# ON THE OPTIMALITY OF LEPT AND $c\mu$ RULES FOR MACHINES IN PARALLEL


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Sept., 1989
revised Nov., 1990 and March, 1991
final June, 1991



**Abstract**

We consider scheduling problems with $m$ machines in parallel and $n$ jobs. The machines are subject to breakdown and repair. Jobs have exponentially distributed processing times and possibly random release dates. For cost functions that only depend on the set of uncompleted jobs at time $t$ we provide necessary and sufficient conditions for the LEPT rule to minimize the expected cost at all $t$ within the class of preemptive policies. This encompasses results that are known for makespan, and provides new results for the work remaining at time $t$. An application is that if the $c\mu$-rule has the same priority assignment as the LEPT rule then it minimizes the expected weighted number of jobs in the system for all $t$ . Given appropriate conditions, we also show that the $c\mu$ rule minimizes the expected value of other objective functions, such as (i) weighted sum of job completion times, and (ii) weighted number of late jobs, or (iii) weighted sum of job tardinesses, when jobs have a common random due date.



*This research was partially supported by the NSF under grant ECS 86-14689

# 1 Introduction

Consider $m$ machines that operate in parallel and are subject to breakdown and repair. Their up times and down times are arbitrarily distributed. The machines are used to process $n$ jobs, whose release dates, $R_1$, ..., $R_n$, are also arbitrarily distributed. Job $j$ has a processing time $P_j$, that is an exponentially distributed random variable (r.v.) with rate $\mu_j$. Each processing time is distributed independently of all other random variables in the model. Without loss of generality, we assume that

**(A1)** $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$.

Let $Q_{j,t}$ be the indicator function for the event that job $j$ has been released by time $t$ but is not yet complete, i.e. $Q_{j,t}$ is 1 or 0 as job $j$ is or is not in the system at time $t$. Define $\underline{Q}_t = (Q_{1,t}, Q_{2,t}, \ldots, Q_{n,t})$ as the state of the jobs at time $t$. A cost $g(\underline{Q}_t)$ is associated with the system at time $t$, where $g : \{0,1\}^n \mapsto \mathcal{R}$. This cost depends only on the set of jobs in the system at $t$.

Our main result is the following theorem.

**Theorem 1.1** *Suppose (A1) holds. Let $\pi$ be the policy that schedules jobs according to priorities that are decreasing in their indices. Then within the class of preemptive policies, $\pi$ minimizes $Eg(\underline{Q}_t)$, the expected cost at time $t$, for all $t$, and all processes of arrivals, machine breakdowns and repairs,* if and only if *the cost function $g(\underline{x})$ satisfies*

**(A2)** $g(\underline{x}) \geq g(\underline{x} - e_\beta)$,

**(A3)** $\mu_\alpha g(\underline{x} - e_\alpha) \geq \mu_\beta g(\underline{x} - e_\beta) + (\mu_\alpha - \mu_\beta) g(\underline{x})$ *for all* $\alpha > \beta$,

*where $e_j$ is a row vector of $n$ components, whose $j$-th component is 1 and others components are 0.*

Note that since (A1) is assumed to hold, $\pi$ is equivalent to the Longest Expected job Processing Time (LEPT) first rule. This rule is well-known to stochastically minimize the *makespan* of jobs with exponentially distributed processing times that are processed on parallel machines; moreover, it is known to hold for an arbitrary arrival process (see Van Der Heyden (1981), Frederickson, Bruno, and Downey (1981), Weiss (1982) and Weber (1982, 1983)). This result reappears as an application of Theorem 1.1 at the start of section 2. Our proof of Theorem 1.1 is based on certain coupling techniques and is similar to the approach of Van Der Heyden.

In section 2 we shed some light on conditions (A2)–(A3), by providing further examples of cost functions for which they hold.

In section 3, we consider the special case $g(\underline{x}) = \sum_{j=1}^{n} c_j x_j$. This is the case of weighted holding cost, in which a cost $c_j$ is incurred for each unit time job $j$ remains in the system. It is easy to check that this cost function satisfies (A2)–(A3) *if and only if*

**(A4)** $\mu_1 c_1 \geq \mu_2 c_2 \geq \ldots \mu_n c_n \geq 0$.

Condition (A4) says that the order of increasing expected processing times is the same as the order of increasing values of $c\mu$. So if both (A1) and (A4) hold, $\pi$ and LEPT are the same and equivalent to the $c\mu$-rule, which is defined as the preemptive rule that always processes a set of jobs whose values of $c\mu$ are greatest. In the literature, a combination of conditions such as (A1) and (A4) is often called an "agreeability condition". In section 3 we show that the $c\mu$-rule minimizes various objective functions, including

**(i)** the expected weighted number of jobs in system at an arbitrary time $t$,

**(ii)** the expected weighted sum of job completion times,

**(iii)** the expected weighted number of late jobs when the jobs have a common random due date,

**(iv)** the expected weighted sum of job tardinesses when the jobs have a common due date.

The LEPT and $c\mu$ rules have received a great deal of attention in the stochastic scheduling literature. If there is only a *single machine*, then there is no need for an agreeability condition. Various authors have considered the single machine case and shown that the $c\mu$-rule minimizes objective functions such as (i)–(iv) above; see Pinedo (1983), Baras, Dorsey and Makowski (1985), Buyukkoc, Varaiya and Walrand (1985) and Shanthikumar and Yao (1991).

Optimality of the $c\mu$-rule for parallel machines requires an agreeability condition. Ross (1983) considered two machines in parallel, $n$ jobs available at time zero and no arrivals afterwards. For this setup he showed that under the agreeability conditions (A1) and (A4), the $c\mu$-rule minimizes the expected weighted sum of completion times. Kämpke (1987) extended Ross' result to $m$ machines, and weakened the agreeability conditions to $c_1 \geq \ldots \geq c_n$ and (A4). Under the agreeability conditions $c_1 \geq \ldots \geq c_n$ and $\rho_1(t_1)c_1 \geq \ldots \rho_n(t_n)c_n$ for all $t_1, \ldots, t_n$, Weber (1988) generalized Kämpke's result to models in which the job processing times have hazard rates, $\rho_1(t), \ldots, \rho_n(t)$. Corollary 3.1 in this paper generalizes Ross' result to $m$ machines and an arbitrarily arrival process.

For more general cost functions, conditions (A2) and (A3) quite arise naturally. They have been considered by Weiss and Pinedo (1980), who investigated similar scheduling problems to those in this paper, but under the assumption that all jobs are present at the start. They showed that a policy minimizes the total holding cost incurred by time $t$ if the expected total cost under that policy, say $G(\underline{x})$, satisfies (A2)–(A3), with $g$ replaced by $G$. They stated conditions on $g$ that would be sufficient to guarantee optimality of the LEPT rule; however, these were incomplete and corrected by Kämpke (1987, 89) through the addition of (A3). (We note that Kämpke was concerned with general list policies, not only LEPT. He therefore also required a supermodularity condition that is not needed in Theorem 1.1.) The contribution of the present paper is to show that conditions (A2)–(A3) are sufficient to guarantee optimality of LEPT when there are arrivals and machine breakdowns and repairs. Also, conditions (A2)–(A3) are necessary, in the sense that they must hold if LEPT is to be optimal for all $t$ and for every possible process of machine breakdown and repair. Our results also apply to models in which machines have different speeds, since this scenario can be approximated by rapidly alternating periods of breakdown and repair. This generalization has previously been made by Weiss and Pinedo, and also Kämpke.

## 2 The optimality of the LEPT rule

To shed some light on the conditions (A2)–(A3) in Theorem 1.1, we first provide some examples that satisfy these conditions. We then prove Theorem 1.1.

**Example 2.1** Consider the indicator function

$$g_j(\underline{x}) = \mathbf{1}_{\{\sum_{i=1}^{j} x_i > 0\}}.$$

It is clear that this function satisfies (A2)–(A3). It then follows from Theorem 1.1 that the LEPT rule minimizes $P(\sum_{i=1}^{j} Q_{i,t} > 0)$. Now let $Z_j$ be the completion time of job $j$. Conditioning on the event $\{R_j = r_j, j = 1, 2, \ldots, n\}$, we note that

$$\begin{aligned} P(\sum_{i=1}^{j} Q_{i,t} > 0) &= 1 - P(Q_{i,t} = 0,\ i = 1, 2, \ldots, j) \\ &= 1 - P(\max_{1 \le i \le j} (Z_i \mathbf{1}_{\{r_i \le t\}}) \le t), \end{aligned}$$

where $\mathbf{1}_{\{r_i \le t\}}$ equals 1 or 0 as the event $[r_i \le t]$ does or does not occur. Considering the case $j = n$, we have that the LEPT rule stochastically minimizes the makespan of jobs arriving before $t$. Note that since the makespan is larger than $t$ if there is a job arriving after $t$,

$$P(\max_{1\le i\le n}(Z_i)\le t) = P(\max_{1\le i\le n}(Z_i)\le t,\ t\ge \max_{1\le i\le n}(r_i)).$$

Unconditioning the event $\{R_j = r_j, j = 1,2,\ldots n\}$ yields that the LEPT rule stochastically minimizes the makespan.

**Example 2.2** Let $V_1,\ldots,V_n$, be independent exponential random variables with mean $1/\mu_1,\ldots,1/\mu_n$. Consider the function

$$g(\underline{x}) = Ef(V_1x_1, V_2x_2,\ldots,V_nx_n),$$

where $f : \mathcal{R}_+^n \cup \{0\} \mapsto R$.

**Theorem 2.3** *If $f$ is*

**(i)** *increasing, i.e. $f(\underline{z}^1) \le f(\underline{z}^2)$ for all $\underline{z}^1 \le \underline{z}^2$ componentwise,*

**(ii)** *arrangement increasing, i.e. $f(\ldots,z_i,\ldots,z_j,\ldots) \ge f(\ldots,z_j,\ldots,z_i,\ldots)$ for all $i,j$ and $z_i \ge z_j$,*

**(iii)** *convex in each variable, i.e. $f(\underline{z}+(\delta_1+\delta_2)e_i) + f(\underline{z}) \ge f(\underline{z}+\delta_1 e_i) + f(\underline{z}+\delta_2 e_i)$ for all $i$ and $\delta_1,\delta_2 \ge 0$,*

**(iv)** *submodular, i.e. $f(\underline{z}+\delta_1 e_i+\delta_2 e_j) + f(\underline{z}) \le f(\underline{z}+\delta_1 e_i) + f(\underline{z}+\delta_2 e_j)$ for all $i \ne j$ and $\delta_1,\delta_2 \ge 0$,*

*then $g(\underline{x})$ satisfies (A2)-(A3).*

*Proof.* It is clear that the increasing property of $f$ implies condition (A2). Chang (1990) has shown that (i)–(iv) imply (A3) when $f$ is symmetric. His proof is easily adapted to the case that $f$ is arrangement increasing. ■

Let $V_{i,t}$ be the remaining work of job $i$ at time $t$. Since the job processing times are memoryless, $V_{i,t}$ has the same distribution as the quantity $V_iQ_{i,t}$. Using Theorem 2.3, we have the following corollary for the work remaining at time $t$.

**Corollary 2.4** *Suppose (A1) holds and $f$ satisfies (i)–(iv) of Theorem 2.3. Then the LEPT rule minimizes $Ef(V_{1,t},V_{2,t},\ldots,V_{n,t})$.*

Examples of functions that satisfy these four conditions are (i) $f(\underline{z}) = \max(a_1 z_1, \ldots, a_n z_n)$, with $a_1 \geq a_2 \geq \ldots \geq a_n$, and $z_i \geq 0$ for all $i$, and (ii) $f(\underline{z}) = \sum_{i=1}^{n} h_i(z_i)$, with $h_i \overset{\geq}{s} h_{i+1}$ (see Definition 3.4) and $h_i(z)$ increasing convex for all $i$. For more examples, see Chang (1990) and Chang and Yao (1990).

In fact, the LEPT rule not only minimizes the total amount of work at time $t$, $\sum_{i=1}^{n} V_{i,t}$, in *expectation*, but also in the sense of *stochastic ordering*. The argument goes as follows. For a given problem, consider an auxiliary problem in which the number of available machines is truncated to 1 past $t$. Let $M$ be the makespan of the auxiliary problem. Clearly, the total amounts of work are the same at time $t$, if in both problems the same scheduling policy has been used up to time $t$. Moreover, the total amount of work remaining at time $t$ is $\max(M - t, 0)$. Since the LEPT rule stochastically minimizes the makespan for the auxiliary problem, it also stochastically minimizes the total amount of work remaining at time $t$ for the original problem, taking into account that the maximum function is increasing.

The rest of this section is devoted to the proof for Theorem 1.1. The proof takes the same approach as Van Der Heyden. It uses the uniformization technique and an inductive method that has been called "forward induction" (see Walrand (1988) section 8.3). Using the well known uniformization technique, our continuous-time optimization problem is transformed into a discrete-time one. We first show that $\pi$ is optimal in one step and then take as an inductive hypothesis that $\pi$ is optimal in $k-1$ steps. A problem with $k$ steps has $k$ decision epochs. Denote the policies applied between these epochs as ($\sigma_0$, $\sigma_1$, ..., $\sigma_{k-1}$). If we are able to show that ($\pi$, $\pi$, ..., $\pi$) is better than ($\sigma$, $\pi$, ..., $\pi$), for any admissible policy $\sigma$ and any initial state, it follows from the induction hypothesis that $\pi$ is optimal over $k$ steps. As a result of applying different policies $\pi$ and $\sigma$ at time 0, the states at step 1 are different. To show that ($\pi$, $\pi$, ..., $\pi$) is better than ($\sigma$, $\pi$, ..., $\pi$), we must show that starting from time 0 the states reached after applying $\pi$ are better than those reached after applying $\sigma$. Thus, the proof requires a partial ordering amongst the states. The appropriate partial ordering is contained in the following definition. Using it, we can establish inequalities that are similar to (3.3) and (3.4) in Van Der Heyden's paper.

**Definition 2.5** *(Partial Sum Ordering) Let $\underline{x}^i = (x_1^i, x_2^i, \ldots, x_n^i)$, $i = 1, 2$, be two vectors. We say that $\underline{x}^1$ is smaller than $\underline{x}^2$ under partial sum, and denote this $\underline{x}^1 \leq_{ps} \underline{x}^2$, if $\sum_{j=1}^{l} x_j^1 \leq \sum_{j=1}^{l} x_j^2$, for all $l = 1, 2, \ldots, n$.*

The partial sum ordering is very similar to the weak majorization ordering (Marshall and Olkin (1979)). However, the weak majorization ordering of $x^1$ and $x^2$ requires their components to be in decreasing order. The following property of the partial sum ordering is easy to prove (see Ross (1983), Lemma 3.4). Note that it does not require $x_j$ to be either 0 or 1.

**Lemma 2.6** *If* $\underline{x}^1 \leq_{ps} \underline{x}^2$, $\sum_{j=1}^n x_j^1 = \sum_{j=1}^n x_j^2$ *and* $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$, *then* $\sum_{j=1}^n \mu_j x_j^1 \geq \sum_{j=1}^n \mu_j x_j^2$.

The next lemma gives a constructive characterization of the partial sum ordering. Consider two integer-valued vectors that are partial sum ordered. We show that the greater of the two vectors can be transformed into the lesser by a number of canonical steps, each of which either reduces some component by 1, or makes a transfer of 1 from a lesser to a greater component. In the first case, we think of 1 as being "transfered out" of the vector; in the second case 1 is transfered between two components. Informally, we call this the "transfer property" of the partial sum ordering. A similar property holds for the weak majorization ordering (Muirhead (1903)).

**Lemma 2.7** *If* $\underline{x}^i$ *and* $\underline{x}^2$ *are two integer-valued vectors and* $\underline{x}^1 \leq_{ps} \underline{x}^2$, *then* $\underline{x}^2$ *can be transformed into* $\underline{x}^1$ *by a finite number of applications of the following two types of transfers:*

**(i)** $\underline{x} \mapsto \underline{x} - e_\beta$, *where* $x_\beta > 0$,

**(ii)** $\underline{x} \mapsto \underline{x} + e_\alpha - e_\beta$, *where* $\alpha > \beta$ *and* $x_\beta > 0$.

*Moreover, if* $T$ *is the combination of a finite number of transfers of the forms in (i) and (ii), then* $T(\underline{x}) \leq_{ps} \underline{x}$.

*Proof.* That $T(\underline{x}) \leq_{ps} \underline{x}$ is clear. Conversely, the transfer property is clear for $n = 1$, since the partial sum ordering in one dimension means $x_1^1 \leq x_1^2$, and we need only make reductions of 1 to $x_1^2$ to obtain $x_1^1$. Thus, we can take as our induction hypothesis that $\underline{x}^2$ can be transformed into $\underline{x}^1$ when these are vectors of $n-1$ components. If $\sum_{i=1}^n x_i^1 < \sum_{i=1}^n x_i^2$, then there is a constant $l$ such that $\sum_{i=1}^{l-1} x_i^2 \leq \sum_{i=1}^n x_i^1 < \sum_{i=1}^l x_i^2$. We can use the transfer $\underline{x} \mapsto \underline{x} - e_\beta$, $\beta \geq l$ until $\sum_{i=1}^n x_i^1 = \sum_{i=1}^l x_i^2$. Thus, we only need to consider the case that $\sum_{i=1}^n x_i^1 = \sum_{i=1}^n x_i^2$. Note that $\underline{x}^1 \leq_{ps} \underline{x}^2$ implies $x_n^1 \geq x_n^2$. If $x_n^1 = x_n^2$, then the induction hypothesis can be applied to the first $n-1$ components. If $x_n^1 > x_n^2$, then we can find a constant $l$ such that $x_l^2 > 0$ and $x_i^2 = 0$, $i = l+1, \ldots, n$ and repeatedly apply the transfer $\underline{x}^2 \mapsto \underline{x}^2 + e_n - e_l$ until the $n$-th component of the resulting vector equals $x_n^1$. Once again, this leaves a case to which the induction hypothesis for $n-1$ applies. This completes the proof. ∎

In the following lemma, we show that the partial sum ordering is preserved under $\pi$ (the LEPT rule).

**Lemma 2.8** *($\pi$-monotone) Consider two systems with different initial conditions. Let $Q^i_{j,t}$ be the indicator function for the event that job $j$ is in the system $i$ at time $t$, and $\underline{Q}^i_t = (Q^i_{1,t}, Q^i_{2,t}, \dots, Q^i_{n,t})$, $i = 1, 2$. Assume (A1) and $\underline{Q}^1_0 \leq_{ps} \underline{Q}^2_0$. Then there exist two random vectors $\hat{\underline{Q}}^1_t$ and $\hat{\underline{Q}}^2_t$ such that under $\pi$ we have (i) $\hat{\underline{Q}}^i_t =_{st} \underline{Q}^i_t$, $i = 1, 2$, and (ii) $\hat{\underline{Q}}^1_t \leq_{ps} \hat{\underline{Q}}^2_t$.*

*Proof.* Let $m(t)$ denote the number of machines available at time $t$ and let $\{M_k, k \geq 1\}$ be the set of epochs that $dm(t) \neq 0$. At each time $M_k$, one machine breaks down or one machine becomes available again following its repair. Consider the event $\{R_j = r_j, j = 1, \dots, n$ and $M_k = m_k,\ k \geq 1\}$. Let $t_0 = 0$ and $\{t_k,\ k \geq 1\}$ be the sequence of $\{r_j,\ j = 1, \dots, n$ and $m_k,\ k \geq 1\}$ after sorting in time. Clearly, between $t_k$ and $t_{k+1}$ there are no arrivals and the number of machines is constant. First, we show that we can construct two processes between $t_0$ and $t_1$ such that they satisfy conditions (i) and (ii) of this lemma. Using the standard coupling and uniformization technique (Keilson (1979)), we generate a Poisson process with rate $\Delta = m(t_0)\mu_n$. Let $\{\tau_k, k \geq 1\}$ be its arrival epochs and define $\tau_0 = 0$. Generate a sequence of independent and identically distributed (i.i.d.) r.v.'s $\{U_k, k \geq 1\}$ uniformly distributed over [0,1]. Construct the uniformized Markov chains as follows:

$$\hat{\underline{Q}}^i_0 = \underline{Q}^i_0, \quad i = 1, 2,$$

and

$$\hat{Q}^i_{j,\tau_{k+1}} = \hat{Q}^i_{j,\tau_k} - \delta^i_{j,k} \quad \text{for all } \tau_{k+1} < t_1$$

where

$$\delta^i_{j,k} = \mathbf{1}\left\{ \frac{\sum_{l=1}^{j-1} \mu_l X^i_{l,k}}{\Delta} \leq U_k < \frac{\sum_{l=1}^{j} \mu_l X^i_{l,k}}{\Delta} \right\},$$

and $X^i_{j,k} = 1$ if job $j$ is served in system $i$ at time $\tau_k$ and $X^i_{j,k} = 0$ otherwise. As desired, it is clear that $\hat{\underline{Q}}^i_t =_{st} \underline{Q}^i_t$, $i = 1, 2$. Under $\pi$, $X^i_{j,k} = \hat{Q}^i_{j,\tau_k}$ if $\sum_{l=1}^j \hat{Q}^i_{l,\tau_k} \leq m(t_0)$ and $X^i_{j,k} = 0$ otherwise. Thus, $\sum_{l=1}^j X^2_{l,k} = \min\{m(t_0), \sum_{l=1}^j \hat{Q}^2_{l,\tau_k}\}$. ¿From the induction hypothesis $\sum_{l=1}^j \hat{Q}^1_{l,\tau_k} \leq \sum_{l=1}^j \hat{Q}^2_{l,\tau_k}$, it follows that $\sum_{l=1}^j X^1_{l,k} \leq \sum_{l=1}^j X^2_{l,k}$, $j = 1, \dots, n$. Since there is at most one departure at $\tau_{k+1}$, $\sum_{l=1}^j \hat{Q}^1_{l,\tau_{k+1}} \leq \sum_{l=1}^j \hat{Q}^2_{l,\tau_{k+1}}$ if $\sum_{l=1}^j \hat{Q}^1_{l,\tau_k} < \sum_{l=1}^j \hat{Q}^2_{l,\tau_k}$. Therefore, we need only consider the case that $\sum_{l=1}^j \hat{Q}^1_{l,\tau_k} = \sum_{l=1}^j \hat{Q}^2_{l,\tau_k}$. In such case, $\sum_{l=1}^j X^1_{l,k} = \sum_{l=1}^j X^2_{l,k}$ and $\sum_{l=1}^p X^1_{l,k} \leq \sum_{l=1}^p X^2_{l,k}$, $p = 1, \dots, j-1$. ¿From Lemma 2.6, it follows that $\sum_{l=1}^j \mu_l X^1_{l,k} \geq \sum_{l=1}^j \mu_l X^2_{l,k}$ and thus $\sum_{l=1}^j \delta^1_{l,k} \geq \sum_{l=1}^j \delta^2_{l,k}$. Therefore, we conclude that $\hat{\underline{Q}}^1_{\tau_{k+1}} \leq_{ps} \hat{\underline{Q}}^2_{\tau_{k+1}}$. Observe that the partial sum ordering is also preserved when there is an arrival of the same type at both systems. By induction, we can construct two processes such that $\hat{\underline{Q}}_t^1 \leq_{ps} \hat{\underline{Q}}_t^2$. ■

Let $J_t(\underline{Q}_0) \stackrel{\text{def}}{=} E(g(\underline{Q}_t)|\underline{Q}_0)$ be the expected cost at time $t$ from the initial state $\underline{Q}_0$ under the policy $\pi$.

**Corollary 2.9** *Assume (A1)–(A3) hold, and $\underline{Q}_0^1 \leq_{ps} \underline{Q}_0^2$. Then scheduling under $\pi$ we have, $J_t(\underline{Q}_0^1) \leq J_t(\underline{Q}_0^2)$ for all $t \geq 0$.*

*Proof.* ¿From (A2) and (A3), it follows that $g(\underline{x} - e_\alpha) \geq g(\underline{x} - e_\beta)$ for all $\alpha > \beta$. Using the transfer property in Lemma 2.7 yields that $g(\underline{x}^1) \leq g(\underline{x}^2)$ if $\underline{x}^1 \leq_{ps} \underline{x}^2$. It then follows from the $\pi$-monotone property in Lemma 2.8 that $J_t(\underline{Q}_0^1) \leq J_t(\underline{Q}_0^2)$ if $\underline{Q}_0^1 \leq_{ps} \underline{Q}_0^2$ . ■

**Lemma 2.10** *Under (A1)–(A3) and $\pi$,*

$$\mu_\alpha J_t(\underline{Q}_0 - e_\alpha) \geq \mu_\beta J_t(\underline{Q}_0 - e_\beta) + (\mu_\alpha - \mu_\beta) J_t(\underline{Q}_0), \tag{1}$$

*for any initial state $\underline{Q}_0$ with $Q_{\alpha,0} = Q_{\beta,0} = 1$ and $\alpha > \beta$.*

*Proof.* To simplify the notations in the proof, we denote $e_{i,j} = e_i + e_j$. We use the same construction as in the proof of Lemma 2.8. First, we show that (1) holds for all $t \in [t_0, t_1)$. ¿From (A3), it follows that (1) is satisfied at $\tau_0 = 0$. Now take as an induction hypothesis that (1) holds for $t \in [\tau_0, \tau_k]$. Let $S$ (resp. $S_\alpha, S_\beta$) be the set of jobs served at time 0 under $\pi$ given the initial state $\underline{Q}_0$ (resp. $\underline{Q}_0 - e_\alpha, \underline{Q}_0 - e_\beta$). Note that $\{\tau_{k+1} - \tau_k\}$ is a sequence of i.i.d. exponential r.v.'s. Analogous to Kolmogorov's backward equation, we have

$$J_{\tau_{k+1}}(\underline{Q}_0) = \sum_{j \in S} J_{\tau_k}(\underline{Q}_0 - e_j) \frac{\mu_j}{\Delta} + J_{\tau_k}(\underline{Q}_0) \frac{\Delta - \sum_{j \in S} \mu_j}{\Delta}.$$

Similarly,

$$J_{\tau_{k+1}}(\underline{Q}_0 - e_\alpha) = \sum_{j \in S_\alpha} J_{\tau_k}(\underline{Q}_0 - e_{\alpha,j}) \frac{\mu_j}{\Delta} + J_{\tau_k}(\underline{Q}_0 - e_\alpha) \frac{\Delta - \sum_{j \in S_\alpha} \mu_j}{\Delta},$$

and

$$J_{\tau_{k+1}}(\underline{Q}_0 - e_\beta) = \sum_{j \in S_\beta} J_{\tau_k}(\underline{Q}_0 - e_{\beta,j}) \frac{\mu_j}{\Delta} + J_{\tau_k}(\underline{Q}_0 - e_\beta) \frac{\Delta - \sum_{j \in S_\beta} \mu_j}{\Delta}.$$

Consider the following four cases:

*Case* 1. $\sum_{j=1}^{\beta} Q_{j,0} > m(t_0)$

In this case, $S = S_\alpha = S_\beta$. Thus,

$$
\begin{aligned}
& \mu_\alpha J_{\tau_{k+1}}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_{k+1}}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_{k+1}}(\underline{Q}_0) \\
= & \sum_{j \in S} \frac{\mu_j}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_{\alpha,j}) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_{\beta,j}) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0 - e_j) \Big) \\
& + \frac{\Delta - \sum_{j \in S} \mu_j}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0) \Big) \\
\geq & \ 0.
\end{aligned}
$$

*Case* 2. $\sum_{j=1}^{\beta} Q_{j,0} \leq m(t_0)$ and $\sum_{j=1}^{\alpha} Q_{j,0} > m(t_0)$

In this case, there exists a $\gamma$ such that $\beta \leq \gamma < \alpha$ and $\sum_{j=1}^{\gamma} Q_{j,0} \leq m(t_0)$, $\sum_{j=1}^{\gamma+1} Q_{j,0} > m(t_0)$. Clearly, $S = S_\alpha = S_\beta \cup \{\beta\} \setminus \{\gamma + 1\}$. Thus,

$$
\begin{aligned}
& \mu_\alpha J_{\tau_{k+1}}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_{k+1}}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_{k+1}}(\underline{Q}_0) \\
= & \sum_{j \in S \setminus \{\beta\}} \frac{\mu_j}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_{\alpha,j}) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_{\beta,j}) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0 - e_j) \Big) \\
& + \frac{\Delta - \sum_{j \in S} \mu_j}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0) \Big) \\
& + \frac{\mu_\beta}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_{\alpha,\beta}) - \mu_{\gamma+1} J_{\tau_k}(\underline{Q}_0 - e_{\beta,\gamma+1}) - (\mu_\alpha - \mu_{\gamma+1}) J_{\tau_k}(\underline{Q}_0 - e_\beta) \Big) \\
\geq & \ 0.
\end{aligned}
$$

*Case* 3. $\sum_{j=1}^{\alpha} Q_{j,0} \leq m(t_0)$ and $\sum_{j=1}^{n} Q_{j,0} > m(t_0)$

Analogous to the proof of Case 2, there exists a $\gamma$ such that $\alpha \leq \gamma < n$ and $\sum_{j=1}^{\gamma} Q_{j,0} \leq m(t_0)$, $\sum_{j=1}^{\gamma+1} Q_{j,0} > m(t_0)$. Clearly, $S = S_\alpha \cup \{\alpha\} \setminus \{\gamma + 1\} = S_\beta \cup \{\beta\} \setminus \{\gamma + 1\}$. Thus,

$$
\begin{aligned}
& \mu_\alpha J_{\tau_{k+1}}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_{k+1}}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_{k+1}}(\underline{Q}_0) \\
= & \sum_{j \in S \setminus \{\alpha,\beta\}} \frac{\mu_j}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_{\alpha,j}) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_{\beta,j}) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0 - e_j) \Big) \\
& + \frac{\Delta - \sum_{j \in S_\beta} \mu_j}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0) \Big) \\
& + \frac{\mu_{\gamma+1}}{\Delta} \Big( \mu_\alpha J_{\tau_k}(\underline{Q}_0 - e_{\alpha,\gamma+1}) - \mu_\beta J_{\tau_k}(\underline{Q}_0 - e_{\beta,\gamma+1}) - (\mu_\alpha - \mu_\beta) J_{\tau_k}(\underline{Q}_0 - e_{\gamma+1}) \Big)
\end{aligned}
$$

$$+\frac{(\mu_\alpha-\mu_\beta)}{\Delta}\Big(\mu_{\gamma+1}J_{\tau_k}(\underline{Q}_0-e_{\gamma+1})-\mu_\beta J_{\tau_k}(\underline{Q}_0-e_\beta)-(\mu_{\gamma+1}-\mu_\beta)J_{\tau_k}(\underline{Q}_0)\Big)$$
$$\geq\ 0.$$

*Case* 4. $\sum_{j=1}^{n} Q_{j,0} \leq m(t_0)$

Clearly, all the jobs are served and $S = S_\alpha \cup \{\alpha\} = S_\beta \cup \{\beta\}$. Thus,

$$\begin{aligned}
&\mu_\alpha J_{\tau_{k+1}}(\underline{Q}_0-e_\alpha)-\mu_\beta J_{\tau_{k+1}}(\underline{Q}_0-e_\beta)-(\mu_\alpha-\mu_\beta)J_{\tau_{k+1}}(\underline{Q}_0)\\
&= \sum_{j\in S\setminus\{\alpha,\beta\}}\frac{\mu_j}{\Delta}\Big(\mu_\alpha J_{\tau_k}(\underline{Q}_0-e_{\alpha,j})-\mu_\beta J_{\tau_k}(\underline{Q}_0-e_{\beta,j})-(\mu_\alpha-\mu_\beta)J_{\tau_k}(\underline{Q}_0-e_j)\Big)\\
&\quad+\frac{\Delta-\sum_{j\in S}\mu_j}{\Delta}\Big(\mu_\alpha J_{\tau_k}(\underline{Q}_0-e_\alpha)-\mu_\beta J_{\tau_k}(\underline{Q}_0-e_\beta)-(\mu_\alpha-\mu_\beta)J_{\tau_k}(\underline{Q}_0)\Big)\\
&\quad+\frac{\mu_\alpha\mu_\beta}{\Delta}\Big(J_{\tau_k}(\underline{Q}_0-e_\alpha)-J_{\tau_k}(\underline{Q}_0-e_\beta)\Big).
\end{aligned}$$

Since $\underline{Q}_0-e_\beta \leq_{ps} \underline{Q}_0-e_\alpha$, $J_{\tau_k}(\underline{Q}_0-e_\alpha) \geq J_{\tau_k}(\underline{Q}_0-e_\beta)$ from Corollary 2.9. It then follows from these four cases that (1) holds for $t \in [t_0, t_1)$.

Now take as the induction hypothesis that (1) holds for all $t \in [t_0, t_k)$. If $t_k$ is an epoch that $dm(t) \neq 0$, then it follows from the same argument, with $m(t_0)$ being replaced by $m(t_k)$, that (1) holds for all $t \in [t_0, t_{k+1})$. If $t_k$ is the arrival epoch of job $j$, then it follows from the same argument, with $\underline{Q}_0$ replaced by $\underline{Q}_0 + e_j$, that (1) holds for all $t \in [t_0, t_{k+1})$. Thus, (1) holds for any $t \geq 0$. ■

*Proof of Theorem 1.1.* (Sufficient condition) Again, we transform the continuous-time problem into a discrete-time one by using uniformization. Let $\{t_k, k \geq 0\}$ be defined as in the proof of Lemmas 2.8 and 2.10. Recall that $t_k$ is either an arrival epoch or an epoch that a machine breaks down or becomes available. Generate a sequence of independent Poisson processes, $N_l(t), l \geq 0$ with rates $\Delta_l(y) = ym(t_l)\mu_n$ for some $y \geq 1$. Let $\{\tau_{l,k}, k \geq 1\}$ be the arrival epochs of the $l$-th Poisson processes and define $\tau_{l,0} = 0$. Observe that the $\tau_{l,k}$'s are epochs that a job completion may occur in the coupling scheme we presented in the proof of Lemmas 2.8 and 2.10. Analogous to the proof of Pinedo (1983), Theorem 1, we first show that $\pi$ is optimal within the class of policies $\mathcal{G}(y)$, where $\mathcal{G}(y)$ consists of all policies that allow preemption only at time epochs $\{t_l + \tau_{l,k}, k \geq 0, l \geq 0\}$ with $t_l + \tau_{l,k} < t_{l+1}$. Then it follows from a continuity argument as $y \to \infty$ that $\pi$ is optimal among all the policies.

To simplify the notations, let $\{\tau'_k, k \geq 0\}$ be the sequence of $\{t_l + \tau_{l,k}, k \geq 0, l \geq 0\}$ after sorting. We will use induction on $\{\tau'_k\}$. First, we show $\pi$ is optimal in a single step for all initial states $\underline{Q}_0$. If $\tau'_1$ is the arrival epoch of job $j$, then the state of jobs at $\tau'_1$ is $\underline{Q}_0 + e_j$ for any policy. This implies the costs at $\tau_1'$ are the same for any policy. If $\tau_1'$ is an epoch that a machine breaks down or becomes available, i.e. $dm(t) \neq 0$, then the state of jobs at $\tau_1'$ is $\underline{Q}_0$ for any policy. Again, this implies the costs at $\tau_1'$ are the same for any policy. Thus, it suffices to consider the case that $\tau_1'$ is not a point of $\{t_k, k \geq 1\}$. Consider a policy $\sigma \in \mathcal{G}(y)$. Let $X_{j,0}^{\sigma}$ be the indicator function of the event that job $j$ is served at time 0 under a policy $\sigma$ and let $\underline{X}_0^{\sigma} = (X_{1,0}^{\sigma}, \ldots, X_{n,0}^{\sigma})$. Let $\underline{X}_0^{\pi}$ be the corresponding quantity under $\pi$. Since the policy $\pi$ schedules jobs according to priority of the lowest index, we have that $\underline{X}_0^{\sigma} \leq_{ps} \underline{X}_0^{\pi}$ for any policy $\sigma$ in $\mathcal{G}(y)$. ¿From Lemma 2.7, it follows that $\underline{X}_0^{\sigma}$ can be reached from $\underline{X}_0^{\pi}$ through a finite number of applications of the following two types of transfers: (i) $\underline{x} \mapsto \underline{x} - e_\beta$ and (ii) $\underline{x} \mapsto \underline{x} + e_\alpha - e_\beta$, $\alpha > \beta$. Thus, we only need to compare the expected cost at $\tau_1'$ for two policies $\sigma^1$ and $\sigma^2$ with $\underline{X}_0^{\sigma^1} = T(\underline{X}_0^{\sigma^2})$, where $T$ is of the form (i) or (ii). Now let $\underline{Q}_{\tau_1'}^{\sigma^i}$ be the states of jobs at $\tau_1'$ under the policy $\sigma^i$, $i = 1, 2$. It suffices to show that

$$E(g(\underline{Q}_{\tau_1'}^{\sigma^1})|\underline{Q}_0) \geq E(g(\underline{Q}_{\tau_1'}^{\sigma^2})|\underline{Q}_0). \tag{2}$$

*Case* 1: The first type of transfer

In this case, $\underline{X}_0^{\sigma^1} = \underline{X}_0^{\sigma^2} - e_\beta$ for some $\beta$. This is equivalent to inserting idle time to a machine. Clearly, we have $\underline{Q}_{\tau_1'}^{\sigma^1} \geq \underline{Q}_{\tau_1'}^{\sigma^2}$. It then follows from (A2) that (2) holds.

*Case* 2: The second type of transfer

In this case, $\underline{X}_0^{\sigma^1} = \underline{X}_0^{\sigma^2} + e_\alpha - e_\beta$ for some $\alpha > \beta$ (Clearly, we assume $X_{\alpha,0}^{\sigma^1} = 1$ and $X_{\beta,0}^{\sigma^1} = 0$). The only difference between $\sigma^1$ and $\sigma^2$ is that $\sigma^2$ puts the low index job $\beta$ into service instead of the high index job $\alpha$. Now define $S^i = \{j : X_{j,0}^{\sigma^i} = 1\}$ as the set of jobs served at time 0 under policy $\sigma^i$. Then for $i = 1, 2$,

$$P(\underline{Q}_{\tau_1'}^{\sigma^i} = \underline{Q}_0 - e_j|\underline{Q}_0) = \frac{\mu_j}{\Delta_0(y)}, \quad j \in S^i \tag{3}$$

$$P(\underline{Q}_{\tau_1'}^{\sigma^i} = \underline{Q}_0|\underline{Q}_0) = \frac{\Delta_0(y) - \sum_{j \in S^i} \mu_j}{\Delta_0(y)}, \quad j \notin S^i. \tag{4}$$

Thus, we have from (A3) that

$$\begin{aligned}
& E(g(\underline{Q}_{\tau_1'}^{\sigma^1})|\underline{Q}_0) - E(g(\underline{Q}_{\tau_1'}^{\sigma^2})|\underline{Q}_0) \\
= \ & \frac{1}{\Delta_0(y)} \Big( \mu_\alpha g(\underline{Q}_0 - e_\alpha) - \mu_\beta g(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) g(\underline{Q}_0) \Big) \\
\geq \ & 0
\end{aligned}$$

Therefore, $\pi$ is optimal in a single step.

Take as the induction hypothesis that $\pi$ is optimal over $k-1$ steps. Now we would like to show that the policy that applies $\sigma^2$ at time 0 and then applies $\pi$ after $\tau_1'$ is better than the policy that applies $\sigma^1$ at time 0 and then applies $\pi$ after $\tau_1'$. Now let $\underline{Q}^{\sigma^i}_{\tau_k'}$, $i=1,2$, be the states of jobs at $\tau_k'$ under these two policies. It suffices to show that

$$E(g(\underline{Q}^{\sigma^1}_{\tau_k'})|\underline{Q}_0) \geq E(g(\underline{Q}^{\sigma^2}_{\tau_k'})|\underline{Q}_0). \tag{5}$$

Again, we only have to consider the case that $\tau_1'$ is not a point of $\{t_k, k \geq 1\}$. For the first type of transfer, we have that $\underline{Q}^{\sigma^1}_{\tau_1'} \geq \underline{Q}^{\sigma^2}_{\tau_1'}$. This implies $\underline{Q}^{\sigma^1}_{\tau_1'} \geq_{ps} \underline{Q}^{\sigma^2}_{\tau_1'}$. It then follows from Corollary 2.9 that (5) holds. Analogous to (4), we have from the transition probabilities of the uniformized Markov chains that for the second type of transfer,

$$\begin{aligned}
&E(g(\underline{Q}^{\sigma^1}_{\tau_k'})|\underline{Q}_0) - E(g(\underline{Q}^{\sigma^2}_{\tau_k'})|\underline{Q}_0) \\
&= \frac{1}{\Delta_0(y)}\Big(\mu_\alpha J_{\tau_1',\tau_k'}(\underline{Q}_0 - e_\alpha) - \mu_\beta J_{\tau_k',\tau_1'}(\underline{Q}_0 - e_\beta) - (\mu_\alpha - \mu_\beta) J_{\tau_1',\tau_k'}(\underline{Q}_0)\Big).
\end{aligned}$$

where $J_{\tau_1',\tau_k'}(\underline{Q}_0) = E(g(\underline{Q}_{\tau_k'})|\underline{Q}_{\tau_1'})$. ¿From Lemma 2.10 and the induction hypothesis, it follows that $\pi$ is optimal within $\mathcal{G}(y)$. A continuity argument then completes the proof that the conditions in theorem 1.1 are sufficient for optimality of $\pi$.

(Necessary condition) We would like to show that if the policy $\pi$ minimizes the expected cost for all $t$ and any realization of machine breakdowns and repairs then (A2) and (A3) must hold for all initial states $\underline{Q}_0$. This means we may consider a problem with $l$ machines available at time 0, where $l = \sum_{i=1}^{\beta} Q_{i,0}$. Clearly, the policy $\pi$ begins by assigning to machines all jobs whose index is not larger than $\beta$. Call this set of jobs $S$. Then the expected cost of the policy $\pi$ after a very small amount of time $\delta t$ is

$$\sum_{i\in S} g(\underline{Q}_0 - e_i)\mu_i \delta t + \Big(1 - \sum_{i \in S} \mu_i \delta t\Big) g(\underline{Q}_0) + o(\delta t). \tag{6}$$

Now consider a policy $\sigma^1$ which schedules jobs having indices smaller than $\beta$ onto $l-1$ machines and keeps the $l$-th machine idle. Let $\sigma^1$ be the policy that is the same as $\pi$ but schedules job $\alpha$ instead of job $\beta$, $\alpha > \beta$. The expected costs of the policies $\sigma^1$ and $\sigma^2$, after a very small amount of time $\delta t$, can be computed similarly. It is easy to see that the difference between the expected costs of the policies $\pi$ and $\sigma^1$ at time $\delta t$ is

$$g(\underline{Q}_0 - e_\beta)\mu_\beta \delta t - g(\underline{Q}_0)\mu_\beta \delta t + o(\delta t). \tag{7}$$

The optimality of $\pi$ for all $t$, when there are $l$ machines available at time at time 0, implies that the quantity in (7) must not be greater than 0. Letting $\delta t \to 0$ yields (A2). Similarly, the difference between the expected costs of the policy $\pi$ and $\sigma^2$ at time $\delta t$ is

$$g(\underline{Q}_0 - e_\beta)\mu_\beta \delta t - g(\underline{Q}_0)\mu_\beta \delta t - g(\underline{Q}_0 - e_\alpha)\mu_\alpha \delta t + g(\underline{Q}_0)\mu_\alpha \delta t + o(\delta t) \tag{8}$$

The optimality of $\pi$ implies that the quantity in (8) must not be greater than 0. Letting $\delta t \to 0$ yields (A3). ■

## 3 The optimality of the $c\mu$-rule

In this section, we consider the cost function $g(\underline{x}) = \sum_{j=1}^{n} c_j x_j$. In other words, for each unit time that job $j$ remains in the system, a cost $c_j$ is incurred. Assuming the agreeability condition (A4) of section 1, we use Theorem 1.1 to establish optimality criteria for the $c\mu$-rule. Recall that the $c\mu$-rule is the preemptive policy that orders jobs in increasing priority according to increasing values of $c\mu$. Under (A4), the $c\mu$-rule results in the same order as the LEPT rule. The objective functions that are considered in this section include the expected values of (i) the weighted number of jobs in system at an arbitrary time $t$, (ii) the weighted sum of job completion times, (iii) the weighted number of late jobs when the jobs have a common random due date, and (iv) the weighted sum of job tardinesses when the jobs have a common due date.

**Corollary 3.1** *(Minimization of the Expected Weighted Number of Jobs at Arbitrary Time $t$) Assume (A1) holds. Let $\pi$ be the policy that schedules jobs according priorities that are decreasing in their indices. Then $\pi$ minimizes $E[\sum_{j=1}^{n} c_j Q_{j,t}]$, the expected weighted number of jobs, for all $t$ ($t > 0$), and all processes of arrivals, breakdowns and repairs,* if and only if *(A4) is satisfied. Preemption need only occur at the release of a new job.*

*Proof.* For $g(\underline{x}) = \sum_{i=1}^{n} c_i x_i$, it is easy to see that (A2) $\Leftrightarrow c_i \geq 0$ and that (A3) $\Leftrightarrow c_i\mu_i \geq c_j\mu_j$, $i < j$. ■

In fact, Corollary 3.1 also follows immediately from the known result that LEPT minimizes the makespan *stochastically*. Recall, $V_{j,t} = Q_{j,t}/\mu_j$. Observe that the objective function is an arrangement increasing function of the expected work remaining at time $t$, i.e.

$$E\left[\sum_{j=1}^{n} c_j Q_{j,t}\right] = E\left[\sum_{j=1}^{n} c_j \mu_j V_{j,t}\right] = \sum_{i=1}^{n}(c_i\mu_i - c_{i+1}\mu_{i+1})E\left[\sum_{j=1}^{i} V_{j,t}\right],$$

where we take $c_{n+1}\mu_{n+1} = 0$. Since LEPT minimizes the expected work remaining at every time $t$, it is clear that the summation on the right hand side above is minimized by LEPT.

It is clear that without the agreeability condition (A1), the $c\mu$-rule itself cannot be optimal. Counterexamples can be found, even under Kämpke's weaker agreeability condition of $c_1 \geq \cdots \geq c_n$ and (A4). Consider the following set of jobs: $c_j\mu_j = 1$, $j = 1, \ldots, n-1$, $\mu_j = n-1$, $j = 1, \ldots, n-1$, $\mu_n = \frac{1}{2}$ and $c_j = 1/(n-1)$, $j = 1, 2, \ldots, n$. There are two machines and all $n$ jobs have a common (fixed) due date at 2. Take $n$ very large. The first $n-1$ jobs require a total amount of processing equal to 1. The variance in this total amount goes to zero as $n$ tends to infinity. It is clear that job $n$ will start under the $c\mu$-rule at time $\frac{1}{2}$. It is also clear that starting job $n$ (the large job) at time zero significantly increases the probability that it is completed by time 2, and therefore maximizes the expected number of jobs completed by the due date. For an example in which the $c\mu$-rule does not minimize the expected sum of the weighted completion times, consider again two machines and the same set of jobs as the one described above and all $n$ jobs available at time zero. Now, instead of a due date at time 2, a second batch of jobs arrives at time 2. It is clear that the $c\mu$-rule does not minimize the expected sum of the weighted completion times. It is necessary to start the long job at time 0 in order to maximize the expected machine utilization before time 2.

**Corollary 3.2** *(Minimization of the Expected Weighted Number of Late Jobs) Assume (A1) and (A4) hold, and the jobs have a common due date, $D$. Let $Z_j$ be the completion time of job $j$. Let $U_j$ be the indicator function for the event $[Z_j \geq D_j]$. Then $\pi$ minimizes $E[\sum_{j=1}^n c_j U_j]$.*

*Proof.* Consider the event $\{R_j = r_j, j = 1, \ldots, n, D = d\}$. On this event, $U_j = Q_{j,d}$ if $r_j < d$ and 1 if $r_j \geq d$. Applying Corollary 3.1 completes the proof. ∎

**Remark 3.3** Corollary 3.2 is an extension of Pinedo and Rammouz (1988), Theorem 2, to parallel machines. Pinedo (1983), Theorem 8, also showed that if $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, $c_1 \geq c_2 \geq \ldots \geq c_n$, and the common due date $D$ has a concave distribution function, then the SEPT rule minimizes $E[\sum_{j=1}^n c_j U_j]$. Corollary 3.2, states a different set of conditions under which the LEPT rule is optimal.

The following definition is useful in obtaining a result for more general weighted functions of completion times. The subsequent corollary extends Pinedo and Rammouz (1988), Theorem 1, to parallel machines.

**Definition 3.4** *(Pinedo and Rammouz (1988)) Let $F_1(t)$ and $F_2(t)$ be two increasing functions. $F_1$ is said to be steeper than $F_2$ if $F_1(t_2) - F_1(t_1) \geq F_2(t_2) - F_2(t_1)$ for all $t_1 < t_2$. We denote this by $F_1 \overset{\geq}{s} F_2$.*

**Corollary 3.5** *(Minimization of the Expected Weighted Sum of Job Completion Times) Suppose (A1) and (A4) hold, and $F_j(t), j = 1, \ldots, n$ are increasing functions, such that $F_1 \overset{>}{s} F_2 \overset{>}{s} \ldots \overset{>}{s} F_n$. Then $\pi$ minimizes $E[\sum_{j=1}^{n} c_j F_j(Z_j)]$.*

*Proof.* It is noted in Pinedo and Rammouz (1988) that Corollary 3.2 is equivalent to minimizing $E[\sum_{j=1}^{n} c_j F(Z_j)]$, where $F(t)$ is the distribution function of the common due date $D$. Thus, $\pi$ minimizes $E[\sum_{j=1}^{n} c_j F(Z_j)]$ for all $F$ increasing. Note that $E[\sum_{j=1}^{n} c_j F_j(Z_j)] = \sum_{i=1}^{n} B_i$ with $B_i = \sum_{j=1}^{n+1-i} c_j E[F_{n+1-i}(Z_j) - F_{n+2-i}(Z_j)]$, adn taking $F_{n+1}(t) = 0$. Now $F_j \overset{>}{s} F_{j+1}$ implies that $F_j - F_{j+1}$ is increasing. Combined with the fact that $F_n$ is increasing, this implies that $\pi$ minimizes $B_i$, $i = 1, \ldots, n$, and thus that it minimizes the sum of the $B_i$'s. ■

**Corollary 3.6** *(Minimization of the Expected Weighted Sum of Job Tardiness) The tardiness of job $j$, $T_j$, is defined as $max(Z_j - D_j, 0)$. Suppose (A1) and (A4) hold, and $D_1 \leq D_2 \leq \ldots \leq D_n$ a.s. Then $\pi$ minimizes $E[\sum_{j=1}^{n} c_j T_j]$.*

*Proof.* Consider the event $\{D_j = d_j, j = 1, \ldots, n\}$. The objective function is equivalent to $E[\sum_{j=1}^{n} c_j F_j(Z_j)]$, with $F_j(t) = \max(t - d_j, 0)$ (Pinedo and Rammouz (1988)). Clearly, the $F_j(t)$'s are increasing and $F_j \overset{>}{s} F_{j+1}$. An application of Corollary 3.5 completes the proof. ■

**Remark 3.7** Pinedo (1983), Theorem 3, showed that if $D_1 \leq D_2 \leq \ldots \leq D_n$ *a.s.* and (A4) is satisfied, then $\pi$ minimizes $E[\sum_{j=1}^{n} c_j T_j]$ for a single machine problem with all the jobs present at time 0. Corollary 3.6 is an extension of that theorem to parallel machines with release dates.

In the following two corollaries, we consider the special case that all jobs are present at time 0. A static list policy is one that assigns jobs to machines in the order of a fixed permutation of their indices. Clearly, the set of static list policies is a subset of the dynamic policies we have considered thus far, and $\pi$, which under (A1) assigns jobs in the order $1, \ldots, n$, is optimal amongst all static list policies when (A1) and (A4) are satisfied. In the following two corollaries the conditions needed in Corollaries 3.2 and 3.6 are relaxed from the strong sense (*a.s.*) to the weak sense (distribution). These corollaries generalise Pinedo (1983), Theorems 4 and 2, to parallel machines. Recall that a r.v. $X$ is stochastically smaller $\leq_{st}$ than $Y$ if $\mathrm{P}(X > t) \leq \mathrm{P}(Y > t)$ for all $t$.

**Corollary 3.8** *(Minimization of the Expected Weighted Number of Late Jobs) Assume (A1) and (A4) hold, and due dates $D_j$, $j = 1, \ldots, n$, have a common distribution function $F(t)$. Let*

*$U_j$ be the indicator function for the event that job $j$ is late. Then $\pi$ minimizes $E[\sum_{j=1}^n c_j U_j]$ amongst all the static list policies.*

*Proof.* Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a permutation of $\{1, 2, \ldots, n\}$ and suppose the jobs are scheduled according to the static list policy that assigns jobs to machines in priority order $\sigma_1, \sigma_2, \ldots, \sigma_n$. Under these circumstances, let $Q^\sigma_{j,t}$ be the indicator function for the event that job $j$ is in the system at time $t$ and let $U^\sigma_j$ be the indicator function that job $j$ is late. ¿From Corollary 3.1, it follows that under (A1) and (A4)

$$\sum_{j=1}^n E c_j Q^\pi_{j,t} \leq \sum_{j=1}^n E c_j Q^\sigma_{j,t}. \tag{9}$$

All the jobs are present at time 0, so $U^\sigma_j = Q^\sigma_{j,D_j}$. Since the order in which jobs will be assigned to machines is determined at time 0, it follows that $Q^\sigma_{j,t}$ is independent of the due date $D_j$. (This is not true if the policy is dynamic.) We have

$$E Q^\sigma_{j,D_j} = \int_0^\infty E Q^\sigma_{j,t}\, dF(t). \tag{10}$$

Combining (9) and (10) completes the proof. ■

**Corollary 3.9** *(Minimization of the Expected Weighted Sum of Job Tardinesses) Assume (A1) and (A4) hold and due dates satisfy $D_1 \leq_{st} D_2 \leq_{st} \ldots \leq_{st} D_n$. Let $T_j$ be the tardiness of job $j$. Then $\pi$ minimizes $E[\sum_{j=1}^n c_j T_j]$ among all static list policies.*

*Proof.* Again, let $\sigma$ be a permutation of $\{1, 2, \ldots, n\}$ and define $Q^\sigma_{j,t}$ as above. Let $T^\sigma_j$ be the tardiness of job $j$ when jobs are processed under the static list policy defined by $\sigma$. All the jobs are present at time 0, so $T^\sigma_j = \int_{D_j}^\infty Q^\sigma_{j,t}\, dt$. Again, $Q^\sigma_{j,t}$ is independent of $D_j$ and we have

$$\begin{aligned} E T^\sigma_j &= \int_0^\infty \int_s^\infty E Q^\sigma_{j,t}\, dt\, dF_j(s) \\ &= \int_0^\infty F_j(t) E Q^\sigma_{j,t}\, dt, \end{aligned}$$

where $F_j(t)$ is the distribution function of the due date of job $j$, $D_j$. Analogous to the proof of Corollary 3.5, $E[\sum_{j=1}^n c_j T^\sigma_j] = \sum_{i=1}^n B_i$, where

$$B_i = \int_0^\infty (F_{n+1-i}(t) - F_{n+2-i}(t)) \sum_{j=1}^{n+1-i} c_j E Q^\sigma_{j,t}\, dt$$

and we define $F_{n+1}(t) = 0$. The assumption $D_1 \leq_{st} D_2 \leq_{st} \ldots \leq_{st} D_n$ implies $F_1(t) \geq F_2(t) \geq \ldots \geq F_n(t) \geq 0$. It follows from Corollary 3.1 that $\pi$ minimizes $B_i$, $i = 1, \ldots, n$, and thus minimizes the sum of the $B_i$'s. ■

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