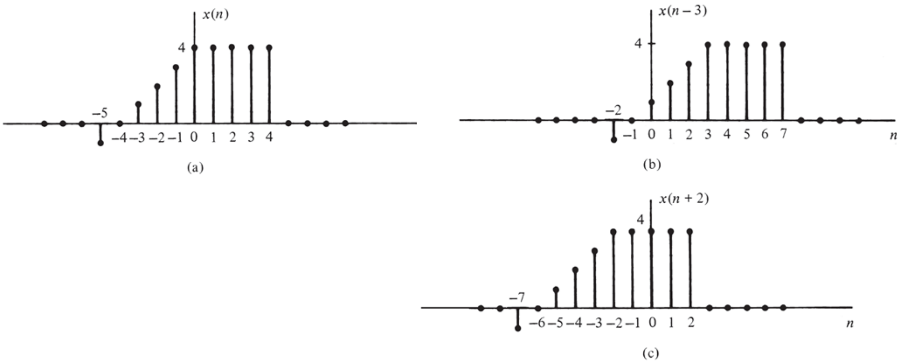
  - Example - The sequence

$$h[n] = \frac{\sin(0.4\pi)}{\pi n}$$

is square-summable but not absolutely summable

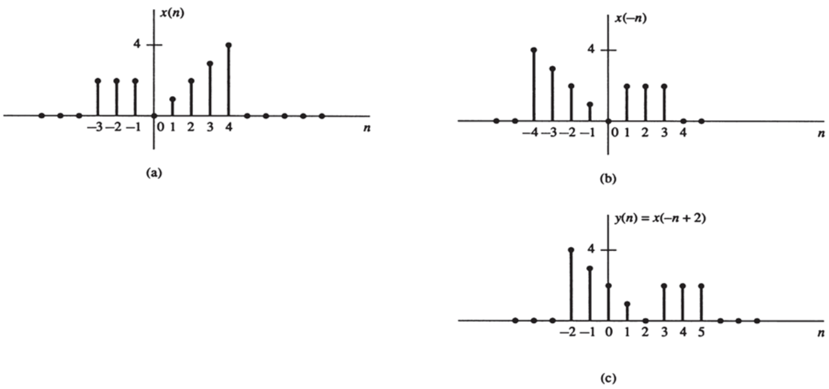
Manipulation of Discrete-Time Signals (1/5)

- Transformation of independent variable (time)
  - **Time shifting**: A signal x[*n*] may be shifted in time by replacing the independent variable *n* by *n* – *k*



Manipulation of Discrete-Time Signals (2/5)

- Transformation of independent variable (time)
  - **Folding/Reflection**: A signal x[*n*] may be folded in time by replacing the independent variable *n* by –*n*



Manipulation of Discrete-Time Signals (3/5)

- The operations of folding and time delaying (or advancing) a signal are **NOT** commutative
- Denote the time-delay operation by TD and the folding operation by FD

$$TD_k\{x[n]\} = x[n-k], \quad k > 0$$

$$FD\{x[n]\} = x[-n]$$

Now

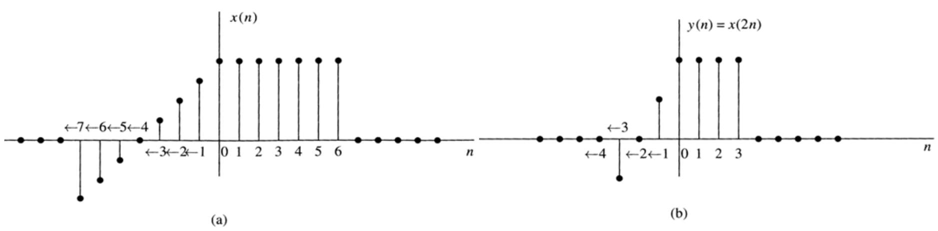
$$TD_k\{FD\{x[n]\}\} = TD_k\{x[-n]\} = x[-n+k]$$

whereas

$$FD\{TD_k\{x[n]\}\} = FD\{x[n-k]\} = x[-n-k] \neq TD_k\{FD\{x[n]\}\}$$

Manipulation of Discrete-Time Signals (4/5)

- Transformation of independent variable (time)
  - **Time Scaling** or **down-sampling**: A signal $x[n]$ may be scaled in time by replacing n by μn



Manipulation of Discrete-Time Signals (5/5)

- Transformation of independent variable (time)
  - **Addition**, **multiplication**, and **scaling** of sequences: Amplitude modifications include addition, multiplication, and scaling of discrete-time
  - **Amplitude scaling** of a signal by a constant :

$$y[n] = Ax[n], \qquad -\infty < n < \infty$$

  - **Sum** of two signals:

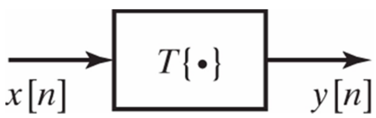
$$y[n] = x_1[n] + x_2[n], \qquad -\infty < n < \infty$$

  - **Product** of two signals:

$$y[n] = x_1[n]x_2[n], \qquad -\infty < n < \infty$$

Discrete-Time Systems

- **Discrete-time system**: A device or an algorithm that performs some prescribed operation on a discrete-time signal (**input** or **excitation**) to produce another discrete-time signal (**output** or **response**)



- We say that the input signal $x[n]$ is "transformed" by the system into a signal $y[n]$ as expressed below

$$y[n] = T\{x[n]\}$$

Input-Output Description of Systems

- The **input-output description** or a discrete-time system consists of a mathematical expression or a rule, which explicitly defines the relation between the input and output signals

$$x[n] \xrightarrow{T} y[n]$$

- **Example**: Determine the response of the following systems to the input signal

$$x[n] = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) $y[n] = x[n-1]$ (b) $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$

(c) $y[n] = \text{median}\{x[n-1], x[n], x[n+1]\}$ (d) $y[n] = \sum_{k=-\infty}^{n} x[k]$

Linear Systems: Accumulator

- **Accumulator -** $y[n] = \sum_{l=-\infty}^{n} x[l]$

$$= \sum_{l=-\infty}^{n-1} x[l] + x[n] = y[n-1] + x[n]$$

- The output $y[n]$ is the sum of the input sample $x[n]$ and the previous output $y[n-1]$
- The system cumulatively adds, i.e., it accumulates all input sample values
- Input-output relation can also be written in the form

$$y[n] = \sum_{l=-\infty}^{-1} x[l] + \sum_{l=0}^{n} x[l] = y[-1] + \sum_{l=0}^{n} x[l], \quad n \ge 0$$

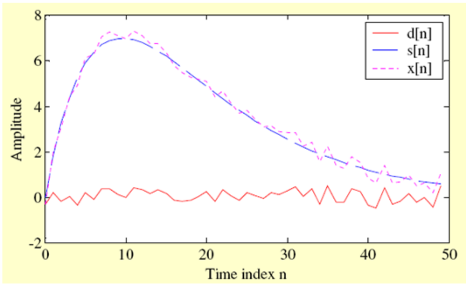
- The second form is used for a causal input sequence, in which case $y[-1]$ is called the **initial condition**

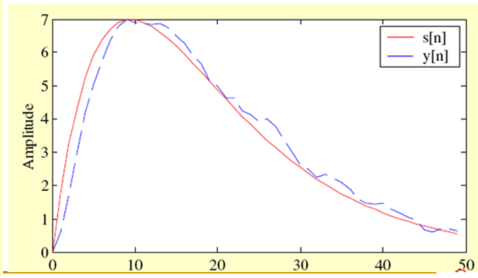
Linear Systems: Moving Average

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

$$= \frac{1}{M_1 + M_2 + 1} \{x[n+M_1] + x[n+M_1-1] + \ldots + x[n] + x[n-1] + \ldots + x[n-M_2]\}$$

- An application: Consider $x[n] = s[n] + d[n]$ where $s[n] = 2[n(0.9)^n]$ is the signal corrupted by a random noise $d[n]$





Nonlinear Systems: Median Filter (1/3)

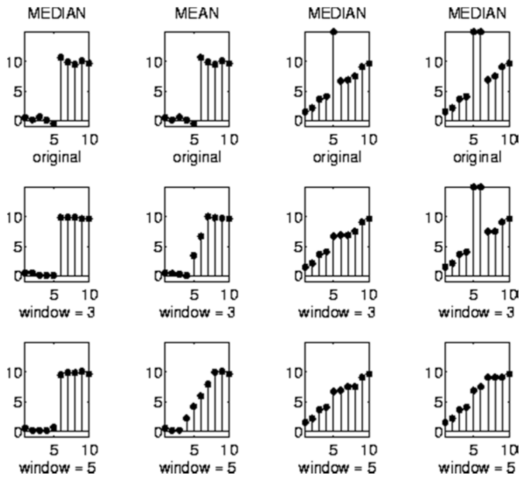
- The median of a set of $(2K+1)$ numbers *is* the number such that K numbers from the set have values greater than this number and the other K numbers have values smaller
- Median can be determined by rank-ordering the numbers in the set by their values and choosing the number at the middle
- Example: Consider the set of numbers

  {2, −3, 10, 5, −1}
- Rank-order set is given by

  {−3 , −1, 2 , 5, 10}
- median{2, −3, 10, 5, −1} = 2

Nonlinear Systems: : Median Filter (2/3)

- **Median Filtering Example**



Nonlinear Systems: Median Filter (3/3)

- **Median Filtering Example**

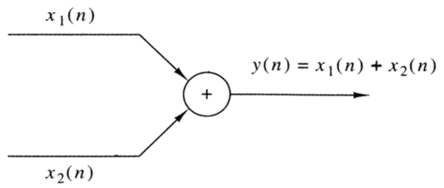
Original Image

Noisy Image
(pepper-and-salt noise)

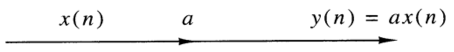
Filtered Image

Block Diagram Representation of Discrete-Time Systems

- Adder

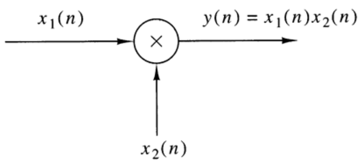


- Constant multiplier

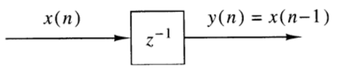


Block Diagram Representation of Discrete-Time Systems

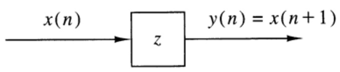
- Signal multiplier/Modulator



- Unit delay element

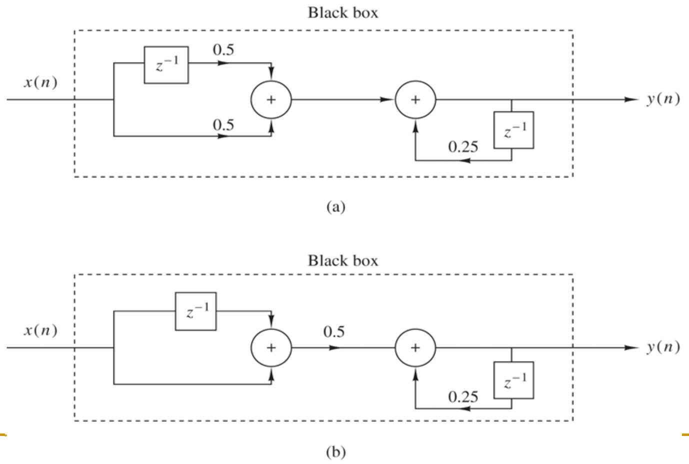


- Unit advance element



Block Diagram Representation of Discrete-Time Systems

- **Example**: $y[n]=\frac{1}{4}y[n-1]+\frac{1}{2}x[n]+\frac{1}{2}x[n-1]$



Static vs. Dynamic Systems

- A discrete-time system is called **static** or **memoryless** if its output at any time instant depends at most on the input sample at the same time $y[n]=\mathcal{T}\{x[n],n\}$

$$y[n]=ax[n]$$
$$y[n]=nx[n]+bx^3[n]$$

- If a discrete-time system is not static, it is said to be **dynamic** or to **have memory**

$$y[n]=x[n]+3x[n-1] \quad \text{(finite memory)}$$
$$y[n]=\sum_{k=0}^{\infty}x[n-k] \quad \text{(infinite memory)}$$

Time (Shift) Invariance

- Time-invariant vs. time-variant systems
  - A system is called **time-invariant** if its input-output characteristics do not change with time $y[n] = \mathcal{T}\{x[n]\}$
  - **Definition**: A relaxed system $\mathcal{T}$ is **time-invariant** or shift-invariant if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

Implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k)$$

For every input signal *x*(*n*) and every time shift *k*.

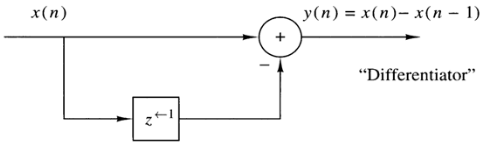
  - In general, we can write the output of a time-invariant system as

$$y(n,k) = \mathcal{T}[x(n-k)]$$

Time (Shift) Invariance

- Examples

$$y[n] = \mathcal{T}\{x[n]\} = x[n] - x[n-1]$$



"Differentiator"

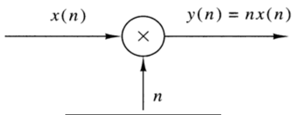
$$y[n,k] = x[n-k] - x[n-k-1]$$

$$y[n-k] = x[n-k] - x[n-k-1]$$

$$y[n,k] = y[n-k]$$

time invariant

$$y[n] = \mathcal{T}\{x[n]\} = nx[n]$$



"Time" multiplier

$$y[n,k] = nx[n-k]$$

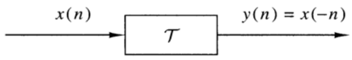
$$y[n-k] = [n-k]x[n-k]$$

$$y[n,k] \neq y[n-k]$$

time variant

Time (Shift) Invariance

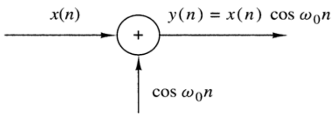
- Examples



"Folder"

time variant

$$y[n] = \mathcal{T}\{x[n]\} = x[-n]$$
$$y[n,k] = x[-n-k]$$
$$y[n-k] = x[-n+k]$$
$$y[n,k] \neq y[n-k]$$



Modulator

time variant

$$y[n] = \mathcal{T}\{x[n]\} = x[n]\cos\omega_0 n$$
$$y[n,k] = x[n-k]\cos\omega_0 n$$
$$y[n-k] = x[n-k]\cos\omega_0[n-k]$$
$$y[n,k] \neq y[n-k]$$

Linearity (1/3)

- A **linear system** is one that satisfies the **superposition principle**
- **Definition**: A system $\mathcal{T}$ is **linear** if and only if

$$\mathcal{T}\{a_1x_1[n] + a_2x_2[n]\} = a_1\mathcal{T}\{x_1[n]\} + a_2\mathcal{T}\{x_2[n]\}$$

for any arbitrary input sequences $x_1[n]$ and $x_2[n]$, and any arbitrary constants a_1 and a_2.

- **Multiplicative/scaling property**: Suppose that $a_2 = 0$

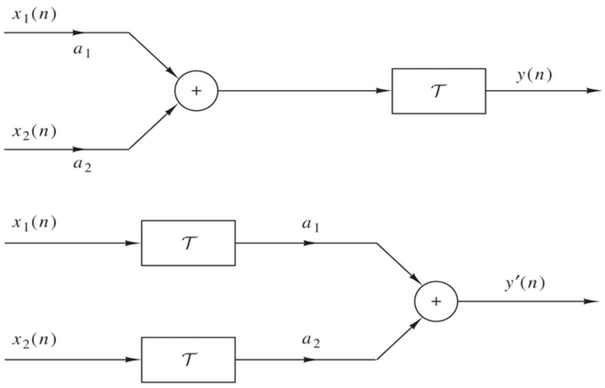
$$\mathcal{T}\{a_1x_1[n]\} = a_1\mathcal{T}\{x_1[n]\} = a_1y_1[n]$$

- **Additivity property**: Suppose that $a_1 = a_2 = 1$

$$\mathcal{T}\{x_1[n] + x_2[n]\} = \mathcal{T}\{x_1[n]\} + \mathcal{T}\{x_2[n]\} = y_1[n] + y_2[n]$$

Linearity (2/3)

- Graphical representation of the superposition principle



$\mathcal{T}$ is **linear** if and only if $y[n] = y'[n]$

Linearity (3/3)

- Linear vs. non-linear systems
  - The linear condition can be extended arbitrarily to any weighted linear combination of signals

$$x[n] = \sum_{k=1}^{M-1} a_k x_k[n] \xrightarrow{\mathcal{T}} y[n] = \sum_{k=1}^{M-1} a_k y_k[n]$$

where

$$y_k[n] = \mathcal{T}\{x_k[n]\}, \quad k = 1, 2, \ldots, M-1$$

  - If a system produces a nonzero output with a zero input, it may be either non-relaxed or nonlinear
  - **Examples**: (a) $y[n] = nx[n]$, (b) $y[n] = x[n^2]$, (c) $y[n] = x^2[n]$, (d) $y[n] = Ax[n] + B$, (e) $y[n] = e^{x[n]}$

Causality

- Causal vs. non-causal systems
  - **Definition**: A system is said to be **causal** if the output of the system at any time n depends only on present and past inputs, but does not depend on future inputs

$$y[n] = \mathcal{F}\{x[n], x[n-1], x[n-2], \ldots\}$$

    where $\mathcal{F}\{\cdot\}$ is some arbitrary function.
  - **Noncausal** vs. **anticausal**
  - If a system produces a nonzero output with a zero input, it may be either non-relaxed or nonlinear
  - **Examples**: (a) $y[n] = x[n] - x[n-1]$, (b) $y[n] = x[n] + 3x[n+4]$, (c) $y[n] = x[n^2]$, (d) $y[n] = x[2n]$, (e) $y[n] = x[-n]$

Stability

- **Bounded-Input, Bounded Output (BIBO)** stability

  If $y[n]$ is the response to an input $x[n]$ and if

$$|x[n]| \le B_x \quad \text{for all values of } n$$

  then

$$|y[n]| \le B_y \quad \text{for all values of } n$$

- **Example** – the M-point moving average filter is BIBO stable

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

- With a bounded input $|x[n]| \le B_x$

$$|y[n]| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right| \le \frac{1}{M}\sum_{k=0}^{M-1} |x[n-k]|$$

$$\le \frac{1}{M}(MB_x) = B_x$$

Passive & Lossless Systems

- A discrete-time system is defined to be **passive** if, for every finite-energy input *x*[*n*], the output *y*[*n*] has, at most, the same energy

$$\sum_{n=-\infty}^{\infty}|y[n]|^2 \le \sum_{n=-\infty}^{\infty}|x[n]|^2$$

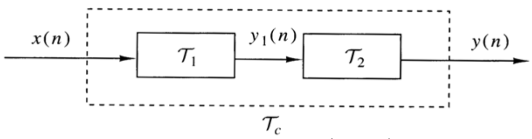
- For a **lossless system**, the above inequality is satisfied with an equal sign for every input
- Example - Consider the discrete-time system defined by $y[n] = \alpha\, x[n-N]$ with N a positive integer
- Its output energy is given by

$$\sum_{n=-\infty}^{\infty}|y[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty}|x[n]|^2$$

passive system if $|\alpha| < 1$, and lossless if $|\alpha| = 1$

Interconnection of Discrete-Time Systems

- **Cascade interconnection**



$$y_1[n] = \mathcal{T}_1\{x[n]\}$$

$$y[n] = \mathcal{T}_2\{y_1[n]\} = \mathcal{T}_2\{\mathcal{T}_1\{x[n]\}\}$$

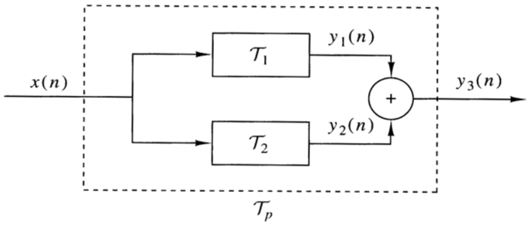
- Systems $\mathcal{T}_1$ and $\mathcal{T}_2$ can be combined or consolidated into a single overall system

$$y[n] = \mathcal{T}_c\{x[n]\} \quad \text{where } \mathcal{T}_c \equiv \mathcal{T}_2\mathcal{T}_1$$

- In general $\mathcal{T}_1\mathcal{T}_2 \ne \mathcal{T}_2\mathcal{T}_1$. However, if systems $\mathcal{T}_1$ and $\mathcal{T}_2$ are LTI, then (a) is time invariant and (b) $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1$

Interconnection of Discrete-Time Systems

- **Parallel interconnection**



$$
\begin{aligned}
y_3[n] &= y_1[n] + y_2[n] \\
&= \mathcal{T}_1\{x[n]\} + \mathcal{T}_2\{x[n]\} \\
&= (\mathcal{T}_1 + \mathcal{T}_2)\{x[n]\} \\
&= \mathcal{T}_p\{x[n]\}
\end{aligned}
$$

- We can use parallel and cascade interconnection of systems to construct larger, more complex systems

Techniques for the Analysis of Linear Systems

- Two basic methods for analyzing the behavior of a linear system:
  - The first is based on the direct solution of the input-output equation

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

  - The second method is to decompose or resolve the input signal into a sum of elementary signals. Then, using the linearity of the system, the response of the system to the elementary signals are sum to obtain the total response

Techniques for the Analysis of Linear Systems

- Suppose the input signal is resolved into a weighted sum of elementary signals

$$x[n] = -\sum_k c_k x_k[n]$$

- The response $y_k[n]$ of the system to the component $x_k[n]$ is

$$y_k[n] \equiv \mathcal{T}\{x_k[n]\}$$

- If the system is linear, we have

$$y[n] = \mathcal{T}\{x[n]\} = \mathcal{T}\left\{\sum_k c_k x_k[n]\right\}$$

$$= \sum_k c_k \mathcal{T}\{x_k[n]\} = \sum_k c_k y_k[n]$$

Why & how to do the signal decomposition?

Resolution of a Discrete-Time Signal into Impulses

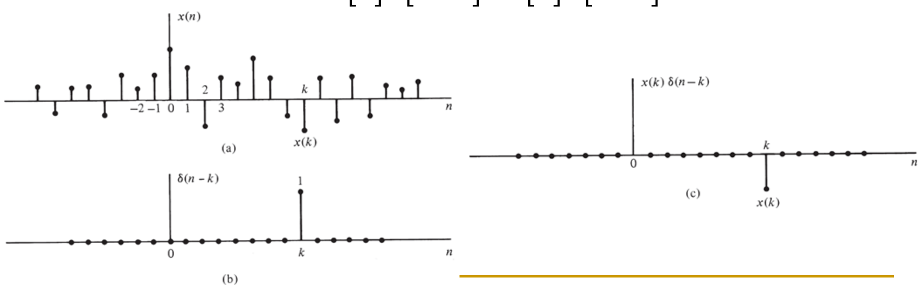
- Select the elementary signals $x_k[n]$ to be

$$x_k[n] = \delta[n-k]$$

where k represents the delay of the unit sample sequence

- Multiply the two sequences $x[n]$ and $\delta[n-k]$?

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$



Resolution of a Discrete-Time Signal into Impulses

- Consequently

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- **Example** - Consider a finite-duration sequence given as

$$x[n] = \{2, \underline{4}, 0, 3\}$$

The sequence can be resolved as

$$x[n] = 2\delta[n+1] + 4\delta[n] + 3\delta[n-2]$$

Resolution of a Discrete-Time Signal into Impulses

- The response of a relaxed linear system to the unit sample sequence input:

$$y[n,k] \equiv h[n,k] = \mathcal{T}\{\delta[n-k]\}$$

- If the impulse at the input is scaled by as

$$c_k h[n,k] = x[k]h[n,k]$$

- If the input is expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The output becomes

$$y[n] = \mathcal{T}\{x[n]\} = \mathcal{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\mathcal{T}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h[n,k]$$

Response of LTI Systems to Arbitrary Inputs

- If the system is time invariant, and denote the response of the LTI system to the unit sample sequence as

$$h[n] \equiv \mathscr{T}\{\delta[n-k]\}$$

- The response of the system to $\delta[n-k]$ is

$$h[n-k] = \mathscr{T}\{\delta[n-k]\}$$

- Consequently

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The relaxed LTI system is completely characterized by a single function *h*[*n*], the impulse response.
- Convolution is commutative

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Computing the Convolution Sum

- The output of an LTI system at $n = n_0$ is given by

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0-k]$$

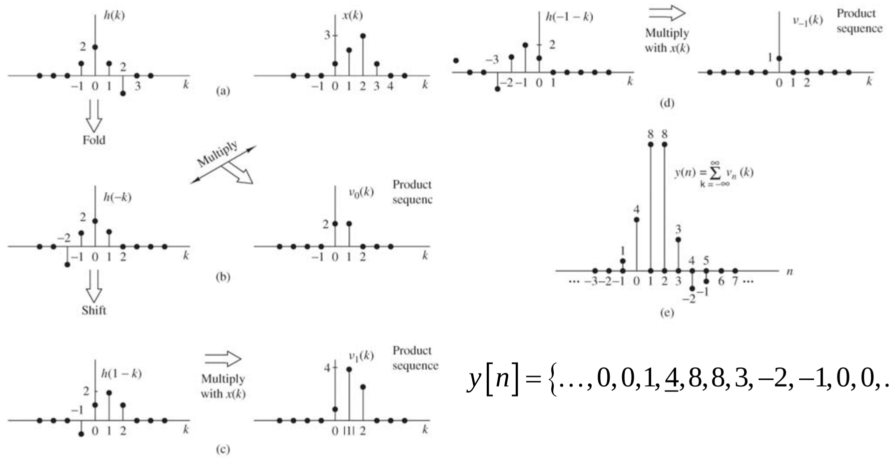
- To compute $y[n_0]$
  - *Folding*. Fold *h*[*k*] about *k* = 0 to obtain *h*[–*k*]
  - *Shifting*. Shift *h*[–*k*] by n_0 to the right (left) if is positive (negative), to obtain $h[n_0–k]$
  - *Multiplication*. Multiply *x*[*k*] by $h[n_0–k]$ to obtain the product sequence

$$v_{n_0}[k] \equiv x[k]h[n_0-k]$$

  - *Summation*. Sum all the values of $v_{n_0}[k]$ to obtain $y[n_0]$

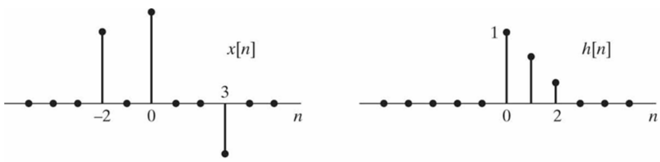
Computing the Convolution Sum

$$x[n]=\{\underline{1},2,3,1\} \qquad h[n]=\{1,2,\underline{1},-1\}$$



$$y[n]=\{\ldots,0,0,1,\underline{4},8,8,3,-2,-1,0,0,\ldots\}$$

Computing the Convolution Sum

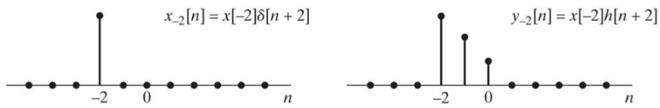


$$y[n]=x[n]*h[n]=\left(\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]\right)*h[n]=\left(x_{-2}[n]+x_0[n]+x_3[n]\right)*h[n]$$

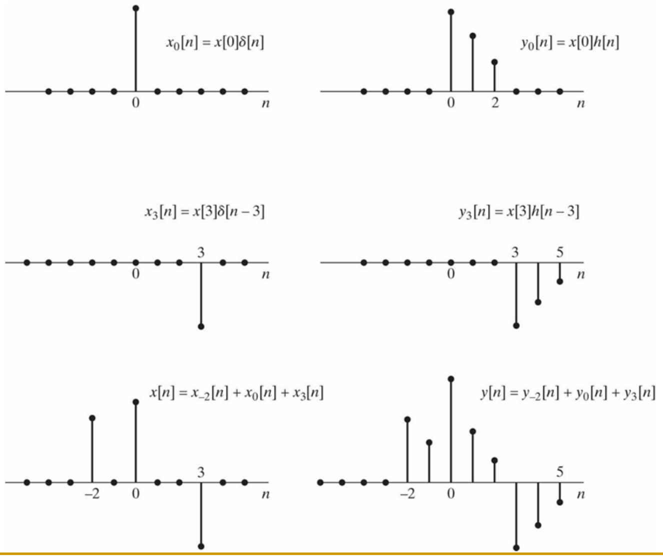
$$=\left(x[-2]\delta[n+2]+x[0]\delta[n]+x[3]\delta[n-3]\right)*h[n]$$

$$=x[-2](\delta[n+2]*h[n])+x[0](\delta[n]*h[n])+x[3](\delta[n-3]*h[n])$$

$$=x[-2]h[n+2]+x[0]h[n]+x[3]h[n-3]=y_{-2}[n]+y_0[n]+y_3[n]$$



Computing the Convolution Sum



Tabular Method of Convolution Sum Computation

$$y[n] = g[n] * h[n] = \sum_{k=N_1}^{N_2} g[n-k]h[k]$$

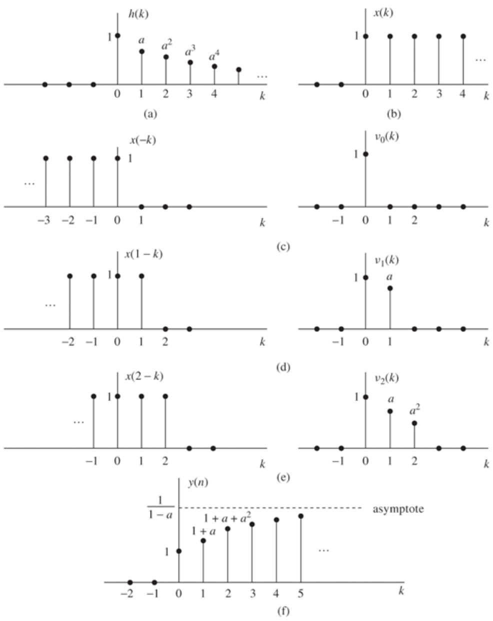
$$= \sum_{k=n-N_1}^{n-N_2} g[k]h[n-k] = \sum_{k=n-N_1}^{n-N_2} g[k]h[-(k-n)]$$

n :	0	1	2	3	4	5
$g[n]$:	$g[0]$	$g[1]$	$g[2]$	$g[3]$		
$h[n]$:	$h[0]$	$h[1]$	$h[2]$	-		
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$		
	-	$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$	$g[3]h[1]$	
	-	-	$g[0]h[2]$	$g[1]h[2]$	$g[2]h[2]$	$g[3]h[2]$
$y[n]$:	$y[0]$	$y[1]$	$y[2]$	$y[3]$	$y[4]$	$y[5]$

Computing the Convolution Sum

- **Example**:

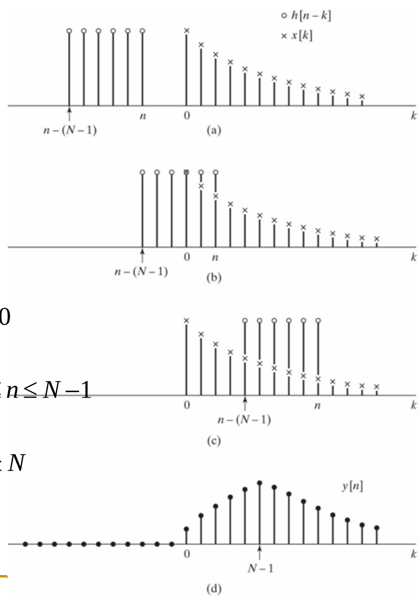
$$h[n]=a^n u[n], \quad |a|<1$$
$$x[n]=u[n]$$
$$y[0]=1$$
$$y[1]=1+a$$
$$y[2]=1+a+a^2$$
$$y[n]=1+a+a^2+\cdots+a^n$$
$$=\frac{1-a^{n+1}}{1-a}$$
$$y[\infty]=\lim_{n\to\infty} y[n]=\frac{1}{1-a}$$



Computing the Convolution Sum

$$h[n]=u[n]-u[n-N]$$
$$x[n]=a^n u[n]$$
$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$
$$=\sum_{k=-\infty}^{\infty} a^{n-k}u[n-k]\big(u[k]-u[k-N]\big)=\sum_{k\le n,k\ge 0,k<N} a^{n-k}$$
$$=\begin{cases} 0, & \text{if } n<0 \\ \sum_{k=0}^{n} a^{n-k}=\frac{a^n(1-a^{-(n+1)})}{1-a^{-1}}=\sum_{k=0}^{n} a^k & \text{if } 0\le n\le N-1 \\ \sum_{k=0}^{N-1} a^{n-k}=\frac{a^n(1-a^{-N})}{1-a^{-1}}=a^{n-N+1}\left(\frac{1-a^N}{1-a}\right), & \text{if } n\ge N \end{cases}$$

$$\text{Note: } \sum_{k=N}^{M} r^k=\frac{r^N-r^{M+1}}{1-r}$$

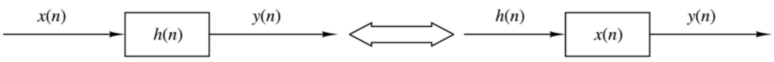


Properties of Convolution (1/2)

- **Commutative Property**

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$



- **Identity and Shifting Properties**

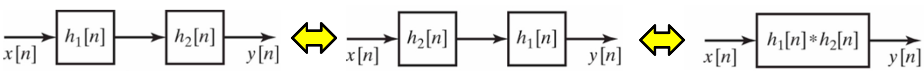
$$y[n] = x[n] * \delta[n] = x[n]$$

$$x[n] * \delta[n-k] = y[n-k] = x[n-k]$$

Properties of Convolution (2/2)

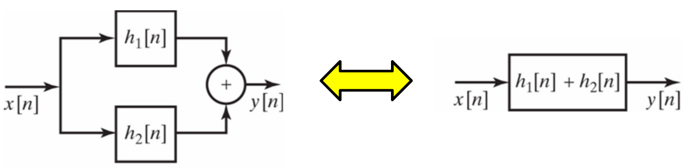
- **Associative Property**

$$(x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n] = x[n] * (h_1[n] * h_2[n])$$



- **Distributive Property**

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$



Causality of LTI Systems (1/2)

- The output of an LTI system at $n = n_0$ is given by

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0 - k]$$

- Divide the sum into two sets of terms:

$$y[n_0] = \sum_{k=0}^{\infty} h[k]x[n_0 - k] + \sum_{k=-\infty}^{-1} h(k)[k]x[n_0 - k]$$

$$= \left[\underbrace{h[0]x[n_0] + h[1]x[n_0 - 1] + \cdots}_{\text{depend on present and past inputs}}\right] + \left[\underbrace{h[-1]x[n_0 + 1] + h[-2]x[n_0 + 2] + \cdots}_{\text{depend on future inputs}}\right]$$

- For a causal system, $h[n] = 0$ for $n < 0$
- Since $h[n]$ is the response of the relaxed LTI system to a unit impulse sequence at $n = 0$, an LTI system is causal **if and only if** its impulse response is zero for negative values of n

Causality of LTI Systems (2/2)

- The output of an causal LTI system becomes

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

- A sequence $x[n]$ is called a **causal sequence** if $x[n] = 0$ for $n < 0$; otherwise, it's a **noncausal sequence**
- If the input to a causal LTI system is a causal sequnce, the input-output equation reduces to

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} x[k]h[n-k]$$

- **Example**: Determine the unit step response of the LTI system with impulse response

$$h[n] = a^n u[n], \qquad |a| < 1$$

$$y[n] = \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}$$

Stability of LTI Systems (1/3)

- **BIBO Stability Condition** - A discrete-time system is BIBO stable if and only if the output sequence {$y[n]$} remains bounded for all bounded input sequence {$x[n]$}
- An LTI discrete-time system is BIBO stable if and only if its impulse response sequence {$h[n]$} is absolutely summable, i.e.

$$B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- Proof: Assume $h[n]$ is a real sequence

  Sufficient condition: Since the input sequence $x[n]$ is bounded we have $|x(n)| \le B_x < \infty$
  therefore

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x B_h < \infty$$

Stability of LTI Systems (2/3)

- Thus, $B_h < \infty$ implies $|y[n]| \le B_x B_h < \infty$, indicating that $y[n]$ is also bounded
- To prove the necessary condition, assume $y[n]$ *is* bounded, i.e., $|y[n]| \le B_y$
- Consider the bounded input given by

$$x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|}, & h[n] \ne 0 \\ 0, & h[n] = 0 \end{cases}$$

- For this input, $y[n]$ at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} x[-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} = B_h$$

- Therefore, if $B_h = \infty$, then {$y[n]$} is not a bounded sequence

Stability of LTI Systems (3/3)

- **Example** - Consider a causal LTI discrete-time system with an impulse response

$$h[n] = a^n u[n]$$

- For this system

$$B_h = \sum_{k=-\infty}^{\infty} \left|a^k\right| u[k] = \sum_{k=0}^{\infty} |a|^k = \frac{1}{1-|a|}, \quad \text{if } |a| < 1$$

- Therefore $B_h < \infty$ *if* $|a| < 1$, for which the system is BIBO stable
- If $|a| = 1$, the system is not BIBO stable