Homework Solutions #3

1) Let $F_x\{u(x,t)\} = U(\xi;t) = U(t)$

PDE: $F_x\{u_t\} = \alpha^2 F_x\{u_{xx}\} \Rightarrow U'(t) = -\alpha^2\xi^2 U$

IC: $F_x\{u(x,0)\} = F_x\{\phi(x)\} = F_x\left\{e^{-(x/L)^2}\right\} \Rightarrow U(0) = \Phi(\xi) = F_x\left\{e^{-(x/L)^2}\right\}$

Solving the ODE+IC: $U(\xi;t) = \Phi(\xi)\cdot e^{-\alpha^2\xi^2 t}$

$$\Rightarrow u(x,t) = F_\xi^{-1}\{U(\xi,t)\} = F_\xi^{-1}\{\Phi(\xi)\}\otimes F_\xi^{-1}\left\{e^{-(\alpha^2 t)\xi^2}\right\} = \phi(x)\otimes G(x,t)$$

$$= e^{-(x/L)^2}\otimes\frac{1}{2\alpha\sqrt{\pi\cdot t}}e^{-\frac{x^2}{4\alpha^2 t}} = \frac{1}{2\alpha\sqrt{\pi\cdot t}}\int_{-\infty}^{\infty}e^{-\left(\frac{\xi}{L}\right)^2}\cdot e^{-\frac{(x-\xi)^2}{4\alpha^2 t}}d\xi \quad \text{(Green's formula)}$$

$$= \sqrt{\frac{L^2}{4\alpha^2 t + L^2}}\cdot e^{\left(\frac{x}{L}\right)^2\left[\frac{4\alpha^2 t}{\left(4\alpha^2 t + L^2\right)} - 1\right]} \quad \text{(See problem 2 in HW2)}$$

2) PDE: $\dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial t} = x$

BC: $w(x,0) = 1$ & IC: $w(0,t) = 1$

$$L_t\left\{\frac{\partial w}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial w}{\partial x}dt = \frac{\partial}{\partial x}L\{w\} = \frac{\partial W}{\partial x}$$

$$L_t\left\{x\frac{\partial w}{\partial t}\right\} = xL_t\left\{\frac{\partial w}{\partial t}\right\} = x[sL_t\{w\} - w(x,0)] = xsL_t\{w\} - x = xsW - x$$

$$L_t\{x\} = \frac{x}{s}$$

$$\Rightarrow \frac{\partial W}{\partial x} + xsW - x = \frac{x}{s} \Rightarrow \frac{\partial W}{\partial x} + xsW = x\left(\frac{1}{s}+1\right), \quad \Rightarrow W = Ce^{-\frac{s}{2}x^2} + \frac{1}{s^2} + \frac{1}{s}.$$

By BC: $L_t\{w(0,t)\} = \dfrac{1}{s} = W(0,s) = C + \dfrac{1}{s^2} + \dfrac{1}{s} \Rightarrow C = -\dfrac{1}{s^2}$

$$\Rightarrow W(x,s) = -\frac{1}{s^2}e^{-\frac{s}{2}x^2} + \frac{1}{s^2} + \frac{1}{s}$$

$$\Rightarrow w(x,t) = L_s^{-1}\{W(x,s)\} = -\left(t - \frac{x^2}{2}\right)u\left(t - \frac{x^2}{2}\right) + t + 1$$

3a) PDE: $u_t = \alpha^2 u_{xx}$, $\{\,0 < x < \infty,\ \ 0 < t < \infty\,\}$

BC: $u(0,t) = \delta(t)$

IC: $u(x,0) = 0$

Step1: 對 PDE 做 Laplace transform $\Rightarrow sU(x,s)-u(x,0)=\dfrac{\partial^2 U(x,s)}{\partial x^2}$

Step2: 解上式 ODE: $U(x,s)=c_1(s)e^{\frac{\sqrt{s}}{\alpha}x}+c_2(s)e^{-\frac{\sqrt{s}}{\alpha}x}$

因為 $x\geq 0$ 時，$U(x,s)$ 需為有限值，故 $c_1(s)=0$

又 $L\{u(0,t)\}=L\{\delta(t)\}=1=U(0,s)=c_2(s)$

$\Rightarrow U(x,s)=e^{-\frac{\sqrt{s}}{\alpha}x}$

Step3: 做 Inverse Laplace transform 即可求的 $u(x,t)$

$$u(x,t)=L^{-1}\left\{e^{-\frac{\sqrt{s}}{\alpha}x}\right\}=\frac{x}{2\alpha\sqrt{\pi}}\frac{e^{-\frac{x^2}{4\alpha^2 t}}}{t^{\frac{3}{2}}}$$

<Proof> $F(s)=e^{-\sqrt{s}}$

$$\frac{dF}{ds}=\frac{1}{-2\sqrt{s}}e^{-\sqrt{s}}$$

$$\frac{d^2F}{ds^2}=\frac{1}{-2s}\frac{dF}{ds}+\frac{1}{4s}F \quad \Rightarrow 4s\frac{d^2F}{ds^2}+2\frac{dF}{ds}-F=0$$

先做 Inverse Laplace transform: $4\left\{\dfrac{d}{dt}[t^2 f(t)]\right\}+2\{-tf(t)\}-f(t)=0$

$\Rightarrow 4t^2\dfrac{df}{dt}+(6t-1)f=0$ (解一階 ODE)

$$\Rightarrow f(t)=ce^{-\int\frac{6t-1}{4t^2}dt}=c\frac{e^{-\frac{1}{4t}}}{t^{\frac{3}{2}}}$$

3b) <MATLAB code>

```
x=linspace(0,15,1001);
t=[1 2 4];
afa=1;
uxt=zeros(length(t), length(x));
for tt=1:length(t)
uxt(tt,:)=x./(2.*afa.*sqrt(pi)).*( exp(-(x.^2)./(4.*(afa.^2).*t(tt)))./(t(tt).^1.5) );
end
figure; plot(x, uxt(1,:), 'r', x, uxt(2,:), 'g', x, uxt(3,:), 'b');
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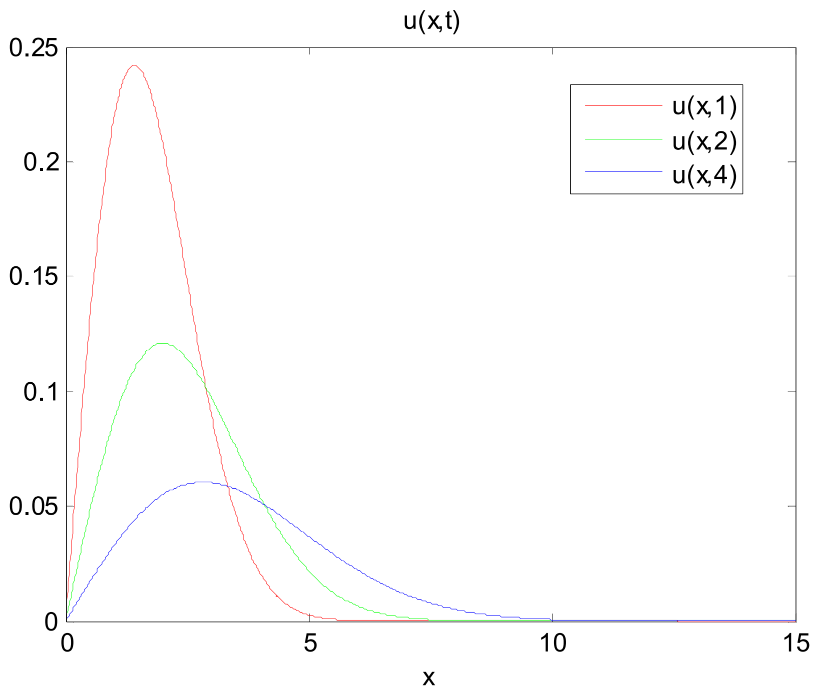


4) PDE: $u_{tt}=c^2u_{xx}\quad \{0<x<\infty,\ 0<t<\infty\}$

BC: $u(0,t)=\delta(t)$

ICs: $u(x,0)=0,\quad u_t(x,0)=0$

Step1: 對 PDE 做 Laplace transform : $s^2U(x,s)-su(x,0)-\dfrac{\partial u(x,0)}{\partial t}=\dfrac{\partial^2U(x,s)}{\partial x^2}$

$$\Rightarrow s^2U(x,s)=\frac{\partial^2U(x,s)}{\partial x^2}$$

Step2: 解 ODE: $U(x,s)=c_1(s)e^{\frac{s}{c}x}+c_2(s)e^{-\frac{s}{c}x}$

因為 $x\ge 0$ 時，$U(x,s)$ 需為有限值，故 $c_1(s)=0$

又 $L\{u(0,t)\}=L\{\delta(t)\}=1=U(0,s)=c_2(s)$

$$\Rightarrow U(x,s)=e^{-\frac{s}{c}x}$$

Step3: 做 Inverse Laplace transform 即可求的 $u(x,t)$

$$u(x,t)=L_s^{-1}\{U(x,s)\}=\delta\left(t-\frac{x}{c}\right).$$

5a) Let $u(x,y,t) = X(x)Y(y)T(t)$, and substitute into PDE

We obtain

$$\frac{T'}{T} = \alpha^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

Let $\frac{X''}{X} = -p^2$ and $\frac{Y''}{Y} = -q^2$. p and q are constant.

So $\frac{T'}{T} + \alpha^2\left(p^2 + q^2\right) = 0$

$$\Rightarrow \boxed{\begin{cases} X'' + p^2 X = 0 \\ Y'' + q^2 Y = 0 \\ T' + \alpha^2 (p^2 + q^2) T = 0 \end{cases}}$$

5b) According to the figure

BCs: $X(0) = 0, \quad X(a) = 0, \quad Y(0) = 0, \quad \frac{dY(b)}{dy} = 0$

<Part 1>

$X'' + p^2 X = 0, \quad X(0) = 0, \quad X(a) = 0$

$\Rightarrow X = c_1 \cos(px) + c_2 \sin(px)$, substitute the BCs:

$\Rightarrow \boxed{p = \frac{m\pi}{a}}, \boxed{X_m = \sin\left[\frac{m\pi}{a} x\right], \quad m = 1,2,3\cdots}$ (m starts with 1)

<Part 2>

$Y'' + q^2 Y = 0, \quad Y(0) = 0, \quad \frac{dY(b)}{dy} = 0$

$\Rightarrow Y = c_3 \cos(qy) + c_4 \sin(qy)$, substitute the BCs:

$\Rightarrow \boxed{q = \frac{(2n+1)\pi}{2b}}, \boxed{Y_n = \sin\left[\frac{(2n+1)\pi}{2b} y\right], \quad n = 0,1,2,3\cdots}$

<Part 3>

$T' + \alpha^2 (p^2 + q^2) T = 0$

$\Rightarrow T = c_5 \exp\left[-\alpha^2\left(p^2 + q^2\right)t\right]$

By $p = \frac{m\pi}{a}$ and $q = \frac{(2n+1)\pi}{2b}$ substitute into T

$$\Rightarrow \boxed{T_{mn} = \exp\left\{-\alpha^2 \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{2n+1}{2b}\right)^2\right] t\right\}}$$

5c) According to (5b), We obtain the fundamental mode (m=1 & n=0) is:

$$u_{mn=10}(x,y,t) = \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{2b} y\right) \exp\left\{-\alpha^2 \pi^2 \left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{2b}\right)^2\right] t\right\}$$

If $t \to \infty$, $\boxed{u_{10}(x,y,t \to \infty) = 0}$

5d) According to (5c)

$$u(x,y,t=0)=100=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}\sin\left[\frac{m\pi}{a}x\right]\sin\left[\frac{(2n+1)\pi}{2b}y\right]$$

$$\Rightarrow C_{mn}=\frac{\left\langle 100,\sin\left(\frac{m\pi}{a}x\right)\right\rangle\left\langle 100,\sin\left[\frac{(2n+1)\pi}{2b}y\right]\right\rangle}{\left\|\sin\left(\frac{m\pi}{a}x\right)\right\|^2\cdot\left\|\sin\left[\frac{(2n+1)\pi}{2b}y\right]\right\|^2}$$

where $\langle f(x),g(x)\rangle\equiv\int_{-\infty}^{\infty}f(x)g(x)dx$, and $\|f(x)\|^2\equiv\langle f(x),f(x)\rangle$.

$$\Rightarrow C_{mn}=\frac{\left(100\int_0^a\sin\left(\frac{m\pi}{a}x\right)dx\right)\cdot\left(100\int_0^b\sin\left[\frac{(2n+1)\pi}{2b}y\right]dy\right)}{\int_0^a\sin^2\left(\frac{m\pi}{a}x\right)dx\cdot\int_0^b\sin^2\left[\frac{(2n+1)\pi}{2b}y\right]dy}$$

$$=\frac{100^2\cdot\frac{a}{m\pi}\left[1-(-1)^m\right]\cdot\frac{2b}{(2n+1)\pi}}{\frac{a}{2}\cdot\frac{b}{2}}=\begin{cases}0, & \text{if } m=\text{even}\\ \frac{160000}{\pi^2 m(2n+1)}, & \text{if } m=\text{odd}\end{cases}$$

So, we obtain

$$\boxed{u(x,y,t)=\frac{160000}{\pi^2}\sum_{m=1,3,5\ldots}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m(2n+1)}\sin\left(\frac{m\pi}{a}x\right)\sin\left[\frac{(2n+1)\pi}{2b}y\right]\cdot\exp\left\{-\alpha^2\pi^2\left[\left(\frac{m}{a}\right)^2+\left(\frac{2n+1}{2b}\right)^2\right]t\right\}}$$

6) $w_{mn}=\sqrt{k_{x,m}^2+k_{y,m}^2}=c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$ u_{11} => $w_{11}=c\pi\sqrt{\frac{1^2}{a^2}+\frac{1^2}{b^2}}$

另一邊長度為 a，另一邊長度為 A/a，對其微分取極值 =>

$$\frac{dw_{11}}{da}=\frac{d}{da}\left[c\pi\sqrt{\frac{1}{a^2}+\frac{a^2}{A^2}}\right]=\frac{\frac{1}{2}c\pi\left[\frac{-2}{a^3}+\frac{2a}{A^2}\right]}{\left[\frac{1}{a^2}+\frac{2a}{A^2}\right]}\overset{\text{if } A=a^2}{=}0$$

可知 $A=a^2$ 有極值，再任意代入其他值可發現此極值為最小值，即可證明當二邊長度相等時有最低頻率。

7a) 由講義6.6式可得此PDE解為

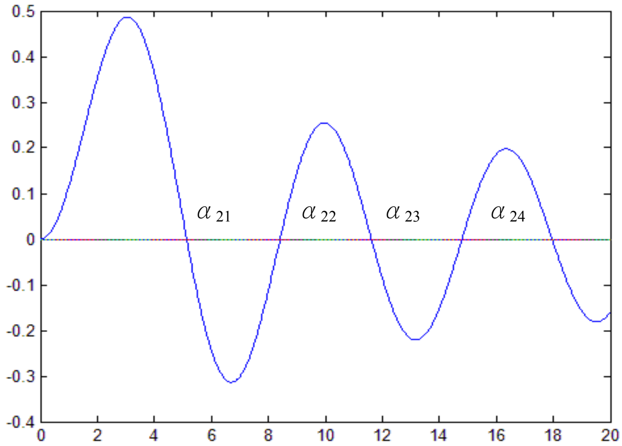
$u(r,\theta,t)=[A_{mn}\cos(\omega_{mn}t)+B_{mn}\sin(\omega_{mn}t)]J_m(\lambda_{mn}r)\cos(m\theta)$

其中 $\lambda_{mn}=\dfrac{\alpha_{mn}}{\rho}$，求 nodal circles 則需使用以下方程式

$$\lambda_{mn} r=\frac{\alpha_{mn}}{\rho} r=\alpha_{mn'} \rightarrow r_{n'}=\left(\frac{\alpha_{mn'}}{\alpha_{mn}}\right)\rho<\rho$$

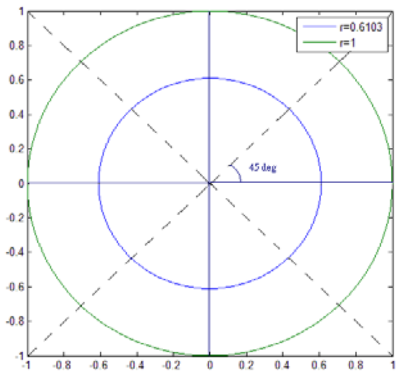
帶入條件 $\rho=1$，$c=1$，u_{22} => $r_{n'}=\left(\dfrac{\alpha_{2n'}}{\alpha_{22}}\right)<1 \Rightarrow \alpha_{2n'}<\alpha_{22}$

我們可畫出 J_2 的圖如下：



由上圖可看出滿足 $\alpha_{2n'}<\alpha_{22}$ 只有 $\alpha_{21}\approx 5.136 \Rightarrow r=\left(\dfrac{5.136}{8.4155}\right)\rho=0.6103\rho$

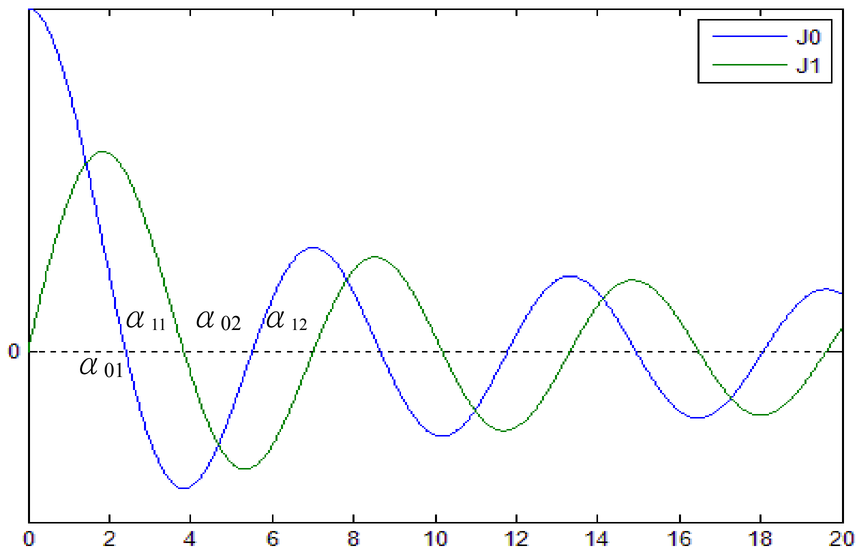
$cos(2\theta)=0$ => $\theta=\dfrac{2p+1}{4}\pi$， $p=0,1,2,3\ldots$



即為所求(dashed lines should end on the green circle)。

7b) 由講義 6.7 式可得 $\dfrac{v_{mn}}{v_{01}}=\dfrac{\alpha_{mn}}{\alpha_{01}}$

下圖為 J_0，J_1 的圖形(藍線為 J_0，綠線為 J_1)：



由圖可知 α_{01}= 2.405，α_{02}=5.52，α_{11}=3.83，α_{12}=7.015

$\frac{v_{02}}{v_{01}} = 2.2952$ ，$\frac{v_{11}}{v_{01}} = 1.5925$ ，$\frac{v_{12}}{v_{01}} = 2.9168$