Lesson 04 Nonhomogeneous PDEs and BCs

■ Overview

This lesson introduces two methods to solve PDEs with nonhomogeneous BCs or driving source, where separation of variables fails to deal with.

(∗) Transformation of Nonhomogeneous BCs (SJF 6)

■ Problem: heat flow in a rod with two ends kept at constant nonzero temperatures T_1, T_2.

Homogeneous PDE: $u_t = \alpha^2 u_{xx}$,

Nonhomogeneous BCs: $u(0,t)=\boldsymbol{T_1}$, $u(L,t)=\boldsymbol{T_2}$

IC: $u(x,0)=\phi(x)$

Separation of variables fails, for nonhomogeneous BCs of $u(x,t)$ cannot be transformed into BCs of $X(x)$: $\{u(0,t)=X(0)T(t)=T_1,\ u(L,t)=X(L)T(t)=T_2\}$ does not mean $\{X(0)=T_1,\ X(L)=T_2\}$.

We can separate the solution into two parts: $u(x,t)=f(x)$ (steady state) + $U(x,t)$ (transient state), such that:

1) $f(x)$ is the solution to an ODE obtained by setting time derivative of the PDE as zero (**e.g.** $f''(x)=0$ in this problem). The ODE's BCs can be nonhomogeneous.

2) $U(x, t)$ is the solution to a new PDE with homogeneous BCs: $\{U(0,t)=0,\ U(L,t)=0\}$.

The remaining conditions are found by examining the original PDE, BCs, and ICs:

PDE: $u_t = \alpha^2 u_{xx} \Rightarrow f_t + U_t = \alpha^2[f''(x) + U_{xx}]$; by $f_t=0$, $f''(x)=0$, $\Rightarrow$ new PDE: $U_t = \alpha^2 U_{xx}$

BCs: (1) $u(0,t)=T_1 \Rightarrow f(0)+U(0,t)=T_1$; by $U(0,t)=0$, $\Rightarrow$ BC of ODE: $f(0)=T_1$;

(2) $u(L,t)=T_2 \Rightarrow f(L)+U(L,t)=T_2$, by $U(L,t)=0$, $\Rightarrow$ BC of ODE: $f(L)=T_2$;

IC: $u(x,0)=\phi(x) \Rightarrow f(x)+U(x,0)=\phi(x)$, $\Rightarrow$ modified IC for new PDE: $U(x,0)=\phi(x)-f(x)$.

Now we have two sub-problems:

1) ODE: $f''(x)=0$, BCs: $\{f(0)=T_1, f(L)=T_2\}$; $\Rightarrow$ solution $f(x)=\left(\frac{T_2-T_1}{L}\right)x+T_1$.

2) PDE: $U_t=\alpha^2 U_{xx}$, homogeneous BCs: $\{U(0,t)=0,\ U(L,t)=0\}$; IC: $U(x,0)=\phi(x)-f(x)$

Fortunately, the new PDE happens to be homogeneous, we can apply separation of variables to solve it.

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1) The most general linear BCs:

$$\begin{cases} a_1 u_x(0,t)+b_1 u(0,t)=T_1(t) \\ a_2 u_x(L,t)+b_2 u(L,t)=T_2(t) \end{cases} \tag{4.1}$$

can always be transformed into homogeneous ones (of new PDE), but the resulting new PDE may be nonhomogeneous and disable the use of separation of variables.

2) For methods permitting nonhomogeneous BCs (like eigenfunction expansion or integral transforms), it is not necessary to perform preliminary transformation.

Solving Nonhomogeneous PDEs by Eigenfunction Expansion

■ Problem: heat flow in a rod in the presence of heat source:

PDE: $u_t=\alpha^2 u_{xx}+f(x,t)$, $\{0<x<L,\ t>0\}$

BCs: $\begin{cases} a_1 u_x(0,t)+b_1 u(0,t)=0 \\ a_2 u_x(L,t)+b_2 u(L,t)=0 \end{cases}$ [as indicated before, linear nonhomogeneous BCs can always be transformed into homogeneous ones of a new PDE with a modified source $f(x,t)$].

IC: $u(x,0)=\phi(x)$

■ Idea

Without the heat source $f(x,t)$, separation of variables gives rise to the general solution (see the cases in Lesson 3): $u(x,t)=\sum_{n=1}^{\infty} A_n X_n(x)\cdot\exp\left(-\alpha^2 k_n^2 t\right)$, where A_n are constants to be determined by IC; k_n and $X_n(x)$ are eigenvalues and eigenfunctions of the Sturm-Liouville problem:

$$X''(x)+k^2 X(x)=0\,,\quad \text{BCs: } \begin{cases} a_1 X'(0)+b_1 X(0)=0 \\ a_2 X'(L)+b_2 X(L)=0 \end{cases}$$

The time dependence $\exp\left(-\alpha_n^2 k_n^2 t\right)$ of normal modes is a result of a source-free system. In the presence of heat source $f(x,t)$, the modal time dependence should be generalized to $T_n(t)$, and the eigenfunction expansion of the solution becomes:

$$u(x,t)=\sum_{n=1}^{\infty} T_n(t)\cdot X_n(x) \tag{4.2}$$

Our goal is solving $T_n(t)$ for all n.

■ Procedures

For simplicity, we assume type-1 BCs: $\{u(0,t)=0,\ u(L,t)=0\}$:

1) Solving $X_n(x)$ by the homogeneous PDE and BCs: $X_n(x)=\sin(k_n x)$, $k_n=\dfrac{n\pi}{L}$ (Lesson 3).
2) Expanding $f(x,t)$ in terms of spatial eigenfunctions $X_n(x)$:

$$f(x,t)=\sum_{n=1}^{\infty} F_n(t)\cdot\sin(k_n x) \tag{4.3}$$

where $F_n(t)$ can be derived by orthogonality of $\{X_n(x)\}$:

$$F_n(t)=\frac{2}{L}\int_0^L f(x,t)\sin(k_n x)dx \tag{4.4}$$

3) Substituting the eigenfunction expansions of $u(x,t)$ and $f(x,t)$ into the (original) nonhomogeneous PDE, and determining $T_n(t)$ by IC:

Substitute eq's (4.2-3) into $u_t=\alpha^2 u_{xx}+f(x,t)$,

$$\Rightarrow\ \sum_{n=1}^{\infty} T_n'(t)\sin(k_n x)=-\alpha^2\sum_{n=1}^{\infty} k_n^2 T_n(t)\sin(k_n x)+\sum_{n=1}^{\infty} F_n(t)\sin(k_n x),$$

$\Rightarrow \sum_{n=1}^{\infty}\left[T_n'(t)+(\alpha k_n)^2 T_n(t)-F_n(t)\right]\sin(k_n x)=0.$

By IC: $u(x,0)=\sum_{n=1}^{\infty}T_n(0)\sin(k_n x)=\phi(x) \Rightarrow T_n(0)=\frac{2}{L}\int_0^L \phi(x)\sin(k_n x)dx \equiv \boldsymbol{A_n}$

$\Rightarrow$ For each n, we get an ODE + IC:

$$\begin{cases} \text{ODE}: T_n'(t)+\alpha^2 k_n^2 T_n(t)=F_n(t) \\ \text{IC}: T_n(0)=\frac{2}{L}\int_0^L \phi(x)\sin(k_n x)dx = A_n \end{cases} \tag{4.5}$$

Solving eq. (4.5), we have:

$$T_n(t)=A_n\cdot \exp\left(-\alpha_n^2 k_n^2 t\right)+\int_0^t \exp\left[-\alpha^2 k_n^2(t-\tau)\right]\cdot F_n(\tau)d\tau \tag{4.6}$$

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The solution: $u(x,t)=\sum_{n=1}^{\infty}\left(A_n e^{-\alpha^2 k_n^2 t}\right)\sin(k_n x)+\sum_{n=1}^{\infty}\left(\int_0^t e^{-\alpha^2 k_n^2(t-\tau)}F_n(\tau)d\tau\right)\sin(k_n x)$ consists of:

1) **Transient state** due to **IC** [~$\phi(x)$, A_n], which will decay to zero

2) **Steady state** due to the **source** $f(x,t)$, not necessarily comes to rest.