Chapter 1
Introduction

- Usefulness of ODEs
- Fundamentals
- Physical modeling

Edited by Shang-Da Yang (楊尚達)
National Tsing Hua University, Taiwan

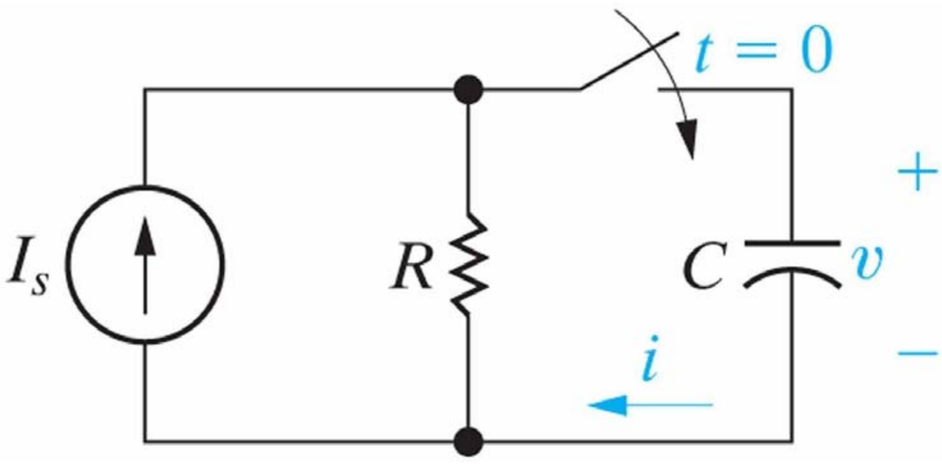
Usefulness of ODEs

- Electric circuits
- Signal processing
- Quantum physics

Why are you here?

(vectortoons.com/)

Circuit analysis



- By Kirchhoff current law:

$$I_s = \frac{v(t)}{R} + C\frac{d}{dt}v(t),$$

$$\Rightarrow \frac{dv}{dt} = -\frac{v}{RC} + \frac{I_s}{C},$$

which is a linear, 1st-order ordinary differential equation (ODE) of constant coefficients.

- Under the initial condition of $v(0) = V_0$, we can get a unique solution: $v(t) = V_f + (V_0 - V_f)\times\exp(-t/\tau)$, where $V_f = I_sR$ (final voltage), $\tau = RC$ (time constant).

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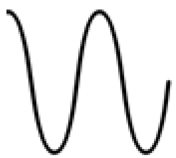
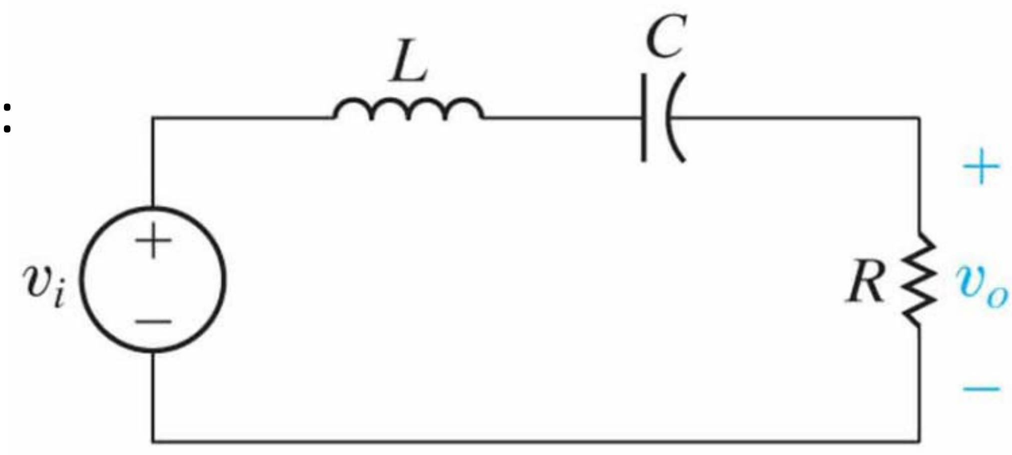
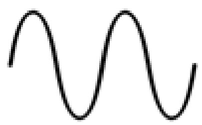
Signal filter (1)

Input voltage:
$V_i \times \cos\omega t$





Output voltage:
$V_o \times \cos(\omega t+\phi)$



- Output voltage $v_0(t)$ must have the same angular frequency ω as the input $v_i(t)$, but different amplitude V_o and phase ϕ.
- V_o and ϕ can be obtained by solving a linear, 2nd-order ODE of constant coefficients.

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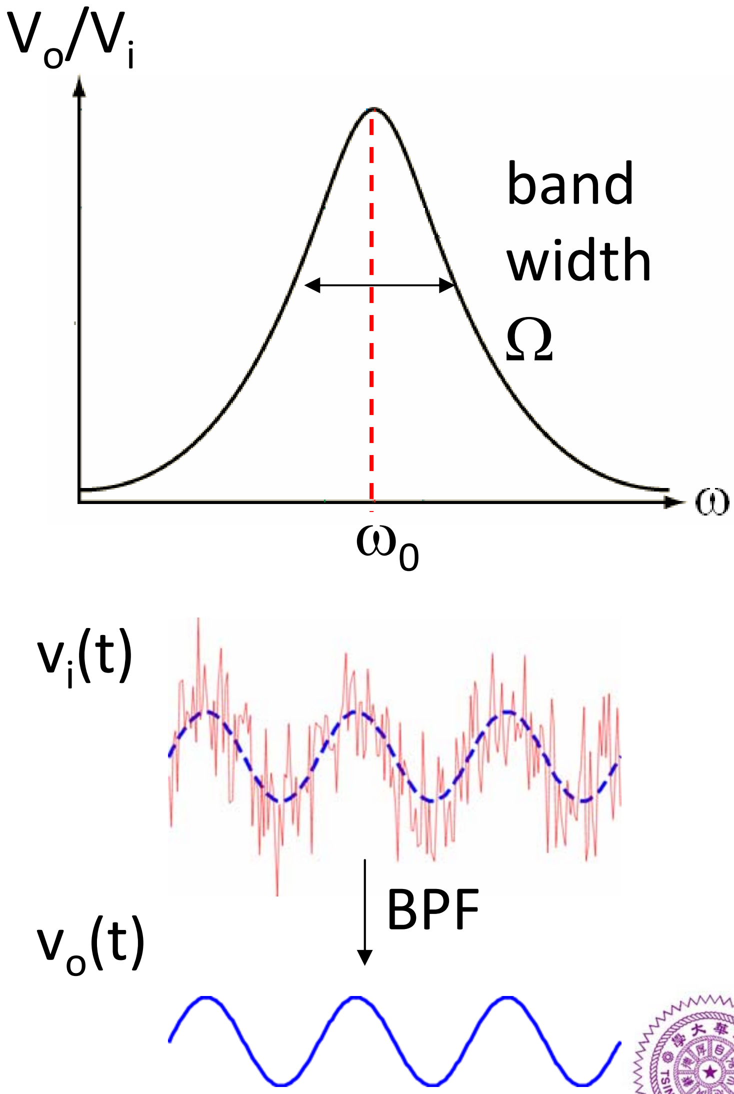
Signal filter (2)

- In fact, V_o/V_i varies with ω (Fourier transform).
- Only frequencies in the "pass-band", $|\omega-\omega_0| < \Omega/2$, can pass through the band-pass filter (BPF).
- White noise (consisting of many ω's) can be rejected by BPF, recovering a clear sinusoid at $\omega \approx \omega_0$.



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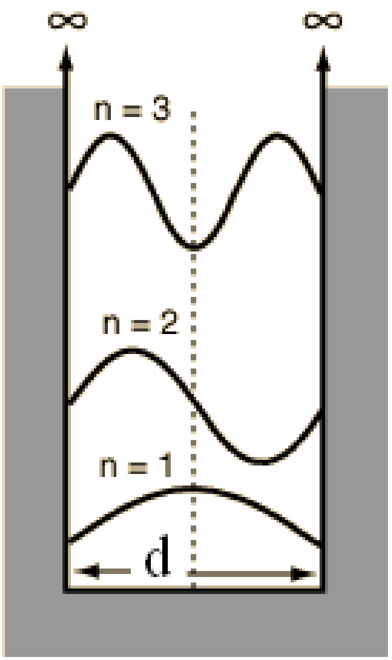
Quantum physics: 1D potential well

- ODE: $-A\psi''(x) = E\psi(x)$; where $\psi(x)$ is the "wavefunction" of a particle (i.e. electron) of energy E.
- Boundary conditions: $\psi(0) = 0$, $\psi(d) = 0$, for particle can only stay in the well.
- Eigen-(wave)functions: $\psi_n(x) = \{\sin(k_n x)\}$, where $k_n = n\pi/d$, $n = 1, 2, \ldots$
- Eigen-values (quantized energies): $E_n = \{(\hbar k_n)^2/(2m)\} \propto n^2$.



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- Fundamentals

What's algebraic equation?

- E.g. $x^2 + 5x + 4 = 0$
- Feature: Unknown x is a "number".

What's differential equation?

- E.g. $y''(x) + 2y'(x) + y(x) = 0$
- Feature: Unknown $y(x)$ is a "function"

DEFINITION 1.1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

Ordinary vs. Partial DEs?

- E.g. $x'(t) + y'(t) = 2x(t) + y(t)$
- It's an ODE for there is only one independent variable "t".

- E.g. $\partial^2 u/\partial t^2 = c^2 \times (\partial^2 u/\partial x^2)$, where $u = u(t,x)$.
- It's a partial DE (PDE) for there are two independent variables "t" (time) and "x" (space).
- Actually, this is the "1D wave equation".
- PDEs are beyond the scope of this course.

Order of ODEs?

- E.g. $y''(x) + 5\times[y'(x)]^3 - 4y(x) = e^x$
- It's a 2nd-order ODE, for the highest order of derivative is 2 (y'').

Linear vs. nonlinear ODEs?

- E.g. $a_1(x) \times y'(x) + a_0(x) \times y(x) = g(x)$
- It's a linear 1st-order ODE.
- Features:
  - Dependent variable y and its derivative y' are of first degree.
  - Coefficients a_0, a_1 do not depend on "y"
- E.g. $(1-y) \times y'(x) + 2y(x) = e^x$, and $d^4y/dx^4 + y^2 = 0$ are nonlinear ODEs. (Why?)
- Nonlinear ODEs behave wildly.

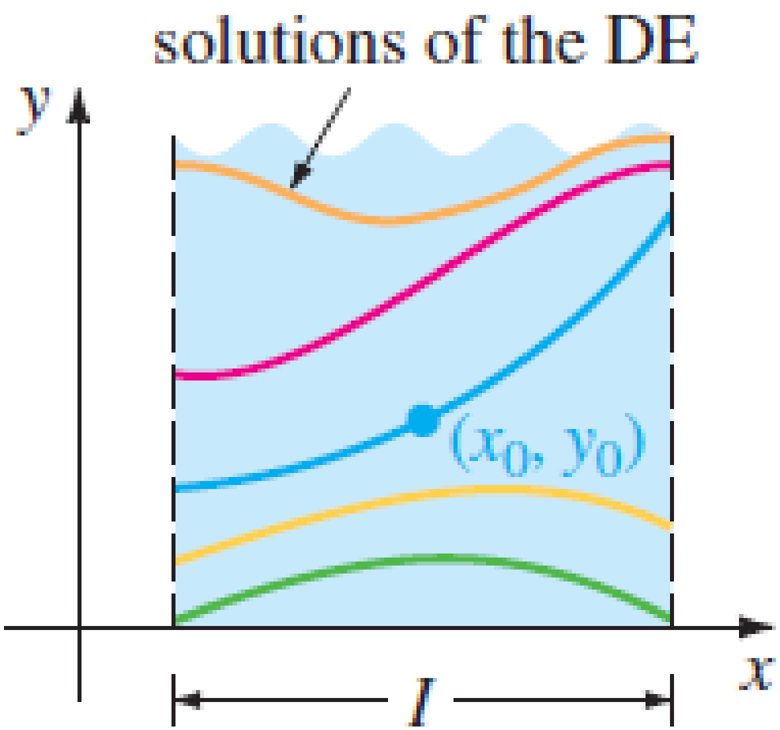
Nonlinear ODEs: butterfly effect

- An infinitesimal displacement of initial condition may cause drastically different solutions.

(dip9.aaschool.ac.uk)

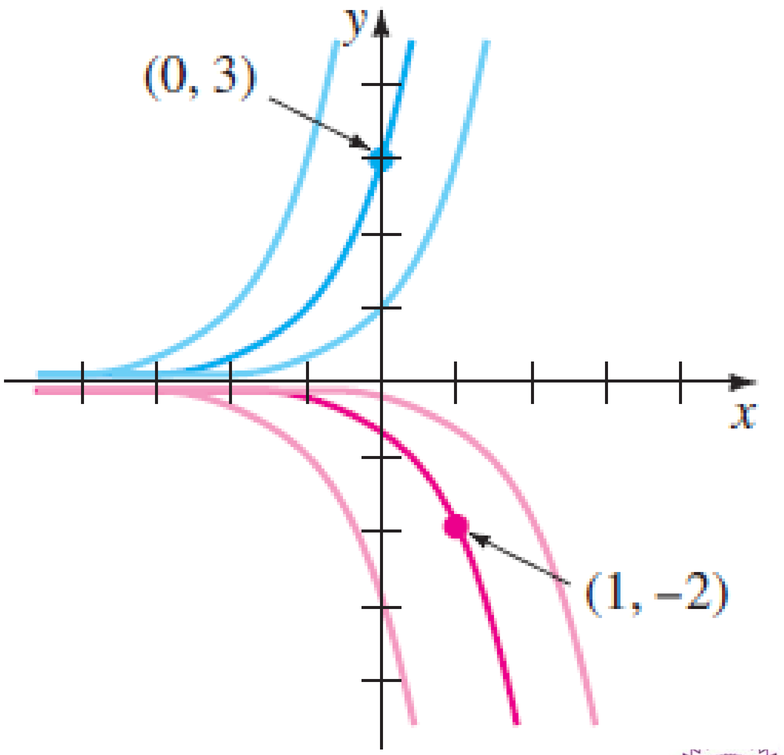
1st-order initial-value problems (IVPs)

- E.g. $\begin{cases} \dfrac{dy}{dx} = f(x, y) & \text{... 1st-order ODE} \\ y(x_0) = y_0 & \text{... need one initial condition (IC)} \end{cases}$



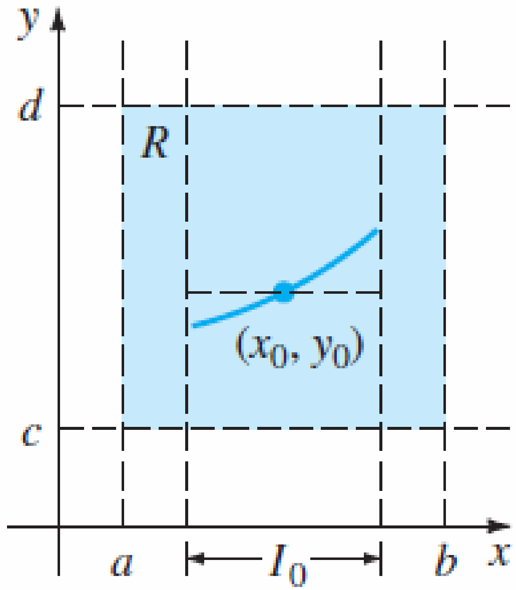
Example: Two 1st-order IVPs

- E.g. ODE: $y' = y$, $\Rightarrow$ family of solutions $y = c \times e^x$. (verify it now, to be solved systematically latter)
- IC-1: $y(0) = 3$, $\Rightarrow$ $c = 3$;
- IC-2: $y(1) = -2$, $\Rightarrow$ $c = -2/e$;



Existence of a unique solution

- 1st-order IVP $\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$

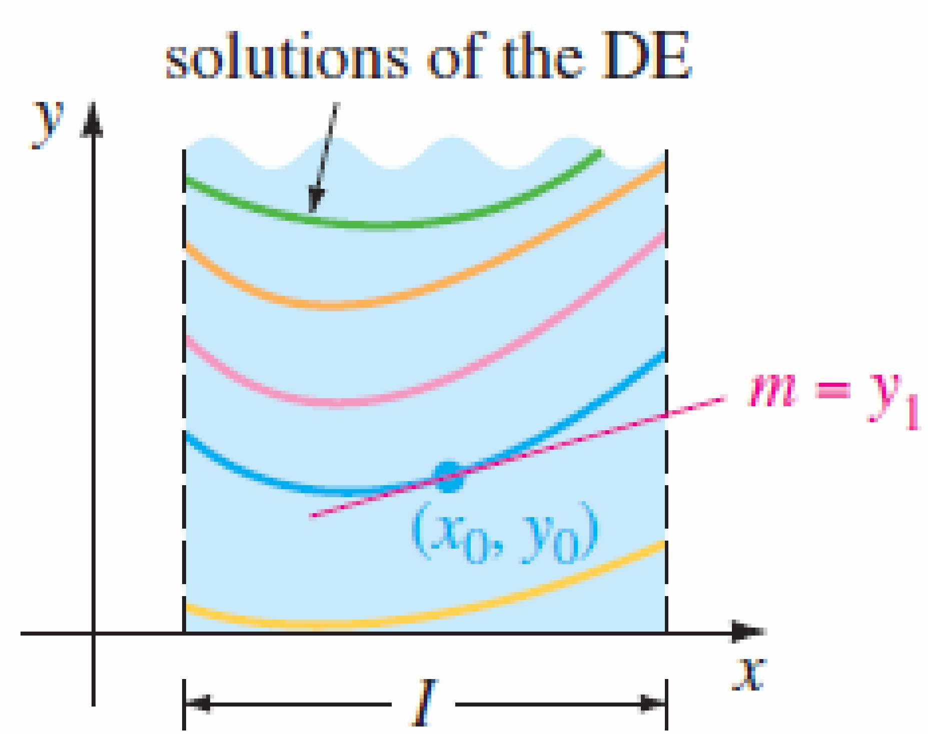


THEOREM 1.2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f / \partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem (2).

2nd-order IVPs

- E.g. $\begin{cases} \dfrac{d^2y}{dx^2} = f(x, y, y') & \text{... 2nd-order ODE} \\ y(x_0) = y_0,\ y'(x_0) = y_1 & \text{... need two ICs} \end{cases}$

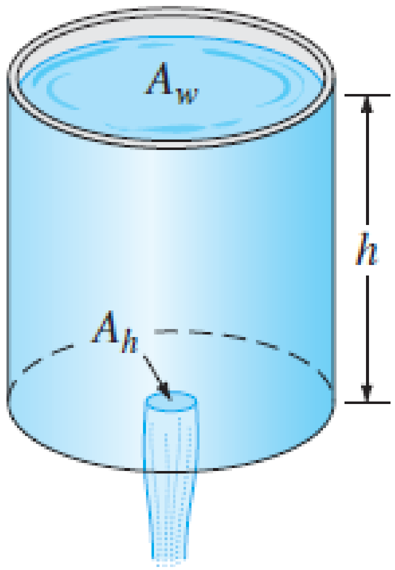


❑ Physical modeling

- Draining of a water tank
- Falling bodies

Modeling of physical problems

- E.g. Draining a tank.
- Torricelli's law: the speed v of efflux of water is the same as that of a free drop, $\Rightarrow$ $v = \sqrt{(2gh)}$ (why?)



- Volume of water $V(t) = A_w \times h(t)$ satisfies with:

$$V'(t) = -A_h \times \sqrt{(2gh)} \text{ (why?).}$$

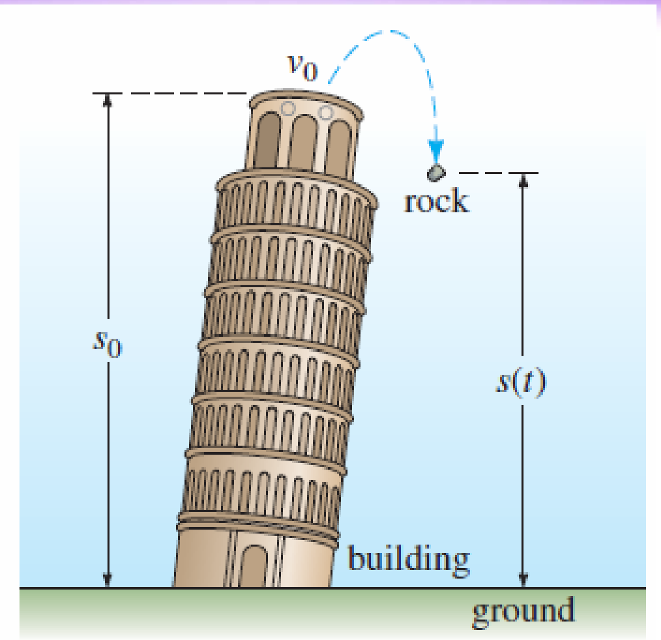
- We got an ODE of unknown h(t):

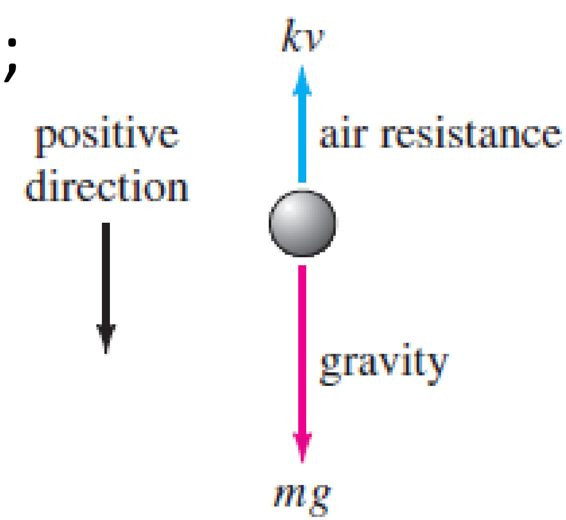
$$h'(t) = -(A_h/A_w) \times \sqrt{(2gh)}.$$

Falling bodies

- Newton's law: net force $F = m \times a$.
- Gravity force: $F_1 = m \times g$.
- Air resistance: $F_2 = -k \times v$.
- $F = F_1 + F_2 = mg - kv = ma$.
- By $a = v'(t)$, $\Rightarrow$ 1st-order ODE:

  $m \times v'(t) = mg - kv$; $v'(t) = -(k/m)v + g$;
- By $v = s'(t)$, $a = s''(t)$, $\Rightarrow$ 2nd-order ODE: $s''(t) = -(k/m) \times s'(t) + g$





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