Twister Networks and Their Applications to Load-Balanced Switches

Ching-Min Lien, Cheng-Shang Chang, Jay Cheng, Duan-Shin Lee and Jou-Ting Liao
Institute of Communications Engineering
National Tsing Hua University
Hsinchu 300, Taiwan, R.O.C.
E-mail: keiichi@gibbs.ee.nthu.edu.tw; cschang@ee.nthu.edu.tw;
jcheng@ee.nthu.edu.tw; lds@cs.nthu.edu.tw; jtliao@gibbs.ee.nthu.edu.tw

***Abstract*—Inspired by the recent development of optical queueing theory, in this paper we study a class of multistage interconnection networks (MINs), called *twister networks*. Unlike the usual recursive constructions of MINs (either by two-stage expansion or by three-stage expansion), twister networks are constructed *directly* by a concatenation of bipartite networks. Moreover, the biadjacency matrices of these bipartite networks are sums of subsets of the powers of the circular shift matrix. Though MINs have been studied extensively in the literature, we show there are several *distinct* properties for twister networks, including routability and conditionally nonblocking properties. In particular, we show that a twister network satisfying (A1) in the paper is routable, and packets can be self-routed through the twister network by using the $\mathcal{C}$-transform developed in optical queueing theory. Moreover, we define an N-modulo distance and use it to show that a twister network satisfying (A2) in the paper is conditionally nonblocking if the N-modulo distance between any two outputs is not greater than two times of the N-modulo distance between the corresponding two inputs. Such a conditionally nonblocking property allows us to show that a twister network with N inputs/outputs can be used as a $p \times p$ rotator and a $p \times p$ symmetric TDM switch for any $2 \leq p \leq N$. As such, one can use a twister network as the switch fabric for a two-stage load balanced switch that is capable of providing incremental update of the number of linecards.**

I. Introduction

Multistage Interconnection Networks (MINs) are commonly used for the constructions of switch fabric and they have been studied extensively in the literature (see e.g., the books [18], [16], [7]). In particular, it is well known that an $N \times N$ Benes network [3] can be constructed recursively by the three-stage Clos network [8] and it is capable of realizing all the $N!$ permutations between its N inputs and its N outputs. As such, an $N \times N$ Benes network is known as a nonblocking switch like an $N \times N$ crossbar switch fabric. On the other hand, an $N \times N$ banyan network, constructed by using the two-stage expansion, can only realize a subset of the $N!$ permutations and it is known as a conditionally nonblocking switch [16].

Inspired by the recent development of optical queueing theory [9], [6], in this paper we consider a class of MINs *directly* constructed by a concatenation of bipartite networks (CBN). *Twister networks* form a subclass of CBN, where the biadjacency matrices of the bipartite graphs used for characterizing the interconnections between stages are sums of subsets of the powers of the circular shift matrix. In spite of the similarity to chordal rings [1], circulant [10, p. 8], and data vortex [19], [11] discussed in the literature, there are several *distinct* properties for the twister networks, including routability and conditionally nonblocking properties. In particular, we show that twister networks satisfying (A1) in Section II are routable, and packets can be self-routed through such networks by using the $\mathcal{C}$-transform developed in [9], [6]. Moreover, we define an N-modulo distance and use that to show that a special type of twister networks that satisfy (A2) in Section II are conditionally nonblocking if the N-modulo distance between any two outputs is not greater than two times of the N-modulo distance between the corresponding two inputs.

One of the most important applications of our development is that twister networks can be used for providing incremental update of the number of linecards in the two-stage load balanced switches [4], [14], [15], [13]. In the literature, there are two well-known conditionally nonblocking switches, rotators (that implements all the powers of the circular shift matrix) and symmetric TDM switches, that can be used for generating the needed connection patterns for the switch fabrics in the two-stage load balanced switches. In this paper, we show that a twister network with N inputs/outputs can be used for exact emulation of a $p \times p$ rotator and a $p \times p$ symmetric TDM switch for any $2 \leq p \leq N$. Moreover, we depicts a placement rule for adding a new linecard to a twister network so that all the existing linecards need not be changed. As such, twister networks can be used for solving the incremental update problem for an arbitrary number of linecards in [15]. As twister networks are capable of self-routing, the new routing paths after an incremental update can be easily determined by the $\mathcal{C}$-transform. This is much better than using the Benes networks as the new routing paths in the Benes networks cannot be easily determined.

This paper is organized as follows. In Section II, we introduce a class of MINs by a concatenation of bipartite networks. We formally define twister networks in Section III and address its routability problem. In Section IV, we show the conditionally nonblocking property for twister networks. Then we show they can be used for exact emulation of rotators and symmetric TDM switches in Section V. We conclude the paper in Section VI by addressing possible extensions of our work.

II. Bipartite Network and Its Concatenation

A graph $G(V, E)$ (see Fig. 1(a)) is called a *directed bipartite graph* if all the nodes V can be partitioned into two sets: the set of input nodes X and the set of output nodes Y. Moreover, all the nodes in the same set are not adjacent to each other, and all the directed edges in E are from the input set X to the output set Y. A directed bipartite graph can be characterized by its biadjacency matrix [2]. Specifically, let $|X| = N$ (resp. $|Y| = L$) and all the nodes in X are indexed from 0 to $N-1$ (resp. all the nodes in Y are indexed from 0 to $L-1$). Hence, the links connecting X and Y can be represented by the biadjacency matrix $A = [a_{ij}]_{N\times L}$, where $a_{ij} = 1$ if there is a directed edge from node i in X to node j in Y and $a_{ij} = 0$ otherwise [2]. We assume that there is no multiple edges connecting each pair of nodes, and all the links have the same weight in this paper.

Switching networks consisting of a set of switches and links are commonly used to route packets from input ports to output ports. It is well known that a switching network can be represented by a directed graph by viewing each switch (resp. link) in the switching network as a node (resp. directed edge) in the graph. In this paper, we are particularly interested in the switching networks that can be represented by directed bipartite graphs. Such switching networks are called *bipartite networks* in this paper. As a directed bipartite graph, a bipartite network can also be characterized by its biadjacency matrix.

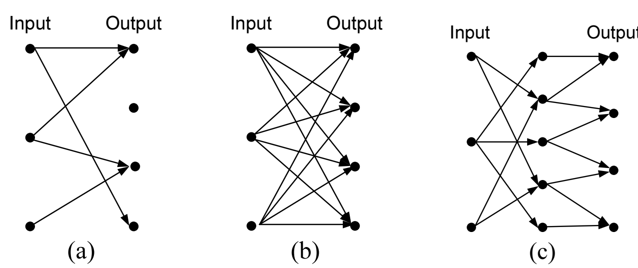


Fig. 1. (a) A directed bipartite graph (b) A complete bipartite graph (c) A three-stage concatenation of bipartite networks

In a bipartite network, one may not be able to route a packet from any input node to any output node. To route packets for all possible input/output pairs, the bipartite network has to form a complete bipartite graph as shown in Fig. 1(b). However, we can alternatively route packets through a concatenation of bipartite networks as shown in Fig. 1(c), whereas none of the bipartite networks is complete. Hence, to provide better routing capability, we are motivated to concatenate various bipartite networks to form a multistage interconnection network (MIN). Such a MIN is called *a concatenation of bipartite networks* (CBN). Specifically, in a CBN with $M+1$ stages, all the nodes in the CBN can be partitioned into $M+1$ independent sets for some $M \geq 1$. These independent sets are called stages in this paper, and are indexed from 0 to M. Each link in a CBN lies only between two consecutive stages. Thus, every two consecutive stages and the associated links between these stages constitute a directed bipartite graph that can be characterized by a biadjacency matrix. As such, a CBN with $M + 1$ stages and N_k, $k = 0, 1, 2, \dots, M$, nodes at the k^{th} stage is characterized by M biadjacency matrices $A^{(k)}$, $1 \leq k \leq M$, where $A^{(k)}$ is the $N_{k-1} \times N_k$ biadjacency matrix for the bipartite network between the $(k-1)^{th}$ stage and the k^{th} stage.

A CBN can be used as a switching network to route packets from a subset of nodes at stage 0 to another subset of nodes at stage M. For this, we name the i^{th} node at stage 0 as input i and the j^{th} node at stage M as output j of the CBN, respectively. The j^{th} node at the k^{th} stage is denoted as node (k, j), and these two representations will be used interchangeably in this paper. As each node in a CBN is in fact a switch, the links from the nodes at the stage right before (resp. to the nodes at the stage right after) are its input links (resp. output links).

We assume all the nodes in a CBN are *nonblocking* switches. Specifically, if there are m input links and n output links for a node, then that node is an $m \times n$ switch that can implement all the $m \times n$ sub-permutation matrices. Note that all the elements in a sub-permutation matrix are either 0 or 1 and there is at most a 1 in each row or column. As such, each input link can only be connected to at most one output link, and vice versa.

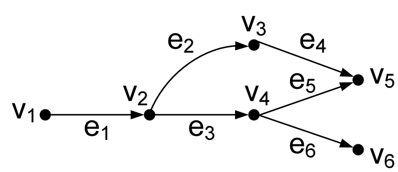


Fig. 2. A directed graph with six nodes and six edges.

As a CBN is used as a switching network, the first question about a CBN is whether it is able to route packets to the corresponding output ports. In other words, one wonders whether there exists a *routing path* for any input/output pair (i, o) or not. For a directed graph, a routing path can be described by a sequence of consecutive vertices and edges. For example, $(v_1, e_1, v_2, e_2, v_3, e_4, v_5)$ is a routing path from node v_1 to v_5 in Fig. 2. As we assume there is at most one edge between each pair of nodes in this paper, the routing path can be simply written as (v_1, v_2, v_3, v_5). Then, the routability of a CBN can be defined as follows.

Definition 1 *A CBN is routable if there exists at least one routing path for any input/output pair* (i, o).

For a CBN, we state in the following proposition a method to compute the number of routing paths between each input/output pair (i, o). The argument (omitted here) is the same as the well-known method for computing the number of walks of a certain length between two nodes in a graph (see e.g., Lemma 2.5 [2, p. 11]).

Proposition 2 *Consider a CBN with $M+1$ stages. Let N_k be the number of nodes at the k^{th} stage and $A^{(k)}$ be the biadjacency matrix for the bipartite network between stages $k-1$ and k, $k=1,2,\ldots,M$.*

(i) *The $(i,j)^{th}$ element in the product $\prod_{k=1}^{M} A^{(k)}$ of all the biadjacency matrices is the number of paths that a packet from input i to output j can be routed through.*

(ii) *Under the Boolean algebra (for matrix multiplication), the $(i,j)^{th}$ element in the product $\prod_{k=1}^{M} A^{(k)}$ of all the biadjacency matrices is 1 if and only if there exists a routing path for the packet from input i to output j and is 0 otherwise.*

According to Proposition 2, we have the following corollary for the routability of a CBN.

Corollary 3 *Consider a CBN with $M+1$ stages. Let N_k be the number of nodes at the k^{th} stage and $A^{(k)}$ be the biadjacency matrix for the bipartite network between stages $k-1$ and k, $k=1,2,\ldots,M$. The following three statements are equivalent.*

(i) *The CBN is routable.*

(ii) *Each element in the $N_0 \times N_M$ matrix $\prod_{k=1}^{M} A^{(k)}$ is nonzero.*

(iii) *Each element in the $N_0 \times N_M$ matrix $\prod_{k=1}^{M} A^{(k)}$ is one under the Boolean algebra (for matrix multiplication).*

III. Twister Network

In this section, we introduce a special class of CBNs, which is called *twister network*. In this paper, an $N \times N$ twister network is constructed according to the circular shift matrix P, which is defined as

$$p_{ij} = \begin{cases} 1 & \text{, if } j = (i+1) \bmod N \\ 0 & \text{, otherwise} \end{cases}, \tag{1}$$

where $0 \le i,j \le N-1$. Notice that $\sum_{n=0}^{N-1} P^n$ is an $N \times N$ matrix with all its elements being one. If one constructs a CBN such that the product of all the biadjacency matrices under the Boolean algebra contains the summation $\sum_{n=0}^{N-1} P^n$, then the CBN is a routable switching network from Corollary 3. That is the motivation for us to define twister networks below.

Definition 4 ***((r,d)-twister network)*** *Let $\mathbf{r} = (r_1, r_2, \ldots, r_M)$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_M)$ be two M-vectors. A CBN with $M+1$ stages is called an $N \times N$ **(r,d)**-twister network if there are N nodes in each stage and the biadjacency matrix for the bipartite network between stages $k-1$ and k is*

$$A^{(k)} = \sum_{j=0}^{r_k-1} P^{j \cdot d_k} = I + P^{d_k} + P^{2d_k} + \cdots + P^{(r_k-1)d_k}$$

for all $1 \le k \le M$, where P is the circular shift matrix.

In Fig. 3(a), we show a 6×6 $(\mathbf{r},\mathbf{d})$-twister network with $\mathbf{r} = (2,2)$ and $\mathbf{d} = (1,2)$. If we place *evenly* all the nodes at the same stage on a circle, the twister network can be viewed as a cylinder, as shown in Fig. 3(b). The links induced by nonzero multiples of circular shift matrices constititue the screw threads on the cylinder, which is the motivation for the name *twister*.

We note that twister networks bear some similarity to chordal rings [1], circulant [10, p. 8], and data vortex [19], [11] discussed in the literature. In particular, if we fold all the nodes with the same index in all the stages into a single node, then an $N \times N$ (**r**,**d**)-twister network becomes a circulant $X(Z_N, C)$ [10, p. 8], where $Z_N = \{0,1,\ldots,N-1\}$ and C is the union of all the C_k's and $C_k = \{j \cdot d_k | 0 \le j \le r_k - 1\}$ for all $1 \le k \le M$. Note that a circulant is a graph with its adjacency matrix being cyclo-symmetric. The data vortex studied in [19], [11] has several parallel cylinders and it is mainly for deflection routing. In such a MIN, all the nodes are 2×2 switches and there is a one unit delay in every link.

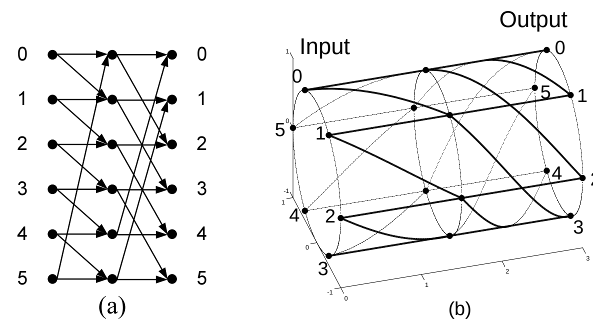


Fig. 3. (a) A 6×6 (**r**,**d**)-twister network with $\mathbf{r} = (2,2)$ and $\mathbf{d} = (1,2)$ (b) Another point of view of this 6×6 twister network.

A. Routability and Generalized $\mathcal{C}$-transform

Notice that an $N \times N$ (**r**,**d**)-twister network may not be routable. Consider the 6×6 twister network shown in Fig. 3(a). The packet from input 0 cannot be routed to output 4 and output 5 in this twister network. For the routability of a twister network, we introduce the generalized $\mathcal{C}$-transform for integer representation in [6].

Definition 5 ***(Generalized $\mathcal{C}$-transform)*** *Consider two M-vectors $\boldsymbol{r} = (r_1, r_2, \ldots, r_M)$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_M)$ with $r_i \in \{2,3,4\ldots\}$ and $d_i \in \{1,2,3\ldots\}$, $i = 1,2\ldots,M$. Define a mapping $C^g : \{0,1,2,\ldots\} \to \{0,1,\ldots,r_1-1\} \times \{0,1,\ldots,r_2-1\} \times \cdots \times \{0,1,\ldots,r_M-1\}$ as follows:*

$$C^g(x) = (I_1^g(x), I_2^g(x), \ldots, I_M^g(x)) \tag{2}$$

where $I_M^g(x), I_{M-1}^g(x), \ldots, I_1^g(x)$ (in that order) are given recursively by

$$I_i^g(x) = \begin{cases} r_i - 1, & \text{if } x - \sum_{k=i+1}^{M} I_k^g(x) \cdot d_k \\ & \quad \ge (r_i-1)d_i, \\ j, & \text{if } jd_i \le x - \sum_{k=i+1}^{M} I_k^g(x) \cdot d_k \\ & \quad < (j+1)d_i \\ & \quad \text{for some } 0 \le j \le r_i - 2, \end{cases} \tag{3}$$

where we adopt the convention that the sum in (3) is zero if the upper index is smaller than its lower index.

The M-vector $C^g(x)$ is called the generalized $\mathcal{C}$-transform of x with respect to $\mathbf{r} = (r_1, r_2, \ldots, r_M)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_M)$. Intuitively, one can view $C^g(x)$ as the "quotients" obtained by the "long division" of x with respect to $\mathbf{d} = (d_1, d_2, \ldots, d_M)$. In the special case that $r_i = 2$ and $d_i = 2^{i-1}$ for all $1 \leq i \leq M$, then the generalized $\mathcal{C}$-transform is simply the usual *binary* representation. If we choose $d_1 = 1$ and $d_i = r_{i-1} d_{i-1} = \prod_{k=1}^{i-1} r_k$ for all $2 \leq i \leq M$, the generalized $\mathcal{C}$-transform yields the so-called *generalized* $\mathbf{r}$*-ary representation* with respect to $\mathbf{r} = (r_1, r_2, \ldots, r_M)$.

Here, we list the two conditions that will be used for the selection of the M-vectors $\mathbf{r}$ and $\mathbf{d}$ in this paper.

(A1) Assume that $d_1 = 1$, $1 \leq d_k \leq \sum_{\ell=1}^{k-1} (r_\ell - 1) d_\ell + 1$ for all $2 \leq k \leq M$ and $r_k \geq 2$ for all $1 \leq k \leq M$.

(A2) Assume that $d_1 = 1$, $d_k = r_{k-1} d_{k-1}$ for all $2 \leq k \leq M$ and $r_k \geq 2$ for all $1 \leq k \leq M$.

Note that every selection of $\mathbf{r}$ and $\mathbf{d}$ in (A2) is also in (A1). Thus, (A1) contains a broader class of selections.

One of the key properties of the generalized $\mathcal{C}$-transform is the *complete decomposition property* in Proposition 6(v) in [6].

Proposition 6 ***(Complete decomposition property [6])*** *Suppose that (A1) holds. Then for any integer* $0 \leq n \leq \sum_{k=1}^{M} (r_k - 1) \cdot d_k$*, it can be represented by the generalized* $\mathcal{C}$*-transform, i.e.,* $n = \sum_{k=1}^{M} I_k^g(n) \cdot d_k$.

Theorem 7 ***(Routability)*** *Assume that* $\mathbf{r}$ *and* $\mathbf{d}$ *satisfy (A1). An* $N \times N$ *(**r**,**d**)-twister network is* routable *if and only if*

$$N \leq \sum_{k=1}^{M} (r_k - 1) d_k + 1.$$

Proof: We first prove the "if" part. Consider an $N \times N$ ($\mathbf{r}$,$\mathbf{d}$)-twister network with $N \leq \sum_{k=1}^{M} (r_k - 1) d_k + 1$. The product of all the biadjacency matrices in this ($\mathbf{r}$,$\mathbf{d}$)-twister network can be written as

$$\begin{aligned} \prod_{k=1}^{M} A^{(k)} &= \prod_{k=1}^{M} \sum_{j=0}^{r_k - 1} P^{j \cdot d_k} \\ &= \sum_{(I_1, I_2, \ldots, I_M) \in S_{\mathbf{r}}} P^{I_1 d_1 + I_2 d_2 + \ldots + I_M d_M}, \quad (4) \end{aligned}$$

where $S_{\mathbf{r}} = \{(I_1, I_2, \ldots, I_M) | 0 \leq I_k \leq r_k - 1, \forall 1 \leq k \leq M\}$.

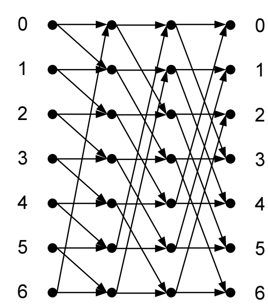
From the complete decomposition property in Proposition 6 and the assumption $N \leq \sum_{k=1}^{M} (r_k - 1) d_k + 1$, we know that any integer $n \in \{0, 1, \ldots, N-1\}$ can be decomposed through the generalized C-transform with respect to $\mathbf{r}$ and $\mathbf{d}$ as $n = \sum_{k=1}^{M} I_k^g(n) d_k$. Since $0 \leq I_k^g(n) \leq r_k - 1$ for all $1 \leq k \leq M$, the generalized $\mathcal{C}$-transform $C^g(n) = (I_1^g(n), I_2^g(n), \ldots, I_M^g(n))$ is definitely contained in the set $S_{\mathbf{r}}$ for all $0 \leq n \leq N-1$. Thus, P^n, $0 \leq n \leq N-1$, are contained in the summation in (4). As such, each element of $\prod_{k=1}^{M} A^{(k)}$ is nonzero. According to Corollary 3, we see that an $N \times N$ ($\mathbf{r}$,$\mathbf{d}$)-twister network is routable.

We prove the "only if" part by contradiction. Suppose there is a *routable* $N \times N$ ($\mathbf{r}$,$\mathbf{d}$)-twister network with $N > \sum_{k=1}^{M} (r_k - 1) d_k + 1$. Again, the product of all the biadjacency matrices can be obtained as in (4). Since the twister network is routable, there exists a routing path from input 0 to output N-1. According to Proposition 2(i), the element at the $(0, N-1)$ position in the matrix $\prod_{k=1}^{M} A^{(k)}$ is positive. In view of (4), we know that P^{N-1} is contained in the summation in (4). Thus, $N - 1 = \sum_{i=1}^{M} I_i d_i$ for some $(I_1, I_2, \ldots, I_M) \in S_{\mathbf{r}}$. This implies $N - 1 \leq \sum_{k=1}^{M} (r_k - 1) d_k$, which contradicts the assumption that $N > \sum_{k=1}^{M} (r_k - 1) d_k + 1$. ■

B. Routing paths

Instead of using a detailed description of all the nodes traversed, the routing path for a packet from an input i to an output o in a twister network with $M + 1$ stages can be simply described by the $(M+1)$-tuple $\mathbf{v} = (v_0, v_1, \cdots, v_M)$, where v_j is the index of the node traversed by the packet at the j^{th} stage for all $0 \leq j \leq M$. That is, the packet traverses node (j, v_j) at the j^{th} stage, where $v_0 = i$ and $v_M = o$.

Notice that there may be multiple routing paths for an input/output pair (i, o) in a routable CBN. For example, consider the 7×7 $(\mathbf{r}, \mathbf{d})$-twister network as shown in Fig. 4, where $\mathbf{r} = (2, 2, 2)$ and $\mathbf{d} = (1, 2, 3)$. In this 7×7 twister network, the packet from input 1 to output 4 can be routed either through the path $\mathbf{v} = (1, 1, 1, 4)$ or through the path $\mathbf{v}' = (1, 2, 4, 4)$.



(c)

Fig. 4. A 7×7 ($\mathbf{r}$,$\mathbf{d}$)-twister network with $\mathbf{r} = (2, 2, 2)$ and $\mathbf{d} = (1, 2, 3)$.

In this paper, we specify the routing path for a routable twister network according to the generalized C-transform.

Definition 8 ***(Routing path)*** *Consider a routable* $N \times N$ *(**r**,**d**)-twister network as described in Theorem 7, i.e., the* M*-vectors* $\mathbf{r}$ *and* $\mathbf{d}$ *satisfy (A1) and* $N \leq \sum_{k=1}^{M} (r_k - 1) d_k + 1$. *Define the space delay* x *for the input/output pair* (i, o) *as*

$$x = (o - i) \text{ mod } N. \quad (5)$$

The routing path $v = (v_0, v_1, \ldots, v_M)$ *for the input/output pair* (i, o) *in the* $N \times N$ *(**r**,**d**)-twister network is defined recursively by*

$$v_j = \left(v_{j-1} + I_j^g(x) d_j\right) \text{ mod } N \quad (6)$$

for all $1 \leq j \leq M$*, where* $v_0 = i$ *and* $C^g(x) = (I_1^g(x), I_2^g(x), \ldots, I_M^g(x))$ *is the generalized C-transform of the space delay* x*.*

Such a routing path is feasible because we know from the biadjacency matrix $A^{(k)}$ between stages $k-1$ and k that node $(k-1, j)$ at the $(k-1)^{th}$ stage is connected to the set of nodes $\{(k, j_n), n = 0, 1, \ldots, r_k - 1\}$ at the k^{th} stage, where $j_n = (j + n \cdot d_k) \bmod N$.

On the other hand, we have from the recursive definition in (6) that the routing path $\mathbf{v} = (v_0, v_1, \cdots, v_M)$ can be written alternatively by

$$v_j = \left(i + \sum_{k=1}^{j} I_k^g(x) d_k \right) \bmod N. \quad (7)$$

From the completion decomposition property in Proposition 6, we have $x = \sum_{k=1}^{M} I_k^g(x) d_k$. Thus,

$$v_M = \left(i + \sum_{k=1}^{M} I_k^g(x) d_k \right) \bmod N = (i + x) \bmod N = o, \quad (8)$$

and a packet from input i can indeed be self-routed to output o by using the routing path determined by the generalized $\mathcal{C}$-transform. For example, consider routing a packet from input 1 to output 4 in the 7×7 $(\mathbf{r}, \mathbf{d})$-twister network in Fig. 4. For this case, the space delay $x = 3$, $C^g(x) = (I_1^g(x), I_2^g(x), I_3^g(x)) = (0, 0, 1)$, and the routing path $\mathbf{v} = (1, 1, 1, 4)$.

IV. Conditional Nonblocking Properties

Like a switch or a switching network, a connection matrix for an (**r**,**d**)-twister network is a sub-permutation matrix that specifies the connections from a subset of its inputs to a subset of its outputs. Also, two routing paths are said to have a *conflict* if they share a common link. As we assume that every switch inside an (**r**,**d**)-twister network is nonblocking, a connection matrix is *feasible* for an (**r**,**d**)-twister network if we can find non-conflicting routing paths for all the input/output pairs specified in that connection matrix. In other words, every pair of the routing paths for that connection matrix have to be *link-disjoint* and they do not share any common link. In this section, we will show that a connection matrix that satisfies a certain property is feasible for an (**r**,**d**)-twister network that satisfies the assumption in (A2). As such, twister networks are conditionally nonblocking switches.

To prove the conditionally nonblocking properties for twister networks, we introduce the N*-modulo distance* defined below.

Definition 9 *The* N*-modulo distance* $d_N(i, j)$ *between two integers* i *and* j *is defined as*

$$d_N(i, j) = \min\left[(i - j) \bmod N, (j - i) \bmod N\right]. \quad (9)$$

The distance can be alternatively defined as

$$d_N(i, j) = \min\left[|i - j| \bmod N, -|i - j| \bmod N\right]. \quad (10)$$

One can easily see that the two definitions above are equivalent. In the special case that $0 \leq i, j \leq N - 1$, (10) can be rewritten as

$$d_N(i, j) = \min[|i - j|, N - |i - j|]. \quad (11)$$

To gain more intuition on the N*-modulo distance*, let us place all the input nodes (and output nodes) of a twister network on a circle as shown in Fig. 3(b). Then for $0 \leq i, j \leq N - 1$, the N*-modulo distance* $d_N(i, j)$ is the length of the shorter arc between nodes i and j on the circle of circumference N.

One can easily verify the following properties for the N-modulo distance. To simplify the notation, we say that $i =_N j$ if $i \bmod N$ equals $j \bmod N$.

Property 10 *Let* i*,* j *and* k *be all integers.*

(i) *(Nonnegativity)* $d_N(i, j) \geq 0$.
(ii) $d_N(i, j) = 0$ *if and only if* $i =_N j$.
(iii) *(Symmetry)* $d_N(i, j) = d_N(j, i)$.
(iv) *(Triangle Inequality)* $d_N(i, j) \leq d_N(i, k) + d_N(k, j)$.
(v) *(Translation Invariance)* $d_N(i, j) = d_N(i + k, j + k)$.
(vi) $d_N(i, j) = d_N(-i, -j)$.
(vii) $d_N(i, j) = d_N(i, k)$ *if* $j =_N k$.
(viii) $d_N(i, j) = d_N(k, \ell)$ *if* $i + k =_N j + \ell$.

In the following theorem, we show a conditionally nonblocking property for a twister network. The proof of Theorem 11 is given in Appendix A.

Theorem 11 *Consider an* $N \times N$ *(**r**,**d**)-twister network with the* M*-vectors **r** and **d** satisfying the assumption in (A2). Let* $N = r_M d_M = \prod_{k=1}^{M} r_k$ *and* $\gamma = \min_{2 \leq k \leq M-1} r_k$*. If the connection matrix has the property that*

$$d_N(o_1, o_2) \leq \gamma d_N(i_1, i_2) \quad (12)$$

for arbitrary input/output pairs (i_1, o_1) *and* (i_2, o_2)*, then the routing paths specified by the generalized* $\mathcal{C}$*-transform in (6) are link-disjoint and the connection matrix is thus feasible.*

Since (A2) is a stronger assumption than (A1), we know from Theorem 7 that the $N \times N$ (**r**,**d**)-twister network considered in this theorem is routable. Also, as there are $\prod_{k=1}^{M} r_k = N$ distinct paths from an input node to the N output nodes in this twister network, the routing path for each input/output pair (i, o) is unique and it is specified by the generalized $\mathcal{C}$-transform in (6). Note that under (A2), we have $d_k = r_{k-1} d_{k-1} = \prod_{\ell=1}^{k-1} r_\ell$ for all $2 \leq k \leq M$. The generalized $\mathcal{C}$-transform in this case is simply the generalized **r**-ary representation.

V. Rotator and Symmetric TDM Switch

As we mentioned in the Introduction, one of the most important applications of twister networks is to provide incremental update of the number of linecards in the two-stage load balanced switches [4], [14], [15], [13]. To explain this, we now introduce two kinds of switches that are commonly used as the switch fabrics for the two-stage load balanced switches.

Definition 12 ***(Rotator)*** *Let P be the $N \times N$ circular shift matrix as defined in (1). An $N \times N$ switch (or switching network) is called a rotator if it can realize the N permutations, P^n, $n = 0, 1, 2, \ldots, N-1$.*

Note that each pair of the input/output ports (i, o) in the permutation matrix P^n satisfies $o =_N (i + n)$.

Definition 13 ***(Symmetric TDM Switch)*** *For $0 \leq n \leq N-1$, let $\tilde{P}_n$ be the permutation such that each input/output pair (i, o) satisfies $(i + o) =_N n$. An $N \times N$ switch (or switching network) is called a symmetric TDM switch if it can realize the N permutations, $\tilde{P}_n$, $n = 0, 1, 2, \ldots, N-1$.*

Rotators and symmetric TDM switches (also known as reflectors in [17]) are special classes of conditionally non-blocking switches. Like banyan networks, they can be easily constructed by using the two-stage expansion [17], [5]. However, the problem of using the two-stage expansion is that it does not provide the flexibility for any number of linecards [14]. Specifically, an $N \times N$ rotator (resp. symmetric TDM switch) generated by the two-stage expansion cannot be used as a $p \times p$ rotator (resp. symmetric TDM switch) for all $p \leq N$. In this section, we will show that an $N \times N$ (**r**,**d**)-twister network satisfying the assumption in (A2) can be used as both $p \times p$ rotators and $p \times p$ symmetric TDM switches for any $p \leq N$. This is done by placing all the p linecards (with each linecard for an input/output pair) *evenly* among the N input/output ports of the $N \times N$ (**r**,**d**)-twister network.

In the following theorem, we first show that an $N \times N$ twister network can be used as a $p \times p$ rotator and a $p \times p$ symmetric TDM switch if the placement of the p linecards satisfies a certain property. The proof of Theorem 14 is given in Appendix B.

Theorem 14 *Consider an $N \times N$ (**r**,**d**)-twister network with the M-vectors **r** and **d** satisfying the assumption in (A2) and $N = r_M d_M = \prod_{k=1}^{M} r_k$. Index the N input/output ports of the $N \times N$ (**r**,**d**)-twister network from 0 to $N-1$. Suppose that there are p linecards, indexed from 0 to $p-1$, and the i^{th} linecard is placed in the $f(i)^{th}$ input/output port of the $N \times N$ (**r**,**d**)-twister network. Without loss of generality, assume that the mapping f is strictly increasing. Let*

$$x(i) = (f((i+1) \text{ mod } p) - f(i)) \text{ mod } N. \quad (13)$$

*Then, the $N \times N$ (**r**,**d**)-twister network can be used as a $p \times p$ rotator and a $p \times p$ symmetric TDM switch if*

$$\max_{0 \leq j \leq p-1} x(j) \leq \gamma \cdot \min_{0 \leq j \leq p-1} x(j) \quad (14)$$

where $\gamma = \min_{2 \leq k \leq M-1} r_k$ as defined in Theorem 11.

Note that $x(i)$'s are simply the gaps between the placement of two consecutive linecards. The intuition of the condition in (14) is that the maximum gap cannot be larger than γ times of the minimum gap.

For the special case that $N = 2^M$, in the following corollary we provide an explicit way to place the linecards.

Corollary 15 *Consider the $N \times N$ (**r**,**d**)-twister network in Theorem 14. Assume that $N = 2^M$ and $2 \leq p \leq N$. Then there exists $0 \leq m \leq M-1$ such that $2^m < p \leq 2^{m+1}$, and p can be written as $2^m + \ell$, where $1 \leq \ell \leq 2^m$. Place the i^{th} linecard in the $f(i)^{th}$ input/output port of the $2^M \times 2^M$ (**r**,**d**)-twister network, where*

$$f(i) = \begin{cases} i \cdot 2^{M-m-1} & \text{, for } 0 \leq i \leq 2\ell - 1 \\ (i - \ell) \cdot 2^{M-m} & \text{, for } 2\ell \leq i \leq p-1 \end{cases} . \quad (15)$$

Then, a $2^M \times 2^M$ $(\mathbf{r}, \mathbf{d})$-twister network can be used as a $p \times p$ rotator and a $p \times p$ symmetric TDM switch for these p linecards.

Proof: Note that the gap between the placement of two consecutive linecards is either 2^{M-m-1} or 2^{M-m}. Thus, we have that $\max_j x(j) = 2 \min_j x(j)$ and the condition (14) is thus satisfied because of the assumption that $r_k \geq 2$ for all k. The result then follows from Theorem 14. ∎

Consider a $2^M \times 2^M$ twister network in Corollary 15. Suppose that there are already p linecards placed by the method indicated in Corollary 15. As such, it can be used as a $p \times p$ rotator.

Now we show how one can add one more linecard in the $2^M \times 2^M$ twister network to make it a $(p+1) \times (p+1)$ rotator. This will be done without repositioning the existing p linecards.

As shown in Corollary 15, p can be written as $2^m + \ell$, where $1 \leq \ell \leq 2^m$. Notice that the first 2ℓ linecards are placed in the $2^M \times 2^M$ twister network every 2^{M-m-1} nodes, and the rest of them are placed every 2^{M-m} nodes.

If $\ell < 2^m$, the additional linecard is placed in the $((2\ell+1)2^{M-m-1})^{th}$ input/output port of the $2^M \times 2^M$ twister network. That is, the additional linecard is placed in the middle of the first pair of linecards that are separated by 2^{M-m} nodes. On the other hand, if $\ell = 2^m$, then $p = 2^{m+1}$. In view of Corollary 15, all the p linecards are placed evenly (i.e., every 2^{M-m-1} nodes). Thus, the additional linecard is placed in the $(2^{M-m-2})^{th}$ input/output port. Relabel the $p+1$ linecards in the ascending order, and the $2^M \times 2^M$ twister network can now be used as a $(p+1) \times (p+1)$ rotator. This shows that one can incrementally update the number of linecards in an $2^M \times 2^M$ twister network without repositioning the existing linecards. For example, for an 8×8 twister network, the order of placing new linecards in the input/output ports is 0,4,2,6,1,3,5,7.

For the $2^M \times 2^M$ twister network with $r_k = 2$ and $d_k = 2^{k-1}$ for all $1 \leq k \leq M$, all the nodes can be made by 2×2 switches (with 2^M 1×2 switches for the 2^M inputs and 2^M 2×1 switches for the 2^M outputs). To use the twister network as a $p \times p$ rotator, we first place the p linecards according to Corollary 15. For each (mapped) $p \times p$ connection matrix needed for a $p \times p$ rotator, we can find the routing paths according to the routing rule in (6). As such, all the connection patterns of the 2×2 switches in a twister network can be determined accordingly. Specifically, consider node (k, j) for some $2 \leq k \leq M-1$. That is, the node is neither an input node nor an output node. If nodes $(k-1, j)$

and $(k+1,j)$ are connected through node (k,j), we say that the 2×2 switch for node (k,j) is in the "bar" state and in the "cross" state otherwise. Also, an input node $(0,j)$ (resp. output node (M,j)) is said to be in the "bar" state if it connects to node $(1,j)$ (resp. node $(M-1,j)$) and in the "cross" state otherwise. A switch is said to be in the state "don't care" if the connection matrix of the twister network is implemented no matter which state the switch is in.

We illustrate how one uses an 8×8 twister network with **r**=(2,2,2) and **d**=(1,2,4) as a 5×5 rotator (see Fig. 5). According to Corollary 15, the five linecards are placed in the 0^{th}, 1^{st}, 2^{nd}, 4^{th} and 6^{th} input/output ports of the 8×8 twister network. The five connection matrices that need to be implemented are P^n, $n=0,1,\cdots,4$, where P is a 5×5 circular shift matrix. In Table I, we show all the states of nodes in this twister network. The element in the m^{th} row and n^{th} column represents the states of switches with the same index m for the connection matrix P^n, where the states are represented as a sequence of "bar" (b), "cross" (x) and "don't care" (z), in the increasing order of their stages (from left to right). For example, the sequence xzxb in the 0^{th} row and the 1^{st} column of Table I indicates that the 2×2 switch for node (0,0) (resp. (1,0), (2,0) and (3,0)) should be set to the cross (resp. don't care, cross, bar) state for the 5×5 circular shift matrix P. Moreover, Table II shows all the states of nodes in this twister network if they are used as a 5×5 symmetric TDM switch.

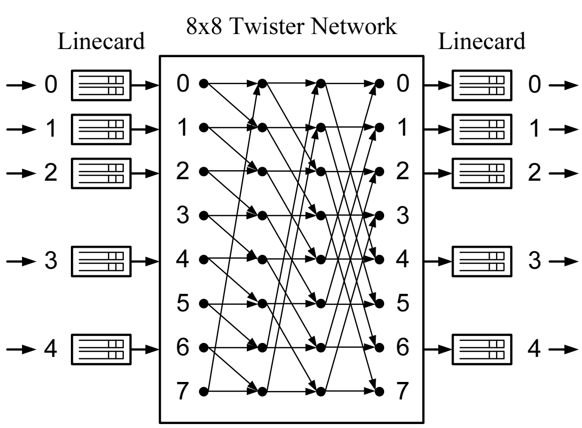


Fig. 5. Using an 8×8 (**r**,**d**)-twister network with **r**=(2,2,2) and **d**=(1,2,4) as a 5×5 rotator

VI. Conclusion

In this paper we studied a class of multistage interconnection networks (MINs), called *twister networks*, that are constructed directly by a concatenation of bipartite networks. We showed there are several *distinct* properties for twister networks, including routability in Theorem 7 and conditionally nonblocking properties in Theorem 11. As an application, we showed that a twister network with N inputs/outputs can be used as a $p\times p$ rotator and a $p\times p$ symmetric TDM switch for any $2\le p\le N$. As such, one can use a twister network as the switch fabric for a two-stage load balanced switch that is capable of providing incremental update of the number of linecards.

Twister	0	1	2	3	4
0	bbbb	xzxb	bxzx	bbxx	bxbx
1	bbbb	xxbb	xzxb	xzzx	xzzx
2	bbbb	bxbb	bbxb	bxxx	xbbx
3	zzzz	zzzz	zzzz	zzzz	zbzz
4	bbbb	bxxb	bbxb	xzbx	bxbx
5	zzzz	zzzz	zzzz	zxxz	zzbz
6	bbbb	bxxb	xzzx	bbxx	bxbx
7	zzzz	zzzz	zbzz	zzzz	zzzz

TABLE I
States of switches in an 8×8 (**r**,**d**)-twister network with $\mathbf{r}=(2,2,2)$ and $\mathbf{d}=(1,2,4)$ for a 5×5 rotator.

Twister	0	1	2	3	4
0	bbbb	xzzx	bxbx	bbxx	bxxb
1	xzxb	xxbb	bbbb	xzzx	xzzx
2	bxxx	bbxx	bxxb	xxbb	bbbb
3	zzzz	zzzz	zzzz	zbzz	zzzz
4	bxxb	bbbb	bxbx	bbxx	xzxb
5	zzzz	zzzz	zzzz	zzbz	zxxz
6	xzbx	bbxx	bxxb	bbbb	bxzx
7	zbzz	zzzz	zzzz	zzzz	zzzz

TABLE II
States of switches in an 8×8 (**r**,**d**)-twister network with $\mathbf{r}=(2,2,2)$ and $\mathbf{d}=(1,2,4)$ for a 5×5 symmetric TDM switch.

There are several interesting developments of twister networks that are not reported here due to space limitation.

(i) Here we only considered link-disjoint routing paths for connection matrices in twister networks. In fact, we have developed results for node-disjoint (or known as crosstalk-free) paths as in [6]. Note that the $2^M\times 2^M$ twister network in Corollary 15 can be used as a $2^M\times 2^M$ crosstalk-free rotator [17].

(ii) We have defined an (**r**,**d**)-banyan network directly by a concatenation of bipartite networks. Unlike the classical banyan network, an (**r**,**d**)-banyan network only uses 1×2 switches for its input nodes and 2×1 switches for its output nodes. Our preliminary result shows that such an (**r**,**d**)-banyan network can also be used as a rotator and a symmetric TDM switch with an arbitrary number of linecards.

Appendix A

In this appendix, we prove Theorem 11.

We prove the theorem by contradiction. Suppose that the routing paths of the pair of input/output ports (i_1,o_1) and (i_2,o_2) share a link between stages $j-1$ and j. Note that the shared link cannot be a link connected to an input node or an output node as otherwise we would have either $i_1=i_2$ or $o_1=o_2$ that contradicts to the assumption that a connection matrix is a sub-permutation matrix. Thus, we know that $2\le j\le M-1$. As these two routing paths must traverse the same nodes at stage $j-1$ and j, we have from the routing rule in

(7) that

$$i_1+\sum_{k=1}^{n} I_k^g(x_1)d_k =_N i_2+\sum_{k=1}^{n} I_k^g(x_2)d_k. \tag{16}$$

for both $n=j-1$ and j. In view of Property 10(viii) of the N-modulo distance, we have that

$$d_N(i_1,i_2)=d_N\left(\sum_{k=1}^{j-1} I_k^g(x_1)d_k,\sum_{k=1}^{j-1} I_k^g(x_2)d_k\right). \tag{17}$$

Since $N=r_M d_M \geq 2d_M$ and $d_k=r_{k-1}d_{k-1}$ in (A2) for $2\leq k\leq M$,

$$\begin{aligned}&\left|\sum_{k=1}^{j-1} I_k^g(x_1)d_k-\sum_{k=1}^{j-1} I_k^g(x_2)d_k\right|\\ &\leq \sum_{k=1}^{j-1}\left|I_k^g(x_1)-I_k^g(x_2)\right|d_k \leq \sum_{k=1}^{j-1}(r_k-1)d_k\\ &= d_j-1<d_M\leq N/2.\end{aligned} \tag{18}$$

From (17), (18) and the definition of the N-modulo distance in (10), we have that

$$\begin{aligned}d_N(i_1,i_2)&=d_N\left(\sum_{k=1}^{j-1} I_k^g(x_1)d_k,\sum_{k=1}^{j-1} I_k^g(x_2)d_k\right)\\ &\leq\left|\sum_{k=1}^{j-1} I_k^g(x_1)d_k-\sum_{k=1}^{j-1} I_k^g(x_2)d_k\right|\leq d_j-1.\end{aligned} \tag{19}$$

Since $\left(i+\sum_{k=1}^{M} I_k^g(x)d_k\right)=_N o$ in (8) for each input/output pair (i,o) in an (**r**,**d**)-twister network, (16) can be rewritten as

$$o_1-\sum_{k=n+1}^{M} I_k^g(x_1)d_k =_N o_2-\sum_{k=n+1}^{M} I_k^g(x_2)d_k \tag{20}$$

for both $n=j-1$ and j. Also, from (20) and Property 10(vi) and (viii), we have that

$$d_N(o_1,o_2)=d_N\left(\sum_{k=j+1}^{M} I_k^g(x_1)d_k,\sum_{k=j+1}^{M} I_k^g(x_2)d_k\right).$$

Similar to (18), we also have

$$\begin{aligned}&\left|\sum_{k=j+1}^{M} I_k^g(x_1)d_k-\sum_{k=j+1}^{M} I_k^g(x_2)d_k\right|\\ &\leq\sum_{k=j+1}^{M}(r_k-1)d_k=N-d_{j+1}.\end{aligned} \tag{21}$$

On the other hand, we have from (A2) that

$$\begin{aligned}&\left|\sum_{k=j+1}^{M} I_k^g(x_1)d_k-\sum_{k=j+1}^{M} I_k^g(x_2)d_k\right|\\ &=d_{j+1}\left|\sum_{k=j+1}^{M}\left(I_k^g(x_1)-I_k^g(x_2)\right)\prod_{\ell=j+1}^{k-1} r_\ell\right|,\end{aligned} \tag{22}$$

where by convention we let $\prod_{\ell=j+1}^{k-1} r_\ell=1$ for $k=j+1$. Notice that $\sum_{k=j+1}^{M}(I_k^g(x_1)-I_k^g(x_2))\prod_{\ell=j+1}^{k-1} r_\ell$ cannot be zero as otherwise we have $o_1=o_2$ from (22) and (20). Thus, $|\sum_{k=j+1}^{M}(I_k^g(x_1)-I_k^g(x_2))\prod_{\ell=j+1}^{k-1} r_\ell|$ is greater or equal to one, and

$$\left|\sum_{k=j+1}^{M} I_k^g(x_1)d_k-\sum_{k=j+1}^{M} I_k^g(x_2)d_k\right|\geq d_{j+1}.$$

In conjunction with (21),

$$\begin{aligned}d_N(o_1,o_2)&=d_N\left(\sum_{k=j+1}^{M} I_k^g(x_1)d_k,\sum_{k=j+1}^{M} I_k^g(x_2)d_k\right)\\ &\geq d_{j+1}.\end{aligned} \tag{23}$$

From (19) and (23), it then follows that

$$d_N(o_1,o_2)\geq d_{j+1}=r_j d_j>r_j d_N(i_1,i_2), \tag{24}$$

which contradicts to the assumption in (12).

Appendix B

In this appendix, we prove Theorem 14.

Let P be the $p\times p$ circular shit matrix. For the connection matrix P^n in a $p\times p$ rotator, the output port o for input i can be written as $o=(i+n)$ mod p, $n=0,1,2,\cdots,p-1$. The space delay for the mapped input/output pair $(f(i_1),f(o_1))$ and $(f(i_2),f(o_2))$ are $x_1=(f(o_1)-f(i_1))$ mod N and $x_2=(f(o_2)-f(i_2))$ mod N, respectively. In view of Theorem 11, it suffices to show that

$$d_N(f(o_1),f(o_2))\leq\gamma d_N(f(i_1),f(i_2)) \tag{25}$$

for any mapped input/output pair $(f(i_1),f(o_1))$ and $(f(i_2),f(o_2))$ for each $0\leq n\leq p-1$ in a $p\times p$ rotator.

Without loss of generality, we assume that $i_2>i_1$ and thus $f(i_2)>f(i_1)$. Clearly, we have from (13) that

$$f(i_2)-f(i_1)=\sum_{i=i_1}^{i_2-1} x(i).$$

We first show that $f(o_2)-f(o_1)$ can be represented in a similar form as follows:

$$f(o_2)-f(o_1)=_N\sum_{i=i_1}^{i_2-1} x(i+n). \tag{26}$$

Note that $x(i)$ can be equivalently written as follows:

$$x(i)=(f((i+1)\bmod p)-f(i\bmod p))\bmod N. \tag{27}$$

As such, $x(i)=x(j)$ if $i=_p j$. Then one can easily verify that $\sum_{i=k}^{k+p-1} x(i)=N$, for all k, as $x(i)$'s are the gaps between two consecutive linecards.

Now, we consider three cases.

Case (i) If $0\leq i_1+n<i_2+n\leq p-1$, then $o_2=i_2+n$ and $o_1=i_1+n$. Thus,

$$f(o_2)-f(o_1)=\sum_{i=o_1}^{o_2-1} x(i)=\sum_{i=i_1}^{i_2-1} x(i+n).$$

Case (ii) If $i_1+n \le p-1$ and $p \le i_2+n \le i_1+p$, then $o_2 = i_2+n-p$ and $o_1 = i_1+n$. Thus, we have that $f(o_2)-f(o_1) =_N N-(f(i_1+n)-f(i_2+n-p))$. Notice that $N=\sum_{i=k}^{k+p-1} x(i)$ for arbitrary integer k, and thus

$$\begin{aligned} f(o_2)-f(o_1) &=_N \sum_{i=i_2+n-p}^{i_2+n-1} x(i) - \sum_{i=i_2+n-p}^{i_1+n-1} x(i) \\ &= \sum_{i=i_1}^{i_2-1} x(i+n). \end{aligned}$$

Case (iii) If $p \le i_1+n < i_2+n \le 2p-1$, then $o_2 = i_2+n-p$ and $o_1 = i_1+n-p$. Thus,

$$\begin{aligned} f(o_2)-f(o_1) &= f(i_2+n-p)-f(i_1+n-p) \\ &= \sum_{i=i_1+n-p}^{i_2+n-p-1} x(i) = \sum_{i=i_1}^{i_2-1} x(i+n). \end{aligned}$$

Hence, (26) holds in all the three cases. It then follows from (14) that

$$\begin{aligned} f(o_2)-f(o_1) &=_N \sum_{i=i_1}^{i_2-1} x(i+n) \le \sum_{i=i_1}^{i_2-1} \max_j x(j) \\ &\le \gamma \sum_{i=i_1}^{i_2-1} \min_j x(j) \le \gamma \sum_{i=i_1}^{i_2-1} x(i) \\ &= \gamma(f(i_2)-f(i_1)). \end{aligned} \tag{28}$$

Following a similar procedure, one can also easily show

$$\begin{aligned} N-(f(i_2)-f(i_1)) &= \sum_{i=i_2}^{i_1+p-1} x(i), \quad \text{and} \\ N-(f(o_1)-f(o_2)) &=_N \sum_{i=i_2}^{i_1+p-1} x(i+n). \end{aligned}$$

Thus,

$$\begin{aligned} &N-(f(o_1)-f(o_2)) =_N \sum_{i=i_2}^{i_1+p-1} x(i+n) \\ &\le \gamma \sum_{i=i_2}^{i_1+p-1} x(i) = \gamma(N-(f(i_2)-f(i_1))). \end{aligned} \tag{29}$$

From (28) and (29), it follows that

$$\begin{aligned} &d_N(f(o_1),f(o_2)) \\ &= \min[(f(o_2)-f(o_1)) \bmod N, \\ &\quad (N-(f(o_2)-f(o_1))) \bmod N] \\ &\le \gamma \min\left[(f(i_2)-f(i_1)),(N-(f(i_2)-f(i_1)))\right] \\ &= \gamma d_N(f(i_1),f(i_2)). \end{aligned}$$

This shows (25) and we complete the proof for a $p\times p$ rotator.

For a $p\times p$ symmetric TDM switch, we have for any fixed $0 \le n \le p-1$ that $(i_1+o_1) =_p (i_2+o_2) =_p n$ for any input/output pairs (i_1,o_1) and (i_2,o_2). Without loss of generality, we also assume that $i_2 > i_1$ and thus $f(i_2) > f(i_1)$. Let $\tilde{n} = n-i_1-i_2$. By considering the three cases, (i) $0 \le i_1 < i_2 \le n$ (ii) $n+1 \le i_1 < i_2 \le p-1$ and (iii) $0 \le i_1 \le n$ and $n+1 \le i_2 \le p-1$, one can show that $f(i_2)-f(i_1) = \sum_{i=i_1}^{i_2-1} x(i)$ and

$$f(o_1)-f(o_2) =_N \sum_{i=i_1}^{i_2-1} x(i+\tilde{n}) \tag{30}$$

by using similar procedures as in the proof for a rotator. Notice that (30) has the same form as (26) except that the positions of o_1 and o_2 are interchanged. Hence, according to the definition in (10), the condition in (25) is also satisfied for any mapped input/output pair $(f(i_1),f(o_1))$ and $(f(i_2),f(o_2))$ of a symmetric TDM switch if the mapping f satisfies the condition in (14).

References

[1] B. W. Arden and Hikyu Lee, "Analysis of chordal ring network," *IEEE Transactions on Computers*, vol. C-30, pp. 291-295, 1981.

[2] A. S. Asratian, T. M. J. Denley, and R. Haggkvist, *Bipartite Graphs and Their Applications.* Cambridge University Press, 1998.

[3] V. E. Benes. *Mathematical Theory of Connecting Networks and Telephone Traffic*. New York: Academic Press, 1965.

[4] C. -S. Chang, D. -S. Lee and Y. -S. Jou, "Load balanced Birkhoff-von Neumann switches, part I: one-stage buffering," *Computer Communicaitons,* vol. 25, pp. 611-622, 2002.

[5] C.-S. Chang, D.-S. Lee, Y.-J. Shih and C.-L. Yu, "Mailbox switch: a scalable two-stage switch architecture for conflict resolution of ordered packets," *IEEE Transactions on Communications*, vol. 56, pp. 136-149, 2008.

[6] C. -S. Chang, J. Cheng, and D. -S. Lee, "SDL constructions of FIFO, LIFO and absolute contractors, " *INFOCOM 2009,* pp. 738-746, Rio de Janeiro, Brazil, Apr. 2009.

[7] H. J. Chao, C. H. Lam and E. Oki. *Broadband Packet Switching Technologies: A Practical Guide to ATM Switches and IP Routers*. John Wiley & Sons, Inc., 2001.

[8] C. Clos, "A study of nonblocking switching networks," *BSTJ*, Vol. 32, pp. 406-424, 1953.

[9] C.-C. Chou, C.-S. Chang, D.-S. Lee and J. Cheng, "A necessary and sufficient condition for the construction of 2-to-1 optical FIFO multiplexers by a single crossbar switch and fiber delay lines," *IEEE Transactions on Information Theory*, Vol. 52, pp. 4519-4531, 2006.

[10] C. Godsil and G. Royle, *Algebraic Graph Theory.* Springer-Verlag New York, Inc., 2001.

[11] C. Hawkins, B. A. Small, D. S. Wills, and K. Bergman, "The data vortex, an all optical path multicomputer interconnection network," *IEEE Transactions on Parallel and Distributed Systems*, vol. 18, pp. 409-420, 2007.

[12] F. K. Huang and P. E. Wright, "Survival reliability of some double-loop networks and chordal rings," *IEEE Transactions on Computers*, vol.44, pp. 1468-1471, 1995.

[13] J. -J. Jaramillo, F. Milan and R. Srikant, "Padded frames: A novel algorithms for stable scheduling in load-balanced switches, " *IEEE/ACM Transactions on Networking,* vol. 16, no. 5, pp. 1212-1225, Oct. 2008.

[14] I. Keslassy, S. -T. Chung, K. Yu, D. Miller, M. Horowitz, O. Sloggard, and N. McKeown, "Scaling internet routers using optics, " *ACM SIGCOMM 2003*, Karlsruhe, Germany, Sep. 2003.

[15] I. Keslassy, S. -T. Chung, N. McKeown, "A load-balanced switch with an arbitrary number of linecards," *Proc. IEEE INFOCOM*, 2004.

[16] S.-Y. R. Li. *Algebraic Switching Theory and Broadband Applications*. Academic Press, 2001.

[17] S.-Y. R. Li and X. J. Tan, "Preservation of conditionally nonblocking switches under two-stage interconnection," *IEEE Transactions on Communications*, vol. 55, pp. 973-980. 2007.

[18] M. Schwartz. *Broadband Integrated Networks*. Prentice Hall, 1996.

[19] Q. Yang and K. Bergman, "Performances of the data vortex switch architecture under nonuniform and bursty traffic," *J. Lightwave Technology*, vol. 20, no. 8, pp. 1242-1247, Aug. 2002.