Lesson 05 Solving PDEs by Integral Transforms

Introduction

■ Definition

An integral transform of function $f(t)$ is defined as:

$$F(s)=\int_A^B K(s,t)f(t)dt \tag{5.1}$$

where $K(s,t)$ is the kernel of the transformation, which is chosen such that the transformation has certain desired properties.

E.g. $F\{f''(t)\}=-\omega^2F(\omega)$, derivative → algebraic multiplication.

■ General strategy

A "suitable" integral transform is used to change partial derivatives in one domain into algebraic manipulations in the other domain, therefore, reducing the number of "effective" variables (the variables involving with derivatives in the PDE) by one. The process is repeated until only one "effective" variable is left, i.e. PDE → ODE.

E.g. By applying a transform with respect to t on the PDE: $u_t= u_{xx}$, we can arrive at an ODE of variable x. By applying two transforms with respect to x and y subsequently on the PDE: $u_{xx}+u_{yy}+u_{zz}=0$, we can arrive at an ODE of variable z.

An integral transform is "suitable" in solving a PDE problem if the interval of integration $[A,B]$ is consistent with the region of interest (ROI) of the problem.

(∗) Solving PDEs by Fourier sine-cosine Transforms

■ Review of Fourier since-cosine transforms (EK 11.8)

$$F_s\{f(t)\}=F_s(\omega)=\frac{2}{\pi}\int_0^{\infty} f(t)\sin(\omega t)dt\,;\quad F_s^{-1}\{F_s(\omega)\}=f(t)=\int_0^{\infty} F_s(\omega)\sin(\omega t)d\omega \qquad (5.2)$$

$$F_c\{f(t)\}=F_c(\omega)=\frac{2}{\pi}\int_0^{\infty} f(t)\cos(\omega t)dt\,;\quad F_c^{-1}\{F_c(\omega)\}=f(t)=\int_0^{\infty} F_c(\omega)\cos(\omega t)d\omega \qquad (5.3)$$

Since $F_s\{f'(t)\}=-\omega\cdot F_c(\omega)$, $F_s\{\ \}$, $F_c\{\ \}$ are inappropriate to handle variables associated with 1st-order partial derivatives.

$$F_s\{f''(t)\}=-\omega^2 F_s(\omega)+\frac{2}{\pi}f(0)\cdot\omega\,;\ F_c\{f''(t)\}=-\omega^2 F_c(\omega)-\frac{2}{\pi}f'(0)\cdot\omega \qquad (5.4)$$

Eq. (5.4) implies that $F_s\{\ \}$, $F_c\{\ \}$ are appropriate to handle variables associated with 2nd-order partial derivatives and ranging from 0 to ∞.

■ Problem: heat diffusion in a **semi-infinite** rod with fixed temperature at one end

PDE: $u_t=\alpha^2 u_{xx}$ $\{0<x<\infty,\ 0<t<\infty\}$

BC: $u(0,t)=T$

IC: $u(x,0)=0$

■ Procedures

1) Eliminate "effective" variable ***x*** by transforming the PDE and IC:

   Let $F_s\{u(x,t)\}=U(\xi;t)=U(t)$, where ξ is regarded as a "constant" (for there is no derivative with respect to ξ in the transformed equation).

   PDE: $F_s\{u_t\}=\alpha^2 F_s\{u_{xx}\} \Rightarrow \dfrac{d}{dt}U=\alpha^2\left(-\xi^2 U+\dfrac{2}{\pi}u(0,t)\cdot\xi\right)=\alpha^2\left(-\xi^2 U+\dfrac{2T}{\pi}\xi\right)$;

   IC: $F_s\{u(x,0)\}=U(0)=0$; leading to a 1st-order ODE problem:

$$\begin{cases}\text{ODE}: U'(t)+(\alpha\xi)^2 U(t)=\dfrac{2\alpha^2 T\xi}{\pi}\\ \text{IC}: U(0)=0\end{cases}$$

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(a) The Fourier sine transform integrates from 0 to ∞, therefore, is suitable to handle problems of semi-infinite dimension.

(b) BC: $u(0,t)=T$ has been incorporated into the transformed equation.

2) Solving the resulting ODE+IC:

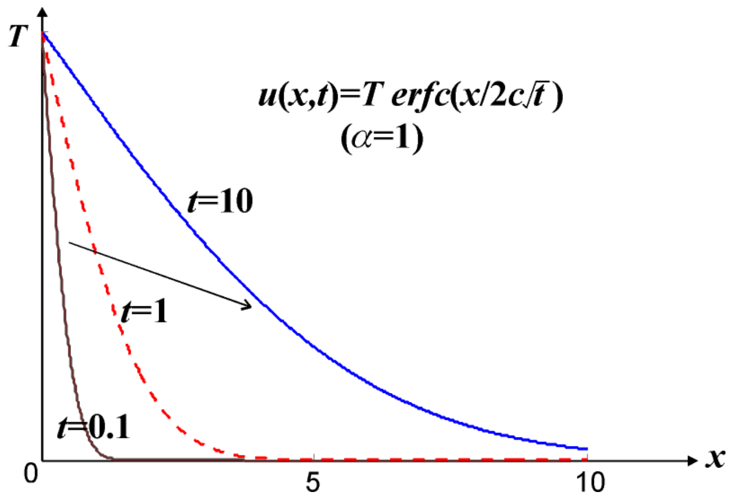
$$U(t)=\frac{2T}{\pi\xi}\left[1-e^{-(\alpha\xi)^2 t}\right]$$

3) Inverse transform to get the exact solution:

By checking the table, we have:

$$u(x,t)=\ F_s^{-1}\{U(\xi,t)\}=T\cdot erfc\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

where $erfc(x)\equiv\frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-t^2}dt$ is the complementary error function.



Solving PDEs by Fourier Transform

■ Review of Fourier transform (EK 11.9)

$$F\{f(t)\}=F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt\,;\ \ F^{-1}\{F(\omega)\}=f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega \quad (5.5)$$

$$F\{f'(t)\}=-j\omega F(\omega) \quad (5.6)$$

$$F\{f''(t)\}=-\omega^2F(\omega) \quad (5.7)$$

Eq's (5.6-7) implies that Fourier transform is OK to handle variables associated with 1st- and 2nd-order partial derivatives, ranging from -∞ to ∞.

■ Problem: heat diffusion in an **infinite** rod (already solved by separation of variables in Lesson 3)

PDE: $u_t=\alpha^2 u_{xx}$ $\{-\infty<x<\infty,\ 0<t<\infty\}$

No BC (two implicit BCs: $u(\pm\infty,t)=0$, otherwise $U=F_x\{u\}$ does not exist)

IC: $u(x,0)=\phi(x)$

■ Procedures

1) Eliminate "effective" variable $\boldsymbol{x}$ by transforming the PDE and IC:

   Let $F_x\{u(x,t)\}=U(\xi;t)=U(t)$, where subscript x means the Fourier transform is performed with respect to variable x, ξ is regarded as a "constant".

   PDE: $F_x\{u_t\}=\alpha^2 F_x\{u_{xx}\}$, $\Rightarrow$ by eq. (5.7), $\Rightarrow U'(t)=-\alpha^2\xi^2 U$;

   IC: $F_x\{u(x,0)\}=F_x\{\phi(x)\}$, $\Rightarrow U(0)=\Phi(\xi)$

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   Since Fourier transform integrates from -∞ to ∞, it cannot be used to transform the time variable t in this problem, where $t<0$ is not defined.

2) Solving the ODE+IC:

$$U(\xi,t)=\Phi(\xi)\cdot\exp[-\alpha^2\xi^2 t]$$

3) Solving the entire problem by inverse transform:

   $\Rightarrow$ $u(x,t)=F_\xi^{-1}\{U(\xi,t)\}$ = $F_\xi^{-1}\{\Phi(\xi)\}\otimes F_\xi^{-1}\{e^{-(\alpha^2 t)\xi^2}\}=\phi(x)\otimes G(x,t)$, where $G(x,t)=$

$\frac{1}{2\alpha\sqrt{\pi\cdot t}}\exp\left(-\frac{x^2}{4\alpha^2 t}\right)$ is Green's function (impulse response) of the system. This procedure gives the same solution as eq. (3.17) derived by separation of variables.

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The major disadvantage of using Fourier transform is that many functions cannot be transformed (**e.g.** $f(x)=k$, e^x, $\sin x$). Only functions that damp to zero fast enough as $|x|\to\infty$ have Fourier transforms (the integral kernel $e^{-ix\xi}$ does not provide damping).

Solving PDEs by Laplace Transform (EK 12.11)

■ Review of Laplace transform (EK 6)

$$\mathscr{L}\{f(t)\}=F(s)=\int_0^\infty f(t)e^{-st}dt\,; \qquad \mathscr{L}^{-1}\{F(s)\}=f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds \tag{5.8}$$

$$\mathscr{L}\{f'\}=sF(s)-f(0) \tag{5.9}$$

$$\mathscr{L}\{f''\}=s^2F(s)-sf(0)-f'(0) \tag{5.10}$$

$F(s)=\mathscr{L}\{f\}$ exists for $\mathrm{Re}\{s\}>a$, if:

1) $f(t)$ is piecewise continuous on the interval $0\le t\le A$ (arbitrary positive number)

2) we can find constants M and a s.t. $|f(t)|\le Me^{at}$ for all sufficiently large t.⇒ The kernel function e^{-st} provides strong damping, ⇒ most functions can have Laplace transform.

■ Problem: heat diffusion of a **semi-infinite rod** ($0<x<\infty$) with one end immersed into some liquid of zero temperature, and has a constant initial temperature T.

PDE: $u_t=u_{xx}$ $\{\alpha=1,\ 0<x<\infty,\ 0<t<\infty\}$

BC: $u_x(0,t)-u(0,t)=0$ (implicit BC: $|u(\infty,t)|<\infty$)

IC: $u(x,0)=T$

■ Procedures

1) Eliminate "effective" variable ***t*** by transforming the PDE and BC:

Let $\mathscr{L}_t\{u(x,t)\}=U(x;s)=U(x)$, where subscript t means the Laplace transform is performed with respect to variable t, s is regarded as a "constant".

PDE: $\mathscr{L}_t\{u_t\}=\mathscr{L}_t\{u_{xx}\}$, $\Rightarrow$ by eq. (5.9), $sU(x)-u(x,0)=U''(x)$; by IC, $U''(x)-sU(x)=-T$

BC: $\mathscr{L}_t\{u_x(0,t)-u(0,t)\}=\mathscr{L}_t\{0\}$, $\Rightarrow$ $U'(0)-U(0)=0$

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Since Laplace transform integrates from 0 to ∞, transforming variable t or x is fine.

2) Solving the ODE+BC:

The general solution (homogeneous + particular solution) is found to be:

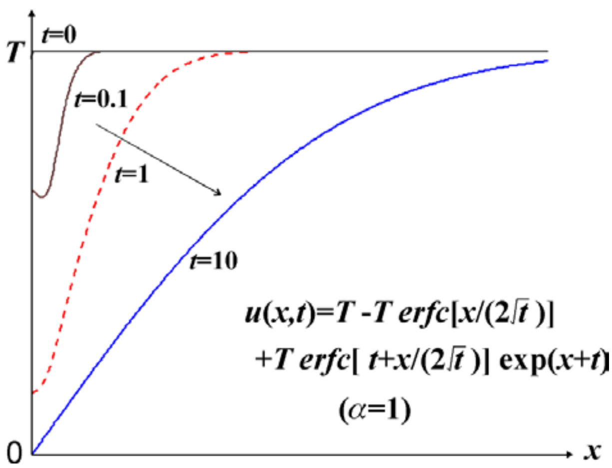
$U(x,s)=\left(c_1e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}\right)+\dfrac{T}{s}$; where coefficients c_1 and c_2 are determined by two BCs.

(i) $|u(\infty,t)|<\infty \Rightarrow U(\infty,s)<\infty$, $\boldsymbol{c_1=0}$; $U(x,s)=c_2e^{-\sqrt{s}x}+\dfrac{T}{s}$.

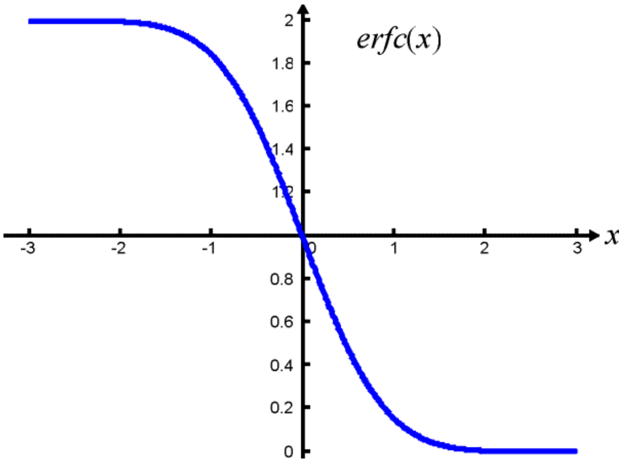
(ii) $U'(0)=U(0) \Rightarrow \boldsymbol{c_2}=\dfrac{-T}{s(1+\sqrt{s})}$; $\Rightarrow U(x,s)=\dfrac{-T}{s(1+\sqrt{s})}e^{-\sqrt{s}x}+\dfrac{T}{s}$

3) Solving the entire problem by inverse transform:

$$\Rightarrow u(x,t)=\mathscr{L}_s^{-1}\{U(x,s)\}=T-T\left[erfc\left(\frac{x}{2\sqrt{t}}\right)-erfc\left(\sqrt{t}+\frac{x}{2\sqrt{t}}\right)e^{(x+t)}\right],\ \text{for } \{-\infty<x<\infty,\ t>0\}.$$



Complementary error function is defined as: $erfc(x) \equiv 1 - erf(x) = \frac{2}{\pi}\int_x^{\infty} e^{-t^2}\,dt$.



E.g. Solving 1-D wave equation with semi-infinite dimension and zero initial displacement & velocity by Laplace transform (Example 1 in EK 12.11).

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	Advantages
Separation of variables	1) Can solve linear PDEs with variable coefficients 2) Derive normal modes (system characteristics)
Integral transforms	Can solve problems described by nonhomogeneous PDE, BCs, ICs