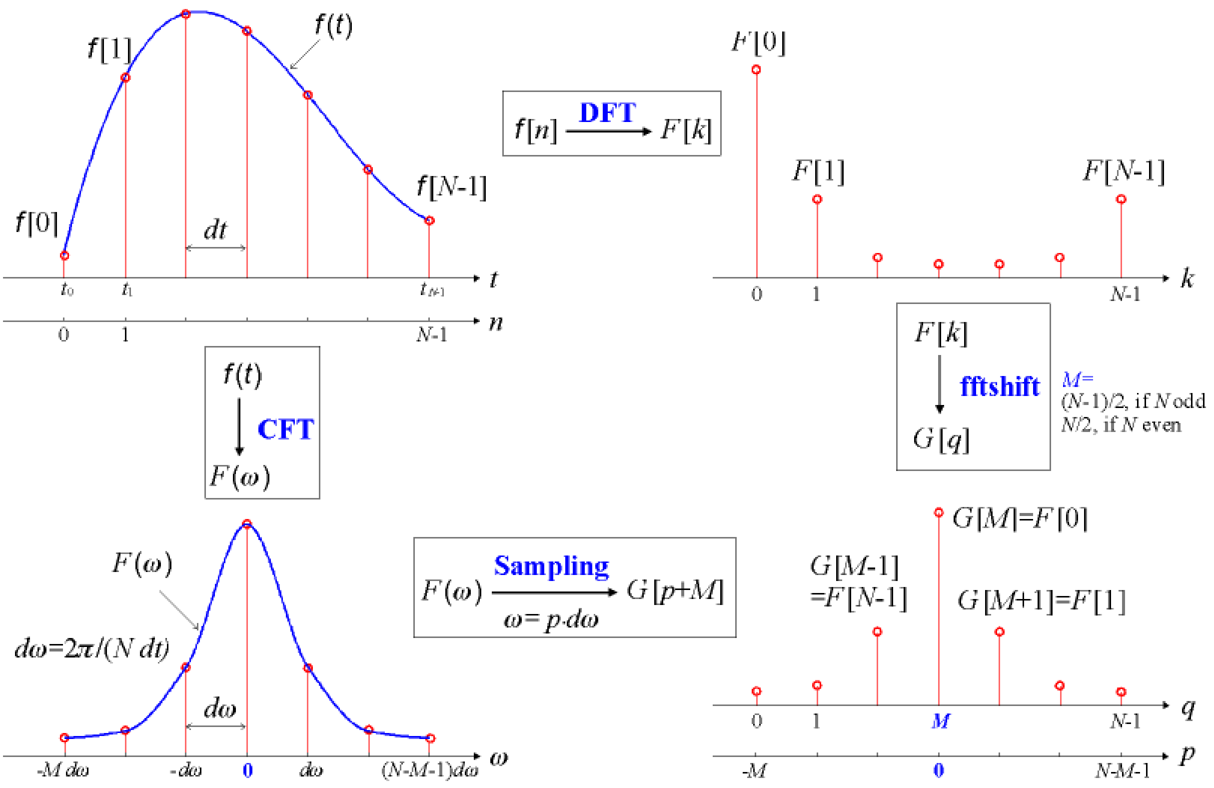
How to simulate CFT by DFT(or FFT)?

■ Definitions

1) CFT: $F(w) = \int_{-\infty}^{\infty} f(t)e^{-jwt}dt$

2) DFT: $F[k] = \sum_{n=0}^{N-1} f[n]\cdot e^{-j\frac{2p}{N}k\cdot n}$, k={0, 1, …, N−1}.

■ Schematic



■ Special cases

1) $F(w\mathbf{=0}) = \int_{-\infty}^{\infty} f(t)dt$, by trapezoidal approximation, $\approx \left(\sum_{n=0}^{N-1} f[n]\right)\cdot dt$;

$F[\mathbf{k=0}]=\sum_{n=0}^{N-1} f[n]$; $\Rightarrow$ $\boxed{F(w{=}0)= dt\cdot F[k{=}0]}$.

(a) The absolute sampling times play no role.

(b) DC spectrum is always sampled.

2) Let spectral resolution $dw=\dfrac{2p}{N\cdot dt}$. $F(w\mathbf{=dw}) = \int_{-\infty}^{\infty} f(t)e^{-jdw\cdot t}dt$, by $\mathbf{t=t_0+n\cdot dt}$ and

trapezoidal approximation, $\approx \sum_{n=0}^{N-1} f[n]\cdot e^{-jdw\cdot(t_0+n\cdot dt)}\cdot dt = dt\cdot e^{-jdw\cdot t_0}\cdot\left(\sum_{n=0}^{N-1} f[n]\cdot e^{-j\frac{2p}{N}n}\right)$;

$F[\boldsymbol{k=1}]=\sum_{n=0}^{N-1} f[n]\cdot e^{-j\frac{2p}{N}n}$; $\Rightarrow$ $\boxed{F(w=dw) = dt\cdot e^{-jdw\cdot t_0}\cdot F[k=1]}$.

Now the absolute sampling times matter.

3) $F(w=\boldsymbol{-dw}) = \int_{-\infty}^{\infty} f(t)e^{jdw\cdot t}dt$, by $t=t_0+n\cdot dt$ and trapezoidal approximation,

$\approx\sum_{n=0}^{N-1} f[n]\cdot e^{jdw(t_0+n\cdot dt)}\cdot dt = dt\cdot e^{jdw\cdot t_0}\cdot\left(\sum_{n=0}^{N-1} f[n]\cdot e^{j\frac{2p}{N}n}\right)$;

$F[\boldsymbol{k=N-1}]=\sum_{n=0}^{N-1} f[n]\times e^{-j\frac{2p}{N}(N-1)n} = \sum_{n=0}^{N-1} f[n]\cdot e^{j\frac{2p}{N}n}$; $\Rightarrow$ $\boxed{F(w=-dw) = dt\cdot e^{jdw\cdot t_0}\cdot F[k=N-1]}$.

■ Conclusion

Since fftshift{$F[k]$}=$G[q]$, q={0, 1, …, M, …, N–1};

where $G[q=M]=F[k=0]$, and $\boxed{\boldsymbol{M}=\begin{cases}(N-1)/2, \text{if } N\in \text{odd}\\ N/2, \text{if } N\in\text{even}\end{cases}}$;

$\Rightarrow$ $\boxed{F(w=p\cdot dw)}$ = $\boxed{dt\cdot e^{-jp\cdot dw\cdot t_0}\cdot G[q=p+M]}$:

1) A multiplicative factor ***dt*** is required.
2) DFT is unaware of absolute time (but index 0, 1, …). It always **assumes an initial time of $\boldsymbol{t=0}$**, which differs from the actual value $\boldsymbol{t=t_0}$. Consequently, a **linear spectral phase** term $e^{-jw\cdot t_0}$ is used to **compensate** this fake delay.

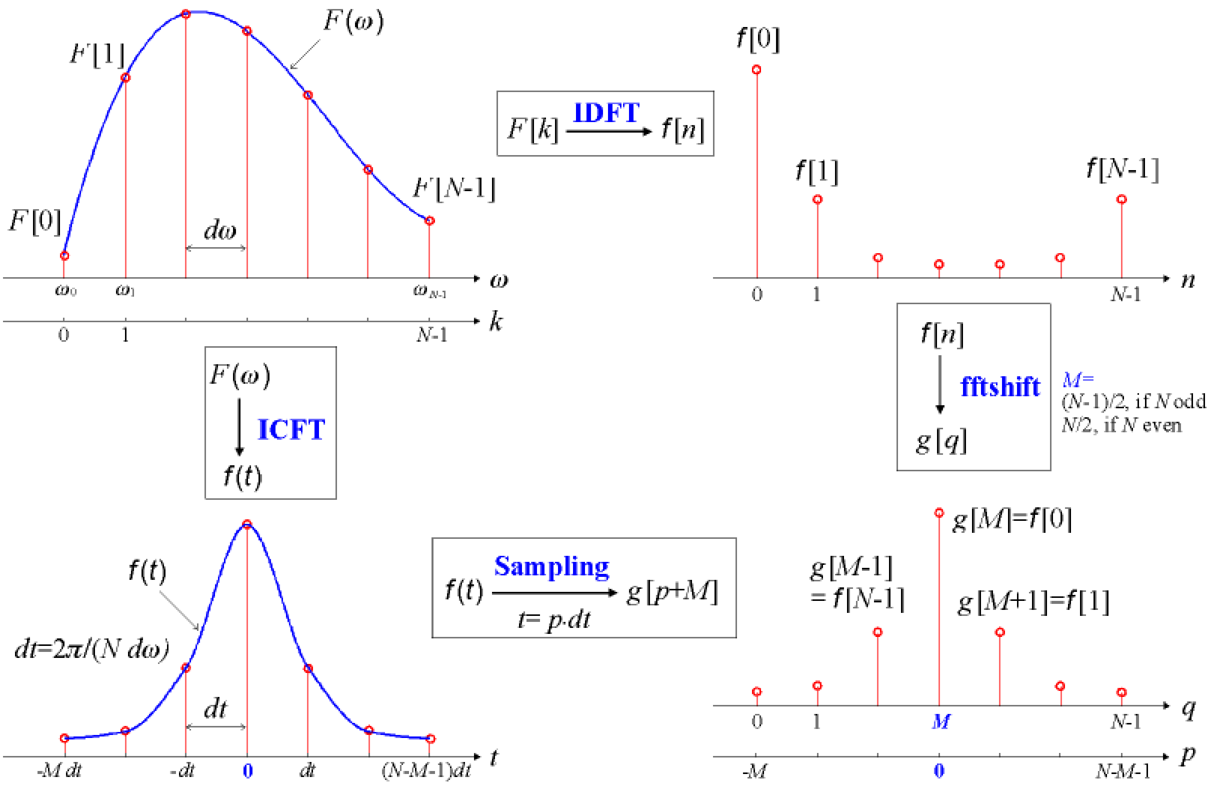
How to simulate ICFT by IDFT(or IFFT)?

■ Definitions

1) ICFT: $f(t) = \frac{1}{2p}\int_{-\infty}^{\infty} F(w)e^{jwt}dw$

2) DFT: $f[n] = \frac{1}{N}\sum_{k=0}^{N-1} F[k]\cdot e^{-j\frac{2p}{N}n\cdot k}$, $n=\{0, 1, \ldots, N-1\}$.

■ Schematic



■ Special cases

1) $f(t=0) = \frac{1}{2p}\int_{-\infty}^{\infty} F(w)dw$, by trapezoidal approximation, $\approx \frac{1}{2p}\left(\sum_{k=0}^{N-1} F[k]\right)\cdot dw$;

$f[n=0] = \frac{1}{N}\sum_{k=0}^{N-1} F[k]; \Rightarrow$ $\boxed{f(t=0) = \frac{N\cdot dw}{2p}\cdot f[n=0]}$.

(a) The absolute sampling frequencies play no role.

(b) Zero-time is always sampled.

2) Let temporal resolution $dt = \frac{2p}{N\cdot dw}$. $f(t=dt) = \frac{1}{2p}\int_{-\infty}^{\infty} F(w)e^{jw\cdot dt}dw$, by $w=w_0+k\times dw$ and

trapezoidal approximation, $\approx \frac{1}{2p}\sum_{k=0}^{N-1} F[k]\cdot e^{j(w_0+k\cdot dw)dt}\cdot dw = \frac{dw}{2p}\cdot e^{jw_0\cdot dt}\cdot\left(\sum_{k=0}^{N-1} F[k]\cdot e^{j\frac{2p}{N}k}\right)$;

$f[\mathbf{n=1}] = \frac{1}{N}\sum_{k=0}^{N-1} F[k]\cdot e^{-j\frac{2p}{N}k}$; $\Rightarrow$ $\boxed{f(t=dt) = \frac{N\cdot dw}{2p}\cdot e^{jw_0\cdot dt}\cdot f[n=1]}$.

Now the absolute sampling frequencies matter.

3) $f(\boldsymbol{t=-dt}) = \frac{1}{2p}\int_{-\infty}^{\infty} F(w)e^{-jw\cdot dt}dw$, by $w=w_0+k\cdot dw$ and trapezoidal approximation,

$$\approx \frac{1}{2p}\sum_{k=0}^{N-1} F[k]\cdot e^{-j(w_0+k\cdot dw)dt}\cdot dw = \frac{dw}{2p}\cdot e^{-jw_0\cdot dt}\cdot\left(\sum_{k=0}^{N-1} F[k]\cdot e^{-j\frac{2p}{N}k}\right);$$

$$f[\boldsymbol{n=N-1}] = \frac{1}{N}\sum_{k=0}^{N-1} F[k]\cdot e^{-j\frac{2p}{N}(N-1)k} = \frac{1}{N}\sum_{k=0}^{N-1} F[k]\cdot e^{j\frac{2p}{N}k};$$

$$\Rightarrow \boxed{f(t=-dt) = \frac{N\cdot dw}{2p}\cdot e^{-jw_0\cdot dt}\cdot f[n=N-1]}.$$

■ Conclusion

Since fftshift{$f[n]$}=$g[q]$, $q=\{0, 1, \ldots, M, \ldots, N-1\}$;

where $g[q=M]=f[n=0]$, and $M=\begin{cases}(N-1)/2, \text{if } N\in \text{odd}\\ N/2, \text{if } N\in\text{even}\end{cases}$;

$$\Rightarrow \boxed{f(t=p\cdot dt)} = \boxed{\frac{N\cdot dw}{2p}\cdot e^{jw_0\cdot p\cdot dt}\cdot g[q=p+M]}:$$

1) A multiplicative factor $\boxed{\frac{N\cdot dw}{2p}=\frac{1}{dt}}$ is required.
2) IDFT is unaware of absolute frequency (but index 0, 1, …). It always **assumes an initial angular frequency of w=0**, which differs from the actual value $w=w_0$. Consequently, a **linear temporal phase** term $e^{jw_0\cdot t}$ is used to **compensate** this fake frequency shift.