Homework Solutions #2

1) PDE: $u_t = \alpha^2 u_{xx}$

BCs: $u(0,t) = 0$, $u_x(L,t) = 0$

IC: $u(x,0) = U_0$

Separation of variables: $X'' + k^2 X = 0$, $\dot{T} + \alpha^2 k^2 T = 0$

Spatial ODE: $X'' + k^2 X = 0 \Rightarrow X = A\cos(kx) + B\sin(kx)$,

$$X' = -k[A\sin(kx) - B\cos(kx)]$$

By BCs: $u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A = 0$,

$$u_x(L,t) = 0 \Rightarrow X'(L) = 0 \Rightarrow k_n = \frac{\left(n - \frac{1}{2}\right)\pi}{L}, \quad n = 1,2,3\ldots$$

$$\Rightarrow X_n(x) = B_n \sin(k_n x)$$

Temporal ODE: $\dot{T} + \lambda_n^2 T = 0$, $\lambda_n = \alpha k_n = \dfrac{\left(n - \frac{1}{2}\right)\pi\alpha}{L} \Rightarrow T_n(t) = \exp(-\lambda_n^2 t)$

$\Rightarrow$ n-th normal mode is $u_n(x,t) = B_n \sin(k_n x)\cdot\exp(-\lambda_n^2 t)$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(k_n x)\cdot\exp(-\lambda_n^2 t)$$

By IC: $u(x,0) = \sum_{n=1}^{\infty} B_n \sin(k_n x) = U_0$.

By Fourier sine series, $\Rightarrow B_n = \dfrac{2U_0}{L}\displaystyle\int_0^L \sin(k_n x)\,dx$

2) From eq.(3.15), $u(x,t) = \displaystyle\int_0^{\infty} [A(k)\cdot\cos(kx) + B(k)\cdot\sin(kx)]\cdot e^{-\alpha^2 k^2 t}\,dk$

$\Rightarrow u(x,0) = \displaystyle\int_0^{\infty} [A(k)\cdot\cos(kx) + B(k)\cdot\sin(kx)]\,dk = e^{-\left(\frac{x}{L}\right)^2}$, find $A(k)$, $B(k)$, by eq.(3.16).

By Green's function formula [eq.(3.17)],

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi\cdot t}}\int_{-\infty}^{\infty} e^{-\left(\frac{\xi}{L}\right)^2}\cdot e^{-\frac{(x-\xi)^2}{4\alpha^2 t}}\,d\xi$$

Let $z = \dfrac{\xi - x}{2\alpha\sqrt{t}}$, $\Rightarrow u(x,t) = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\left(\frac{x+2\alpha z\sqrt{t}}{L}\right)^2}\cdot e^{-z^2}\,dz = \dfrac{e^{-\left(\frac{x}{L}\right)^2}}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{4\alpha^2 t + L^2}{L^2}z^2}\cdot e^{-\frac{4\alpha x\sqrt{t}}{L^2}z}\,dz$, by the integration formula: $\int_{-\infty}^{\infty} e^{-ax^2} \cdot e^{-2bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{a}}$, $\Rightarrow$

$$u(x,t)=\frac{e^{-\left(\frac{x}{L}\right)^2}}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi L^2}{4\alpha^2 t + L^2}} \cdot e^{\frac{4\alpha^2 x^2 t}{L^2(4\alpha^2 t + L^2)}} = \sqrt{\frac{L^2}{4\alpha^2 t + L^2}} \cdot e^{\left(\frac{x}{L}\right)^2\left[\frac{4\alpha^2 t}{(4\alpha^2 t + L^2)} - 1\right]}$$

3a) *Let* $P=u-f(x)$

- ⇨ $P_t = P_{xx} + f_{xx}(x) \Rightarrow f_{xx}(x)=0 \Rightarrow f(x)=cx+n$
- ⇨ BC: $P(0,t)=1-f(0)=0, \Rightarrow f(0)=1, f(x)=cx+1,$
- ⇨ $P_x(1,t)+hP(1,t)=1-f_x(1)+h\,f(1)=0 \Rightarrow \quad c=\frac{1-h}{1+h}, f(x)=\left(\frac{1-h}{1+h}\right)x+1$
- ⇨ IC: $P(x,0)=1-x-f(x)$

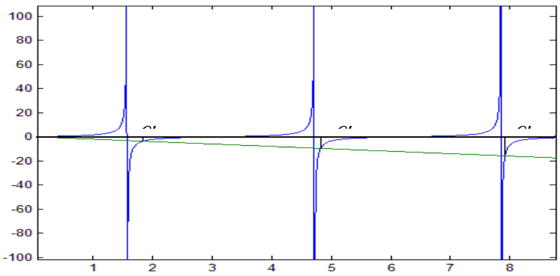
Let $P=X(x)T(t) \Rightarrow X''T=XT' \Rightarrow \frac{T'}{T}=\frac{X''}{X}=\lambda \Rightarrow T'-\lambda T=0$, $X''-\lambda X=0$

1. 由計算可知$\lambda \geq 0$無特徵值
2. $\lambda < 0 \Rightarrow X''+|\lambda| X=0$ *let* $|\lambda|=\beta^2$

   $X=c_1\cos(\beta x)+c_2\sin(\beta x)$

   (1) $P(0,t)=0 \Rightarrow c_1=0$

   (2) $P_x(1,t)+hP(1,t)=0 \Rightarrow \beta c_2\cos(\beta)+hc_2\sin(\beta)=0 \Rightarrow \tan(\beta)=-\frac{\beta}{h}$



$\alpha_n (n=1,2,3...)$ 為 $\tan(\beta)=-\beta/h$ 之解 $\Rightarrow X_n = c_n\sin(\alpha_n x)$

$T'+\alpha_n T=0 \Rightarrow T=k\exp(-\alpha_n^2 t)$

$$P=XT=\sum_{n=1}^{\infty} B_n \sin(\alpha_n x)\cdot\exp(-\alpha_n^2 t)$$

$$u=P+f(x)=\sum_{n=1}^{\infty} B_n \sin(\alpha_n x)\cdot\exp(-\alpha_n^2 t)+\left(\frac{1-h}{1+h}\right)x+1$$

代入 IC 求 $B_n \Rightarrow$ $-(1+c)x=\sum_{n=1}^{\infty} B_n \sin(\alpha_n x)$

$$\Rightarrow B_n = \frac{\int_0^1 -(1+c)x\sin(\alpha_n x)dx}{\int_0^1 \sin^2(\alpha_n x)dx} = \frac{(1+c)[\frac{\cos(\alpha_n)}{\alpha_n} - \frac{\sin(\alpha_n)}{\alpha_n^2}]}{\frac{1}{2} - \frac{\sin(2\alpha_n)}{4\alpha_n}}$$

3b) 當 h=10^3 則 α_n 接近於 $n\pi$ (n=0, 1 , 2 , 3…) =>

$$B_n \approx (1+\frac{1-h}{1+h})\frac{[\frac{(-1)^n}{n\pi}]}{\frac{1}{2}} = 2(1+\frac{1-h}{1+h})\frac{(-1)^n}{n\pi} \quad ,u = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\cdot\exp[-(n\pi)^2 t] + \left(\frac{1-h}{1+h}\right)x+1$$

Type 1 BC: $u_t=u_{xx}$ 令 $u=p+f(x)$,且 $p(0,t) = p(1,t) =0$ (*Type1*) , =>

$u(0,t)=1, u(1,t)=[2/(1+h)]$可解得 $u = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\cdot\exp[-(n\pi)^2 t] + \left(\frac{1-h}{1+h}\right)x+1$, 與 h=10^3 答案相似, 所以在 h=10^3 的情況下近似於 Type 1 BC.

當 h=10^{-3} 則 α_n 接近於 $(2n+1)\pi/2$ (n=0, 1 , 2 , 3…) =>

$$B_n \approx (1+\frac{1-h}{1+h})\frac{[\frac{(-1)^n}{(2n+1)^2\pi^2/4}]}{\frac{1}{2}} = 2\cdot(1+\frac{1-h}{1+h})\frac{(-1)^{n+1}}{(2n+1)^2\pi^2}$$

$$u = \sum_{n=1}^{\infty} B_n \sin[\frac{(2n+1)\pi}{2}x]\cdot\exp[-(\frac{(2n+1)\pi}{2})^2 t] + (\frac{1-h}{1+h})x+1$$

$$= \sum_{n=1}^{\infty} B_n \cos(n\pi x)\cdot\exp[-(\frac{(2n+1)\pi}{2})^2 t] + (\frac{1-h}{1+h})x+1$$

Type 2 BC: $u_t=u_{xx}$ 令 $u=p+f(x)$,且 $p_x(0,t) = p_x(1,t) =0$ (*Type2*) , =>
$u_x(0,t)=u_x(1,t)= (1-h)/(1+h)$
可解得 $u = \sum_{n=1}^{\infty} A_n\cos(n\pi x)\cdot\exp[-(n\pi)^2 t] + (\frac{1-h}{1+h})x+1$, 與 h=10^{-3} 答案相似, 所以在 h=10^{-3} 的情況下近似於 Type 2 BC.

4a) $u(0,t) = S(0,t) = \boxed{A(t) = g_1(t)}$

$$\left.\begin{array}{l} u_x(1,t) = S_x(1,t) = -A(t)+B(t) \\ hu(1,t) = hS(1,t) = hB(t) \end{array}\right\} -A(t)+(1+h)B(t) = g_2(t)$$

$$\Rightarrow \boxed{B(t) = \frac{g_1(t)+g_2(t)}{1+h}}$$

4b) Let $u(x,t) = U(x,t) + S(x,t) = U(x,t) + (1-x)A(t) + xB(t)$

Substituting into PDE, BCs and IC, we obtain

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} = \dfrac{\partial U}{\partial t} + (1-x)A'(t) + xB'(t) \\ BCs: \begin{cases} U(0,t) + A(t) = g_1(t) \\ \dfrac{\partial U(1,t)}{\partial x} + hU(1,t) - A(t) + (1+h)B(t) = g_2(t) \end{cases} \\ IC: U(x,0) + (1-x)A(0) + xB(0) = \phi(x) \end{cases}$$

Separating $U(x,t)$ and $A(t), B(t)$:

$$\boxed{\begin{cases} \dfrac{\partial^2 U}{\partial x^2} = \dfrac{\partial U}{\partial t} + (1-x)A'(t) + xB'(t) \\ BCs: \begin{cases} U(0,t) = 0 \\ \dfrac{\partial U(1,t)}{\partial x} + hU(1,t) = 0 \end{cases} \\ ICs: U(x,0) = \phi(x) - (1-x)g_1(0) \\ \qquad - x\dfrac{g_1(0) + g_2(0)}{1+h} \end{cases}} \& BCs: \begin{cases} A(t) = g_1(t) \\ -A(t) + (1+h)B(t) = g_2(t) \end{cases}$$

4c) No. Because PDE of $U(x,t)$ is nonhomogeneous.

5) 此題為 nonhomogeneous PDE，因此要把因變數轉換掉，變成 homogeneous PDE 再來求解。

Nonhomogeneous PDE: $u_t = c^2 u_{xx} + H,\ H > 0$

BCs: $u(0,\ t) = u(\pi,\ t) = 0$

ICs: $u(x,\ 0) = f(x)$

$L = \pi$

把題目已給的 $u = v - H(\dfrac{x(x-\pi)}{2C^2})$ 代入 nonhomogeneous PDE，可得

Homogeneous PDE: $v_t = c^2 v_{xx}$

BCs: $v(0,\ t) = v(\pi,\ t) = 0$

(令 $u(x,\ t) = v(x,\ t) + w(x)$ 代入 nonhomogeneous PDE，也可以求得 $w(x) = -H(\dfrac{x(x-\pi)}{2C^2})$)

利用分離變數法，得到公式 (3.4)，因此

$$v(x,\ t)=\sum_{n=1}^{\infty}A_n\sin(k_n x)\exp(-\lambda_n{}^2 t)$$

$$=\sum_{n=1}^{\infty}A_n\sin(nx)\exp(-(nc)^2 t)$$

$$u=v-H(\frac{x(x-\pi)}{2C^2})=\sum_{n=1}^{\infty}A_n\sin(nx)\exp(-(nc)^2 t)-H(\frac{x(x-\pi)}{2C^2})\quad ,n=1,2,3\ldots$$

由 IC 可以求得

$$A_n=\frac{2}{\pi}\int_0^{\pi}f(x)+H(\frac{x(x-\pi)}{2C^2})\sin(nx)dx$$

6) 令 $u(x,\ t)=v(x,\ t)+w(x)$ 代入

PDE: $v_t-c^2v_{xx}-c^2w_{xx}=Ne^{-\alpha x}\Rightarrow -c^2w_{xx}=Ne^{-\alpha x}$

BCs: $v(0,\ t)=v(L,\ t)=0$

$$\Rightarrow\begin{cases}u(0,\ t)=v(0,\ t)+w(0)\Rightarrow u(0,\ t)=w(0)\\ u(L,\ t)=v(L,\ t)+w(L)\Rightarrow u(L,\ t)=w(L)\end{cases}$$

IC: $v(x,\ 0)=f(x)-w(x)$

$$\Rightarrow\begin{cases}v(0,\ 0)=f(0)-w(0)=0\Rightarrow f(0)=w(0)\\ v(L,\ 0)=f(L)-w(L)=0\Rightarrow f(L)=w(L)\end{cases}$$

由 $-c^2w_{xx}=Ne^{-\alpha x}$ ，可得

$$w(x)=-\frac{N}{\alpha^2C^2}e^{-\alpha x}+ax+b$$

$$f(0)=w(0)=-\frac{N}{\alpha^2C^2}+b\Rightarrow b=f(0)+\frac{N}{\alpha^2C^2}$$

$$f(L)=w(L)=-\frac{N}{\alpha^2C^2}e^{-\alpha L}+aL+b$$

$$\Rightarrow a=\frac{1}{L}\left[\frac{N}{\alpha^2C^2}e^{-\alpha L}-f(0)+\frac{N}{\alpha^2C^2}+f(L)\right]$$

$$w(x)=-\frac{N}{\alpha^2C^2}e^{-\alpha x}+\frac{1}{L}\left[\frac{N}{\alpha^2C^2}e^{-\alpha L}-f(0)+\frac{N}{\alpha^2C^2}+f(L)\right]x+f(0)+\frac{N}{\alpha^2C^2}$$