# A Mathematical Theory for Multistage Battery Switching Networks

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**Abstract**—In this paper, we propose a mathematical theory for multistage battery switching networks. The theory aims to address several design issues in managing a large-scale battery system, including flexibility, reliability, efficiency, complexity (scalability) and sustainability. Our multistage battery switching network is constructed by a concatenation of various rectangular "shapes" of battery packs. The shape of each battery pack is specified by its voltage and its capacity. We show that our multistage battery switching network can support a maximum number of $L_{\max}$ loads under the constraint that the total voltages of these loads do not exceed a design constant $V_{\max}$. Moreover, the voltage of each battery pack can be determined *optimally* by solving a Simultaneous Integer Representation (SIR) problem. To determine the capacity of each battery pack, we propose a max-min fairness battery allocation scheme, and show by computer simulations that such a scheme outperforms the uniform battery allocation scheme. We also propose a fault tolerant battery switching network that can still be operated properly even after $F_{\max}$ battery packs fail. Such a fault tolerant battery switching network enables a battery system to implement the Largest Remaining Capacity First (LRCF) policy that does not require the knowledge of the load profile.

**Index Terms**—Large-scale battery systems, fault tolerance, resource management.

## 1 Introduction

Large-scale battery systems with hundreds or thousands of (rechargeable) batteries are commonly adopted in many systems such as electric vehicles (EVs) and smart micro-grids. How to effectively manage large-scale battery systems has received a lot of attention recently (see e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]). One of the key issues of managing large-scale battery systems is *reliability*. Most current battery systems are interconnected in a fixed manner and a single failure of a battery might result in a severe damage of the whole system. As such, dynamically reconfiguring the interconnection of the batteries in a battery system was previously proposed in several early works [6], [7], [8] to handle battery failures. Another issue is how to use the energy *efficiently*. As most battery systems are interconnected in a fixed manner, to support multiple types of loads one has to rely on voltage regulators or DC-DC converters to convert the supplied voltages so as to match the required voltages of loads. Since the efficiency of voltage regulators and DC-DC converters degrades significantly when the difference between the supplied voltage and the required voltage is large [13], [14], [15], it is thus desirable to reconfigure the interconnection of batteries so that the difference between the supplied voltage and the required voltage is small [4], [12]. Also, there is the sustainability issue. In particular, the problem of how to prolong the battery operating time was addressed in [3].

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In addition to the reliability issue, the efficiency issue, and the sustainability issue, we think there are another two issues that also need to be addressed: flexibility and complexity (scalability). The hardware complexity (in terms of the numbers of switches or relays) in a large-scale battery system should be kept low, while the system should be flexible enough to generate any desired configuration (within some design constraints). Though there are several battery systems that allow batteries to be connected in any desired configuration, e.g., the battery switch array system in [5] and the Power Trees (PTs) in [11], the number of switches in those systems are proportional to the number of batteries. That not only poses a scalability problem but also a reliability problem as it is difficult to ensure that a large number of switches can be operated properly. In particular, it was mentioned in [11] the order of turning on/off a large number of switches in their system could be a problem as batteries might be short-circuited due to random delays in the process of turning on/off a large number of switches. In view of all these, our main objective of this paper is to propose a mathematical theory for multistage battery switching networks that aims to address the following five issues:

- ***Flexibility***: The system needs to be flexible enough to generate the desired configuration (output voltages).
- ***Reliability***: The system needs to work properly under nominal load conditions, in spite of potential failure of battery packs.
- ***Efficiency***: The system should be efficient in power usage.
- ***Complexity***: The hardware complexity of switches in the system should be kept low and the computational complexity to control these switches should

also be simple.
- ***Sustainability***: The system should provide a long life span.

Our approach is like Lego. We first connect batteries in a fixed manner into a battery pack to form a building block, and then interconnect various forms of battery packs by switches to form a battery switching network. By doing so, there is no switch inside any battery pack and thus the total number of switches in the system can be reduced. In this paper, we only consider battery packs in the rectangle shape. Specifically, a battery pack with voltage $d$ and capacity $b$ is constructed by connecting $bd$ batteries first in series and then in parallel as shown in Figure 1. The challenge of such an approach is to determine the voltage and the capacity of each building block so that the battery switching network can also address these five issues.

We summarize our contributions as follows:
(i) Flexibility: Using these building blocks, we construct a battery system that can support a maximum number of $L_{\max}$ loads (that might require different voltages) provided that the sum of the voltages of these loads does not exceed a design constant $V_{\max}$. Such a battery system is called a $(V_{\max}, L_{\max})$-battery switching network in Definition 1. The voltage $d$ of each battery pack can be determined *optimally* in Theorem 6 by solving a Simultaneous Integer Representation (SIR) problem (see Definition 2).
(ii) Complexity: The total number of on/off switches in our multistage battery switching network is $4NL_{\max}$, where $N$ is the number of battery packs (stages). The connection patterns of these on/off switches can be determined in $O(NL_{\max})$ steps by using the alternating C-transform in Algorithm 1.
(iii) Efficiency and Sustainability: We propose a max-min fairness battery allocation scheme that can be used for determining the capacity of each battery pack to match its average energy consumption rate. By computer simulations, we show such a scheme significantly outperforms the uniform allocation scheme (that simply assigns each battery pack with the same capacity) in terms of the system life.
(iv) Reliability: We show in Theorem 8 how to construct a fault tolerant battery switching network so that the system can still be operated properly even after $F_{\max}$ battery packs fail. If the system is also able to monitor the state of charge (SOC) of each battery pack, then such a system can only choose the $N - F_{\max}$ battery packs that have the largest remaining capacities to operate. We call such a policy the Largest Remaining Capacity First (LRCF) policy. We also show by computer simulations that the LRCF policy performs very well even without the knowledge of the stochastic load profile.

The paper is organized as follows: in Section 2, we show how to construct a battery switching network that can support a maximum number $L_{\max}$ loads under the constraint that the sum of the voltages of these loads does not exceed $V_{\max}$. By solving the SIR problem, we then show how to choose the optimal voltage of each battery pack. In Section 3, we address the capacity assignment problem and propose a max-min fairness allocation scheme to match the capacity of each battery pack to its average energy consumption rate. In Section 4, we propose fault tolerant battery switching networks and the LRCF policy. We report our simulation results in Section 5. Finally, we address possible extensions of our work in Section 6.

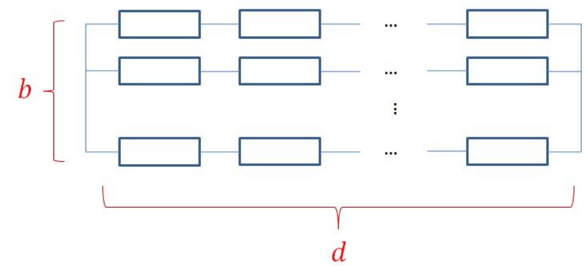



Fig. 1. A battery pack with voltage $d$ and capacity $b$.

## 2 Construction of Battery Switching Networks

In this paper, we consider ideal batteries (with zero internal resistance). Each (fully charged) battery can support one unit of voltage (e.g., 1 Volt) and one unit of capacity (e.g., 1 Amp-hour). A battery pack with voltage $d$ and capacity 1 is constructed by connecting $d$ batteries in series, and a battery pack with voltage $d$ and capacity $b$ (see Fig. 1) is constructed by connecting $b$ battery packs with voltage $d$ and capacity 1 in parallel. As such, there are $b \cdot d$ batteries in a battery pack with voltage $d$ and capacity $b$.

Now we give our definition of a battery switching network.

***Definition 1.*** **(Battery switching network)** *A system is called a $(V_{\max}, L_{\max})$-battery switching network if the system can simultaneously support any $L$ loads with the required integer-valued voltages $X_1, X_2, ..., X_L$ under the following two conditions:*

$$L \le L_{\max}, \tag{1}$$

*and*

$$X_1 + X_2 + ... + X_L \le V_{\max}. \tag{2}$$

The constraint in (1) sets the limit on the maximum number of loads that can be supported by the system, while the constraint in (2) sets the limit on the maximum total voltage that can be supported by the system. In the literature (see e.g., [11]), a load is generally described by two parameters: the required voltage and the required current. However, the required current is not needed for our definition of a battery switching network in Definition 1. Also, for a load that requires a large current, it can be served by treating it as multiple loads with a small current. As such, the constraint on the maximum number of loads indirectly puts a limit on the maximum required current for a load. We also note that in practice one might still need voltage regulators or DC-DC converters as the output voltage of a non-ideal battery is not constant during its discharge. However, as mentioned in the introduction, it is still desirable to configure the interconnection of batteries to keep the difference between the supplied voltage and the required voltage small so that voltage regulators or DC-DC converters can be operated efficiently [4], [12].

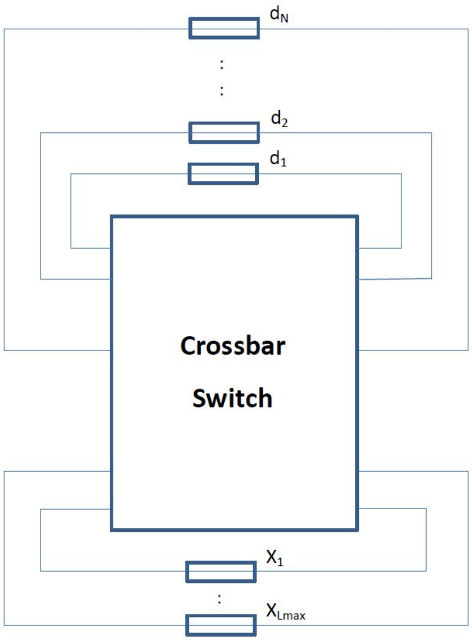



Fig. 2. Construction of a battery switching network by using a crossbar switch.

### 2.1 Constructing a Battery Switching Network by Using a Single Crossbar Switch

The basic idea to construct a battery switching network is to dynamically interconnect battery packs with fixed voltages (either in series or in parallel) to support the desired voltages. This can be done by using a crossbar switch and various battery packs with fixed voltages. In the literature (see e.g., [16], [17]), an $N \times N$ crossbar switch is commonly defined as a network device (with $N$ inputs and $N$ outputs) that is capable of realizing any (sub)permutation that connects the $N$ inputs to the $N$ outputs. In Fig. 2, we consider a network device that consists of an $(N+L_{\max}) \times (N+L_{\max})$ crossbar switch and $N$ battery packs with fixed integer-valued voltages $d_1, d_2, ..., d_N$. In order for such a network device to support $L$ loads with the required integer-valued voltages $X_i$ $i = 1, 2, \ldots, L$, one must be able to find *disjoint* subsets of battery packs $S_i$, $i = 1, 2, \ldots, L$, such that for $i = 1, 2, \ldots, L$,

$$X_i = \sum_{k \in S_i} d_k, \tag{3}$$

and then connect the battery packs in each subset in series to support the desired voltage.

As an illustrating example, consider a $4 \times 4$ crossbar switch and three voltage packs with voltages $d_1 = 1$, $d_2 = 2$ and $d_3 = 4$ in Figure 3. For a load that requires voltage 5, we can use the binary representation to derive

$$5 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2.$$

Now we can set the connection pattern as shown in Figure 3. By doing so, the two voltage packs with voltages $d_1 = 1$ and $d_3 = 4$ are connected in series to yield the desired voltage 5.

The reason that we restrict ourselves to the selection of *disjoint sets* in this paper is to avoid *short circuits* in battery packs. For instance, suppose that three battery packs, say 1,2, and 3, are already connected in series to support a load.

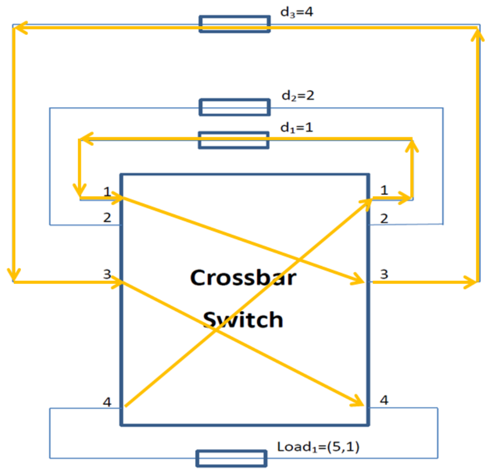



Fig. 3. An illustrating example for the connection pattern to support a load that requires voltage 5.

If we further connect battery packs 1 and 3 in series to support another load, then such a connection creates a short circuit (a loop) in battery pack 2.

The problem to find disjoint subsets of battery packs that satisfy (3) is formulated as the Simultaneous Integer Representation (SIR) problem below.

***Definition 2.*** **(The Simultaneous Integer Representation (SIR) problem)** *Given a set of positive integers $S = \{d_1, d_2, \ldots, d_N\}$ and a set of nonnegative integers $X = \{X_1, X_2, ..., X_L\}$, we say $X$ is* simultaneously representable *by the basis set $S$ if there exist $C_i(k)$, $i = 1, 2, \ldots, L$, $k = 1, 2, \ldots, N$, such that*

$$C_i(k) = 0 \text{ or } 1, \tag{4}$$

$$\sum_{i=1}^{L} C_i(k) \le 1, \quad \textit{and} \tag{5}$$

$$X_i = \sum_{k=1}^{N} C_i(k) \cdot d_k. \tag{6}$$

*The Simultaneous Integer Representation (SIR) problem with the basis set $S$ and the maximum number of representable integers $L_{\max}$, denoted by the $(S, L_{\max})$-SIR problem in this paper, is to find the largest integer $V_{\max}$ so that any set of nonnegative integers $X = \{X_1, X_2, ..., X_L\}$ is simultaneously representable by the basis set $S$ as long as $L \le L_{\max}$ and*

$$X_1 + X_2 + ... + X_L \le V_{\max}.$$

We note that the $(S, L_{\max})$-SIR problem in Definition 2 is different from the postage stamp problem [18]. In the postage stamp problem, there is only one integer that needs to be represented and the elements in the basis set can be repeatedly selected to represent that integer, i.e., $C_i(k)$ can be larger than 1. Also, the total usage of the elements in the basis set in the postage stamp problem is bounded, i.e., $\sum_{k=1}^{N} C_i(k) \le h$ for some constant $h$. The postage stamp problem is known to be NP-hard [19]. However, the $(S, L_{\max})$-SIR problem is much simpler as every element in the basis set can only be used at most once. As such, the solution of the $(S, L_{\max})$-SIR problem is bounded above by the sum of all the elements in the basis set. We will show later in Corollary 5 that the sum of all the elements in the basis set is indeed the solution of the $(S, L_{\max})$-SIR problem if the elements in the basis set satisfy certain inequalities.

In the following proposition, we establish the connection between the simultaneous integer representation problem and the construction of a battery switching network.

***Proposition 3.*** *Given $S = \{d_1, \ldots, d_N\}$ and $L_{\max}$, let $V_{\max}$ be the solution of the $(S, L_{\max})$-SIR problem. Then the network represented in Figure 2 is a $(V_{\max}, L_{\max})$-battery switching network.*

**Proof.** Consider a set of integer-valued loads $X = \{X_1, X_2, ..., X_L\}$ that is simultaneously representable by $S$. Note that the indicator variable $C_i(k)$ indicates whether the battery pack with voltage $d_k$ in Figure 2 should be used for supporting the load that requires voltage $X_i$. Furthermore, the inequalities in (5) imply that each battery pack can be used at most once so that the connection patterns from the $(N + L_{\max})$ inputs to the $(N + L_{\max})$ outputs of the crossbar switch is a (sub)permutation. As the connection pattern is a (sub)permutation, it is realizable by the crossbar switch. The equality in (6) indicates that the load $X_i$ can be supported by connecting the subset of battery packs in $S_i = \{k : C_i(k) = 1\}$ in series. ■

For example, if we choose the basis set $S = \{d_i = 2^{i-1}, i = 1, 2, \ldots, N\}$, then it follows from the binary representation that the solution of the $(S, 1)$-SIR problem with such a basis set is $2^N - 1$. According to Proposition 3, such a system is a $(2^N - 1, 1)$-battery switching network that can support a single load with the maximum voltage $2^N - 1$. To support $L_{\max}$ loads, one can simply replicate each battery packs in the above $(2^N - 1, 1)$-battery switching network $L_{\max}$ times. Specifically, one can construct a $(2^N - 1, L_{\max})$-battery switching network by using an $(NL_{\max} + L_{\max}) \times (NL_{\max} + L_{\max})$ crossbar switch and $NL_{\max}$ battery packs with voltages $d_i = 2^{\lfloor (i-1)/L_{\max} \rfloor}$, $i = 1, 2, \ldots, NL_{\max}$. In particular, for $L_{\max} = 2$, we can choose the basis set

$$\begin{aligned} S &= \{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\} \\ &= \{1, 1, 2, 2, 4, 4, 8, 8\} \end{aligned} \tag{7}$$

for the construction of a $(15, 2)$-battery switching network with a $10 \times 10$ crossbar switch in Figure 2. A careful examination reveals that the solution of the $(S, 2)$-SIR problem for the basis set in (7) is 30. Thus, it is in fact a $(30, 2)$-battery switching network. The question is whether this construction can be further improved by choosing another basis set. We show in the following example that the answer is yes. Suppose now we choose

$$\begin{aligned} S^* &= \{d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*, d_7^*, d_8^*\} \\ &= \{1, 1, 2, 3, 4, 6, 9, 14\}. \end{aligned} \tag{8}$$

Then one can verify that the solution of the $(S^*, 2)$-SIR problem for the basis set $S^*$ in (8) is 40. As such, we now have a $(40, 2)$-battery switching network with the same $10 \times 10$ crossbar switch in Figure 2. Such a $(40, 2)$-battery switching network (with the basis set $S^*$ specified in (8)) is better than the $(30, 2)$-battery switching network (with the basis set $S$ specified in (7)) as the former can support a higher maximum total voltage.

In the classic packet switch network design, crossbar switches (as shown in Figure 2) are generally used for establishing *non-conflicting* routing paths for arriving packets (see e.g., [16], [17]). By viewing $d_k$'s in Figure 2 as the delays through various fiber delay lines, crossbar switches were used in [20] to construct "optical queues" that route packets through non-conflicting routing paths with specified/required delays. Instead of viewing $d_k$'s as delays, the battery switching network in Figure 2 views $d_k$'s as voltages and provides non-conflicting routing paths for loads with specified/required voltages. Both the mathematical theories in optical queues and battery switching networks in this paper are related to the integer representation problem. The new advance of the theory is that the integer representation problem in this paper is for simultaneous representation of *several* integers, instead of a single integer considered in [20], [21] for constructing optical queues.

### 2.2 Optimal Selection of the Basis Set

In Proposition 3, we have shown that a $(V_{\max}, L_{\max})$-battery switching network can be constructed by using a single $(N + L_{\max}) \times (N + L_{\max})$ crossbar switch and $N$ battery packs with voltages $d_1, d_2, \ldots, d_N$, where $V_{\max}$ is the solution of the $(S, L_{\max})$-SIR problem with the basis set $S = \{d_1, d_2, \ldots, d_N\}$. In this section, we address the problem of, given an integer $N$ and the maximum number of loads $L_{\max}$, selecting the basis set $S = \{d_1, \ldots, d_N\}$ of cardinality $N$ to maximize $V_{\max}$. Our specific steps for the optimization problem are as follows: in Theorem 4, we first show a sufficient condition for the basis set so that $L$ integers can be simultaneously representable. We then argue in Corollary 5 that $V_{\max}$ is the sum of all the elements in the basis set under such a sufficient condition for the basis set. Finally, in Theorem 6, we show how one can choose the optimal basis set to maximize $V_{\max}$.

In Theorem 4, we first show that a set of $L$ integers are representable by a basis set $S = \{d_1, d_2, \ldots, d_N\}$ provided that $d_1, d_2, ..., d_N$ are chosen to satisfy certain inequalities.

***Theorem 4.*** *Given an integer $N$, consider a basis set $S = \{d_1, d_2, \ldots, d_N\}$ with cardinality $N$. Suppose that $d_1 = 1$ and for some positive integer $L$*

$$d_k \leq d_{k+1} \leq \left\lfloor \frac{\sum_{\ell=1}^{k} d_\ell}{L} \right\rfloor + 1, \tag{9}$$

*for all $k = 1, 2, ..., N - 1$. If a set $X = \{X_1, X_2, \ldots, X_L\}$ of $L$ nonnegative integers satisfy*

$$X_1 + X_2 + ... + X_L \leq \sum_{k=1}^{N} d_k, \tag{10}$$

*then this set $X = \{X_1, X_2, \ldots, X_L\}$ is simultaneously representable by the basis set $S$.*

**Proof.** We prove this by induction on $N$. For $N = 1$, we have $d_1 = 1$. Then it is trivial that $X$ is simultaneously representable by the basis set $S = \{1\}$ if $X_1 + X_2 + ... + X_L \leq 1$. Now, as the induction hypothesis, we assume that any $X$ satisfying (10) is simultaneously representable by a basis set $S = \{d_1, d_2, \ldots, d_N\}$ that satisfies (9). We now show that the claim holds for a basis set $S = \{d_1, d_2, \ldots, d_N, d_{N+1}\}$ that satisfies (9).

Let $i^*$ be the index of the largest integer in $X$, i.e.,

$$i^* = \arg\max(X_i).$$

We consider the following two cases.

*Case 1.* $X_{i^*} \geq \left\lfloor \frac{\sum_{k=1}^{N} d_k}{L} \right\rfloor + 1$:

It then follows from the assumption in this case and the assumption in (9) that

$$X_{i^*} \geq \left\lfloor \frac{\sum_{\ell=1}^{N} d_\ell}{L} \right\rfloor + 1 \geq d_{N+1}.$$

Let

$$X'_j = \begin{cases} X_j - d_{N+1}, & \text{if } j = i^*, \\ X_j, & \text{if } j \neq i^*. \end{cases} \tag{11}$$

Then $X'_{i^*}$ is still a nonnegative integer. Moreover, for the set of nonnegative integers $X' = \{X'_1, X'_2, \ldots, X'_L\}$,

$$\sum_{i=1}^{L} X'_i = \sum_{i=1}^{L} X_i - d_{N+1}. \tag{12}$$

From the condition in (10), we know (for a basis with $N+1$ elements) that

$$X_1 + X_2 + ... + X_L \leq \sum_{k=1}^{N+1} d_k.$$

Using this in (12) yields

$$\sum_{i=1}^{L} X'_i \leq \sum_{k=1}^{N+1} d_\ell - d_{N+1} = \sum_{k=1}^{N} d_k.$$

It then follows from the induction hypothesis that $X'$ is simultaneously representable by the basis set $\{d_1, \ldots, d_N\}$. In conjunction with (11), $X$ is simultaneously representable by the basis set $\{d_1, \ldots, d_N, d_{N+1}\}$.

*Case 2.* $X_{i^*} < \left\lfloor \frac{\sum_{k=1}^{N} d_k}{L} \right\rfloor + 1$:

For this case, we have

$$X_{i^*} \leq \left\lfloor \frac{\sum_{k=1}^{N} d_k}{L} \right\rfloor \leq \frac{\sum_{k=1}^{N} d_k}{L}. \tag{13}$$

In view of (13), we have that

$$\sum_{i=1}^{L} X_i \leq L \cdot \frac{\sum_{k=1}^{N} d_k}{L} = \sum_{k=1}^{N} d_k.$$

From the induction hypothesis, we know that $X$ is simultaneous representable by the basis set $\{d_1, \ldots, d_N\}$ and thus $X$ is also simultaneous representable by the basis set $\{d_1, \ldots, d_N, d_{N+1}\}$. ■

In Corollary 5, we show that the sum of all the elements in the basis set is indeed the solution of the $(S, L_{\max})$-SIR problem if the elements in the basis set satisfy certain inequalities.

***Corollary 5.*** *Consider a basis set $S = \{d_1, d_2, \ldots, d_N\}$ with $d_1 = 1$ and*

$$d_k \leq d_{k+1} \leq \left\lfloor \frac{\sum_{\ell=1}^{k} d_\ell}{L_{\max}} \right\rfloor + 1, \tag{14}$$

*for all $k = 1, 2, ..., N-1$. Let $V_{\max}$ be the solution of the $(S, L_{\max})$-SIR problem. Then*

$$V_{\max} = \sum_{k=1}^{N} d_k.$$

**Proof.** Clearly, the solution of the $(S, L_{\max})$-SIR problem cannot be larger than the sum of all the elements in $S$, i.e., $V_{\max} \leq \sum_{k=1}^{N} d_k$. On the other hand, we have already shown in Theorem 4 that any set of nonnegative integers $X$ are simultaneously representable by $S$ if

$$\sum_{i=1}^{L_{\max}} X_i \leq \sum_{k=1}^{N} d_k.$$

Thus, we also have $V_{\max} \geq \sum_{k=1}^{N} d_k$. ■

Intuitively, the optimal basis set that maximizes the solution of the $(S, L_{\max})$-SIR problem (among all the basis sets with $N$ elements) can be chosen by using the upper bound in (14). This is formally stated in the following theorem. The key insight of the proof of Theorem 6 is to show that there is a collection of integers (not larger than the sum of all elements in the basis set) that are not representable by the basis set if the condition in (15) is not satisfied.

***Theorem 6.*** *Let $d_1^* = 1$ and*

$$d_{k+1}^* = \left\lfloor \frac{\sum_{\ell=1}^{k} d_\ell^*}{L_{\max}} \right\rfloor + 1, \tag{15}$$

*for all $k = 1, 2, ..., N-1$. Then the solution of the $(S, L_{\max})$-SIR problem for any basis set $S$ with $N$ elements cannot be larger than $\sum_{k=1}^{N} d_k^*$. As such, the basis set $S^* = \{d_1^*, d_2^*, \ldots, d_N^*\}$ is optimal in the sense of achieving the largest solution of the $(S, L_{\max})$-SIR problem among all the basis sets with $N$ elements.*

**Proof.** Consider a basis set $S = \{d_1, d_2, \ldots, d_N\}$. Without loss of generality, assume that $d_1 \leq d_2 \leq ... \leq d_N$. If $d_1 > 1$, then the solution of the $(S, L_{\max})$-SIR problem for the basis set $S$ with $d_1 > 1$ is simply 0 as the integer 1 cannot be represented. Thus, we only need to consider the case with $d_1 = 1$.

If $d_k$, $k = 1, 2, \ldots, N-1$, in this basis set satisfy all the inequalities in (14), then one can easily see by induction and (15) that $d_k \leq d_k^*$ for all $k = 2, 3, \ldots, N$. Thus,

$$\sum_{k=1}^{N} d_k \leq \sum_{k=1}^{N} d_k^*.$$

From Corollary 5, we know the solution of the $(S, L_{\max})$-SIR problem for this basis set $S = \{d_1, d_2, \ldots, d_N\}$ is $\sum_{k=1}^{N} d_k$, which is not larger than $\sum_{k=1}^{N} d_k^*$.

On the other hand, suppose that there exists some $k \in \{1, 2, \ldots, N-1\}$ such that the second inequality in (14) is violated. Let $k^*$ be the smallest integer such that

$$d_{k^*+1} > \left\lfloor \frac{\sum_{\ell=1}^{k^*} d_\ell}{L_{\max}} \right\rfloor + 1. \tag{16}$$

Let

$$r = \sum_{\ell=1}^{k^*} d_\ell - L_{\max} \left\lfloor \frac{\sum_{\ell=1}^{k^*} d_\ell}{L_{\max}} \right\rfloor. \tag{17}$$

Then $0 \leq r < L_{\max}$. Consider the set of integers $X = \{X_1, X_2, \ldots, X_{L_{\max}}\}$, where

$$X_i = \begin{cases} \left\lfloor \frac{\sum_{\ell=1}^{k^*} d_\ell}{L_{\max}} \right\rfloor + 1, & \text{for } 1 \leq i \leq r+1, \\ \left\lfloor \frac{\sum_{\ell=1}^{k^*} d_\ell}{L_{\max}} \right\rfloor, & \text{for } r+2 \leq i \leq L_{\max}. \end{cases}$$

Clearly, for $i = 1, 2, \ldots, L_{\max}$, we have from (16) that

$$X_i \leq \left\lfloor \frac{\sum_{\ell=1}^{k^*} d_\ell}{L_{\max}} \right\rfloor + 1 < d_{k^*+1} \leq \ldots \leq d_N.$$

As such, the set of integers from $d_{k^*+1}$ to $d_N$ cannot be used to represent any $X_i$. In other words, only $d_1, d_2, \ldots, d_{k^*}$ can be used to represent these $X_i$'s. But this is impossible because

$$\begin{aligned} \sum_{i=1}^{L_{\max}} X_i &= L_{\max} \left\lfloor \frac{\sum_{\ell=1}^{k^*} d_\ell}{L_{\max}} \right\rfloor + r + 1 \\ &= \sum_{\ell=1}^{k^*} d_\ell + 1 > \sum_{\ell=1}^{k^*} d_\ell, \end{aligned}$$

where we use (17) in the last identity. This shows that the solution of the $(S, L_{\max})$-SIR problem for this basis set $S$ cannot be larger than $\sum_{\ell=1}^{k^*} d_\ell$. As $k^*$ is the smallest integer such that the second inequality in (14) is not satisfied, we know that for $k = 1, 2, \ldots, k^* - 1$,

$$d_{k+1} \leq \left\lfloor \frac{\sum_{\ell=1}^{k} d_\ell}{L_{\max}} \right\rfloor + 1.$$

This then implies that, $d_k \leq d_k^*$ for $k = 1, 2, \ldots, k^*$. Thus,

$$\sum_{\ell=1}^{k^*} d_\ell \leq \sum_{\ell=1}^{k^*} d_\ell^* \leq \sum_{\ell=1}^{N} d_\ell^*.$$

We then conclude that the solution of the $(S, L_{\max})$-SIR problem for this basis set $S$ cannot be larger than $\sum_{\ell=1}^{N} d_\ell^*$. ∎

As an illustrating example, for $L_{\max} = 2$, the optimal basis set with 8 elements is

$$\begin{aligned} S^* &= \{d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*, d_7^*, d_8^*\} \\ &= \{1, 1, 2, 3, 4, 6, 9, 14\}, \end{aligned}$$

and it is better than the replicated binary basis

$$\begin{aligned} S &= \{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\} \\ &= \{1, 1, 2, 2, 4, 4, 8, 8\}. \end{aligned}$$

To further see the gain from using the optimal selection, we define the efficiency of a basis set $S$ as the ratio of the binary logarithm of the solution of the $(S, L_{\max})$-SIR problem to the size of the basis set, i.e., $\log_2 V_{\max}/|S|$. In Fig. 4, we show the comparison of efficiency between the optimal basis and the replicated binary basis for $L_{\max} = 2$. One can find the optimal basis recursively from (15) and then use that to numerically compute the efficiency. When $N$ is large, the optimal efficiency is approximately 0.585. On the other hand, the efficiency of the replicated binary basis is only 0.5. Clearly, for any given $N$, the efficiency, defined as $\log(V_{\max})/N$, is uniquely maximized by the optimal basis in Theorem 6.

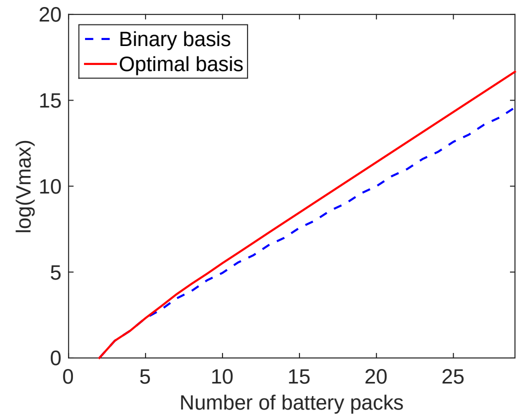



Fig. 4. Efficiency between the optimal basis and the replicated binary basis for $L_{\max} = 2$.

### 2.3 Alternating *C*-transform

We note that Theorem 6 does not illustrate how one might solve the simultaneous integer representation problem (4)–(6). In this section, we present an algorithm, called the alternating $C$-transform in this paper, that takes a basis set $S = \{d_1, d_2, ..., d_N\}$ (satisfying (9) and $d_1 = 1$) and a set of $L$ integers $X_1, X_2, ..., X_L$ (satisfying (10)) as its input. The algorithm will output the indicator variables $C_i(k)$ that satisfy (4)–(6). The detail of the algorithm is shown in Algorithm 1. We note the alternating $C$-transform is a generalization of the $C$-transform in [20], [21] that was proposed to solve the representation problem for a single integer.

Now we describe the basic idea behind the alternating $C$-transform. Consider a basis set $S = \{d_1, d_2, \ldots, d_N\}$ with $d_1 \leq d_2 \leq ... \leq d_{N-1} \leq d_N$. At the beginning, the algorithm takes as an input $L$ nonnegative integers $X_1, X_2, \ldots, X_L$ and the basis set $S$. Then we would like to find the integer representations of these integers by using the elements in the basis set $S$. As suggested in the proof of Theorem 4, the alternating $C$-transform first finds the largest integer, say $X_k$, among all the $L$ integers. If $X_k$ is not smaller than $d_N$, then $d_N$ is included in the subset that will be used for representing $X_k$. We then deduct $d_N$ from $X_k$. By doing so, the problem is then reduced to another simultaneous integer representation problem with the basis set $\{d_1, d_2, \ldots, d_{N-1}\}$ that only has $N-1$ elements. Also, the $L$ integers are now $X_1, X_2, \ldots, X_k - d_N, \ldots, X_L$. On the other hand, if $X_k$ is smaller than $d_N$, then $X_i$, $i = 1, 2, \ldots, L$, are all smaller than $d_N$ (as $X_k$ is the largest integer). Thus, $d_N$ cannot be included in any subset for representing any integer. As such, the problem is then reduced to another simultaneous integer representation problem with the basis set $\{d_1, d_2, \ldots, d_{N-1}\}$ that only has $N-1$ elements. Now the $L$ integers are still $X_1, X_2, \ldots, X_k, \ldots, X_L$. In both cases, the number of elements in the basis set is reduced from $N$ to $N-1$. One can apply the alternating $C$-transform *iteratively* to solve the simultaneous integer representation problem. For instance, there are four elements in the basis set $\{d_1, d_2, d_3, d_4\} = \{1, 1, 2, 3\}$ and two integers $(X_1, X_2) = (X_1(1), X_2(1)) = (3, 3)$. First, we find the largest integer, i.e., $X_1(1) = 3$. Since $X_1(1) \geq d_4 = 3$, $d_4$ is selected to represent $X_1$. We then subtract $d_4$ from $X_1(1)$ and this leads to the updated integers $(X_1(2), X_2(2)) = (0, 3)$. In the second step, the largest integer is $X_2(2) = 3$. As $X_2(2) \geq d_3 = 2$, $d_3$ is selected to represent $X_2$. This further reduces the two integers to $(X_1(3), X_2(3)) = (0, 1)$. In the third step, the largest integer is $X_2(3) = 1$. As $X_2(3) \geq d_2 = 1$, $d_2$ is also selected to represent $X_2$. Now the two integers are reduced to $(X_1(4), X_2(4)) = (0, 0)$ and the algorithm is completed. For this particular example, we then have $X_1 = d_4$ and $X_2 = d_3 + d_2$. Note that the computational complexity of the alternating $C$-transform is $O(NL)$ as it takes $O(L)$ steps to find the largest elements in the loads by using a simple linear search The computational complexity can also be reduced by using a more sophisticated data structure, e.g., a heap, when $L$ is large. As pointed by one of the reviewers, the complexity of the alternative $C$-transform is in fact at most $O(N \log(L))$ if $N \geq L$. To see this, one first sorts $X_1, \ldots, X_L$ in ascending order, which has complexity $O(L \log(L))$. In the $t^{th}$ iteration, once the integers $X_j(t)$, $j = 1, \ldots, L$ have been updated, one can sort these integers in ascending order again. Since the sequence of integers are previously sorted, and each iteration only changes one component of this sequence, this sorting can be done via binary search, which takes $O(\log(L))$ number of steps. It follows that the complexity of this algorithm is at most $O(L \log(L) + N \log(L))$. Thus, if $N \geq L$, the complexity of the alternative $C$-transform is at most $O(N \log(L))$.

**Algorithm 1** Alternating $C$-transform

**Input:** A basis set $S = \{d_1, d_2, ..., d_N\}$ (satisfying (9) and $d_1 = 1$) and a set of $L$ integers $X_1, X_2, ..., X_L$ (satisfying (10));
**Output:** The indicator variables $C_i(k)$ that satisfy (4)–(6);

```
Initially, set X_i(1) = X_i, ∀ i = 1,2,...,L;
for t = 1,2,...,N do
    Let i(t) = arg max(X_i(t));
    if X_{i(t)}(t) ≥ d_{N+1-t} then
        C_{i(t)}(N+1-t) = 1;
        X_{i(t)}(t+1) = X_{i(t)}(t) - d_{N+1-t};
    else
        C_{i(t)}(N+1-t) = 0;
        X_{i(t)}(t+1) = X_{i(t)}(t);
    end if
    for j ≠ i(t) do
        C_j(N+1-t) = 0;
        X_j(t+1) = X_j(t);
    end for
end for
```

### 2.4 Multi-stage Feedforward Battery Switching Network

One of the salient properties of the alternating $C$-transform is that there is an order (from $N$ to 1) of computing the indicator variables $C_i(k)$, $i = 1, 2, \ldots, L$ and $k = 1, 2, \ldots, N$. In view of this, one can construct a battery switching network by using a multistage feedforward network (see Fig. 5). In such a network, there are $N$ stages and the $k^{th}$ stage is made of $L_{\max}$ $1 \times 2$ and $2 \times 1$ switching elements and a battery pack with voltage $d_k$ (see Figure 6). The indicator variables $C_i(k)$, $i = 1, 2, \ldots, L$, and $k = 1, 2, \ldots, N$, obtained from the alternating $C$-transform are then used for setting up the connection patterns in each stage. Specifically, the $i^{th}$ load is connected to the battery pack in the $k^{th}$ stage if $C_i(k) = 1$. By choosing $d_i$, $i = 1, 2, \ldots, N$, to satisfy the condition in Corollary 5, the multistage feedforward network in Figure 5 can then be operated as a $(V_{\max}, L_{\max})$-battery switching network, where $V_{\max} = \sum_{k=1}^{N} d_k$. In comparison with the construction by using a single crossbar switch, the hardware complexity is greatly reduced from $O((N + L_{\max})^2)$ to $O(NL_{\max})$. In fact, if we use 2 on/off switches to implement a $1 \times 2$ switch and a $2 \times 1$ switch in Figure 5, then the total number of on/off switches in an $N$-stage network is only $4NL_{\max}$. Even with the replicated binary basis, we have

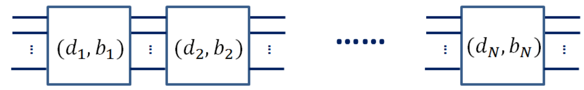


Fig. 5. A multistage feedforward battery switching network.

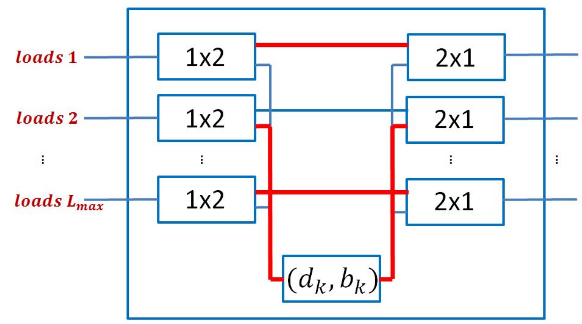


Fig. 6. The detailed implementation of each stage in a multi-stage feedforward battery switching network.

$$V_{\max} = \sum_{k=1}^{N} d_k \approx 2^{N/L_{\max}}$$

for large $N$. Thus, the number of stages $N$ is $O(\log V_{\max})$ with such a basis. For a system with $B$ batteries, if we assign the $k^{th}$ battery pack with the number of batteries proportional to $d_k$ (the uniform battery allocation scheme in the next section), then $V_{\max}$ is proportional to $B$ and the number of stages $N$ is thus $O(\log B)$. In comparison with the $O(B)$ battery switch array system in [5], the saving of the number of on/off switches is very significant in the multistage feedforward network even with the replicated binary basis.

## 3 CAPACITY ASSIGNMENT AND BATTERY ALLOCATION

Once we determine the voltages $d_1, d_2, ..., d_N$ in the multi-stage battery switching network in Figure 5, our next design problem is to determine the capacity of each battery pack. Suppose that we are given $B$ identical batteries. The battery allocation problem is then to allocate these $B$ identical batteries to the $N$ battery packs so that the system life (defined as the first time that one of the $N$ battery packs depletes its energy) is maximized. A naïve allocation is simply to allocate the $B$ batteries *uniformly* to the $N$ battery packs so that every battery pack has roughly the same capacity. However, as the energy consumed by each battery pack is different, such a uniform allocation scheme may not perform well. The key idea to prolong the system life is thus to allocate batteries so that the capacity of each battery pack is roughly proportional its average energy consumption rate. For this, we need to address the problem of computing the average energy consumption rate.

### 3.1 Average Energy Consumption Rate

It is obvious that the average energy consumption rate depends on the load profiles. In this section, we consider stochastic load profiles. Assume there are $J$ types of loads, indexed from $1, 2, \ldots, J$. A type $j$ load is characterized by three parameters, the required voltage $V_j$, the required current $I_j$, and the mean service time $\tau_j$. Type $j$ loads arrive at the system according to a Poisson process with rate $\lambda_j$ and such a Poisson process is independent of everything else. We note that for the operation of the battery switching network, we only need the information of the required voltage, and the other parameters in the stochastic load profiles are used for computing the average energy consumption rates that in turn will be used for battery capacity allocation. Specifically, for a $(V_{\max}, L_{\max})$-battery switching network, an arriving load can only be admitted to the system provided that

$$n_1 + n_2 + \ldots + n_J \le L_{\max}, \tag{18}$$

$$V_1 n_1 + V_2 n_2 + \ldots + V_J n_J \le V_{\max}, \tag{19}$$

where $n_j$ is the number of type $j$ loads in the system. An arriving load that cannot be admitted to the system is simply lost. The condition in (18) is simply the constraint on the maximum number of loads in (1) and the condition in (19) is simply the constraint on the maximum total voltage that can be supported by the system in (2).

Let $\underline{n} = (n_1, n_2, \ldots, n_J)$ be the state vector and $\Gamma$ be the set of admissible states that satisfy (18) and (19). From the classical queueing theory (see e.g., the book [22]), such a queueing system is known as a loss system (or a loss network) in the literature and has the following steady state probabilities for the corresponding Markov chain in the state vector $\underline{n}$:

$$\pi(\underline{n}) = G^{-1} \cdot \Big(\prod_{j=1}^{J} \frac{\rho_j^{n_j}}{n_j!}\Big),$$

where

$$\rho_j = \lambda_j \tau_j,$$

and $G$ is the normalization constant, i.e.,

$$G = \sum_{\underline{n}\in\Gamma} \prod_{j=1}^{J} \frac{\rho_j^{n_j}}{n_j!}.$$

To compute the average energy consumption rate, we note that the connection pattern in a $(V_{\max}, L_{\max})$-battery switching network needs to be reconfigured every time there is a state change, i.e., an arrival or a departure of a load. For every admissible state $\underline{n}$ in $\Gamma$, let

$$X(\underline{n}) = \{\underbrace{V_1, \ldots, V_1}_{n_1}, \underbrace{V_2, \ldots, V_2}_{n_2}, \ldots, \underbrace{V_J, \ldots, V_J}_{n_J}\}$$

be the required voltages of the loads in state $\underline{n}$. The connection pattern in state $\underline{n}$ is determined by applying the alternating $C$-transform for the required voltages in $X(\underline{n})$. Let $C_{j,k}(\underline{n})$ be the indicator variable to indicate whether the $k^{th}$ battery pack is used by a type $j$ load when the system is in state $\underline{n}$, i.e., $C_{j,k}(\underline{n}) = 1$ if the $k^{th}$ battery pack is used by a type $j$ load when the system is in state $\underline{n}$ and 0 otherwise. The average energy consumption rate for the $k^{th}$ battery pack, denoted by $r_k$, can then be computed as follows:

$$r_k = \sum_{\underline{n}\in\Gamma} \sum_{j=1}^{J} C_{j,k}(\underline{n}) \cdot I_j \cdot \pi(\underline{n}), \tag{20}$$

where $I_j$ is the required current for a type $j$ load.

### 3.2 Max-min Fairness Allocation

Once we obtain the voltages $d_k, k = 1, 2, \ldots, N$, and the average energy consumption rates $r_k, k = 1, 2, \ldots, N$, for all the $N$ battery packs, we may formulate the battery allocation problem with a total number of $B$ batteries as the following max-min fairness problem:

$$\max \min_{1\le k\le N} \frac{b_k}{r_k}$$

subject to

$$\sum_{k=1}^{N} b_k d_k \le B,$$

where $b_k$ is the number of batteries allocated to the $k^{th}$ battery pack, $k = 1, 2, \ldots, N$. The intuition for this is to assign the capacity of each pack roughly proportional to its average energy consumption rate. Without the constraint that $b_k$'s are positive integers, the optimal solution can be found as follows:

$$b_k = \frac{B \cdot r_k}{\sum_{\ell=1}^{N} r_\ell \cdot d_\ell}. \tag{21}$$

To see this, note that

$$\Big(\min_{1\le k\le N} \frac{b_k}{r_k}\Big) \sum_{k=1}^{N} r_k d_k \le \sum_{k=1}^{N} \frac{b_k}{r_k} r_k d_k \le B.$$

However, as $b_k$'s need to be positive integers, one can start from the basic allocation $\left\lfloor \frac{B\cdot r_k}{\sum_{\ell=1}^{N} r_\ell \cdot d_\ell} \right\rfloor$ and then repeatedly increase the capacity of the battery pack with the minimum value of $b_k/r_k$ by 1. The details of the heuristic algorithm to approximate the optimal solution in (21) is shown in Algorithm 2.

## 4 Fault Tolerant Battery Switching Networks

In the previous section, the system life of a $(V_{\max}, L_{\max})$-battery switching network is defined as the first time that one of the battery packs has depleted its energy. Such a definition of the system life seems plausible as a $(V_{\max}, L_{\max})$-battery switching network with a depleted battery pack cannot support the loads specified in Definition 1. However, if we choose the basis set properly, we might still be able to operate such a device as another $(V'_{\max}, L_{\max})$-battery

**Algorithm 2** Max-min fairness battery allocation algorithm

**Input:** $B, r_k, d_k, k = 1, 2, \ldots, N$;

**Output:** The capacity of each battery pack $b_1, b_2, ..., b_N$;

1: Initially, start with $b_k^{(0)} = \left\lfloor \frac{B \cdot r_k}{\sum_{\ell=1}^{N} r_\ell \cdot d_\ell} \right\rfloor$;

2: Let $B^{(0)} = B - \sum_{k=1}^{N} b_k^{(0)} d_k$;

3: Set $s = 0$;

4: Find $k^* = \arg\min \frac{b_k^{(s)}}{r_k}$;

5: **if** $B^{(s)} \geq d_{k^*}$ **then**

6: $\quad b_{k^*}^{(s+1)} \leftarrow b_{k^*}^{(s)} + 1$;

7: $\quad b_k^{(s+1)} \leftarrow b_k^{(s)}$ for all $k \neq k^*$;

8: $\quad B^{(s+1)} = B^{(s)} - d_{k^*}$;

9: $\quad s \leftarrow s + 1$;

10: $\quad$ Repeat from step (4.);

11: **end if**

12: **if** $B^{(s)} < d_{k^*}$ **then**

13: $\quad$ Uniformly allocate the remaining batteries to those battery packs with $d_k = 1$;

14: **end if**

switching network with $V'_{\max} \leq V_{\max}$. If this is possible, we might be able to prolong the system life of a battery switching network even though some of its battery packs have depleted their energy. This motivates us to consider fault tolerant battery switching networks.

***Definition 7.*** **(Fault tolerant battery switching network)** *A multistage feedforward battery switching network constructed by using a basis set $S = \{d_1, d_2, \ldots, d_N\}$ is called a $(V_{\max}, L_{\max}, F_{\max})$-battery switching network if it can still be operated as a $(V_{\max}, L_{\max})$-battery switching network even when any $F \leq F_{\max}$ battery packs have depleted their energy.*

In the following theorem, we show how to choose a basis set to construct a fault tolerant battery switching network. The intuition of this theorem is to ensure that the inequalities in (14) still hold after any $F_{\max}$ elements are removed from the basis set.

***Theorem 8.*** *Consider a basis set $S = \{d_1, d_2, \ldots, d_N\}$ with $d_k = 1$, $k = 1, 2, \ldots, F_{\max} + 1$ and*

$$d_k \leq d_{k+1} \leq \left\lfloor \frac{\sum_{\ell=1}^{k-F_{\max}} d_\ell}{L_{\max}} \right\rfloor + 1, \tag{22}$$

*for all $k = F_{\max} + 1, F_{\max} + 2, ..., N - 1$. Then the multistage feedforward battery switching network constructed by using such a basis set $S = \{d_1, d_2, \ldots, d_N\}$ is a $(V_{\max}, L_{\max}, F_{\max})$-battery switching network, where*

$$V_{\max} = \sum_{k=1}^{N-F_{\max}} d_k.$$

**Proof.** Assume that there are $F \leq F_{\max}$ battery packs that have depleted their energy. Let $\tilde{d}_k$, $k = 1, 2, \ldots, N - F$, be the voltages in the remaining $N - F$ battery packs. Without loss of generality, we assume that $\tilde{d}_1 \leq \tilde{d}_2 \leq \ldots \leq \tilde{d}_{N-F}$. Clearly, $\tilde{d}_k = d_j$ for some $k \leq j \leq k + F$. As we assume that $d_k \leq d_{k+1}$ for all $k$ in (22), it follows that

$$d_k \leq \tilde{d}_k \leq d_{k+F}, \tag{23}$$

for all $k = 1, 2, \ldots, M - F$.

We first show that the solution of the $(\tilde{S}, L_{\max})$-SIR problem with the basis set $\tilde{S} = \{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_{N-F}\}$ is $\sum_{k=1}^{N-F} \tilde{d}_k$. For this, it suffices to show that the basis set $\tilde{S}$ satisfies the condition in Corollary 5, i.e., $\tilde{d}_1 = 1$ and

$$\tilde{d}_k \leq \tilde{d}_{k+1} \leq \left\lfloor \frac{\sum_{\ell=1}^{k} \tilde{d}_\ell}{L_{\max}} \right\rfloor + 1, \tag{24}$$

for all $k = 1, 2, ..., N - F - 1$.

As $d_k = 1$ for $k = 1, 2, \ldots, F_{\max} + 1$ and $F \leq F_{\max}$, we have

$$1 = d_1 \leq \tilde{d}_1 \leq d_{F+1} = 1.$$

Thus, $\tilde{d}_1 = 1$. It is also clear that $\tilde{d}_k \leq \tilde{d}_{k+1}$ for all $k = 1, 2, \ldots, N - F - 1$. For $1 \leq k \leq F_{\max} - F$, we have from (23) and $d_k = 1$ for $k = 1, 2, \ldots, F_{\max} + 1$ that

$$\tilde{d}_{k+1} \leq d_{k+1+F} = 1 \leq \left\lfloor \frac{\sum_{\ell=1}^{k} \tilde{d}_\ell}{L_{\max}} \right\rfloor + 1.$$

Similarly, for $F_{\max} - F + 1 \leq k \leq N - F - 1$, we have from (23), (22), and $F \leq F_{\max}$ that

$$\begin{aligned} \tilde{d}_{k+1} &\leq d_{k+1+F} \leq \left\lfloor \frac{\sum_{\ell=1}^{k+F-F_{\max}} d_\ell}{L_{\max}} \right\rfloor + 1 \\ &\leq \left\lfloor \frac{\sum_{\ell=1}^{k} \tilde{d}_\ell}{L_{\max}} \right\rfloor + 1. \end{aligned}$$

Thus, the inequalities in (24) are all satisfied.

Since $F \leq F_{\max}$ and $d_k \leq \tilde{d}_k$ for all $k$ in (23), we have

$$\sum_{k=1}^{N-F} \tilde{d}_k \geq \sum_{k=1}^{N-F_{\max}} \tilde{d}_k \geq \sum_{k=1}^{N-F_{\max}} d_k.$$

This implies that the solution of the $(\tilde{S}, L_{\max})$-SIR problem with the basis set $\tilde{S} = \{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_{N-F_{\max}}\}$ is not smaller than $\sum_{k=1}^{N-F_{\max}} d_k$. As such, the multistage feedforward battery switching network with $F$ depleted battery packs can still be operated as a $(V_{\max}, L_{\max})$-battery switching network with $V_{\max} = \sum_{k=1}^{N-F_{\max}} d_k$. Thus, it is a $(V_{\max}, L_{\max}, F_{\max})$-battery switching network. ■

For example, the basis set $\{1, 1, 1, 2, 2, 3, 4, 6, 8\}$ can be used for constructing a $(20, 2, 1)$-battery switching network and the basis set $\{1, 1, 1, 1, 2, 2, 3, 4, 5, 6, 8\}$ can be used for constructing a $(20, 2, 2)$-battery switching network.

For a $(V_{\max}, L_{\max}, F_{\max})$-battery switching network, the system life is the first time there are $F_{\max} + 1$ battery packs that have depleted their energy. One salient feature of a fault tolerant battery switching network with $N$ battery packs is that we now have the freedom to choose any $N - F_{\max}$ battery packs to operate. In view of this, we propose the Largest Remaining Capacity First (LRCF) policy for scheduling the usage of the $N$ battery packs. For the LRCF policy, we need to monitor the remaining capacity of each battery pack, i.e., the state of charge (SOC) of each battery pack. When there is a state change, we select the $N - F_{\max}$ battery packs that have the largest remaining capacities and apply the alternating $C$-transform by using the basis set from these $N - F_{\max}$ battery packs. For the LRCF policy, we can simply do uniform battery allocation so that each battery pack roughly has the same capacity at the beginning. As such, there is no need to know the load profile. In the simulation section, we will show that the LRCF policy (implemented on a fault tolerant battery switching network) performs well even without the knowledge of the load profile and in some load conditions it even performs better than the max-min fairness battery allocation scheme that requires the knowledge of the load profile.

# 5 SIMULATION

In this section, we perform various computer simulations to compare the performance of three battery allocation schemes: the uniform allocation scheme and the max-min fairness allocation scheme in a battery switching network and the Largest Remaining Capacity First (LRCF) scheme in a fault tolerant battery switching network.

## 5.1 Experimental Setup

We first describe the stochastic load profile used in our simulations. There are two types of loads, i.e., $J = 2$. The three parameters, voltage, current, and mean service time, for the first type are $(V_1, I_1, \tau_1)$=$(5, 1, 1)$ and those for the second type are $(V_2, I_2, \tau_2)$=$(12, 1, 1)$. As in Section 3.1, the arrivals of type 1 (resp. 2) loads are modeled by a Poisson process with arrival rate $\lambda_1$ (resp. $\lambda_2$). In our experiments, we fix the mean service times of both types, i.e., $\tau_1 = 1$ and $\tau_2 = 1$, and vary the corresponding Poisson arrival rates.

The total number of batteries $B$ is 7650 and each is fully charged with one unit of capacity (Ah). We consider a six-stage $(17, 2)$-battery switching network with the basis $d_1 = 1, d_2 = 1, d_3 = 2, d_4 = 3, d_5 = 4$, and $d_6 = 6$. For such a battery switching network, the maximum number of loads $L_{\max}$ is 2 and the maximum supportable total voltage $V_{\max}$ is 17. As such, there are five admissible states for the stochastic load profile, i.e.,

$$\underline{n} = (n_1, n_2) = (0, 0), (0, 1), (1, 0), (1, 1) \text{ and } (2, 0). \quad (25)$$

For the uniform allocation scheme, each battery pack is allocated a fixed 450 units of capacity (Ah).

We also consider the LRCF scheme in several fault tolerant battery switching networks with $F_{\max} = 1, 2, 5,$ and 20, respectively. The basis sets of these four battery switching network are $\{1, 1, 1, 2, 2, 3, 4, 6, 8\}$ for $F_{\max} = 1$, $\{1, 1, 1, 1, 2, 2, 3, 4, 5, 6, 8\}$ for $F_{\max} = 2$, $\{1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 6, 8\}$ for $F_{\max} = 5$, and 22 1's and $\{2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9\}$ for $F_{\max} = 20$. The allocation of the 7650 batteries in these fault tolerant battery switching networks is performed by using the uniform allocation scheme. In order to have a fair comparison with the other two schemes, the admissible states for these fault tolerant battery switching networks are set to be the five states in (25).

In Figure 7, we show the average system life for the uniform allocation scheme, the max-min fairness allocation scheme, and the LRCF scheme in the fault tolerant battery switching networks with $F_{\max} = 1, 2, 5,$ and 20. In Figure 7a, we fix $\rho_2 = 1$ and vary the Poisson arrival rate $\rho_1$ of type 1 loads. On the other hand, we fix $\rho_1 = 1$ and vary the Poisson arrival rate $\rho_2$ of type 2 loads in Figure 7b. Each average system life is measured by averaging over 100 independent experiments.

## 5.2 Comparison Between the Max-min Fairness Allocation Scheme and the Uniform Allocation Scheme

It is clear from Figure 7a that the average system life of the max-min fairness allocation scheme is significantly longer than that of the uniform allocation scheme. A similar conclusion can also be drawn from Figure 7b. The gain of the average system life in Figure 7a can be up to 25%. However, the gain in Figure 7b is not as large as that in Figure 7a.

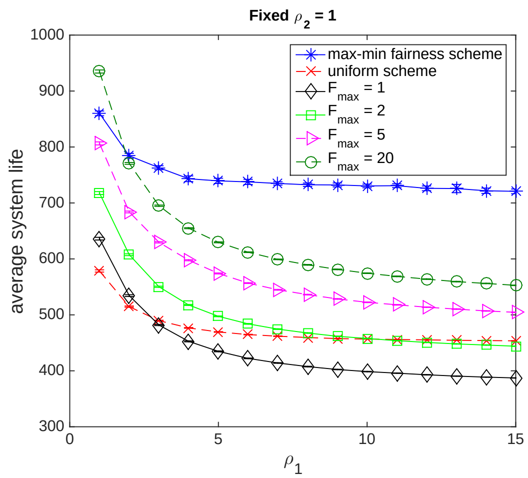
To further understand this, we plot in Figure 8 the average energy consumption rate for each pack by using (20) and in Figure 9 the capacity of each battery pack by using the max-min fairness allocation scheme in Section 3.2. As we can see from Figure 8a and Figure 9a that the average energy consumption rates are roughly proportional to the capacities, in particular, $r_2, r_3, r_4, r_5$ in Figure 8a and $b_2, b_3, b_4, b_5$ in Figure 9a are all increasing with respect to $\rho_1$. On the other hand, we can see from Figure 8b and Figure 9b that $r_1, r_3, r_4, r_5$ and $r_6$ are increasing with respect to $\rho_2$ but $b_4$ and $b_5$ are not increasing with respect to $\rho_2$. This is due to the integer constraint of the max-min fairness allocation scheme. In particular, for the fifth battery pack, its average energy consumption rate $r_5$ is the largest among all the six battery packs. That means the system uses the fifth pack more frequently than the other battery packs. As $b_5$ is not increasing with respect to $\rho_2$, it is expected that the gain of using the max-min fairness allocation scheme in this scenario is not as significant as that in Figure 7a.

We note that our computation of the average energy consumption rates in (20) is based on the steady state probabilities of the corresponding Markov chain for modelling the loss system. Thus, if the average system life is not significantly longer than the mixing time of the corresponding Markov chain, then the actual energy consumption rates in the transient period might be quite different from those obtained from (20). As such, the gain of using the max-min fairness allocation scheme over the uniform allocation scheme might not be that great if the average system life is not significantly longer than the mixing time of the corresponding Markov chain.

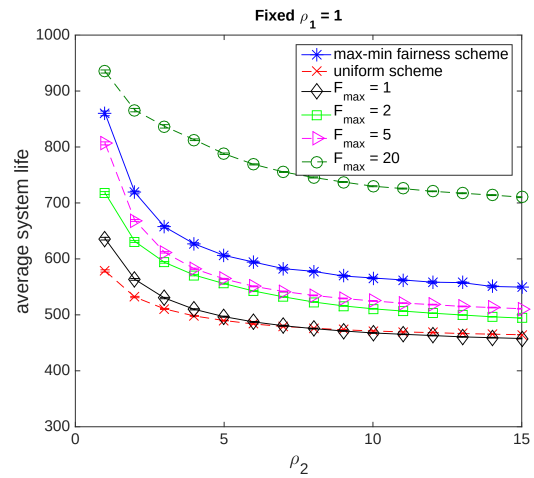
## 5.3 The LRCF Scheme

Recall that in the LRCF scheme, there is no need to know the load profile. However, it is assumed that the remaining capacity of each battery pack can be monitored and it is known to the system. For an $N$-stage fault tolerant battery switching network that can tolerate $F_{\max}$ faulty battery packs, the LRCF scheme only uses the $N - F_{\max}$ battery packs that have the largest remaining capacities. Also, the system life of such a fault tolerant battery switching network is the first time there are $F_{\max} + 1$ battery packs that have depleted their energy.

It is clear from Figure 7 that increasing the fault tolerant capability in general increases the average system life. However, increasing the fault tolerance capability also increases the hardware complexity as we might need more stages for the constructions of fault tolerant battery switching networks that have larger fault tolerance capability. We note the gap between the max-min fairness allocation scheme and the LRCF scheme in the battery switching network with $F_{\max} = 2$ is not that much. Moreover, the gap is narrowed

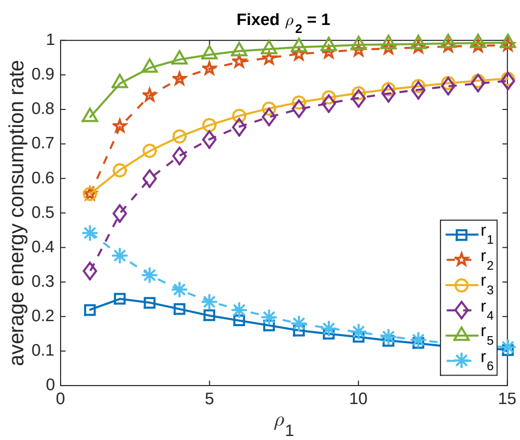



(a) As a function of $\rho_1$ when $\rho_2 = 1$.

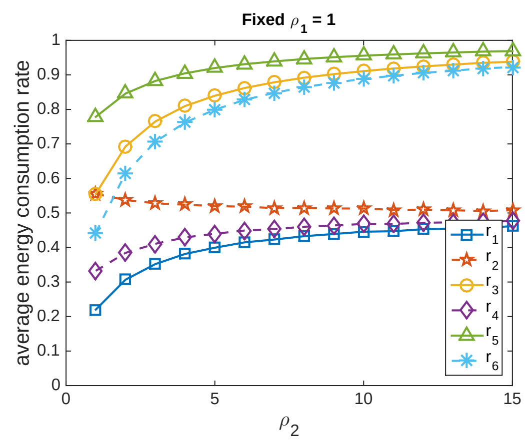



(b) As a function of $\rho_2$ when $\rho_1 = 1$.

Fig. 7. Comparison of average system life.

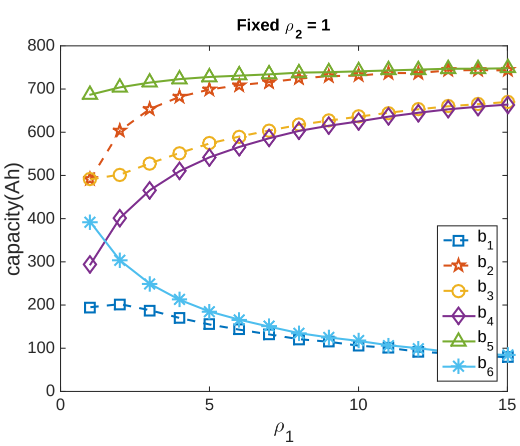



(a) As a function of $\rho_1$ when $\rho_2 = 1$.

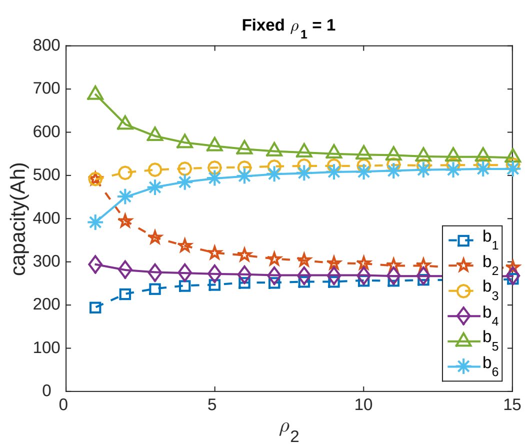



(b) As a function of $\rho_2$ when $\rho_1 = 1$.

Fig. 8. Average energy consumption rate of each battery pack.




(a) As a function of $\rho_1$ when $\rho_2 = 1$.




(b) As a function of $\rho_2$ when $\rho_1 = 1$.

Fig. 9. Capacity of each battery pack.

when we further increase the fault tolerance capability. For $F_{\max} = 20$, the average system life of the LRCF scheme in the fault tolerant battery switching network is even better than that of the max-min fairness allocation scheme for the traffic profiles in Figure 7b. This shows that the knowledge of the load profile may not be needed if one can monitor the remaining capacity of each battery pack.

## 6 Conclusions

In this paper, we proposed and analyzed a multistage battery switching network that can support a maximum number of $L_{\max}$ loads under the constraint that the total voltages of these loads do not exceed a design constant $V_{\max}$. The voltage of each battery pack can be determined *optimally* by solving a Simultaneous Integer Representation (SIR) problem. If the average energy consumption rates are known, we proposed a max-min fairness battery allocation scheme to determine the capacity of each battery pack. If the average energy consumption rates are not known, we also proposed the Largest Remaining Capacity First (LRCF) policy in a fault tolerant battery switching network that can still be operated properly even after $F_{\max}$ battery packs fail. Extensive simulations were carried out to compare the performance of various battery allocation schemes. Our simulation results showed that the max-min fairness battery allocation scheme outperforms the uniform allocation scheme. (that assigns each battery pack with the same capacity) and the LRCF scheme also performs well even without the knowledge of the load profile.

Though it might be still difficult to use our theory to construct large battery systems for practical use by using the current state-of-the-art technology, one can make our multistage battery switching network more realizable by addressing the following research problems: (i) allowing a matching error to compensate the effect of non-ideal batteries, (ii) reducing the manufacturing cost for battery packs by limiting the number of shapes, (iii) minimizing the discharge current of individual batteries to prolong battery life, (iv) determining the order of turning on/off switches to avoid short circuits during the reconfiguration process, and (v)constructing fault tolerant battery packs that allow reconfiguration in battery packs. In addition to these, there are other physical characteristics of real batteries that need to be taken into account (see e.g., the handbook [23]). In particular, the storage capacities of non-ideal batteries vary and might decrease in time. As such, the max-min fairness allocation scheme might not work well as it assumes that the storage capacities are all the same. On the other hand, the LRCF policy does not require that assumption, and it might be a better choice than the max-min fairness allocation scheme in this regard. Another issue is the life time of a battery. A non-ideal battery can only be recharged for a limited amount of cycles, and there is a tradeoff between the depth-of-discharge (DoD) in a cycle and the life time of a battery. In practice, large DoD in a cycle decreases the life time of a battery. Thus, in order to prolong the life time of a battery, one should avoid large DoD. In this regard, the LRCF policy that tries to balance the DoD of batteries has the advantage over the max-min fairness allocation scheme. Finally, we note that for the safety reason, one might need to implement charge equalizer circuits (see e.g., [24], [25]) to ensure the balance of the SOC of the battery cells in a battery pack. The circuit-level problems are beyond the scope of this paper.

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