Homework Solution #1

1.

令方程式為 $a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial u}{\partial x}+d\frac{\partial^2 u}{\partial t^2}+e\frac{\partial u}{\partial t}+fu=0$ ，且 U1,U2 為線性獨立的二解，將 U=C1U1+C2U2 代入方程式：

$$a\frac{\partial^2}{\partial x^2}(c_1u_1+c_2u_2)+b\frac{\partial}{\partial x}(c_1u_1+c_2u_2)+d\frac{\partial^2}{\partial t^2}(c_1u_1+c_2u_2)+e\frac{\partial}{\partial t}(c_1u_1+c_2u_2)+f(c_1u_1+c_2u_2)$$

$$\Rightarrow c_1\left(a\frac{\partial^2 u_1}{\partial x^2}+b\frac{\partial u_1}{\partial x}+d\frac{\partial^2 u_1}{\partial t^2}+e\frac{\partial u_1}{\partial t}+fu_1\right)+c_2\left(a\frac{\partial^2 u_2}{\partial x^2}+b\frac{\partial u_2}{\partial x}+d\frac{\partial^2 u_2}{\partial t^2}+e\frac{\partial u_2}{\partial t}+fu_2\right)$$

$=c_1\cdot 0+c_2\cdot 0=0$ 得證

2a

1) 無 horizontal motion ⇒ $T_2\cos\beta = T_1\cos\alpha = T$
2) 由於為 nonuniform 介質(在 x 和 x+△x 兩點間之 mass density 不同)，因此必須修改成(vertical force) / ρ = (△x)·acceleration ⇒
$$\frac{T_2\sin\beta}{\rho(x+\Delta x)}-\frac{T_1\sin\alpha}{\rho(x)}=(\Delta x)\times u_{tt}$$
3)
$$\frac{(2)}{(1)}\Rightarrow\frac{\tan\beta}{\rho(x+\Delta x)}-\frac{\tan\alpha}{\rho(x)}=\frac{\Delta x}{T}u_{tt}\Rightarrow\frac{C^2(x+\Delta x)\cdot u_x(x+\Delta x)}{\Delta x}-\frac{C^2(x)\cdot u_x(x)}{\Delta x}=u_{tt}$$
$$\Rightarrow u_{tt}=\frac{\partial}{\partial x}\left[C^2(x)\cdot u_x(x)\right],\ \text{其中 } C^2(x)=\frac{T}{\rho(x)}$$

2b

1) 無 horizontal motion ⇒ $T_2\cos\beta = T_1\cos\alpha = T$
2) 考慮 gravitational force (*F=mg*) ⇒
$$T_2\sin\beta - T_1\sin\alpha=(\rho\,\Delta x)\times u_{tt}-(\rho\,\Delta x)\times g$$
3)
$$\frac{(2)}{(1)}\Rightarrow\tan\beta-\tan\alpha=\frac{\rho\Delta x}{T}(u_{tt}-g)\Rightarrow C^2\left[\frac{u_x(x+\Delta x)}{\Delta x}-\frac{u_x(x)}{\Delta x}\right]=u_{tt}-g$$
$$\Rightarrow u_{tt}=C^2u_{xx}-g\ \text{其中 } C^2=\frac{T}{\rho}$$

3a)

Initial displacement $\phi(x) = 0.01\sin(2\pi x)$

Initial velocity $\gamma(x) = 0$

BCs: $u(0,t) = 0$, $u(L,t) = 0$

ICs: $u(x,0) = \phi(x)$, $u_t(x,0) = \gamma(x) = 0$

L=0.5m

Separation of variables

Let $u(x,t) = X(x)T(t)$, substitute it into PDE$\Rightarrow \dfrac{T''}{c^2T} = \dfrac{X''}{X} = a \Rightarrow \begin{cases} X'' - aX = 0 \\ T'' - c^2 aT = 0 \end{cases}$.

Let $a = -k^2 < 0 \Rightarrow X'' + k^2 X = 0 \Rightarrow X_n = A\cos(k_n x) + B\sin(k_n x)$,

$k_n = \dfrac{n\pi}{L}, n = 1.2..$

By BCs: $u(0,t) = X(0)T(t) = 0, u(L,t) = X(L)T(t) = 0 \Rightarrow A = 0$, $X_n = B\sin(k_n x)$

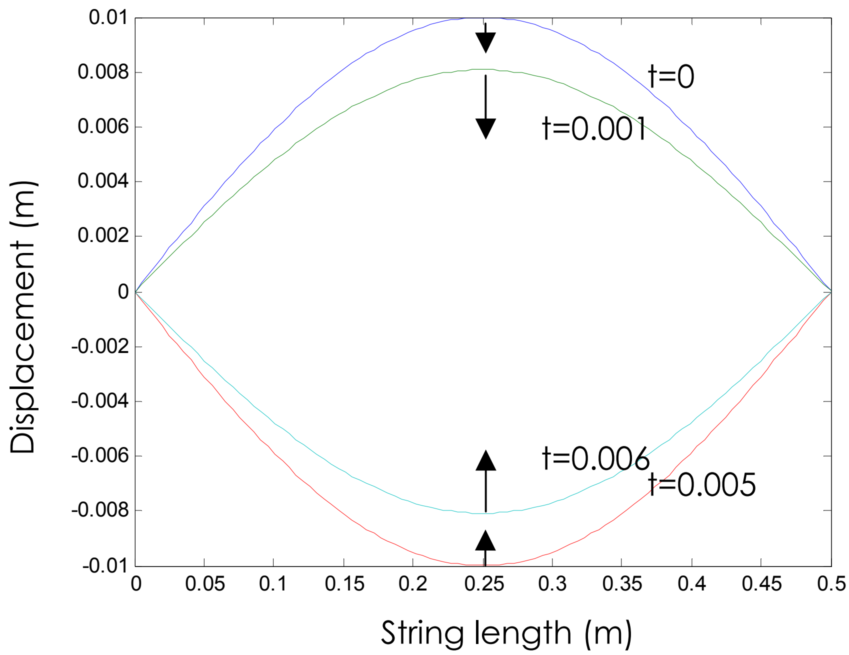
$T'' + \omega^2 T = 0 \Rightarrow T_n = C\cos(\omega_n t) + D\sin(\omega_n t)$, $\omega_n = \dfrac{n\pi c}{L}$

So $u_n(x,t) = X_n T_n = B\sin(k_n x)[C\cos(\omega_n t) + D\sin(\omega_n t)]$

By ICs: $\begin{cases} u(x,0) = X(x)T(0) = B\sin(k_n x)[C\cos 0 + D\sin 0] = \phi(x) \\ u_t(x,0) = X(x)T'(0) = B\sin(k_n x)[-C\sin 0 + D\cos 0] = 0 \end{cases}$

$\Rightarrow C = 1, D = 0 \Rightarrow T = \cos(\omega t) = \cos(2\pi ct)$ & $X = 0.01\sin(2\pi x), n = 1$

$u(x,t) = XT = 0.01\sin(2\pi x)\cos(2\pi ct)$, $c = \sqrt{\dfrac{T}{\rho}} = \sqrt{\dfrac{100kg \cdot m \cdot s^{-2}}{0.01kg \cdot m^{-1}}} = 100\dfrac{m}{s}$



3b)

$$n=1 \Rightarrow \nu_1 = \frac{\sqrt{T/\rho}}{2L} = \frac{\sqrt{100kg\cdot m\cdot s^{-2}/0.01kg\cdot m^{-1}}}{2\times 0.5m} = 100Hz$$

3c)

In (3b), $\nu' = \sqrt{2}\nu_1 = 141Hz$.

So in (3a), $u(x,t) = 0.01\sin\left(2\sqrt{2}\pi x\right)\cos\left(2\sqrt{2}\pi ct\right)$

3d) $u_{tt} = c^2 u_{xx}$, $c^2 = \frac{T}{\rho} = \frac{100\text{Nt}}{10\text{g/m}} = \frac{10^5\,\text{g-m}/s^2}{10\text{g/m}} = 10^4\,\text{m}^2/\text{s}^2$

Two BCs: $u(0,t)=0,\ u(L,t)=0$

Two ICs: $u(x,0)=\phi(x),\ u_t(x,0)=0$, $\phi(x)=0.08x-0.16x^2$

由公式(2-3)令 $u(x,t)=\sum_{n=1}^{\infty}[A_n\cdot\cos(w_n t)+B_n\cdot\sin(w_n t)]\sin(k_n x)$

$$w_n = \frac{n\pi c}{L},\ k_n = \frac{n\pi}{L},\ n=1,2,3\ldots,\ L\text{=0.5m}$$

再由公式(2-4)可得

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cdot\sin(k_n x)dx$$
$$= \frac{2}{L}[\int_0^L 0.08x\cdot\sin(\frac{n\pi}{L}x)dx + \int_0^L 0.16x^2\cdot\sin(\frac{n\pi}{L}x)dx]$$
$$= 4[0.08\cdot(\frac{-x\cos(2n\pi x)}{2n\pi} + \frac{\sin(2n\pi x)}{4n^2\pi^2}\Big|_0^{0.5})$$
$$-0.16\cdot(\frac{-x^2\cos(2n\pi x)}{2n\pi} + \frac{2x\sin(2n\pi x)}{4n^2\pi^2} + \frac{2\cos(2n\pi x)}{8n^3\pi^3}\Big|_0^{0.5})]$$
$$= \frac{0.32}{n^3\pi^3}(n = 1,3,5...)$$

$$B_n = 0$$

把 A_n，B_n 代入公式(2-3)即可求出

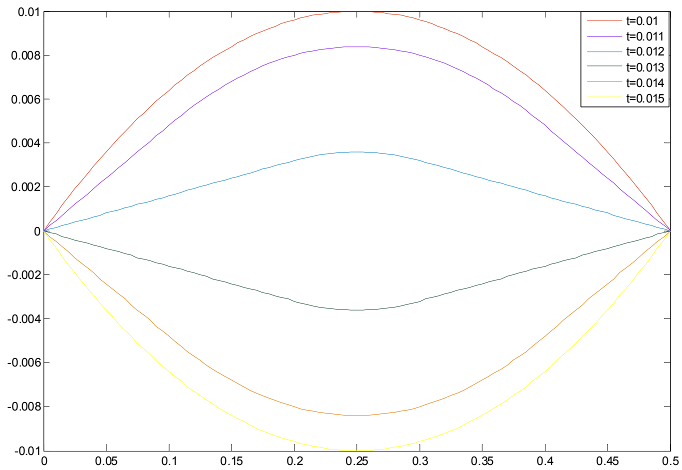
$$u(x,t) = \frac{0.32}{\pi^3}\sum_{n=1}^{\infty}\frac{\cos[200(2n-1)\pi t]}{(2n-1)^3}\sin[2(2n-1)\pi x]$$

Matlab Code:

```
clear all;
clc;
t=linspace(1e-2,1.5e-2,6);            % 要看其它時刻，請自行改參數
n=1:100;           % 請自行選擇要疊合多少個頻率
L=0.5;          % 繩子長度
x=linspace(0,L,101);
phi= 0.16.*x.*(0.5-x);          %IC 給的條件，用來驗證
uxt=zeros(length(t), length(x));
for beta= 1:length(t);
      for afa= 1:length(x);
            uxt(beta,afa)= (0.32/pi^3).*
sum(      ( cos(200.*(2.*n-1).*pi.*t(beta))./((2.*n-1).^3) ).*( sin(2.*(2.*n-1).*p
i.*x(afa)) )      );
      end
end
```

```
% figure; plot(x, phi, x, uxt, 'o'); xlabel('length(m)'); title('u(x,t) at
t=0');legend(['phi(x)'],['u(x,0)']);

for beta= 1:length(t);
    plot(x, uxt(beta,:));
    hold on;
end
xlabel('length(m)'); title('u(x,t)');
```



3e) 有無限多個共振頻率，

$$w_n = 2\pi v_n = 200(2n-1)\pi$$
$$\Rightarrow v_n = 100(2n-1)\ (\text{Hz}) \quad n = 1,2,3...$$

4)

Question:

$$\begin{bmatrix} u_{tt} = c^2 u_{xx} & \\ \begin{cases} u(0,t) = 0 \\ u_x(0,t) = 0 \end{cases} & \& \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases} \end{bmatrix}$$

Let $u(x,t) = X(t)T(t)$, substitute it into $u_{tt} = c^2 u_{xx}$

$$\Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = -p^2(const.)$$

step1. $X''+p^2X = 0 \,\&\, X(0) = 0 \,\&\, X'(L) = 0$ (review ODE)

$$\Rightarrow X_n(x) = \sin p_n x \quad \& \quad \boxed{p_n = \frac{(2n+1)\pi}{2L}, n = 0,1,2,3......}$$

step2. $T''+c^2p^2T = 0$ (review ODE)

$$\Rightarrow T_n = c_1\cos(cp_nt) + c_2\sin(cp_nt)$$

So, $\boxed{u(x,t) = \sum_{n=0}^{\infty}\left[A_n\cos(cp_nt) + B_n\sin(cp_nt)\right]\sin(p_nx)}$

Substituting ICs into $u(x,t)$

$$u(x,0) = \sum_{n=0}^{\infty} A_n\sin(p_nx) = f(x)$$

$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin(p_nx)dx}$ (review Fourier)

$$u_t(x,t) = \sum_{n=0}^{\infty} cp_n\left[-A_n\sin(cp_nt) + B_n\cos(cp_nt)\right]\sin p_nx$$

$$u_t(x,0) = \sum_{n=0}^{\infty} cp_nB_n\sin(p_nx) = 0 \Rightarrow \boxed{B_n = 0}$$

5)

From D'Alembert method, we know the general solution is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

Where $f(x) = 0$ & $g(x) = \begin{cases} 1, -1 \le x \le 1 \\ 0, otherwise \end{cases} = H(x+1) - H(x-1)$, [H is Step Func.]

So $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\left[H(\tau+1) - H(\tau-1)\right]d\tau$

$x - ct \le \tau < -1 \Rightarrow 0$

$\tau = -1 \Rightarrow 0$

$-1 < \tau < 1 \Rightarrow \frac{1}{2c}[\tau + 1]$ (you have to match $u(\tau = -1) = 0$)

$\tau = 1 \Rightarrow \frac{1}{c}$

$1 < \tau \le x + ct \Rightarrow \frac{1}{c}$

So, $u(x,t) = \left\{ \frac{1}{2c}[\tau + 1][H(\tau + 1) - H(\tau - 1)] + \frac{\tau}{c} H(\tau - 1) \right\} \Bigg|_{x-ct}^{x+ct}$

$$\boxed{\begin{aligned} &= \frac{1}{2c}[x + ct + 1][H(x + ct + 1) - H(x + ct - 1)] + \frac{x + ct}{c} H(x + ct - 1) \\ &- \frac{1}{2c}[x - ct + 1][H(x - ct + 1) - H(x - ct - 1)] - \frac{x - ct}{c} H(x - ct - 1) \end{aligned}}$$