Consensus and Polarization of Binary Opinions in Structurally Balanced Networks

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***Abstract*—In this paper we propose a new model for binary opinion dynamics in a (fully-connected) structurally balanced network. In a structurally balanced network, agents are classified into two clusters and two agents in the same cluster (resp. different clusters) are connected with a positive (resp. negative) edge. Initially, every agent is assigned with one of the two opinions randomly. In every time slot, three agents are randomly selected to have their opinions updated. If the three agents belong to the same cluster, the majority rule is used to update their opinions. On the other hand, if the three agents belong to two different clusters, with probability p a consensus is reached by the majority rule, and with probability $1-p$ a polarization (in line with the signs of the three edges) is reached. The probability p, called the rationality probability, plays a significant role for measuring how rational the agents in a network behave when they encounter different opinions. By applying a fluid limit theorem for jump Markov processes, we derive a system of differential equations for the density functions of opinions for large networks. We show that the equilibrium points corresponding to consensus and polarization are the only stable equilibrium points. All other equilibrium points are all unstable. As such, as time goes on the network eventually reaches a consensus or a polarization, depending on the rationality probability and the initial state of the network.**

keywords: consensus, polarization, majority rule, clusterable network, structural balance, social network, opinion dynamics

I. Introduction

With the advent of online social networking services, research on social networks gains popularity beyond social scientists, and receives attention among physicists, mathematicians and computer scientists. Opinion dynamics is one of the most important topics in the study of social networks. Agreement and disagreement occur in our daily life all the time. For instance, we may agree with a friend and adopt his/her opinions. Alternatively, we may disagree with someone whom we are not fond of, and thus keep a different opinion. Research on opinion dynamics ranges from the mechanism of update to the algorithms to maximize the influence. We refer the reader to Castellano *et. al.* [1] for a plethora of studies in this area. We also refer the reader to [2], [3] for recent studies on opinion dynamics.

Recently a model was proposed to study iterative and hierarchical debates on issues [4][5]. In this model n agents, endowed with binary opinions, form a complete graph of social contacts. At each iteration, r agents are randomly selected. These agents form a discussion group and at the end of discussion all agents take the majority opinion in the group. This model was called the majority rule (MR) model and it was solved in the mean field approximation by Krapivsky and Redner [6] in 2003 with a fixed r. Lambiotte [7] investigated the MR model on networks with community structure in 2007. Chen and Redner [8] extended the MR model to multi-state opinions and plurality rule. We refer the reader to Castellano *et. al.* [1] for more extensions of the MR model. In this paper, we extend the MR model to signed structurally balanced networks. We refer the reader to [9], [10], [11] for references on structurally balanced networks. Li *et. al.* [12] modeled the spreading of opinions in structurally balanced networks by a Susceptible-Infectious-Recovery (SIR) epidemic process. Singh *et. al.* [13] proposed a three-state model, in which both opinions and edge signs update with time.

In political science, the term polarization generally refer to the observation that individuals' opinions or view points are often ideologically aligned with a political party [14]. Polarization between two political parties has long been observed in many democratic countries, such as the United States of America [14], [15], [16], [17], [18], [19]. For instance, the Democrat and the Republican parties in the United States of America have relatively opposite ideological point of views. It is often observed that bills are voted in a polarized way in the Congress of the United States. Specifically, nearly all the congressmen in the two political parties vote similarly. However, the vote by the congressmen in one party is different from that by the other party. In fact, in Taiwan nearly all bills are polarized.

In this paper we shall extend the MR model to study political polarization. We consider a static structurally balanced signed network, in which two opinions dynamically update. We shall show that local polarization or consensus in a triad leads to a global polarization or consensus in opinions. To distinguish between polarization and consensus, we introduce a parameter called rationality probability. The larger the rationality probability, the more likely a global consensus is reached eventually. For networks with a finite size, we formulate the extended MR model as a discrete-time Markov chain with absorbing states corresponding to polarization or consensus. We observe a phase transition from polarization to consensus. Specifically, there exists a threshold on the rationality probability. If the rationality probability is less than this threshold, polarization will occur. Otherwise, consensus will occur. The larger the network is, the more obvious the phase transition is. Applying a fluid limit theorem for jump Markov processes, we scale both space and time linearly to obtain a system of differential equations for the density functions of opinions. We show that the equilibrium points of the system corresponding to polarization and consensus can be stable under certain conditions. Other equilibrium points are always nonstable.

The rest of the paper is organized as follows. In Section II we present our model. In Section III we present a Markov chain analysis on networks with a finite size. In Section IV we apply a fluid limit theorem to derive a nonlinear system of differential equations. We present our study of the equilibrium points of this system in this section. In Section VI we present our numerical study on the model. We draw our conclusions in Section VII.

II. The Model

In this paper, we propose a new model for binary opinion dynamics in a fully connected signed network with $n_1 + n_2$ vertices (or agents). Each vertex is associated with one of two opinions, opinion 0 and opinion 1. We assume that this network is clusterable and has two clusters. The sizes of the two clusters are n_1 and n_2 vertices, respectively. Specifically, any edge that connects two vertices in the same cluster is a positive edge. On the other hand, any edge that connects two vertices in different clusters is a negative edge.

For such a network, we say a triad (with three vertices) is *stable* if (i) their opinions are the same, or (ii) their opinions are in line with the signs of the three edges, i.e., two vertices connected by a positive (resp. negative) edge have the same opinion (resp. different opinions). For the first case, the stable triad is said to be in *consensus*. On the other hand, the stable triad for the second case is said to be in *polarization* as their opinions are polarized by the signed edges.

Consider a Poisson process with rate 1. At every arrival epoch of the Poisson process, we randomly select a triad and perform the following update to the opinions of the three vertices in the triad.

(i) The selected triad is a stable triad: nothing is done.

(ii) The selected triad is not a stable triad:

1) The three selected vertices are in the same cluster: The opinions of the three vertices are modified according to the majority rule. By doing so, the triad is in consensus.
2) The three vertices belong to different clusters and the two vertices in the same cluster have different opinions (see panel (a) in Figure 1): With probability p, the opinions in the triad are modified according to the majority rule (see panel (b) in Figure 1). The opinions in the triad in panel (a) then reach a consensus. On the other hand, with probability $1 - p$, the triad is changed in the way that the two vertices in the same cluster have the same opinion that is different from the (original) opinion of the vertex in the other cluster (see panel (c) in Figure 1). By doing so, the opinions in panel (c) reaches polarization.

The reader might wonder why we did not address the case that the three vertices belong to different clusters and the two vertices in the same cluster have the same opinion. This is because in such a case, the triad is stable as it is either in consensus or in polarization. Also, we note from the above micro-rule that there is at most one vertex that changes its opinion in each update of a randomly selected triad. The probability p is called the rationality probability in this paper as it measures how rational the vertices in a network behave when they encounter different opinions. If they are rational, then the majority rule should be used. On the other hand, if they are irrational, then their opinions are influenced by their relationships with other people.

We can extend the concept of *consensus* and *polarization* for a triad to the whole network. Specifically, we say the network is in consensus if all the vertices have the same opinion, and in polarization if all the vertices in one cluster have one opinion while all the vertices in the other cluster have the other opinion. Through the above micro-rule for opinion updates in randomly selected triads, we will show later that the whole network will be either in consensus or in polarization in the long run, depending on the rationality probability p and the initial state of the network.

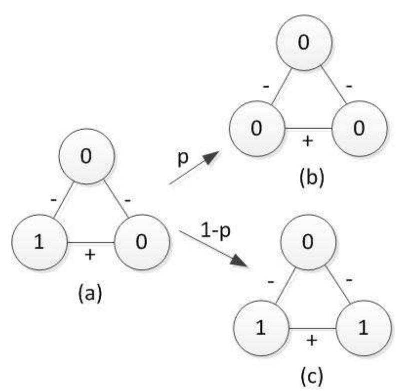


Fig. 1: Opinion update in a triad.

III. The generator of the Markov process

Let $N_1(t)$ denote the number of vertices with opinion 1 in cluster 1 at time t. Similarly, let $N_2(t)$ be the number of such vertices in cluster 2. Clearly, the process $N(t) = (N_1(t), N_2(t))$ is a continuous-time Markov process with the state space $\{(i,j) : 0 \le i \le n_1, 0 \le j \le n_2\}$. This Markov process has four absorbing states, namely states $(0,0), (n_1,0), (0,n_2)$ and (n_1,n_2). States $(0,0)$ and (n_1,n_2) correspond to the case that vertices in the network reach consensus. States $(0,n_2)$ and $(n_1,0)$ imply that the network is polarized. We are particularly interested in the probability that polarization occurs.

Before we derive the generator of this Markov process, we introduce some probabilities. Consider the state (i, j) and randomly pick three vertices from the network. For $\ell = 0, 1, 2$ or 3, let $g_\ell(o_1, o_2, o_3)$ be the probability that the opinions of the three randomly selected vertices are o_1, o_2 and o_3, and the vertices with the opinions $o_1, o_2, \ldots, o_\ell$ are drawn from cluster 1. For example, supposing that the state is (i, j) with $i \geq 3$, $g_3(1, 1, 0)$ is the probability that the three vertices are drawn from cluster 1 and two of them have opinion 1 and the other one has opinion 0. Specifically,

$$g_3(1,1,0) = \frac{\begin{pmatrix} i \\ 2 \end{pmatrix}\begin{pmatrix} n_1 - i \\ 1 \end{pmatrix}}{\begin{pmatrix} n_1 + n_2 \\ 3 \end{pmatrix}}. \tag{1}$$

As another example, in state (i, j)

$$g_2(0,1,0) = \frac{i \cdot (n_1 - i) \cdot (n_2 - j)}{\begin{pmatrix} n_1 + n_2 \\ 3 \end{pmatrix}}.$$

Now we derive the state transition rate from state (i, j) to state $(i+1, j)$. Denote the state transition rate by $q_{(i,j),(i+1,j)}$. Then, since the Poisson process in our opinion dynamics has rate 1, we have

$$q_{(i,j),(i+1,j)} = g_3(1,1,0) + g_2(1,0,1) \cdot p + g_2(1,0,0) \cdot (1-p). \tag{2}$$

To see (2), we note that there are three cases in which the number of vertices with opinion 1 in cluster 1 can increase. First, the three randomly selected three vertices are all from cluster 1 and two of them have opinion 1 and the third vertex has opinion 0. In this case, the opinions are modified according to the majority rule and as a result, the number of opinion 1 in cluster 1 is increased by one. The probability of this case is $g_3(1, 1, 0)$ and it is the first term on the right side of (2). In the second case, two out of the three randomly selected vertices are chosen from cluster 1 and the third vertex is chosen from cluster 2. The opinions of the two vertices in cluster 1 are 0 and 1, while the opinion of the vertex in cluster 2 is 1. In this case, if the vertices behave rationally (with probability p), the vertex with opinion 0 in cluster 1 changes its opinion to 1. The number of vertices with opinion 1 is thus increased by one. The probability of this case is $g_2(1, 0, 1) \cdot p$ and it is the second term on the right side of (2). The third case is similar to the second case, where two vertices are drawn from cluster 1 and one vertex is from cluster 2. However, in the third case, the opinion of the vertex in cluster 2 is 0. If the vertices behave irrationally (with probability $1 - p$), the vertex with opinion 0 in cluster 1 changes its opinion to 1. The probability of this case is $g_2(1, 0, 0) \cdot (1 - p)$ and it is the third term on the right side of (2).

Other transition rates can be similarly derived. We only list them without giving detailed derivations.

$$\begin{aligned} q_{(i,j),(i,j+1)} &= g_0(1,1,0) + g_1(1,1,0) \cdot p \\ &\quad + g_1(0,1,0) \cdot (1-p) && (3) \\ q_{(i,j),(i-1,j)} &= g_3(1,0,0) + g_2(1,0,1) \cdot (1-p) \\ &\quad + g_2(1,0,0) \cdot p && (4) \\ q_{(i,j),(i,j-1)} &= g_0(1,0,0) + g_1(1,1,0) \cdot (1-p) \\ &\quad + g_1(0,1,0) \cdot p. && (5) \end{aligned}$$

As there is at most one vertex that can change its opinion, we then have

$$\begin{aligned} q_{(i,j),(i,j)} &= -q_{(i,j),(i+1,j)} - q_{(i,j),(i,j+1)} \\ &\quad - q_{(i,j),(i-1,j)} - q_{(i,j),(i,j-1)}. \end{aligned} \tag{6}$$

For the Markov processes $N(t)$, there is an embedded Markov chain at each arrival epoch. Let $\boldsymbol{Q}$ be the $(n_1 \cdot n_2) \times (n_1 \cdot n_2)$ generator matrix of $N(t)$ with the states being ordered in the lexicographic order. As the underlining Poisson process is with rate 1, the state transition probability matrix $\boldsymbol{P}$ of the embedded Markov chain is simply $\boldsymbol{I} + \boldsymbol{Q}$, where $\boldsymbol{I}$ is the $(n_1 \cdot n_2) \times (n_1 \cdot n_2)$ identity matrix.

Once the transition probabilities of the embedded Markov chain are obtained, it is a standard procedure to calculate absorption probabilities. We refer the reader to [20] for a discussion of Markov chains with absorbing states. Suppose that a Markov chain has t transient states and r absorbing states and that the one-step transition probability matrix can be written in the form of a block matrix

$$\begin{pmatrix} \boldsymbol{P}_t & \boldsymbol{R} \\ 0 & \boldsymbol{I}_r \end{pmatrix},$$

where the $t \times t$ matrix $\boldsymbol{P}_t$ contains transition probabilities among transient states, matrix $\boldsymbol{R}$ contains transition probabilities from transient states to absorption states. Matrix $\boldsymbol{I}_r$ is an $r \times r$ identity matrix, where r is the number of absorption states. Then, the absorption probability from state k_1 to state k_2 is given in the $(k_1, k_2)^{th}$ entry of the $t \times r$ matrix η, where

$$\eta = (\boldsymbol{I}_t - \boldsymbol{P}_t)^{-1}\boldsymbol{R}. \tag{7}$$

However, computing the absorption probabilities for large networks by using (7) is difficult as it needs to invert a large matrix. To tackle this problem, we shall consider the fluid limit for large networks in the next section.

IV. The Fluid Limit of Large Networks

In this section we derive the fluid limit for the opinion dynamics in large networks. For this, we consider a sequence of networks that scale with n. In the n^{th} network, we set $n_1 = n$ and $n_2 = \lceil an \rceil$, where a is a parameter that denotes the asymptotic ratio of the sizes of these two clusters. Using Kurtz's fluid limit theorem for jump Markov processes [21], we have the following theorem.

Theorem 1. *Suppose that* $n_1 = n$ *and* $n_2 = \lceil an \rceil$ *and that we start the opinion dynamics with* $N_1(0) = \lceil nx_0 \rceil$ *and* $N_2(0) = \lceil any_0 \rceil$ *for some* $0 \leq x_0 \leq 1$ *and* $0 \leq y_0 \leq 1$. *As* $n \to \infty$, *the sequence of stochastic processes* $(N_1(nt)/n_1, N_2(nt)/n_2)$ *converges to* $(x(t), y(t))$ *that is the solution of the following system of first order differential equations with the initial condition* (x_0, y_0)*:*

$$x'(t) = \mu_1(x,y) = \frac{3}{(1+a)^3} x(1-x) \times (2x - 1 + 2a(1-2p)(1-2y)), \tag{8}$$

and

$$y'(t) = \mu_2(x,y) = \frac{3a^2}{(1+a)^3} y(1-y) \times (2y - 1 + 2a^{-1}(1-2p)(1-2x)). \tag{9}$$

Specifically, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathsf{P}\Big(\sup_{0\le s\le t} [|\frac{N_1(nt)}{n_1} - x(t)| + |\frac{N_2(nt)}{n_2} - y(t)|] \ge \epsilon\Big) = 0. \tag{10}$$

Note that $N_1(nt)/n_1$ (resp. $N_2(nt)/n_2$) is the density of the number of vertices that have opinion 1 in cluster 1 (resp. cluster 2) at time nt. The fluid limit theorem shows that the density processes follow a deterministic trajectory characterized by $(x(t), y(t))$ if we scale the space and time properly in large networks.

Proof. To prove (8) and (9), one needs to compute the drift of the process $(N_1(nt)/n_1, N_2(nt)/n_2)$. Specifically, let

$$\mu_n(x,y) = (\mu_{n,1}(x,y), \mu_{n,2}(x,y)), \tag{11}$$

where for $\ell = 1$ and 2,

$$\mu_{n,\ell}(x,y) = \lim_{\Delta t\to 0} \frac{\mathsf{E}[\Delta N_\ell(nt)|N(nt) = (nx, nay)]}{n_\ell \Delta t}. \tag{12}$$

Note that we abuse the notations a bit for clarity and nx and nay in (12) should be $\lceil nx\rceil$, $\lceil nay\rceil$ as $N_\ell(nt)$, $\ell = 1$ and 2 are integers.

In view of Theorem 2.11 in [21] (for density dependent family of Markov chains), it suffices to show that as $n \to \infty$, $\mu_{n,\ell}(x,y)$ converges to $\mu_\ell(x,y)$ for $\ell = 1$ and 2. For the Markov process $N(nt)$, its generator matrix is nQ as it is speeded up n times in comparison with the original Markov process $N(t)$. In view of (2) and (4), it follows that

$$\begin{aligned}&\mathsf{E}[\Delta N_1(nt)|N(nt) = (nx, nay)] \\ &= n(q_{(nx,nay),(nx+1,nay)} - q_{(nx,nay),(nx-1,ny)})\Delta t \\ &\qquad + o((\Delta t)^2) \\ &= n\Big(g_3(1,1,0) - g_3(1,0,0) + p(g_2(1,0,1) - g_2(1,0,0)) \\ &\quad + (1-p)(g_2(1,0,0) - g_2(1,0,1))\Big)\Delta t + o((\Delta t)^2).\end{aligned} \tag{13}$$

Similarly, we have from (3) and (5) that

$$\begin{aligned}&\mathsf{E}[\Delta N_2(nt)|N(nt) = (nx, nay)] \\ &= n(q_{(nx,nay),(nx,nay+1)} - q_{(nx,nay),(nx,nay-1)})\Delta t \\ &\qquad + o((\Delta t)^2) \\ &= n\Big(g_0(1,1,0) - g_0(1,0,0) + p(g_1(1,1,0) - g_1(0,1,0)) \\ &\quad + (1-p)(g_1(0,1,0) - g_1(1,1,0))\Big)\Delta t + o((\Delta t)^2).\end{aligned} \tag{14}$$

Recall that $n_1 = n$ and $n_2 = \lceil an\rceil$. For the state (nx, nay), we have from (1) that as $n \to \infty$

$$g_3(1,1,0) = \frac{\frac{nx(nx-1)}{2}(n_1 - nx)}{\frac{(n_1+n_2)(n_1+n_2-1)(n_1+n_2-2)}{3\cdot 2\cdot 1}} \to \frac{3x^2(1-x)}{(1+a)^3}. \tag{15}$$

Similarly, as $n \to \infty$,

$$g_3(1,0,0) = \frac{nx\frac{(n_1-nx)(n_1-nx-1)}{2}}{\frac{(n_1+n_2)(n_1+n_2-1)(n_1+n_2-2)}{3\cdot 2\cdot 1}} \to \frac{3x(1-x)^2}{(1+a)^3}, \tag{16}$$

$$g_2(1,0,0) = \frac{(nx)(n_1 - nx)(n_2 - nay)}{\frac{(n_1+n_2)(n_1+n_2-1)(n_1+n_2-2)}{3\cdot 2\cdot 1}} \to \frac{6ax(1-x)(1-y)}{(1+a)^3}, \tag{17}$$

and

$$g_2(1,0,1) = \frac{(nx)(n_1 - nx)nay}{\frac{(n_1+n_2)(n_1+n_2-1)(n_1+n_2-2)}{3\cdot 2\cdot 1}} \to \frac{6ax(1-x)y}{(1+a)^3}. \tag{18}$$

Using (15)–(18) in (13) yields

$$\lim_{n\to\infty} \mu_{n,1}(x,y) = \mu_1(x,y).$$

To show that $\lim_{n\to\infty} \mu_{n,2}(x,y) = \mu_2(x,y)$, one first observes that

$$g_0(1,1,0) \to \frac{3a^3y^2(1-y)}{(1+a)^3}, \tag{19}$$

$$g_0(1,0,0) \to \frac{3a^3y(1-y)^2}{(1+a)^3}, \tag{20}$$

$$g_1(0,1,0) \to \frac{6a^2(1-x)y(1-y)}{(1+a)^3}, \tag{21}$$

$$g_1(1,1,0) \to \frac{6a^2xy(1-y)}{(1+a)^3}. \tag{22}$$

Then use these in (14) and (12) to show $\lim_{n\to\infty} \mu_{n,2}(x,y) = \mu_2(x,y)$. ■

V. Stability

In this section we study the stability of the fluid limit of large networks. For this, we need to consider the equilibrium points of the system of first order differential equations in (8) and (9), i.e., the solutions of $\mu_1(x,y)=\mu_2(x,y)=0$. From (8) and (9), it is clear that the equilibrium points are the solutions of

$$\begin{aligned} x(1-x)(2x-1+2a(1-2p)(1-2y)) &= 0, \quad (23)\\ y(1-y)(2y-1+2a^{-1}(1-2p)(1-2x)) &= 0. \quad (24)\end{aligned}$$

Both the identities in (23) and (24) contain three factors. As we can choose a factor from each identity to solve a system of two linear equations, there are 9 solutions of (23) and (24) and thus 9 equilibrium points. Clearly, (0,0), (1,1), (0,1), and (1,0) are solutions of Eqs. (23) and (24). In addition, the solution of the linear system

$$\begin{aligned} x &= 0\\ 2y-1+2a^{-1}(1-2p)(1-2x) &= 0\end{aligned}$$

is a solution of Eqs. (23) and (24). This implies that $(0,1/2-(1-2p)/a)$ is a solution. For this solution to lie in the unit square, p and a must satisfy $0\le 1/2-(1-2p)/a\le 1$. This implies that

$$-\frac{a}{2}\le 1-2p\le\frac{a}{2}. \quad (25)$$

Similarly,

$$\left(1,\frac{1}{2}+\frac{1-2p}{a}\right),\left(\frac{1}{2}-\frac{1-2p}{a},0\right),\text{ and }\left(\frac{1}{2}+\frac{1-2p}{a},1\right)$$

are also equilibrium points. Finally, the solution of the linear system

$$\begin{aligned} 2x-1+2a(1-2p)(1-2y) &= 0 \quad (26)\\ 2y-1+2a^{-1}(1-2p)(1-2x) &= 0 \quad (27)\end{aligned}$$

is an equilibrium point. If $p\ne 1/4$ and $p\ne 3/4$, the above system has a unique solution $(1/2,1/2)$. If $p=1/4$ or $p=3/4$, the two straight lines in (26) and (27) coincide. In this case any point on the straight line (26) is also an equilibrium point.

One common approach for analyzing the stability of an equilibrium point (x^*,y^*) is to use *linearization*. Specifically, one first obtain Taylor's expansions for $\mu_1(x,y)$ and $\mu_2(x,y)$ around the equilibrium point, i.e., for $\ell=1$ and 2,

$$\begin{aligned}\mu_\ell(x,y) &= \mu_\ell(x^*,y^*)+\frac{\partial\mu_\ell(x,y)}{\partial x}\Big|_{x^*,y^*}(x-x^*)\\ &+ \frac{\partial\mu_\ell(x,y)}{\partial y}\Big|_{x^*,y^*}(y-y^*)\\ &+ o(\sqrt{(x-x^*)^2+(y-y^*)^2}). \quad (28)\end{aligned}$$

Note that $\mu_\ell(x^*,y^*)=0$ as (x^*,y^*) is an equilibrium point. Then consider the 2×2 matrix

$$\boldsymbol{A}_{(x^*,y^*)}=\begin{pmatrix}\frac{\partial\mu_1(x,y)}{\partial x}\Big|_{x^*,y^*} & \frac{\partial\mu_1(x,y)}{\partial y}\Big|_{x^*,y^*}\\ \frac{\partial\mu_2(x,y)}{\partial x}\Big|_{x^*,y^*} & \frac{\partial\mu_2(x,y)}{\partial y}\Big|_{x^*,y^*}\end{pmatrix}. \quad (29)$$

The system of first order differential equations in (8) and (9) can be approximated by

$$\begin{aligned}\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix} &= \boldsymbol{A}_{(x^*,y^*)}\begin{pmatrix}x(t)-x^*\\ y(t)-y^*\end{pmatrix}\\ &+o(\sqrt{(x(t)-x^*)^2+(y(t)-y^*)^2}). \quad (30)\end{aligned}$$

Thus, the equilibrium point (x^*,y^*) is *stable* if the real parts of both eigenvalues of the matrix $\boldsymbol{A}_{(x^*,y^*)}$ are *negative*. Such an equilibrium point is called a *node sink*. On the other hand, if the real parts of both eigenvalues of the matrix $\boldsymbol{A}_{(x^*,y^*)}$ are *positive*, then such an equilibrium point is called a *node source* and it is *unstable*. Finally, if the real parts of both eigenvalues of the matrix have *different signs*, then it is called a *saddle point* and it is also *unstable*.

Our main result of this section is the following stability result.

Theorem 2. *Assume that $p\ne 1/4$ and $p\ne 3/4$. The classification of the 9 equilibrium points of the system of first order differential equations in (8) and (9) are shown in Table I. Except the four equilibrium points, (0,0), (1,1), (0,1), and (1,0), the other equilibrium points are not stable for any choices of a and p.*

One important corollary of this theorem is that for large networks, the opinions of all the vertices are either in consensus or in polarization in the long run. In particular, if $1-2p<-(2a)^{-1}$, then the opinions will reach a consensus for every initial state. On the other hand, if $1-2p>(2a)^{-1}$, then the opinions will be polarized for every initial state. For $-(2a)^{-1}<1-2p<(2a)^{-1}$, whether the opinions of all the vertices will be in consensus or in polarization depends on the initial state. Numerical results for this will be given in Section VI-B.

Now we prove Theorem 2.

Proof. First we consider the origin. Note that

$$\boldsymbol{A}_{(0,0)}=\begin{pmatrix}\frac{-3}{(1+a)^3}+\frac{6(1-2p)a}{(1+a)^3} & 0\\ 0 & \frac{-3a^2}{(1+a)^3}+\frac{6(1-2p)a}{(1+a)^3}\end{pmatrix}. \quad (31)$$

Since $\boldsymbol{A}_{(0,0)}$ is diagonal, its eigenvalues are the diagonal entries. It is easy to verify that if $1-2p>a/2$, both eigenvalues are positive. If $1-2p<1/(2a)$, both are negative. If $1/(2a)<1-2p<a/2$, one eigenvalue is positive and the other is negative. In the first case, the origin is a nodal source and the system is unstable. In the second case, the origin is a nodal sink and the system is stable. In the third case, the origin is a saddle point and the system is again unstable.

We now study the stability of points $(1,1)^T$, $(1,0)^T$ and $(0,1)^T$. It is easy to verify that $\boldsymbol{A}_{(1,1)}=\boldsymbol{A}_{(0,0)}$. Thus, the equilibrium point $(1,1)^T$ has identical stability properties as that of the origin. It is also easy to verify that

$$\begin{aligned}&\boldsymbol{A}_{(1,0)}=\boldsymbol{A}_{(0,1)}\\ &=\begin{pmatrix}\frac{-3}{(1+a)^3}-\frac{6(1-2p)a}{(1+a)^3} & 0\\ 0 & \frac{-3a^2}{(1+a)^3}-\frac{6(1-2p)a}{(1+a)^3}\end{pmatrix}. \quad (32)\end{aligned}$$

TABLE I: Stability conditions of the equilibrium points

equilibrium point	nodal sink	nodal source	saddle point
$(1/2,1/2)^T$		$1/4<p<3/4$	$p<1/4$ or $p>3/4$
$(0,0)^T$ and $(1,1)^T$	$1-2p<(2a)^{-1}$	$1-2p>a/2$	$(2a)^{-1}<1-2p<a/2$
$(0,1)^T$ and $(1,0)^T$	$1-2p>-(2a)^{-1}$	$1-2p<-a/2$	$-a/2<1-2p<-(2a)^{-1}$
$(0,1/2-(1-2p)/a)^T$ and $(1,1/2+(1-2p)/a)^T$ and $(1/2-(1-2p)/a,0)^T$ and $(1/2+(1-2p)/a,1)^T$		$3/4<p<\min(1/2+a/4,1)$ or $\max(1/2-a/4,0)<p<1/4$	otherwise

It is easy to verify that the two eigenvalues of the matrix in (32) are both positive if $1-2p<-a/2$. The two eigenvalues are both negative if $1-2p>-1/(2a)$. The two eigenvalues have different signs if

$$-a/2<1-2p<-1/(2a).$$

Suppose that $p\neq 1/4$ and $p\neq 3/4$. We now determine the stability property of the equilibrium point $(1/2,1/2)$.

$$\boldsymbol{A}_{(1/2,1/2)}=\begin{pmatrix} \frac{1}{(1+a)^3}\frac{3}{2} & \frac{-3(1-2p)a}{(1+a)^3} \\ \frac{-3(1-2p)a}{(1+a)^3} & \frac{a^2}{(1+a)^3}\frac{3}{2} \end{pmatrix}.$$

Since $\boldsymbol{A}_{(1/2,1/2)}$ is symmetric, its eigenvalues must be real. The characteristic equation of $\boldsymbol{A}_{(1/2,1/2)}$ is

$$s^2-\frac{3(1+a^2)}{2(1+a)^3}s+\frac{9a^2}{4(1+a)^6}-\frac{9(1-2p)^2a^2}{(1+a)^6}=0. \quad (33)$$

If $p<1/4$ or $p>3/4$, then $(1-2p)^2>1/4$. In this case, the constant term on the left of (33) is negative. Thus, the two roots of (33) must be real and have different signs. It implies that point $(1/2,1/2)^T$ is a saddle point and is unstable. If $1/4<p<3/4$, it is easy to verify that the constant term on the left of (33) is positive. In addition, the derivative of the characteristic polynomial in (33) at $s=0$ is negative. Hence, the two roots are both positive. This implies that the equilibrium point $(1/2,1/2)^T$ is a nodal source and thus, is unstable.

We now analyze the stability property of point $(0,1/2-(1-2p)/a)$. We have

$$\boldsymbol{A}_{(0,1/2-(1-2p)/a)} = \begin{pmatrix} \frac{9-48p+48p^2}{(1+a)^3} & 0 \\ \frac{3(2+a-4p)(-2+a+4p)(-1+2p)}{a(1+a)^3} & \frac{3(a^2-4(1-2p)^2)}{2(1+a)^3} \end{pmatrix}. \quad (34)$$

The two eigenvalues of the matrix in (34) are

$$\lambda_1 = \frac{3(-3+4p)(-1+4p)}{(1+a)^2} \quad (35)$$

$$\lambda_2 = \frac{3(2+a-4p)(-2+a+4p)}{2(1+a)^2}. \quad (36)$$

We now analyze the two eigenvalues. Note that λ_1 is a convex function of p, while λ_2 is a concave function of p. Eigenvalue λ_1 is negative if $1/4<p<3/4$. Eigenvalue λ_2 is positive if

$$\frac{1}{2}-\frac{a}{4}<p<\frac{1}{2}+\frac{a}{4}. \quad (37)$$

Recall that we assume that $a\geq 1$. Thus, the interval $(1/4,3/4)$ is contained in the interval $(1/2-a/4,1/2+a/4)$. Note that condition (37) is identical that (25), which is required in order for the equilibrium point $(0,1/2-(1-2p)/a)$ to lie in the unit square. These facts imply that the two eigenvalues are either both positive, or one negative and one positive. In either case, the equilibrium point $(0,1/2-(1-2p)/a)$ is unstable. If

$$\frac{1}{2}-\frac{a}{4}<p<\frac{1}{4} \text{ or } \frac{3}{4}<p<\frac{1}{2}+\frac{a}{4},$$

the equilibrium point is a nodal source. Otherwise, it is a saddle point.

The stability analysis of points $(1/2-(1-2p)/a,0)$, $(1,1/2+(1-2p)/a)$ and $(1/2+(1-2p)/a,1)$ is similar. We omit the details. ■

In Theorem 2 we did not consider the cases in which $p=1/4$ or $p=3/4$. We now consider these two special cases. Recall that in these special cases the linear system in (26) and (27) is degenerate. We first consider $p=1/4$. We substitute (8) into (9) and obtain

$$\frac{y'}{y(1-y)}=(-a)\frac{x'}{x(1-x)}. \quad (38)$$

Integrate both sides of (38) and we obtain

$$y=\frac{c_1(1-x)^a}{x^a+c_1(1-x)^a}, \quad (39)$$

where constant c_1 is

$$c_1=\frac{y_0}{1-y_0}\left(\frac{x_0}{1-x_0}\right)^a.$$

Eq. (39) describes the trajectory along which the solution of (8) and (9) moves. From (39) it is clear that if the initial point (x_0,y_0) is not on the straight line (26) the solution converges to either (0,1) or (1,0). If the initial condition is on the straight line (26), it stays there forever, since the derivatives x' and y' are both zero. In Figure 2 we show the trajectories corresponding to $p=1/4$, $a=1,2$ and $(x_0,y_0)=(0.2,0.1)$, $(0.2,0.5)$ and $(0.2,0.9)$. The two straight lines correspond to (26) with $a=1$ and 2. For any value of a, the corresponding straight line bisections the unit square into two regions. Any initial point moves along the curve described in (39) to point $(1,0)$ or $(0,1)$ without intersecting with the straight line. Similarly, the trajectories corresponding to $p=3/4$ always reach $(0,0)$ or $(1,1)$, unless the initial points lie on the straight line (26).

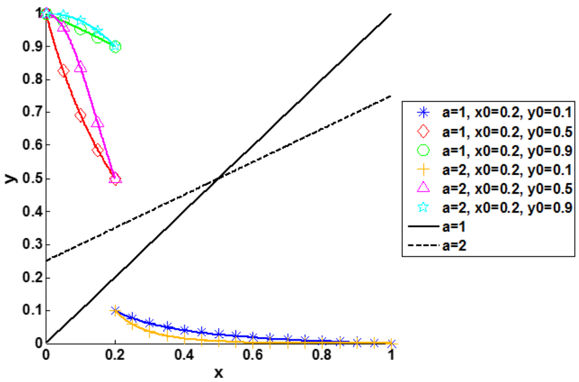


Fig. 2: Trajectories described in (39).

VI. Numerical Results

In this section we present our numerical results. In Section VI-A, we show the numerical results for the polarization probabilities of the Markov process. In Section VI-B, we plot the basins of attraction for the fluid limit of large networks to further understand the effects of parameters and initial states. In Section VI-C, we make an attempt to apply our model for the interaction among members of congress or parliament.

A. Absorption probabilities of the Markov process

In this section, we show various numerical results for the Markov process derived from our model of binary opinion dynamics. We are particularly interested in the probability that polarized opinions occur in a network. In our numerical study we assume that at time zero with probability $p(i,j)$, there are i and j vertices respectively in clusters 1 and 2 that have opinion 1. Suppose that starting from state (i,j) the probability of absorption into state $(0,n_2)$ (*resp.* state $(n_1,0)$) is denoted by $\eta_{(i,j),(0,n_2)}$ (*resp.* $\eta_{(i,j),(n_1,0)}$). We define the probability of polarization

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} p(i,j)\left(\eta_{(i,j),(0,n_2)}+\eta_{(i,j),(n_1,0)}\right). \tag{40}$$

In a special case, in which each vertex independently chooses opinion 1 with probability q and opinion 0 with probability $1-q$, the initial distribution $p(i,j)$ is

$$\binom{n_1}{i} q^i(1-q)^{n_1-i}\binom{n_2}{j} q^j(1-q)^{n_2-j}. \tag{41}$$

We first show the probability of polarization (as a function of the rationality probability p) for five different combinations of n_1 and n_2 in Figure 3. From this figure, it is clear that the smaller the difference between the sizes of the two clusters, the more likely that polarization occurs (when n_1+n_2 is fixed). We then study the probability of polarization with $(n_1,n_2) = (20,30)$. We consider five values of q. The result is presented in Figure 4, which clearly indicates that the smaller the difference between the expected number of opinions initially, the more likely polarization occurs.

We show the probability of polarization for $q=1/2$ and $q=4/5$ in Figure 5 and Figure 6, respectively. In these figures, we consider five network sizes with constant $a=3/2$. From

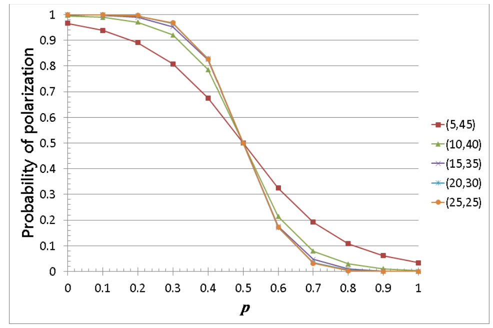


Fig. 3: Probability of polarization with $q = 1/2$ and five different combinations of n_1 and n_2 with $n_1+n_2 = 50$.

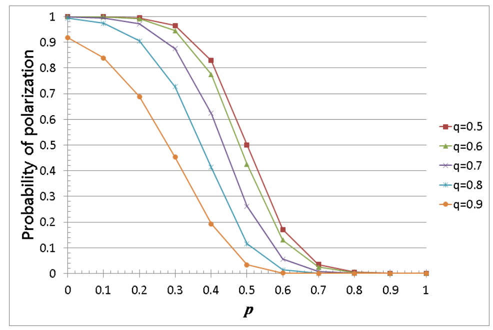


Fig. 4: Probability of polarization with $(n_1,n_2) = (20,30)$ and five different values of q.

these figures, it seems clear that the network exhibits a phase transition where polarization occurs with high probability when the rationality probability p is less than a threshold value. When p is more than this threshold, polarization is unlikely to occur. From the figures, it shows that as the network size grows, this phase transition becomes more eminent and eventually reaches a binary outcome as predicted by the fluid limit of large networks.

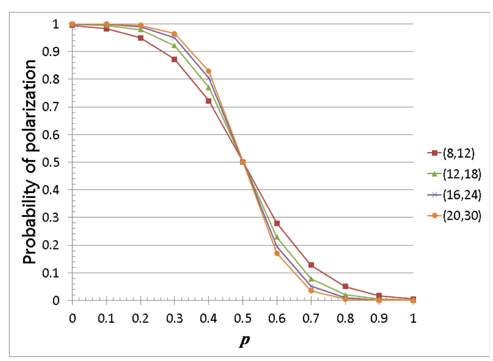


Fig. 5: Probability of polarization with $q = 1/2$ and four different combinations of n_1 and n_2 with $a = 3/2$.

B. Basins of attraction for the fluid limit

In this section, we show numerical results for the fluid limit of large networks. Recall that the parameter a is the the asymptotic ratio of the sizes of these two clusters. To understand the effect of the two parameters a and p, we plot in Figure 7 the basins of attraction for $a=1$ and $p=0.4$, 0.5 and 0.6. Note that the basin of attraction is the set of points in the space of system variables such that initial conditions chosen in

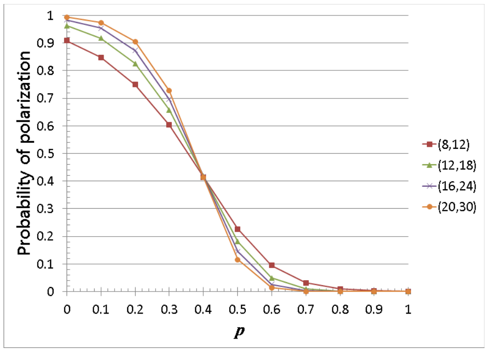


Fig. 6: Probability of polarization with $q = 4/5$ and $a = 3/2$.

this set dynamically evolve to a particular equilibrium point. Each arrow at a point (x, y) in these figures is drawn in line with the direction of $(\mu_1(x, y), \mu_2(x, y))$ so that one can see how an initial state in a basin of attraction is attracted to its corresponding equilibrium point. As shown in this figure, increasing the rationality probability p enlarges the basins of attraction for the equilibrium points $(0, 0)$ and $(1, 1)$. Thus, it is more likely to reach a consensus for a random chosen initial state.

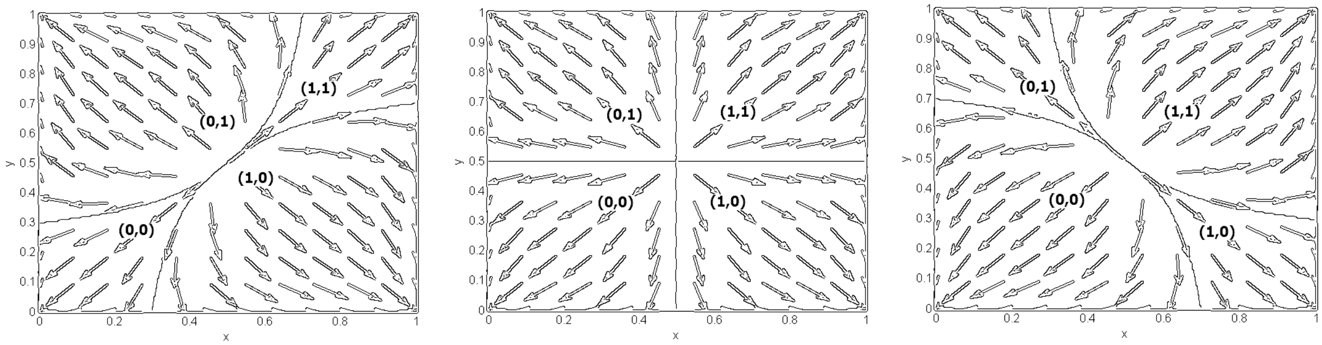
In Figure 8, we plot the basins of attraction for $p = 0.45$ and $a = 2$ and 4. From this figure, one can see that the trajectory from any initial state moves first to the vicinity of a boundary and then converges to one of the stable equilibrium points. For instance, consider a trajectory hits $y = 1$ at some point $\hat{x}$. Then this trajectory will move to its left if $\mu_1(x, y)|_{\hat{x},1} < 0$ and to its right if $\mu_1(x, y)|_{\hat{x},1} > 0$. From (8), we know that this trajectory will move to $(1, 1)$ along the line $y = 1$ if $\hat{x} > (1 + 2a(1-2p))/2$ and $(0, 1)$ if $\hat{x} < (1+2a(1-2p))/2$. For $a = 2$ (resp. $a = 4$) and $p = 0.45$, we have $(1+2a(1-2p))/2 = 0.7$ (resp. 0.9) and that matches extremely well to the boundary between the two basins of attraction for $(1, 1)$ and $(0, 1)$ at $y = 1$ in Figure 8a (resp. Figure 8b). In Figure 8c we further increase a to 8. In this case, equilibrium points $(0, 0)$ and $(1, 1)$ cease to be stable. The only stable equilibrium points are $(0, 1)$ and $(1, 0)$, and the system approaches polarization as time goes to infinity.

C. Applications

Finally, we attempt to model the interaction among members of congress or parliament. In nearly all democratic countries parliament consists of two major political parties. Parliament members who belong to the same political party tend to have similar political opinions, while the two parties tend to have opposite views in most issues. As a result, votes on many bills tend to be polarized. In this section, we use voting records to determine the rationality probability p for two democratic countries and compare the polarization probability obtained by our model with that of the voting records. First we consider the Legislative Yuan in Taiwan. This is the organization that makes laws in Taiwan. It is equivalent to the congress in the United States and the parliament in the United Kingdom. We consider two consecutive sessions of length 3 years each. The first session, denoted by session I from now on, began in February 22, 2008 and ended in January 20, 2012. The second session, denoted by session II from now on, began in January 24, 2012 and is still on going at the time when this paper is being written. For session II we consider bills that were voted before February 24, 2015. There were 128 members in session I. For various reasons, such as lawsuits or taking up new positions in the government, some members in session II left the Legislative Yuan. As a result, the number of members was reduced and became 113 in session II. Similar to those of most democratic countries, the political ecology in Taiwan consists of two major political parties and several small ones. In this study we focus on the two major political parties and ignore the small parties. In session I, the two major parties had 85 and 36 members respectively. In session II, the two major parties had 65 and 40 members as shown in Table II. Some Legislative Yuan members may not appear to cast ballots for some bills and some may even cast invalid ballots. Thus, we consider only those parliament members who cast valid ballots for 80% or more of all bills in the sessions. Those members who meet this criteria are called effective and those who did not meet this criteria are ignored in the study. The numbers of members of the two parties who meet this criteria in session i are denoted by n_1^i and n_2^i respectively for $i = I$ and II. A bill that all effective members have voted is said to be a major bill. In this study we consider only major bills and non-major bills are ignored. In Table II we list the number of effective legislators of the two parties followed by the total number of legislators. In this table, we also list the number of major bills followed by the total number of bills.

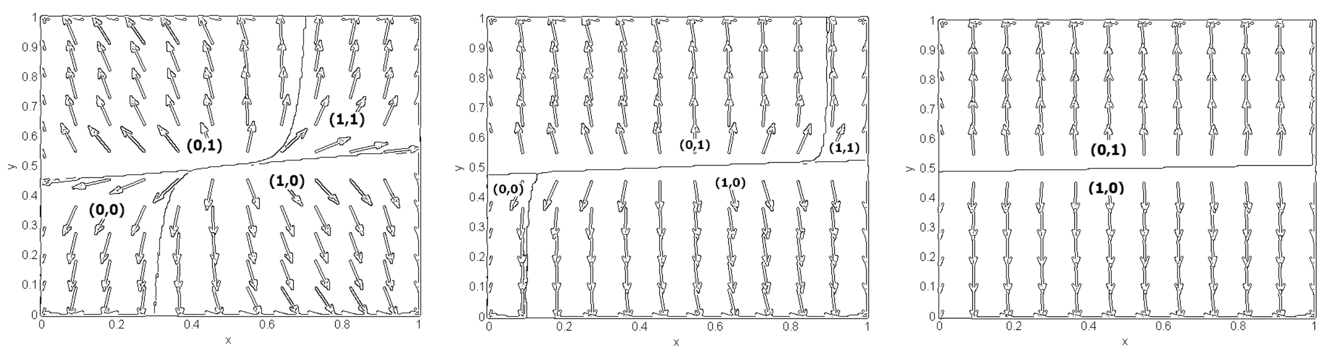
We estimate the rationality probability p in the following way. We consider all combinations of triads, in which two parliament members belong to one party and the other member belongs to a different party. For a particular triad we count the number of major bills that the triad votes unanimously in session II. We also count the number of major bills where the vote of the triad is polarized. We accumulate those two numbers for all the triad combinations. We approximate p by the fraction of unanimous votes to the total number of unanimous votes and polarized votes. The value of p is shown in Table III.

We now describe how the initial distribution of opinions for session II is obtained. Note that we have determined n_1^I and n_2^I. In session I, there were 75 major bills. For any point (i, j) in the rectangle $\mathcal{R}_I = \{(i, j) : i = 0, 1, \ldots, n_1^I, j = 0, 1, \ldots, n_2^I\}$, $p(i, j)$ is the fraction of major bills in the total number of major bills that i and j members in the two parties have voted favorably. We then re-scale the range of the two-dimensional probability mass function $p(i, j)$ from $\mathcal{R}_I$ to rectangle $\mathcal{R}_{II} = \{(i, j) : i = 0, 1, \ldots, n_1^{II}, j = 0, 1, \ldots, n_2^{II}\}$. With the rationality probability p and the initial distribution $p(i, j)$ we compute the probability of polarization using (40) with n_1 and n_2 replaced by n_1^{II} and n_2^{II} respectively. The result is presented in the third column of Table III. We say that a bill is polarized if 80% or more of the effective members in both parties cast the same votes, while the majorities of the two parties vote differently. We say that a bill is unanimous if



(a) $a = 1$ and $p = 0.4$. (b) $a = 1$ and $p = 0.5$. (c) $a = 1$ and $p = 0.6$.

Fig. 7: Basins of attraction for $a = 1$.



(a) $a = 2$ and $p = 0.45$. (b) $a = 4$ and $p = 0.45$. (c) $a = 8$ and $p = 0.45$.

Fig. 8: Basins of attraction for $p = 0.45$.

Country	information source	session I		session II	
		number of legislators	number of bills	number of legislators	number of bills
Taiwan	vote.ly.gov.tw	(32,2)/(85,36)	75/490	(53,38)/(65,40)	53/405
United States	www.govtrack.us	(74,26)/(245,200)	1069/1600	(72,28)/(237,202)	533/762

TABLE II: Information on the Legislative Yuan of Taiwan and the Congress of the United States. The third column and the fifth column are the number of effective legislators of two parties followed by the total number of legislators. The fourth column and the sixth column are the number of major bills followed by the total number of bills.

80% or more of the effective members in both parties cast the same vote. There were totally 405 bills voted in session II, out of which 53 bills were major. Out of the 53 major bills, 52 bills were polarized and one bill was unanimous. Thus, the fraction of polarized bills is $52/53 = 0.9811$. This number is presented in the last column of Table III. Note that we compute the rationality probability and the initial opinion distribution based on statistics collected in session I. Our model then predicts the probability of polarization in session II.

Similarly, we analyzed the voting record of the 112-nd and the 113-rd United States Congress. As before, we consider only a subset of legislators and bill. The number of effective legislators of the United States is very large. There were nearly 400 effective legislators. Recall that the calculation of absorption probabilities in (7) is based on matrix multiplication and inversion. The most efficient matrix multiplication and inversion algorithm has a complexity of $O(n^{2.373})$ [22], where $n = (n_1 + 1)(n_2 + 1)$ is the dimension of the matrix. The storage complexity of matrices is $O(n^2)$. The size of the Congress causes very large matrices in the calculation of absorption probabilities. Either specialized numerical method or special computer with large amount of memory is needed. We decide to reduce the number of effective legislators to 100. Specifically we sort the legislators in descending order of the number of bills that they have voted and we consider the top 100 legislators. Other procedure remains the same. The rationality probability p is 0.2721. The probability of polarization obtained using (7) is 0.89. There were 533 major bills voted in session II, out of which there were 82 unanimous bills and 289 polarized bills. The rest bills are neither unanimous nor polarized. The fraction of polarized bill is 289/533. These numbers are presented in Table III.

Country	p	probability of polarization	fraction of polarized bills
Taiwan	0.0878	0.9867	0.9811
United States	0.2721	0.89	0.779

TABLE III: Rationality probability and polarization probability.

In this paper, we use parliamentary voting records as a case study of opinion spreading in signed networks. We can also study the dynamics of opinions in web news sites or product-rating sites, such as Slashdot and Epinions [11], using our model. We note that our model may also be used to study the phenomenon of consensus verses polarization in other applications, such as spread of technologies or adoption of conventions in a signed network. For instance, it is well known that international relations can be friendly or hostile, and exhibit structural balance [10]. Although there are many technologies that are unanimously adopted by all countries in the world, there are examples where adoption is polarized. For example, NTSC and PAL are two video recording standards on cassettes. Nearly half of the countries adopt NTSC and nearly the other half adopt PAL. As another example, adoption of traffic regulations is polarized between right-hand traffic and left-hand traffic. There are evidences that adoption of technologies or conventions by a country can be affected by its friendly neighbors as well as its hostile neighbors.

VII. Conclusions

In this paper we have proposed a new model for binary opinion dynamics in a static structurally balanced network. For networks with a finite size, we apply the technique of Markov chains with absorbing states to analyze the probability of polarization. We observe that the probability of polarization exhibits a phase transition phenomenon. For networks of an infinite size, we apply a fluid limit theorem to derive a nonlinear system of differential equations. We present an analysis on the stability of the equilibrium points of the system.

During the revision process of this paper, the associate editor pointed out reference [13] to us. We were not aware of this reference during the writing of this paper. We now highlight [13] and briefly discuss the differences between the two papers. We believe that relationship between two actors is relatively stable and can only change in a long time frame. It is less likely that this relationship changes frequently and abruptly in the adoption process of one opinion. Rather, it is more likely that the relationship between two actors is the accumulative result of appreciation/depreciation of many issues. Thus, in this paper we have assumed a static fully-connected structurally balanced network. The update process of opinions in this network randomly chooses triads. Opinions in the chosen triad are updated according to different rules depending on whether members of the triads are friendly or hostile to each other.

Singh *et. al.* [13] propose a signed network model with three types of opinions. Specifically, there are two extreme opinions, the leftists and rightists, and a median opinion, the centrists. Extremists are connected by negative edges and edges connecting other opinion types are all positive. In their model, both the opinions and the edge signs update with time. At each time slot, either a vertex is randomly chosen, or a triad is chosen. If a vertex is chosen and if the vertex is an extremist, it is converted to a centrist. If a triad is chosen and the triad is unbalanced, the authors propose rules to balance the triad by converting either an extremist to a centrist, or vice versa. Edge signs are updated accordingly, since edges between a leftist and a rightist are negative and all other edges are positive. The network converges to either a consensus of centrist state, or a polarization between leftist state and rightist states. The authors derive a system of differential equations for the density of leftists and that of rightists. Depending of the values of the parameters, the system of differential equations can have only one stable equilibrium point, which corresponds to an all-centrist consensus, or three equilibrium points. In the latter case, two are stable and the third point is an unstable saddle point. In the limit of the update process, if a polarization is formed, the two clusters of the structurally balanced network have an equal size. The authors also discuss the convergence time of the update process.

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