Boundary-value Problems (BVPs)

- Eigenvalues & eigenfunctions
- Sturm-Liouville problem (11.4)

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- Eigenvalues & eigenfunctions
  - An easy example
  - Analogy to linear algebra

E.g. A BVP (1)

- 2nd-order LH ODE: $y''(x) + \lambda y(x) = 0$
- Two BCs: $y(0) = 0$, $y(L) = 0$.
- Substituting $y = e^{mx}$ into the ODE gives an auxiliary equation: $m^2 + \lambda = 0$.
- Case 1 ($\lambda = -\alpha^2 < 0$): $m = \pm\alpha$ (distinct real roots).

  $\Rightarrow y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$
- BC1: $y(0) = c_1 + c_2 = 0$ ... (1)
- BC2: $y(L) = c_1 e^{\alpha L} + c_2 e^{-\alpha L} = 0$ ... (2)

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ e^{\alpha L} & e^{-\alpha L} \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ unique trivial solution: $c_1 = c_2 = 0$, $y(x) = 0$.

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E.g. A BVP (2)

- Case 2 ($\lambda = 0$): $m = 0$ (repeated root).

  $\Rightarrow y(x) = (c_1 + c_2 x) \times e^{mx} = c_1 + c_2 x$

- BC1: $y(0) = c_1 = 0$ ... (1)
- BC2: $y(L) = c_1 + c_2 L = 0$ ... (2)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ unique trivial solution: $c_1 = c_2 = 0$, $y(x) = 0$.

E.g. A BVP (3)

- Case 3 ($\lambda = \beta^2 > 0$): $m = \pm j\beta$ (complex conjugate roots).

  $\Rightarrow y(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$
- BC1: $y(0) = c_1 = 0$ … (1)
- BC2: $y(L) = c_2\sin(\beta L) = 0$ … (2)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \sin(\beta L) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  $\Rightarrow$ infinitely many nontrivial solutions ($c_2 \neq 0$) may exist if $\beta L = n\pi$.
- Nontrivial solutions and the ODE's parameter λ are simultaneously solved: (1) $\lambda_n = (\beta_n)^2 = (n\pi/L)^2$ (eigenvalues), (2) $y_n(x) \propto \sin(n\pi x/L)$ (eigenfunctions).

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Remarks

- Solutions $\{y_n(x) \propto \sin(n\pi x/L)\}$ are orthogonal functions on x = [0,L], i.e. $(y_m(x),y_n(x)) = 0$ if $m \neq n$.
- Any function f(x) on the interval $x \in [0,L]$ can be expanded in terms of $\sin(n\pi x/L)$ … Fourier series!
- Eigenvalue problem in linear algebra: given matrix [A], try to find eigenvectors **x** and eigenvalues λ such that $[A]\mathbf{x} = \lambda\mathbf{x}$.
- E.g.

$$[A]=\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}, \Rightarrow \left(\lambda_1 = 6, \vec{x}_1 \propto \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right); \left(\lambda_2 = 1, \vec{x}_2 \propto \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

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- Sturm-Liouville problem
  - Regular problem
  - Singular problem

Regular Sturm-Liouville problem

- 2nd-order LH ODE: $r(x)y'' + r'(x)y' + [q(x)+\lambda p(x)]y = 0$, where $r(x) > 0$, $r'(x)$, $p(x) > 0$, $q(x)$ are continuous on the interval $x \in [a,b]$.
- Homogeneous separated BCs:

$$\begin{cases} A_1 \times y(a) + B_1 \times y'(a) = 0, \\ A_2 \times y(b) + B_2 \times y'(b) = 0, \end{cases} \quad \text{where } A_{1,2}, B_{1,2} \text{ are given.}$$

- E.g. $\{y''(x) + \lambda y(x) = 0,\ y(0) = 0,\ y(L) = 0\}$ corresponds to: $\{r(x) = 1(>0),\ q(x) = 0,\ p(x) = 1(>0),\ a = 0,\ b = L,\ A_1 = A_2 = 1,\ B_1 = B_2 = 0\}$. $\Rightarrow$ Regular Sturm-Liouville problem.

Properties

- There're infinitely many eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, and $\lambda_\infty \to \infty$. E.g. $\lambda_n = (n\pi/L)^2$ in p5.
- Each eigenvalue λ_n corresponds to a unique eigenfunction $y_n(x)$. E.g. $y_n(x) \propto \sin(n\pi x/L)$ in p5.
- Eigenfunctions $\{y_n(x)\}$ are orthogonal with respect to the weight function $p(x)$ (see proof in p418), i.e.

$$\int_a^b p(x) y_m(x) y_n(x) dx = 0, \text{ if } m \neq n.$$

- E.g. $\int_0^L \sin(\frac{m\pi x}{L}) \times \sin(\frac{n\pi x}{L}) dx = 0, \text{ if } m \neq n$ in p5.

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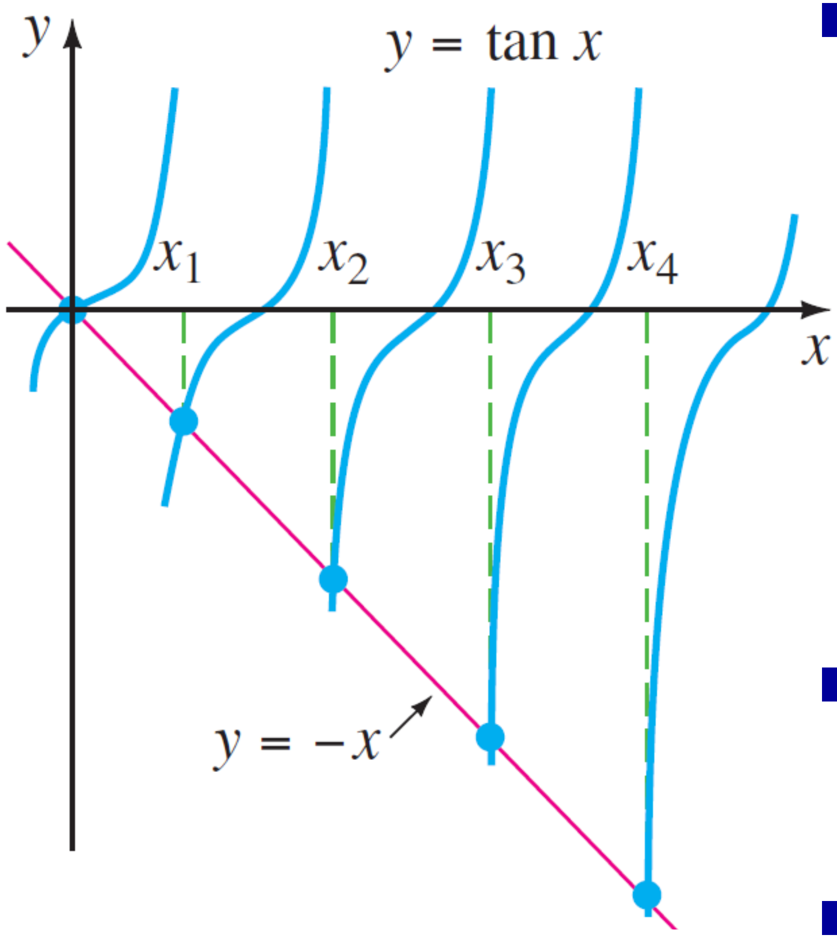
Example 2 (1)

- ODE: $y''(x) + \lambda y(x) = 0; \Rightarrow \{r(x) = 1, p(x) = 1, q(x) = 0.\}$
- BCs: $y(0) = 0$, $y(1) + y'(1) = 0$;
- Auxiliary equation: $m^2 + \lambda = 0$.
- Case 1 or 2 only gives trivial solution $y(x) = 0$.
- Case 3 ($\lambda = \beta^2 > 0$): $m = \pm j\beta$.

  $\Rightarrow y(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$
- BC1: $y(0) = c_1=0$, $\Rightarrow y(x) = c_2\sin(\beta x)$, $y'(x) = \beta c_2\cos(\beta x)$
- BC2: $y(1) + y'(1) = c_2\sin(\beta) + \beta c_2\cos(\beta) = 0$, $\tan\beta = -\beta$

Example 2 (2)



- BCs result in a nonlinear algebraic (eigenvalue) equation: $\tan\beta = -\beta$, whose infinitely many roots $\{\beta_n\}$ can be obtained by graphical method.
- Eigenfunctions: $y_n(x) \propto \sin(\beta_n x)$, orthogonal set
- Eigenvalues $\lambda_n = (\beta_n)^2 = \{2.2088^2, 4.9132^2, \ldots\}$

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Singular Sturm-Liouville problem

- Same ODE: $r(x)y'' + r'(x)y' + [q(x)+\lambda p(x)]y = 0$.
- 4 variations of BCs that can lead to orthogonal eigenfunctions bounded on $x \in [a,b]$, i.e.

$$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \text{ if } \ m \neq n:$$

1) If $r(a) = 0$, $\Rightarrow$ only need one BC at the other boundary point $x = b$: $A_2y(b) + B_2y'(b) = 0$;
2) If $r(b) = 0$, $\Rightarrow$ only need one BC: $A_1y(a) + B_1y'(a) = 0$;
3) If $r(a) = r(b) = 0$, $\Rightarrow$ no BC at $x = a$ or $x = b$ is needed;
4) If $r(a) = r(b)$, $\Rightarrow$ need periodic BCs: $y(a) = y(b)$, $y'(a) = y'(b)$.

Legendre's equation

- Consider $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$:

  $\Rightarrow r(x) = 1 - x^2 > 0$ on $x \in [-1,1]$, $r'(x) = -2x$, $q(x) = 0$, $p(x) = 1 > 0$, $\lambda = n(n+1)$.

- Since $r(\pm 1) = 0$, according to Case 3, no BC is needed at $x = \pm 1$ while the eigenfunctions $\{P_n(x)\}$ (Legendre polynomials) remain orthogonal with respect to $p(x) = 1$, i.e. $\int_a^b P_m(x) \times P_n(x)dx = 0$, if $m \neq n$.

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Bessel's equation (1)

- $x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0$, where n is a given integer, α^2 is some eigenvalue to be determined.
- It looks not a Strum-Liouville problem at the first glance, for $(x^2)' = 2x \neq x$.
- Surprisingly, it's indeed a Strum-Liouville problem if we divide x for both sides of the equality:

$$xy'' + 1y' + (\alpha^2x - n^2/x)y = 0,$$

$\Rightarrow r(x) = x,\ r'(x) = 1,\ p(x) = x,\ q(x) = -n^2/x,\ \lambda = \alpha^2.$

Bessel's equation (2)

- In Ch 6, we have obtained $y(x) = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$.
- Since $\lim_{x\to 0}|Y_n(\alpha x)| = \infty$, bounded eigenfunctions are $\{J_n(\alpha x)\}$.
- Since $r(x) = x$, $\Rightarrow$ $r(0) = 0$, a singular Sturm-Liouville problem of Case 1.
- Eigenvalues $\lambda = \alpha^2$ are determined by eigenvalue equation arising from BC at $x = b$: $A_2 J_n(\alpha b) + B_2 \alpha J'_n(\alpha b) = 0$, $\Rightarrow \lambda_i = (\alpha_i)^2$, $i = 1, 2, 3, \ldots$
- Eigenfunctions $\{J_n(\alpha_i x)\}$ are orthogonal with respect to $p(x) = x$: $\int_0^b x J_n(\alpha_i x) \times J_n(\alpha_j x)\,dx = 0, \text{ if } i \neq j.$