Chapter 14
Fourier Transforms

- Fourier integral (14.3)
- Fourier transforms (14.4)
- Fourier transform properties
- Applications

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❑ Fourier integral

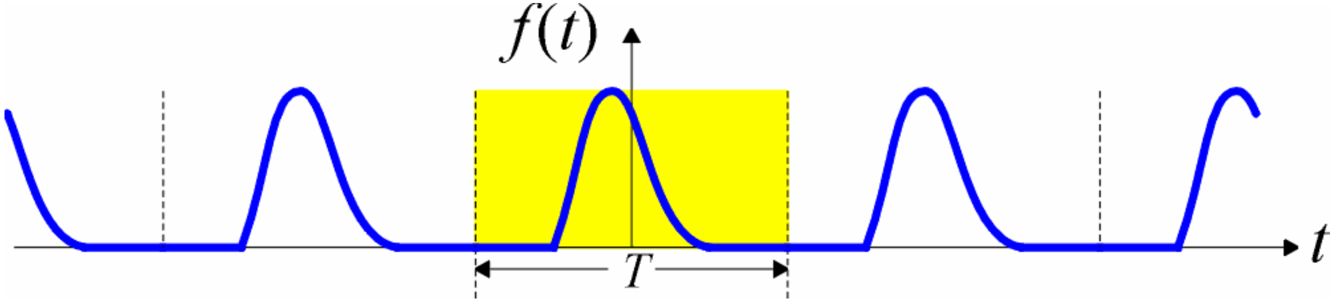
- Generalization of Fourier series
- Examples

Fourier series

- A periodic function f(t) of period T can be expanded by Fourier series:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right],$$

$$\text{where}\ \begin{cases} a_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}\left[f(t)\times\cos(n\omega_0 t)\right]dt, \\ b_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}\left[f(t)\times\sin(n\omega_0 t)\right]dt. \end{cases}$$



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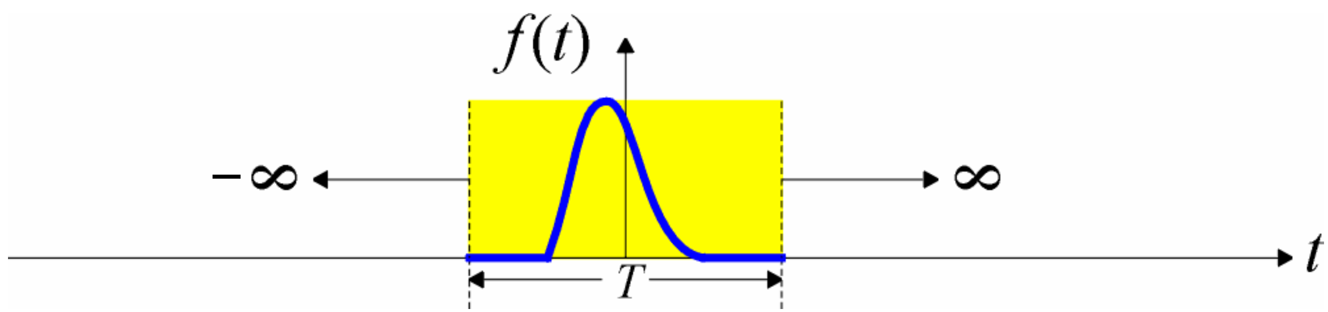
Non-periodic function

- A non-periodic function f(t) can be obtained by increasing the period T to infinity ($T \to \infty$):



- In the limit of $T \to \infty$,

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = 0,$$

as long as f(t) is integrable: $\left|\int_{\infty}^{\infty} f(t)dt\right| < \infty.$

Cosine terms (1)

- By defining $\Delta\omega = \omega_0 = 2\pi/T$, $\omega_n = n\omega_0 = n\Delta\omega$, we can rewrite the cosine terms of the Fourier series as:

$$\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) = \sum_{n=1}^{\infty}\left[\frac{2}{T}\int_{-T/2}^{T/2} f(\tau)\cos(n\omega_0\tau)d\tau\right]\cos(n\omega_0 t)$$

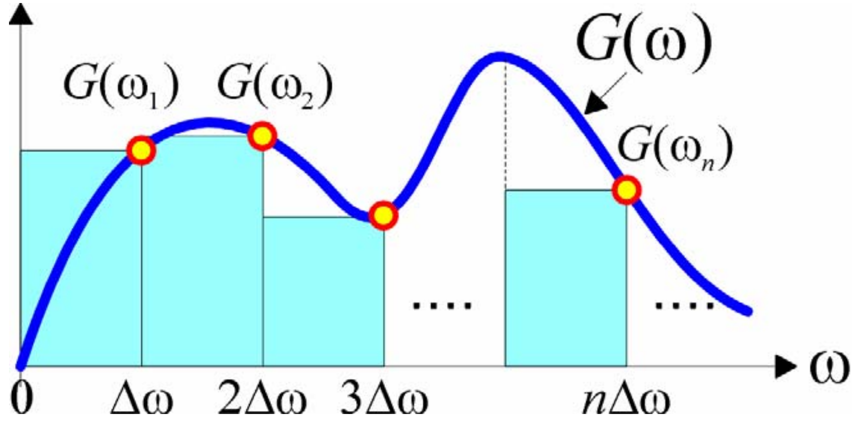
$$= \sum_{n=1}^{\infty}\frac{\Delta\omega}{\pi}\left[\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\tau)\cos(\omega_n\tau)d\tau\right]\cos(\omega_n t) = \frac{1}{\pi}\sum_{n=1}^{\infty} G(\omega_n)\Delta\omega,$$

$$\text{where } G(\omega_n) = \left[\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\tau)\cos(\omega_n\tau)d\tau\right]\cos(\omega_n t),$$

$$\Rightarrow G(\omega) = \left[\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\tau)\cos(\omega\tau)d\tau\right]\cos(\omega t).$$

Cosine terms (2)

- The area under a continuous function G(ω) is the limit of the summation of rectangular areas:



$$\lim_{\Delta\omega \to 0} \sum_{n=1}^{\infty} G(\omega_n)(\Delta\omega)$$

$$= \int_0^{\infty} G(\omega) d\omega$$

- $$\lim_{T \to \infty} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) = \lim_{\Delta\omega \to 0} \frac{1}{\pi} \sum_{n=1}^{\infty} G(\omega_n) \Delta\omega = \frac{1}{\pi} \int_0^{\infty} \boxed{G(\omega)} d\omega$$

$$= \frac{1}{\pi} \int_0^{\infty} \boxed{\left\{ \underbrace{\left[\int_{-\infty}^{\infty} f(\tau) \cos(\omega\tau) d\tau \right]}_{A(\omega)} \cos(\omega t) \right\}} d\omega.$$

Fourier coefficients

- $$\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) \xrightarrow{T\to\infty} \frac{1}{\pi}\int_0^{\infty} A(\omega)\times\cos(\omega t)d\omega,$$

  where $A(\omega) = \int_{-\infty}^{\infty} f(\tau)\times\cos(\omega\tau)d\tau.$

- $$\sum_{n=1}^{\infty} b_n \cos(n\omega_0 t) \xrightarrow{T\to\infty} \frac{1}{\pi}\int_0^{\infty} B(\omega)\times\sin(\omega t)d\omega,$$

  where $B(\omega) = \int_{-\infty}^{\infty} f(\tau)\times\sin(\omega\tau)d\tau.$

- $$f(t) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right],$$

  $$\xrightarrow{T\to\infty} \frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\times\cos(\omega t) + B(\omega)\times\sin(\omega t)\right]d\omega.$$

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Convergence

- If f(t) and f'(t) are piecewise continuous on every finite interval, and f(t) is absolutely integrable on $t \in (-\infty, \infty)$, i.e. $\int_{\infty}^{\infty} \left| f(t) \right| dt < \infty$

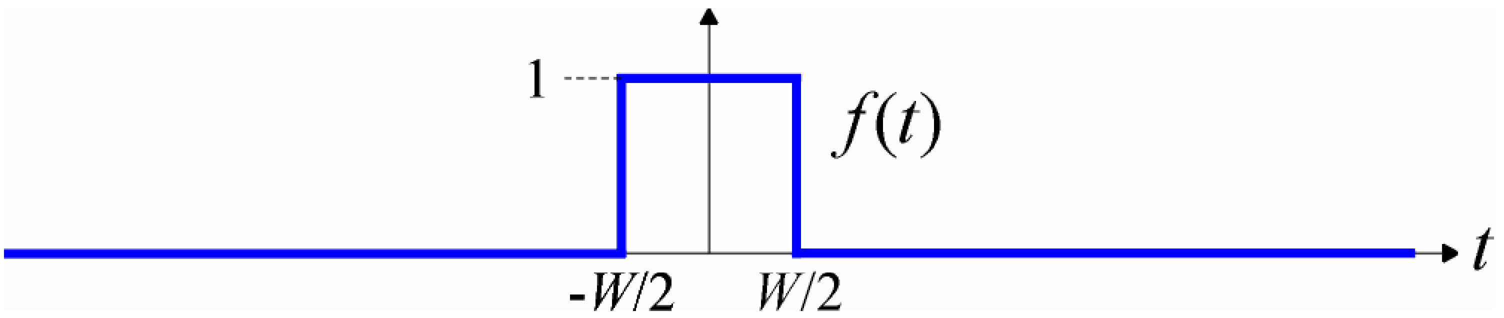
  $\Rightarrow$ Fourier integral converges to

1) f(t) at any point of continuity.

2) $[f(t^+)+f(t^-)]/2$ (average) at a point of discontinuity.

Example: Square pulse (1)

- Let f(t) = $\Pi(t/W)$, $\Pi(t)$ = {1, if |t| < 0.5; 0, otherwise}



- Since f(t) $\in$ even, $\Rightarrow$ no sine terms, B(ω) = 0;

$$A(\omega) = \int_{-\infty}^{\infty} f(\tau) \times \cos(\omega\tau) d\tau = 2\int_{0}^{W/2} 1 \times \cos(\omega\tau) d\tau$$

$$= 2\frac{\sin(\omega\tau)\Big|_0^{W/2}}{\omega} = \frac{2\sin(\omega W/2)}{\omega} = W \times \mathrm{sinc}(\frac{W}{2}\omega),$$

$$\text{where } \mathrm{sinc}(\theta) \equiv \frac{\sin\theta}{\theta}.$$

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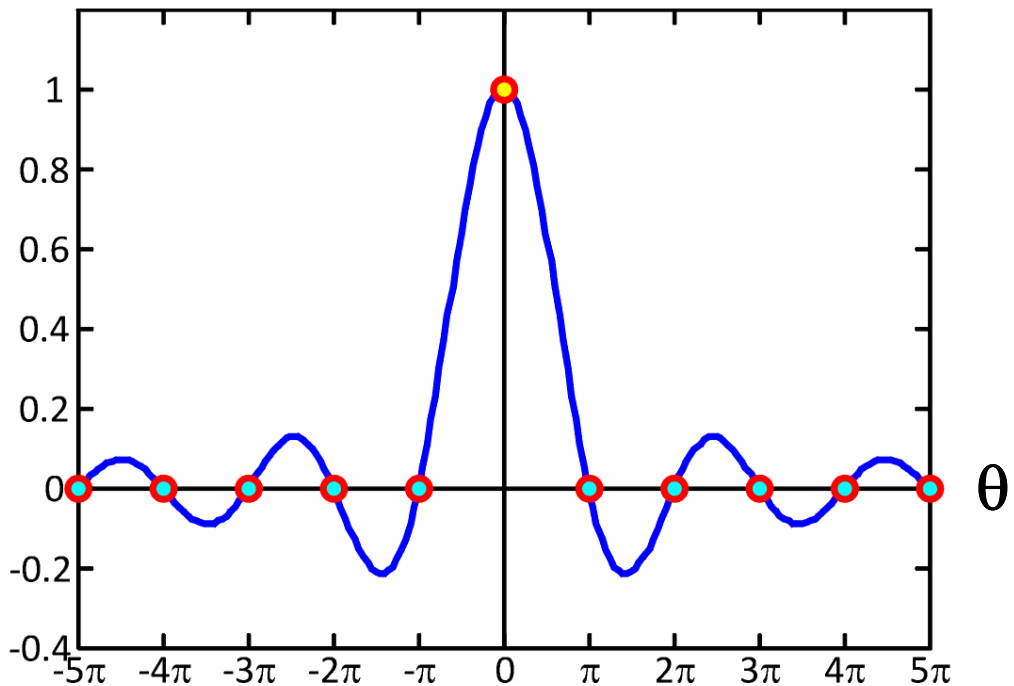
Sinc-function

- $\mathrm{sinc}(\theta) \equiv \sin(\theta)/\theta; \Rightarrow$

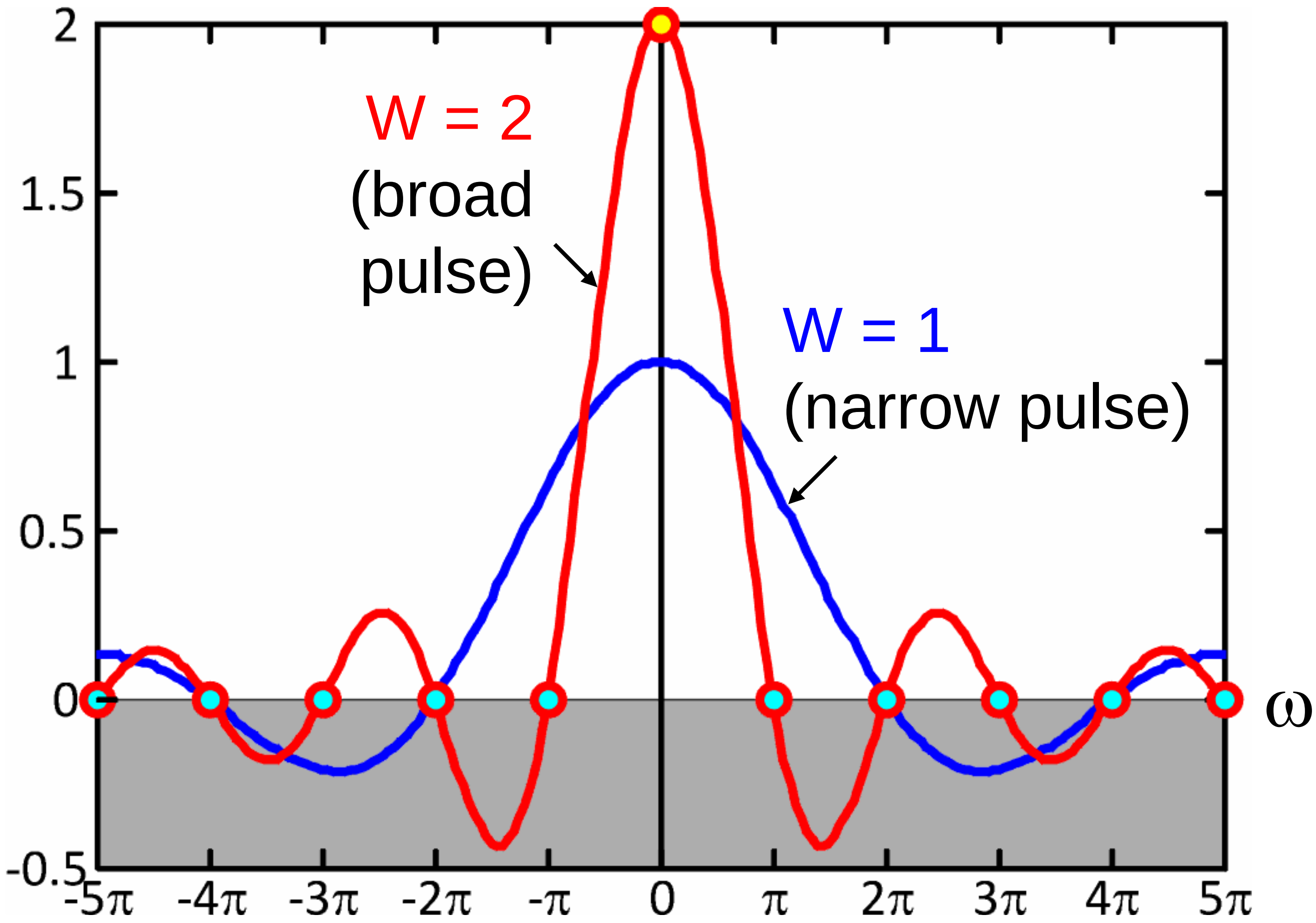
1) $\mathrm{sinc}(n\pi) = \sin(n\pi)/(n\pi) = 0$, except for $n = 0$.

2) $\mathrm{sinc}(0) = \lim_{\theta\to 0}[\sin'(\theta)/\theta'] = \lim_{\theta\to 0}[\cos(\theta)/1] = 1$.



Example: Square pulse (2)

- Fourier cosine coefficient $A(\omega) = W \times \mathrm{sinc}(\omega W/2)$:



Example: Square pulse (3)

- The Fourier integral of a square pulse f(t) = Π(t/W) is:

$$f(t) = \frac{W}{\pi}\int_0^{\infty} \mathrm{sinc}(\frac{W}{2}\omega)\times\cos(\omega t)d\omega.$$

- A higher angular frequency ω corresponds to a smaller amplitude [$\propto$ sinc(Wω/2)].
- Some frequency components $\omega = 2n\pi/W$ are missing.
- The main lobe width Δ of A(ω) is defined as the distance between the first two nulls $\omega = \pm 2\pi/W$, $\Rightarrow$ $\Delta = 4\pi/W \propto W^{-1}$ (broader pulse $\rightarrow$ narrower bandwidth).

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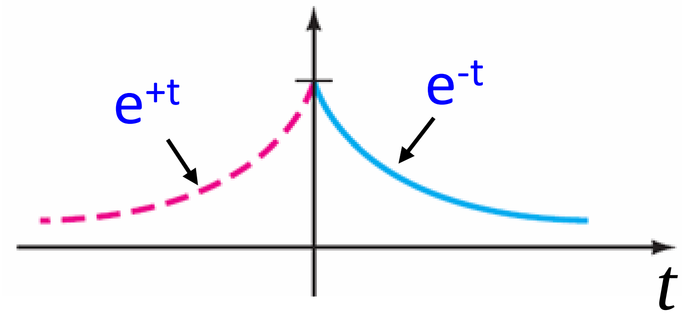
Example: $e^{-|t|}$

- Let f(t) = {e^{t}, if t < 0; e^{-t}, if t > 0}.



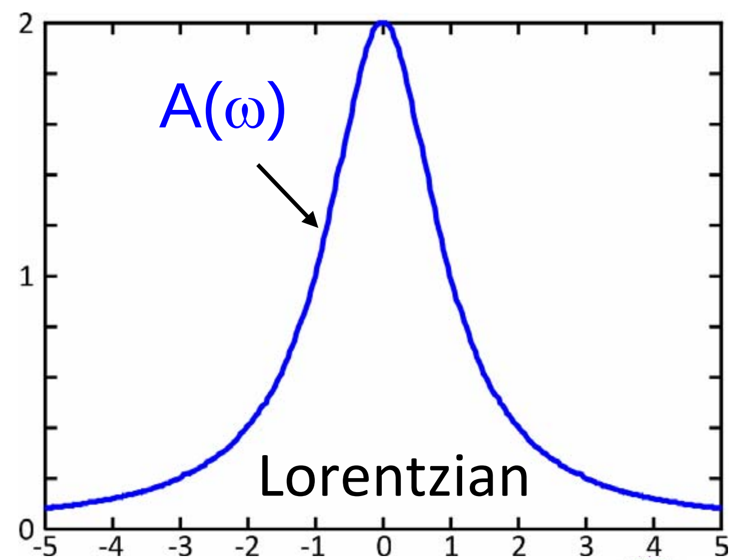
- Since f(t) ∈ even, ⇒ B(ω) = 0;

$$A(\omega)=\int_{-\infty}^{\infty} f(\tau)\cos(\omega\tau)d\tau = 2\int_{0}^{\infty} e^{-\tau}\cos(\omega\tau)d\tau \equiv 2I;$$

$$I=\frac{\cancel{e^{-\tau}\sin(\omega\tau)\Big|_{0}^{\infty}}}{\omega}+\frac{\int_{0}^{\infty} e^{-\tau}\sin(\omega\tau)d\tau}{\omega}$$

$$=\frac{1}{\omega}\left[\frac{e^{-\tau}\cos(\omega\tau)\Big|_{0}^{\infty}}{-\omega}-\frac{I}{\omega}\right];$$

$$I=\frac{1}{1+\omega^{2}},\ \Rightarrow A(\omega)=\frac{2}{1+\omega^{2}}.$$



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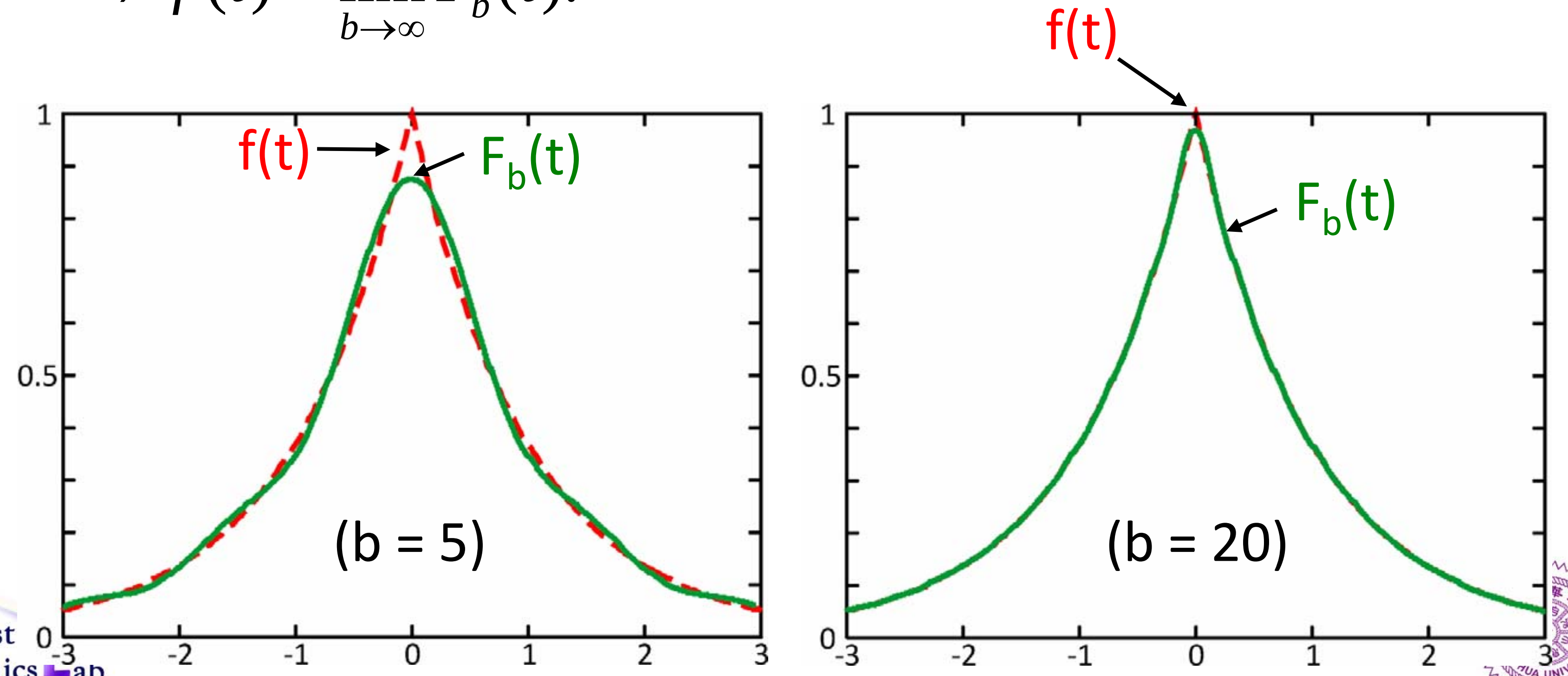
Convergence of Fourier integral

- In this example, the "partial integral" is

$$F_b(t) \equiv \frac{1}{\pi}\int_0^b \left[A(\omega)\times\cos(\omega t)\right]d\omega = \frac{2}{\pi}\int_0^b \left[\frac{\cos(\omega t)}{1+\omega^2}\right]d\omega.$$

$$\Rightarrow f(t) = \lim_{b\to\infty} F_b(t).$$



- Fourier transforms
  - Fourier transform pair
  - Spectrum

Rewrite the Fourier integral formula (1)

- Substitute

$$A(\omega)=\int_{-\infty}^{\infty} f(\tau)\times\cos(\omega\tau)d\tau,\ \ B(\omega)=\int_{-\infty}^{\infty} f(\tau)\times\sin(\omega\tau)d\tau,$$

into $f(t)=\dfrac{1}{\pi}\left[\int_{0}^{\infty} A(\omega)\times\cos(\omega t)d\omega+\int_{0}^{\infty} B(\omega)\times\sin(\omega t)d\omega\right];$

$$\Rightarrow f(t)=\frac{1}{\pi}\left\{\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\underline{\cos(\omega\tau)}d\tau\right]\underline{\cos(\omega t)}d\omega\right.$$

$$\left.+\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\underline{\sin(\omega\tau)}d\tau\right]\underline{\sin(\omega t)}d\omega\right\};$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\boxed{\left\{\int_{-\infty}^{\infty} f(\tau)\times\underline{\cos[\omega(\tau-t)]}d\tau\right\}}d\omega$$

even function of “ω”

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau)\times\cos[\omega(\tau-t)]d\tau\right\}d\omega.$$

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Rewrite the Fourier integral formula (2)

- Introducing a zero equality:

$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\boxed{\left\{\int_{-\infty}^{\infty} f(\tau)\times\sin\left[\omega(\tau-t)\right]d\tau\right\}}d\omega=0.$$

odd function of "ω"

$$\Rightarrow f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau)\times\left\{\underline{\cos}\left[\omega(\tau-t)\right]+\underline{i\sin}\left[\omega(\tau-t)\right]\right\}d\tau\right\}d\omega$$

$\cos\theta + i\times\sin\theta = e^{i\theta}$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\times e^{i\omega(\tau-t)}d\tau\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\times e^{i\omega\tau}d\tau\right]e^{-i\omega t}d\omega.$$

$$\Rightarrow\begin{cases} f(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}F(\omega)\times e^{-i\omega t}d\omega, \\ F(\omega)=\displaystyle\int_{-\infty}^{\infty}f(t)\times e^{i\omega t}dt.\end{cases}$$

Fourier transform in physics convention

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Another convention

- It's remains valid if we employ:

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau)\times\{\cos[\omega(\tau-t)]-j\sin[\omega(\tau-t)]\}d\tau\right\}d\omega$$

cosθ - j×sinθ = $e^{-j\theta}$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\times e^{-j\omega(\tau-t)}d\tau\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\times e^{-j\omega\tau}d\tau\right]e^{+j\omega t}d\omega.$$

$$\Rightarrow\begin{cases} f(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(\omega)\times e^{+j\omega t}d\omega, \\ F(\omega)=\displaystyle\int_{-\infty}^{\infty} f(t)\times e^{-j\omega t}dt. \end{cases}$$

Fourier transform
in EE convention

Interpretations

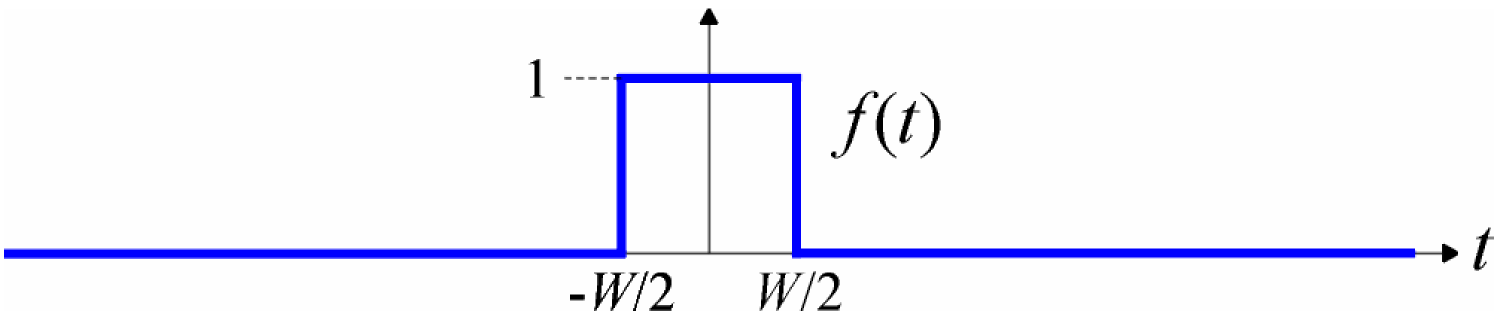
- f(t) ↔ F(ω) are Fourier transform pair:

$$\begin{cases} F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = F\{f(t)\}, & \text{...Fourier transform (FT)} \\ f(t)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega = F^{-1}\{F(\omega)\}. & \text{...Inverse FT} \end{cases}$$

- F(ω) = F{f(t)}: an integral transform of kernel function $e^{-j\omega t}$ (compared with e^{-st} in the LT).
- F(ω) is the coefficient (projection) of f(t) (vector) on a complex sinusoidal function $e^{j\omega t}$ (orthogonal axis).
- E.g. F(Ω) describes the "strength" of f(t) at ω = Ω.

Example: Square function (1)

- Let f(t) = Π(t/W), Π(t) ≡ {1, if |t| < 0.5; 0, otherwise}



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-W/2}^{W/2} 1 \times e^{-j\omega t} dt$$

$$= \frac{e^{-j\omega t}\Big|_{-W/2}^{W/2}}{-j\omega} = \frac{e^{-j\theta} - e^{j\theta}\Big|_{\theta = W\omega/2}}{-j\omega} = \frac{-2j\sin\theta}{-j\omega}$$

ω = 2θ/W

$$= W \times \mathrm{sinc}\left(\frac{W}{2}\omega\right)$$

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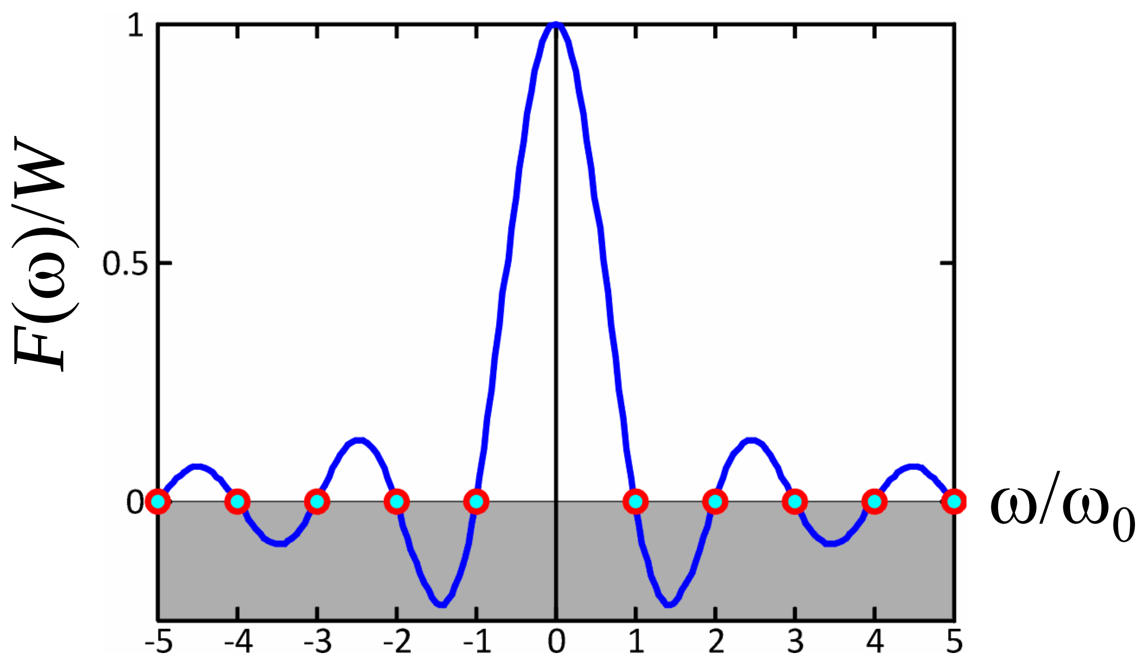
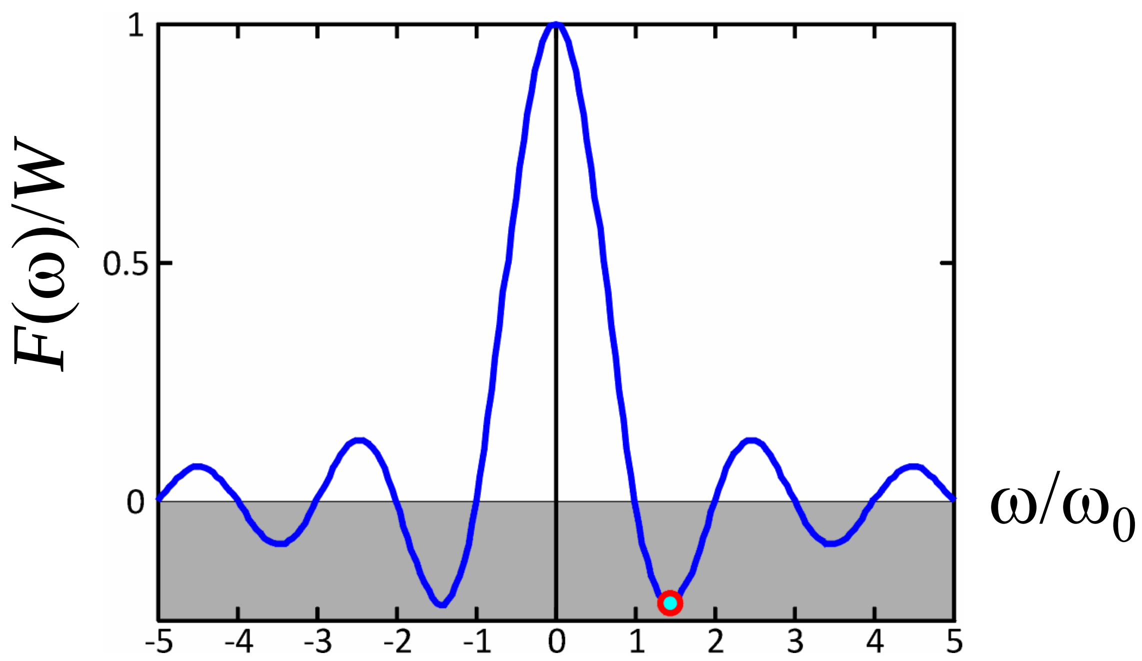
Spectrum (1)

- $F(\omega) = W\times\mathrm{sinc}(W\omega/2) = W\times\mathrm{sinc}(\pi\omega/\omega_0)$, if $\omega_0 \equiv 2\pi/W$.
- $F(\omega=0) = W$, $\Rightarrow$ need $W\times e^{j0t} = W$ (DC term) to synthesize f(t).
- $F(n\omega_0) = 0$, $\Rightarrow$ don't need $e^{jn\omega_0 t}$ to synthesize f(t).



Spectrum (2)

- $F(1.43\omega_0) \approx -0.217W = 0.217W\angle\pi$, $\Rightarrow$ need $0.217W \times e^{j(1.43\omega_0 t+\pi)}$ to synthesize f(t).
- $|F(\omega)|$, $\angle F(\omega)$ specify the magnitude and phase (timing) of the constituent complex sinusoids



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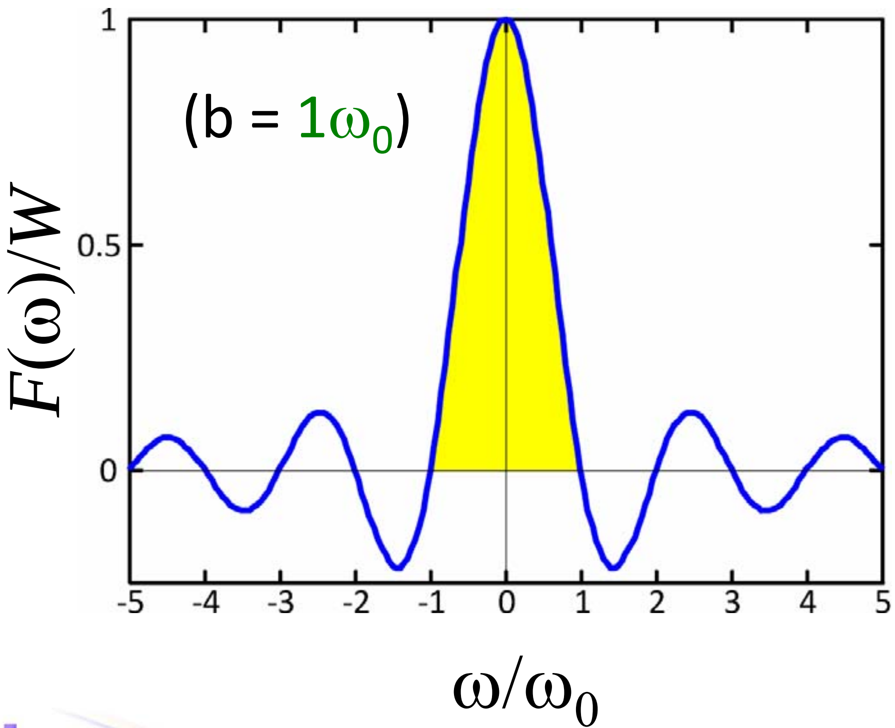
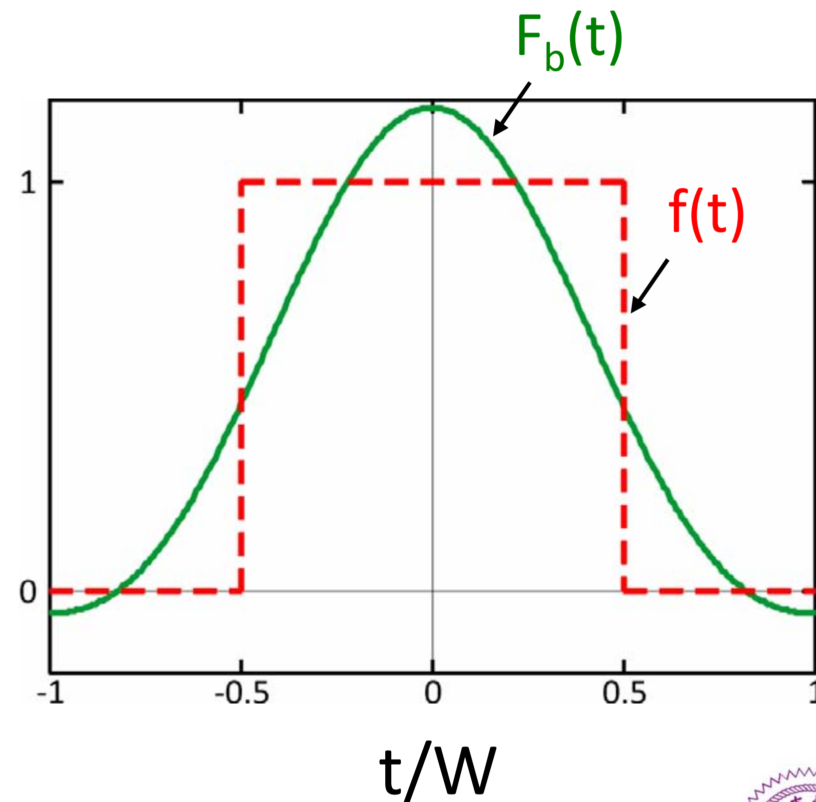
Convergence (1)

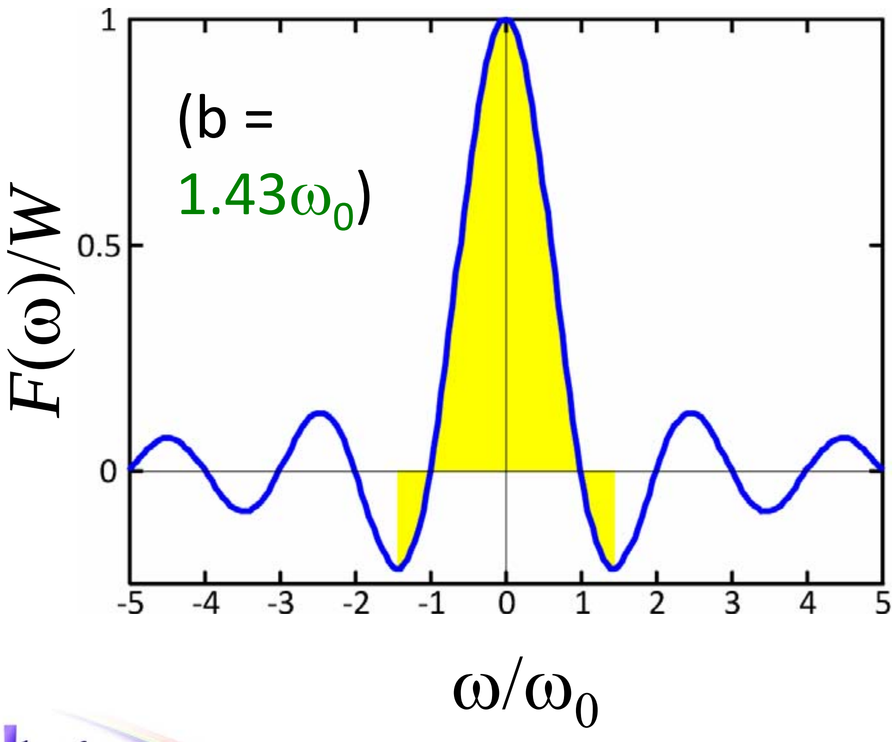
- Let the partial integral of f(t) be $F_b(t) \equiv \frac{1}{2\pi}\int_{-b}^{b} F(\omega)e^{j\omega t}d\omega,$

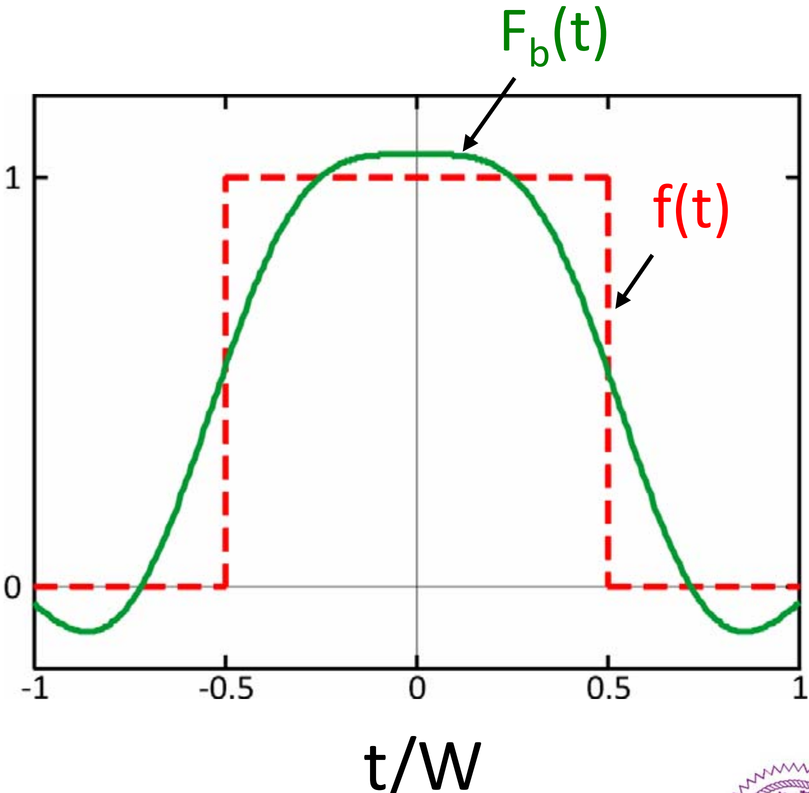
$$\Rightarrow f(t) = \lim_{b\to\infty} F_b(t).$$



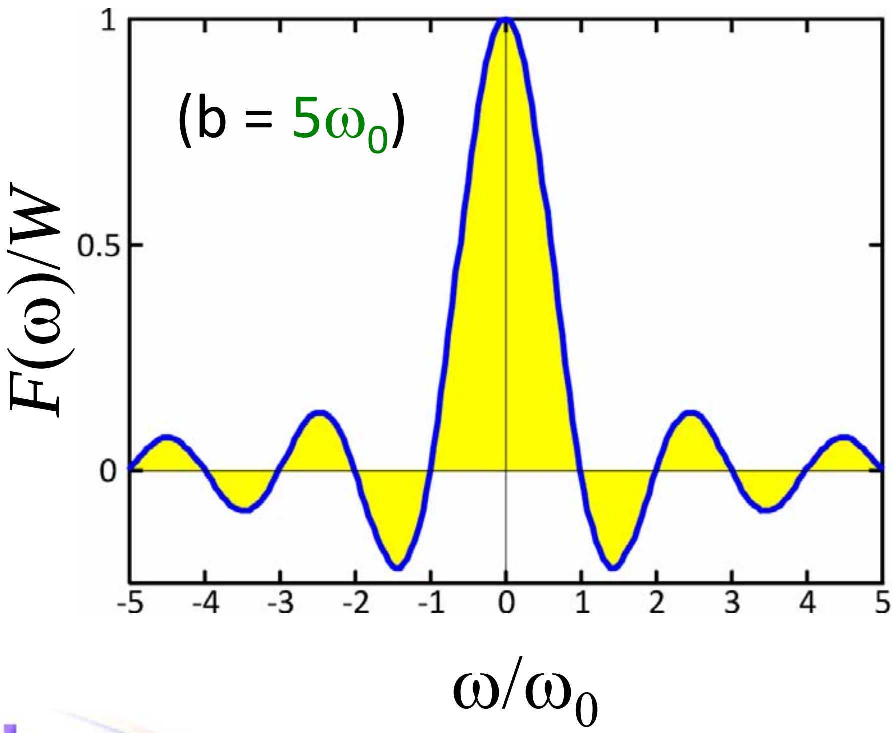


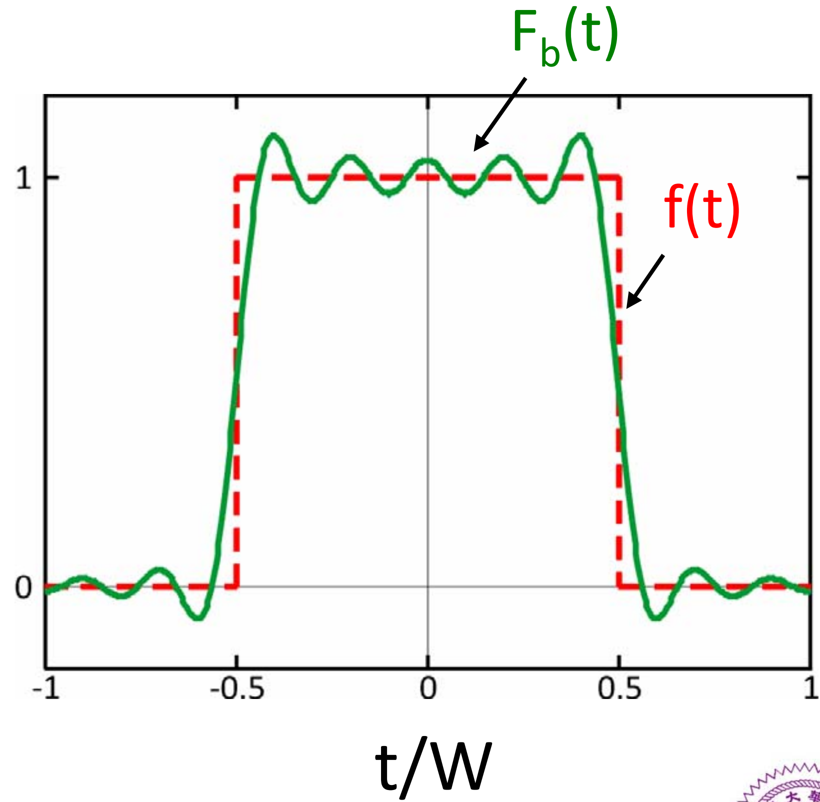
Convergence (2)





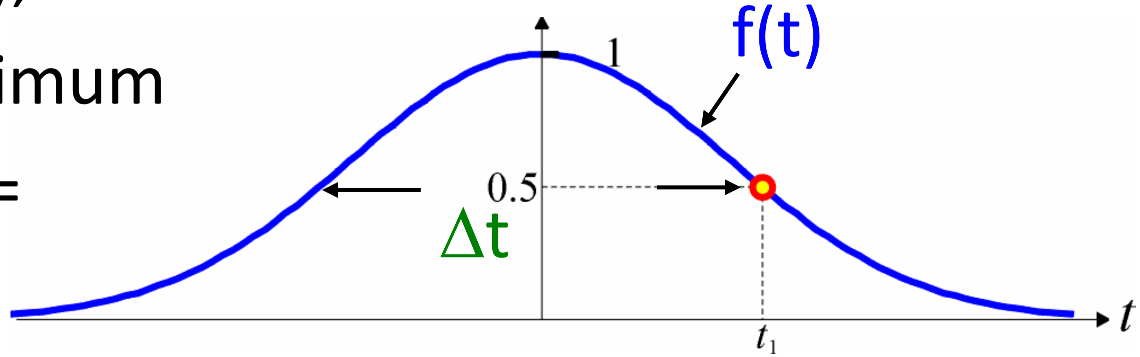
Convergence (3)





Example: Gaussian function (1)

- Let f(t) = exp(-αt^2), $\Rightarrow$ full width at half maximum (FWHM) Δt = 2t$_1$ = 2(ln2/α)$^{1/2}$.



- $$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} e^{-\alpha t^2}\times e^{-j\omega t}dt=\int_{-\infty}^{\infty} e^{-at^2-bt}dt,$$
  where a = α, b = jω in this example.

- -(at^2 + bt) = -a[t + b/(2a)]^2 + b^2/(4a),

$$\Rightarrow \int_{-\infty}^{\infty} e^{-at^2-bt}dt=e^{\frac{b^2}{4a}}\int_{-\infty}^{\infty} e^{-a(t+\frac{b}{2a})^2}dt=e^{\frac{b^2}{4a}}\underbrace{\boxed{\int_{-\infty}^{\infty} e^{-at^2}dt}}_{I}$$

Example: Gaussian function (2)

- $I = \int_{-\infty}^{\infty} e^{-ax^2} dx,$

$$\Rightarrow I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2} dx \right) \times \left(\int_{-\infty}^{\infty} e^{-ay^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dxdy$$

- $\exp[-a(x^2+y^2)] = \exp(-ar^2)$, a circularly symmetric function on the xy-plane, $\Rightarrow$ polar integral is easier:

$$I^2 = \int_0^{2\pi} \left(\int_0^{\infty} e^{-ar^2} rdr \right) d\phi = 2\pi \int_0^{\infty} re^{-ar^2} dr = 2\pi \frac{e^{-ar^2}\Big|_0^{\infty}}{-2a} = \frac{\pi}{a},$$

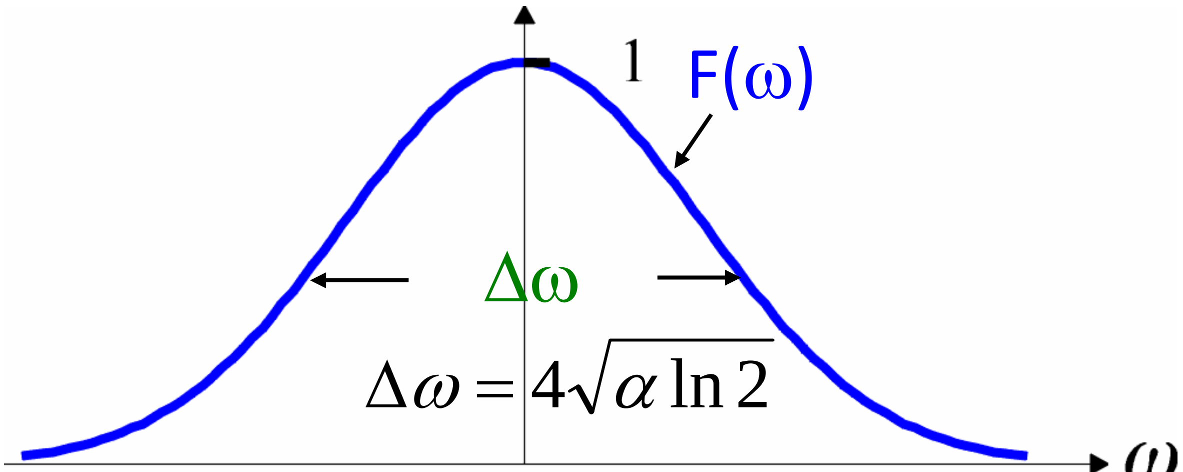
$$\Rightarrow I = \sqrt{\frac{\pi}{a}}.$$

Example: Gaussian function (3)

- $F(\omega)=\int_{-\infty}^{\infty} e^{-at^2-bt}dt \;\; = e^{\frac{b^2}{4a}}\times I=\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}},$

- Substitute a = α, b = jω into the above formula, the FT of a Gaussian function remains Gaussian

$$F(\omega)=\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}$$

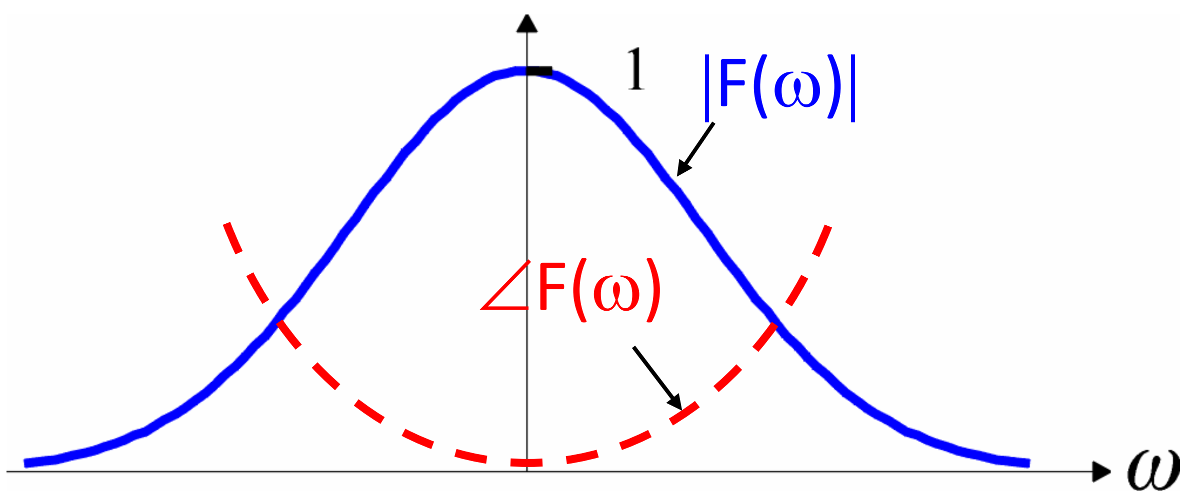


Spectral phase ∠F(ω) matters

- Let $F(\omega) \propto e^{-\frac{\omega^2}{4\alpha}} \times \boxed{e^{j\frac{\beta}{4\alpha}\omega^2}}$, where ∠F(ω) is a quadratic function [β/(4α)]ω².

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \propto \int_{-\infty}^{\infty} e^{-\overset{a}{\boxed{\frac{1-j\beta}{4\alpha}}}\omega^2} \times e^{\overset{b\,=\,-jt}{\boxed{j\omega t}}} d\omega$$

By $\int_{-\infty}^{\infty} e^{-a\omega^2 - b\omega} d\omega = \sqrt{\pi/a} \times e^{-b^2/(4a)}$, $\Rightarrow f(t) \propto e^{-\frac{\alpha}{1+\beta^2}t^2}$.



- Fourier transform properties

1. Reality

$$f(t) \in R, \Rightarrow F(-\omega) = F^*(\omega), \quad \Rightarrow \begin{cases} |F(\omega)| \text{ is even,} \\ \angle F(\omega) \text{ is odd.} \end{cases}$$

- Proof:

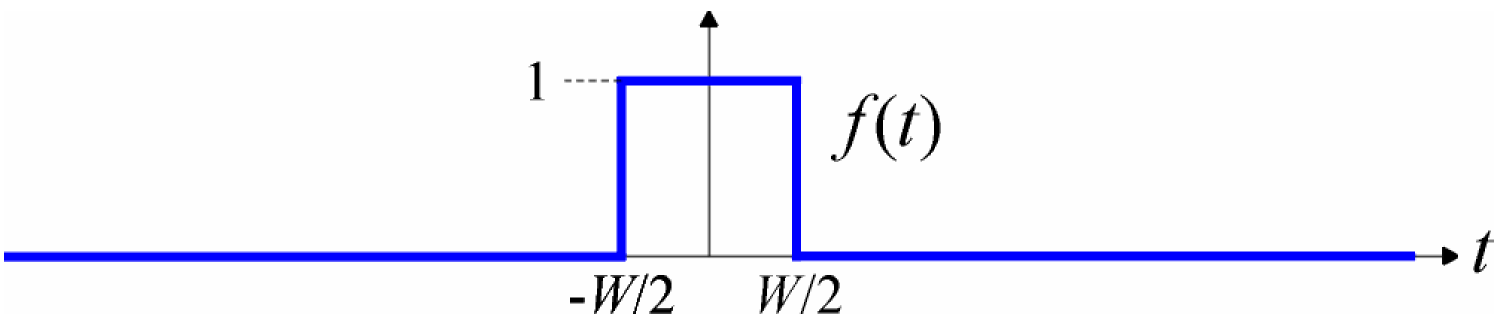
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t)[\cos(\omega t) - j\sin(\omega t)]dt,$$

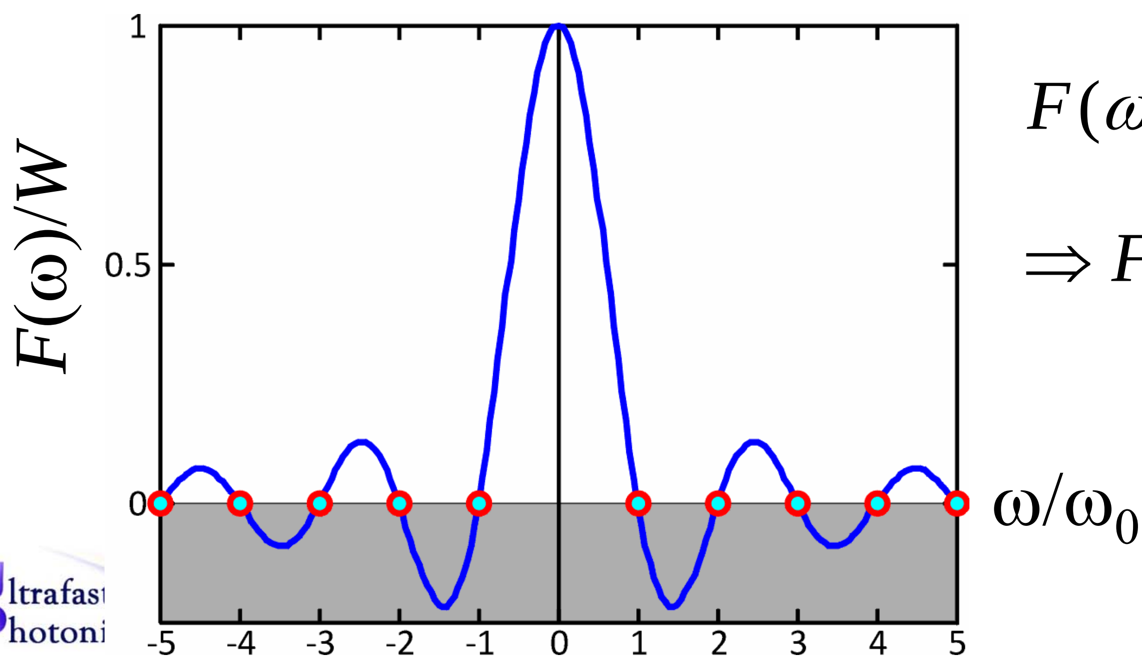
$$\Rightarrow F(-\omega) = \int_{-\infty}^{\infty} f(t)[\cos(\omega t) + j\sin(\omega t)]dt,$$

$$\Rightarrow F^*(\omega) = \int_{-\infty}^{\infty} f^*(t)[\cos(\omega t) - j\sin(\omega t)]^* dt$$

$$= \int_{-\infty}^{\infty} f(t)[\cos(\omega t) + j\sin(\omega t)]dt = F(-\omega).$$

Example: Square function





$$F(\omega) = W \times \text{sinc}(\frac{W}{2}\omega) \in R,$$

$$\Rightarrow F(-\omega) = F(\omega) = F^*(\omega).$$

Fourier integral vs. Fourier transform?

- The corresponding synthesis formulae are:

$$f(t)=\begin{cases}\dfrac{1}{\pi}\displaystyle\int_0^{\infty}\left[A(\omega)\cos(\omega t)+B(\omega)\sin(\omega t)\right]d\omega, \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}F(\omega)e^{-j\omega t}d\omega.\end{cases}$$

- For f(t) ∈ R, $\int_{-\infty}^{\infty}\boxed{F(\omega)}e^{-j\omega t}d\omega=\int_{-\infty}^{\infty}\boxed{\left|F(\omega)\right|e^{j\psi(\omega)}}e^{-j\omega t}d\omega$

$$=\int_{-\infty}^{\infty}\left|F(\omega)\right|\times\left[\cos(\psi-\omega t)+j\sin(\psi-\omega t)\right]d\omega$$

$$=\int_{-\infty}^{\infty}\underset{\text{even}}{\left|F(\omega)\right|}\times\left[\underset{\text{even}}{\cos\psi}\,\underset{\text{even}}{\cos(\omega t)}+\underset{\text{odd}}{\sin\psi}\,\underset{\text{odd}}{\sin(\omega t)}\right]d\omega$$

$$+\int_{-\infty}^{\infty}\underset{\text{even}}{j\left|F(\omega)\right|}\times\left[\underset{\text{odd}}{\sin\psi}\,\underset{\text{even}}{\cos(\omega t)}-\underset{\text{even}}{\cos\psi}\,\underset{\text{odd}}{\sin(\omega t)}\right]d\omega$$

Fourier integral vs. F.T. (2)

$$\Rightarrow \int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega = \int_{-\infty}^{\infty} |F(\omega)| \times \left[\cos\psi \cos(\omega t) + \sin\psi \sin(\omega t)\right]d\omega$$

$$= 2\int_{0}^{\infty} \underbrace{\boxed{|F(\omega)|\cos\psi(\omega)}}_{A(\omega)} \times \cos(\omega t) + \underbrace{\boxed{|F(\omega)|\sin\psi(\omega)}}_{B(\omega)} \times \sin(\omega t)d\omega$$

$$= 2\int_{0}^{\infty} \left[A(\omega) \times \cos(\omega t) + B(\omega) \times \sin(\omega t)\right]d\omega$$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega$$ Inverse FT

$$= \frac{1}{\pi}\int_{0}^{\infty} \left[A(\omega) \times \cos(\omega t) + B(\omega) \times \sin(\omega t)\right]d\omega$$ Fourier integral

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2. Scaling

$$f(at) \leftrightarrow \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

- Proof:

$$F\{f(at)\} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt;$$

Let $at = \tau$, $\Rightarrow dt = d\tau/a$;

$$\Rightarrow F\{f(at)\} = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega(\tau/a)} \frac{d\tau}{a} = \frac{1}{a} \boxed{\int_{-\infty}^{\infty} f(\tau) e^{-j(\frac{\omega}{a})\tau} d\tau}$$

F(ω/a)

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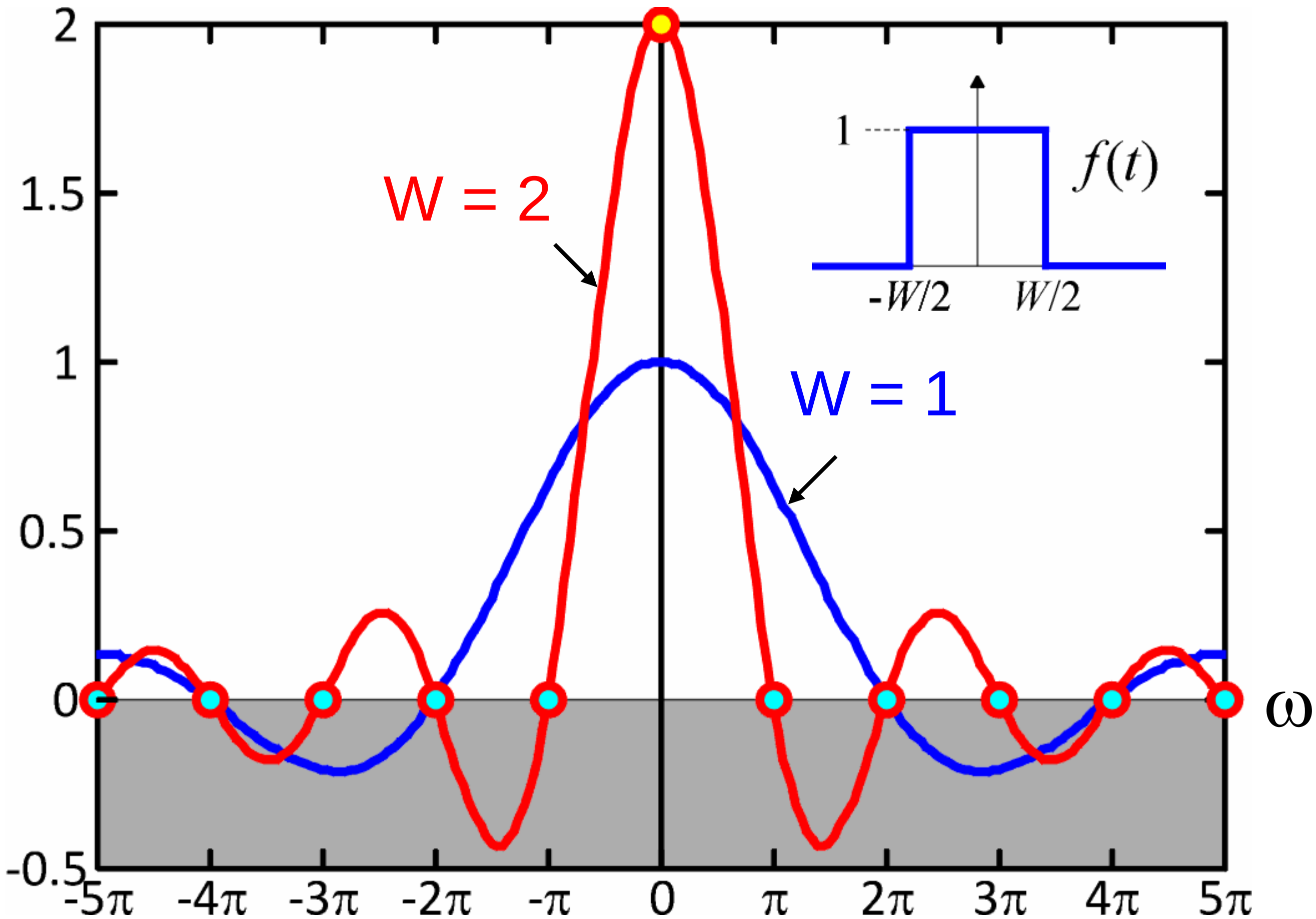
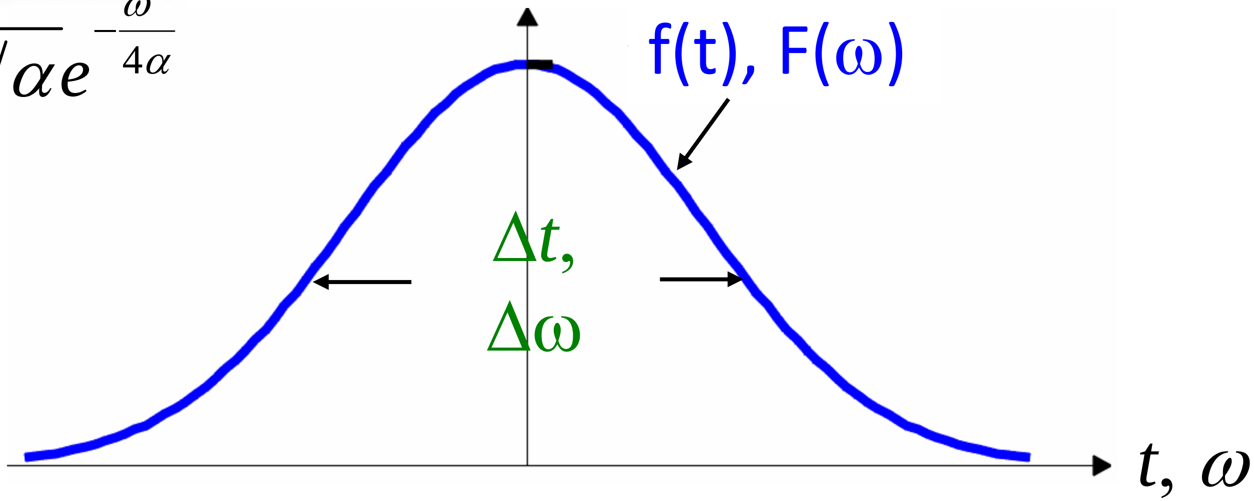
Example: Square function

- $\Pi(t/W) \leftrightarrow W \times \mathrm{sinc}(\omega W/2)$ (i.e. $a = 1/W$):



Example: Gaussian function

- $e^{-\alpha t^2} \leftrightarrow \sqrt{\pi/\alpha}\, e^{-\frac{\omega^2}{4\alpha}}$



- Bigger α, ⇒ shorter Δt, steeper rising/falling edges in the time domain, ⇒ need higher frequency components to synthesize f(t), larger bandwidth Δω.
- "Time-bandwidth product" is independent of α:

$$\Delta t \times \Delta\omega = 2\sqrt{\frac{\ln 2}{\cancel{\alpha}}} \times 4\sqrt{\cancel{\alpha}\ln 2} = 8\ln 2$$

3. Time delay

$$f(t-\tau) \leftrightarrow F(\omega) \times e^{-j\omega\tau}$$

- Proof:

$$F\{f(t-\tau)\} = \int_{-\infty}^{\infty} f(t-\tau) e^{-j\omega t} dt;$$

Let t - τ = t', ⇒ dt = dt';

F(ω)

$$\Rightarrow F\{f(t-\tau)\} = \int_{-\infty}^{\infty} f(t') e^{-j\omega(t'+\tau)} dt' = e^{-j\omega\tau} \boxed{\int_{-\infty}^{\infty} f(t') e^{-j\omega t'} dt'}$$

- Shift in the time domain is equivalent to a linear phase modulation in the frequency domain.

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Physical meaning

- Shifting a complex sinusoid $e^{j\omega t}$ by a time delay τ causes $e^{j\omega(t-\tau)}$, i.e. a frequency-dependent phase shift $\phi(\omega) = -\omega\tau$.
- Different frequencies ω have different periods T, $\Rightarrow$ a common delay τ corresponds different phases ϕ.

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4. Frequency shift

$$f(t)\times e^{j\omega_0 t} \leftrightarrow F(\omega-\omega_0)$$

- Proof:

$$F\left\{f(t)e^{j\omega_0 t}\right\}=\int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}\times e^{-j\omega t}dt=\underbrace{\int_{-\infty}^{\infty} f(t)\times e^{-j(\omega-\omega_0)t}dt}_{F(\omega-\omega_0)};$$

- Shift in the frequency domain is equivalent to a linear phase modulation in the time domain.
- E.g. Let $f(t) = e^{j\Omega t} = \cos(\Omega t)+j\sin(\Omega t)$, $\Rightarrow f(t)\times e^{j\omega_0 t} = e^{j(\Omega+\omega_0)t}$, frequency is shifted by an amount of ω_0.

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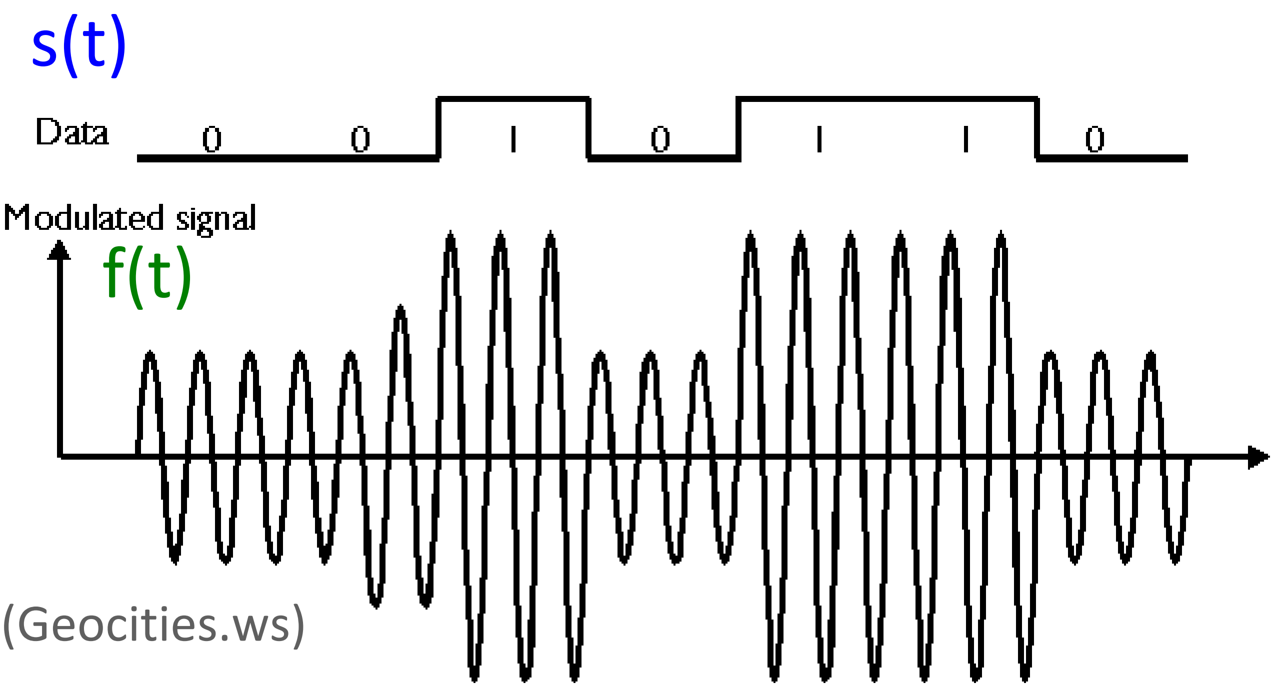
Example: Signal transmission

- Cellular phone signal $s(t)$ is first modulated by a "carrier" $\cos(\omega_c t)$ before transmitted over the air:
- $f(t) = s(t)\times\cos(\omega_c t) = s(t)\times[e^{j\omega_c t} + e^{-j\omega_c t}]/2$, $\Rightarrow F(\omega) = [S(\omega-\omega_c) + S(\omega-\omega_c)]/2$.



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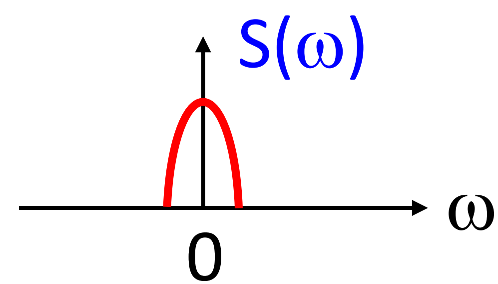

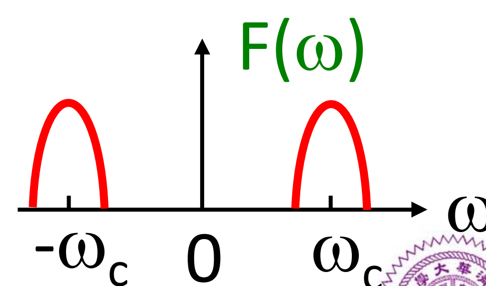

5. Convolution

$$f(t) \otimes g(t) \leftrightarrow F(\omega) \times G(\omega)$$

- Proof:

For -∞ < t < ∞, $f(t) * g(t) \equiv \int_{-\infty}^{\infty} f(\tau) \times g(t-\tau) d\tau,$

$$F\{f * g\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) \times g(t-\tau) d\tau \right] \times e^{-j\omega t} dt,$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(t-\tau) \times e^{-j\omega t} dt \right] d\tau,$$ let t' = t − τ,

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(t') \times e^{-j\omega (t'+\tau)} dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau \left[\int_{-\infty}^{\infty} g(t') e^{-j\omega t'} dt \right] = F(\omega) \times G(\omega).$$

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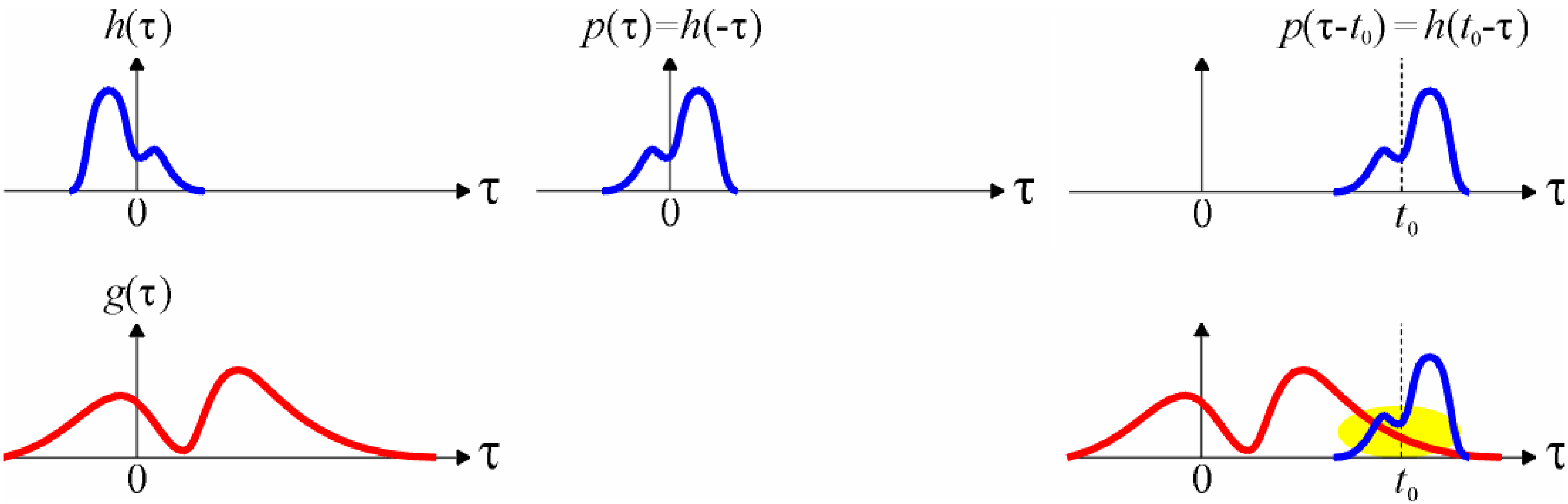
Physical meaning

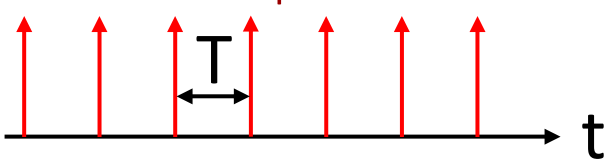
- $f(t) = g * h = \int_{-\infty}^{\infty} g(\tau) \times h(t-\tau) d\tau$ means that $f(t_0)$ results from $g(t \approx t_0)$ over a range determined by the width of $h(t)$.



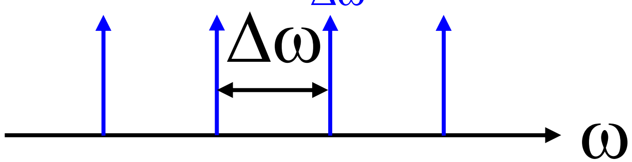
6. Comb function

$$\sum_{n=-\infty}^{\infty}\delta(t-nT)\leftrightarrow\sum_{m=-\infty}^{\infty}\delta(\omega-m\cdot\Delta\omega),\ \Delta\omega=\frac{2\pi}{T}$$

$\mathrm{comb}_T(t)$



$\mathrm{comb}_{\Delta\omega}(\omega)$



- A non-rigorous proof:

$$F\{\delta(t)\}=\int_{-\infty}^{\infty}\delta(t)\times e^{-j\omega t}dt=e^{-j\omega(0)}=1.$$

$$F\{\delta(t-nT)\}=F\{\delta(t)\}\times e^{-j\omega(nT)}=e^{-jnT\omega}.$$

$$F\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\right\}=\sum_{n=-\infty}^{\infty}e^{-jnT\omega}=\begin{cases}\infty,\ \text{if }\ \omega=m\times\Delta\omega,\\ 0,\ \text{otherwise}\end{cases}$$

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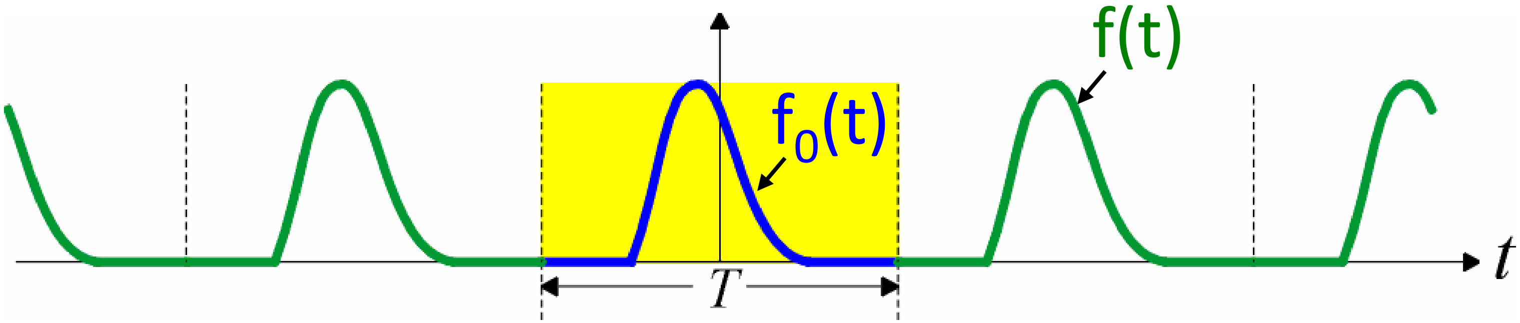
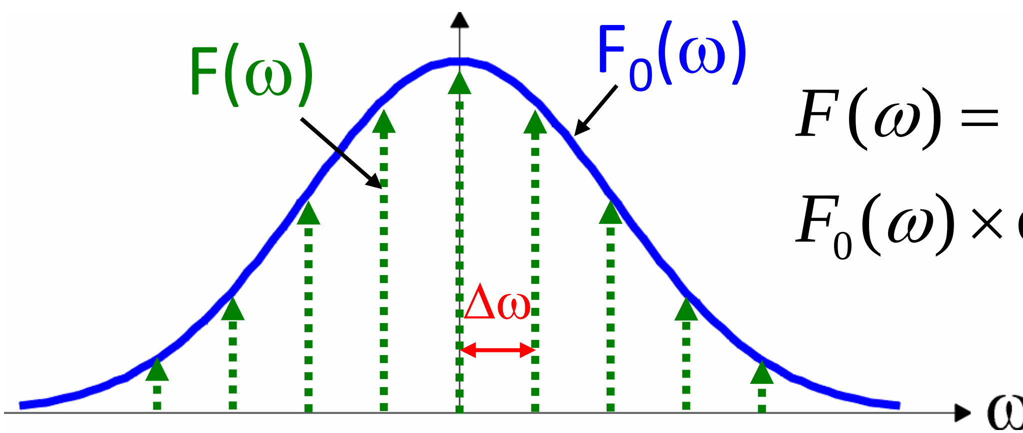
Example: Periodic pulse train

- A periodic function f(t) of period T is formulated as:

$$f(t)=\sum_{n=-\infty}^{\infty} f_0(t-nT)=\sum_{n=-\infty}^{\infty} f_0(t)*\delta(t-nT)=f_0(t)*\mathrm{comb}_T(t),$$





$$F(\omega)=$$
$$F_0(\omega)\times \mathrm{comb}_{\Delta\omega}(\omega)$$

7. Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

- Proof:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \times \boxed{f^*(t)} dt$$

$$\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega\right]^*$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} f(t) \times \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t} d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega) \times \underbrace{\boxed{\int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt}}_{F(\omega)} d\omega$$

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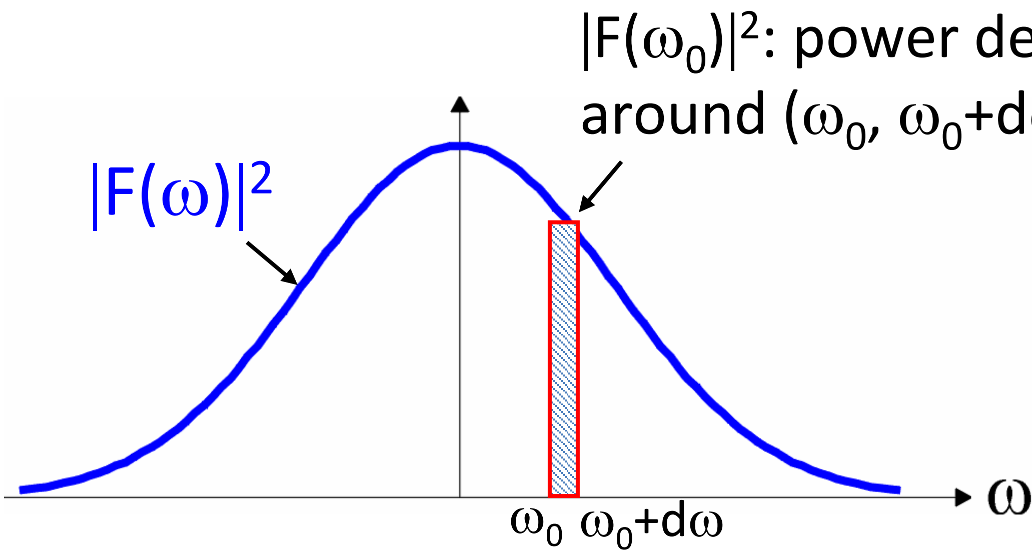
Physical meaning

- If $f(t) \rightarrow$ e-field, $|f(t)|^2 \rightarrow$ power, $\int|f(t)|^2dt \rightarrow$ energy, which is equivalent to $\int|F(\omega)|^2d\omega$.
- Just like power means energy per unit time, $|F(\omega)|^2 \rightarrow$ power spectral density (power per unit frequency).



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8. Derivative

$$\begin{cases} f'(t) \leftrightarrow j\omega \times F(\omega) \\ f''(t) \leftrightarrow (j\omega)^2 \times F(\omega) = -\omega^2 \times F(\omega) \end{cases}$$

- Proof:

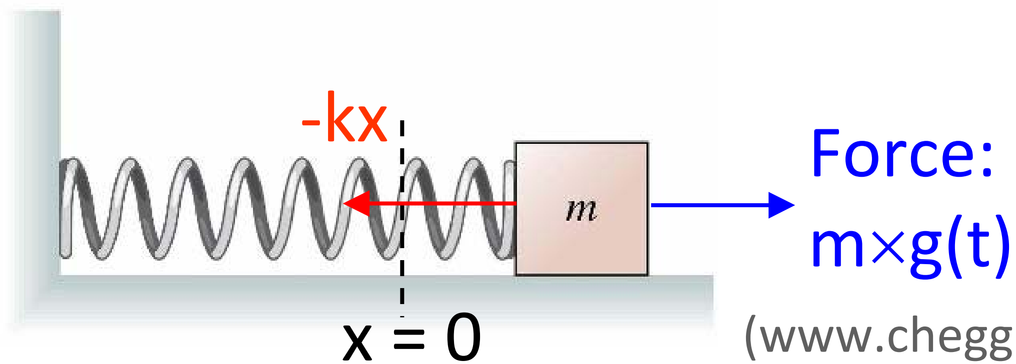
$$F\{f'(t)\} = \int_{-\infty}^{\infty} f'(t) \times e^{-j\omega t} dt;$$

Let u' = f', v = $e^{-j\omega t}$, $\Rightarrow$ u = f, v' = $(-j\omega)e^{-j\omega t}$;

$$F\{f'(t)\} = \cancel{\left[f'(t) \times e^{-j\omega t} \Big|_{-\infty}^{\infty} \right]} + j\omega \underbrace{\left[\int_{-\infty}^{\infty} f(t) \times e^{-j\omega t} dt \right]}_{F(\omega)};$$

Example: Simplifying ODEs

- ODE: $x''(t) + (\omega_0)^2 x(t) = g(t)$, where the resonance frequency is $\omega_0 = \sqrt{(k/m)}$, $g(t)$ is an arbitrary function (not necessarily periodic).



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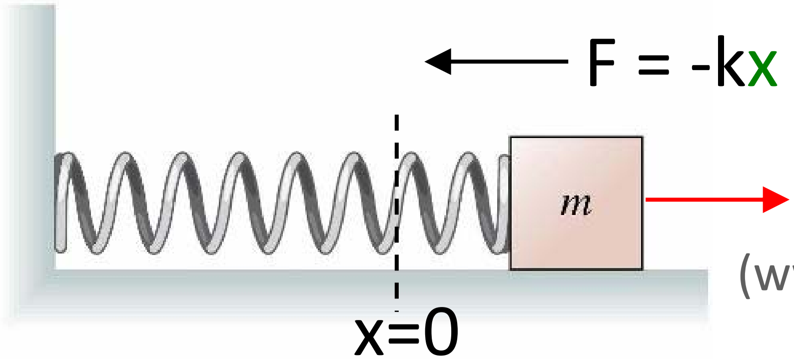
- F{ODE}: $-\omega^2 X(\omega) + (\omega_0)^2 X(\omega) = G(\omega)$ … algebraic eq.

  $\Rightarrow X(\omega) = G(\omega)/[(\omega_0)^2-\omega^2]$, input spectrum $G(\omega)$ is modified by a transfer function $H(\omega) = [(\omega_0)^2-\omega^2]^{-1}$.

Comparison: p57, Ch4

- ODE: $x''(t) + (\omega_0)^2 x(t) = g(t) = (F_0/m)\times\sin(\omega t)$.



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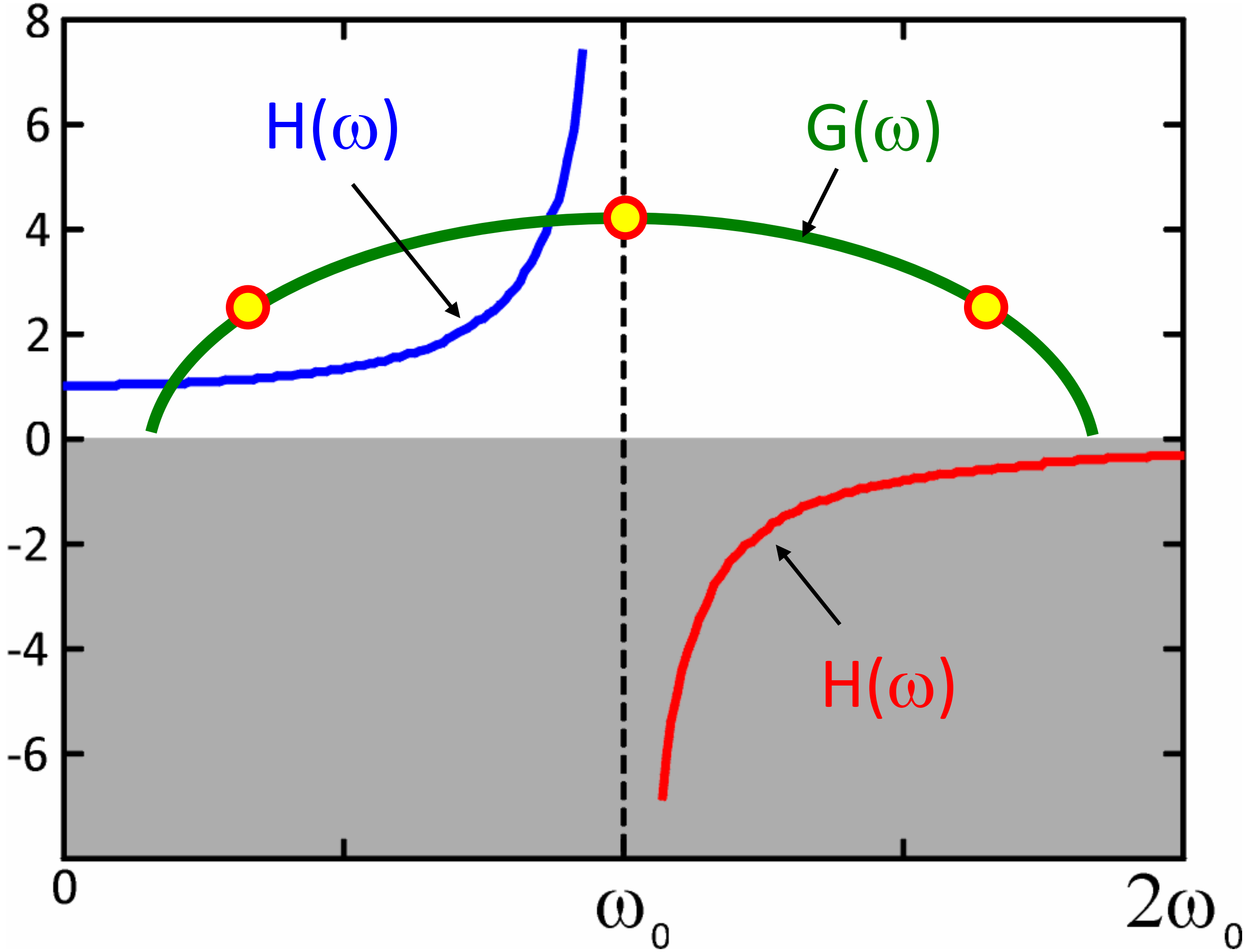
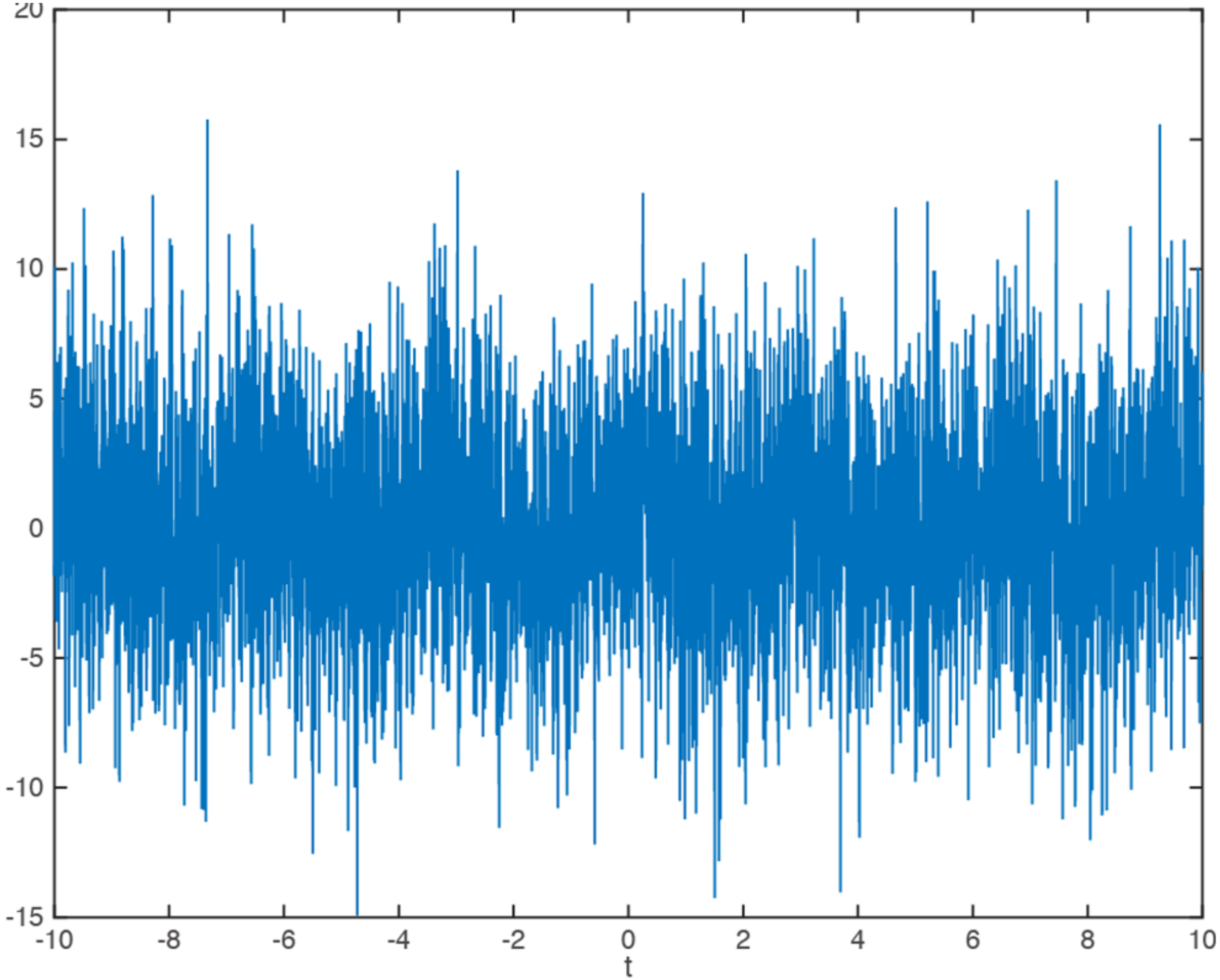
- Substituting $x_p(t) = A\times\cos(\omega t)+B\times\sin(\omega t)$ into the ODE, $\Rightarrow A = 0,\ B = (F_0/m)/[(\omega_0)^2-\omega^2] \propto [(\omega_0)^2-\omega^2]^{-1}$.
- $x(t) = x_c+x_p = A_0\cos(\omega_0 t+\phi)+B(\omega)\sin(\omega t)$.
- $F\{ODE\}$ = analyze $x_p(t)$ frequency by frequency.

Resonance: Spectral picture



- Applications of FT
- Noise rejection

Is there information carried?



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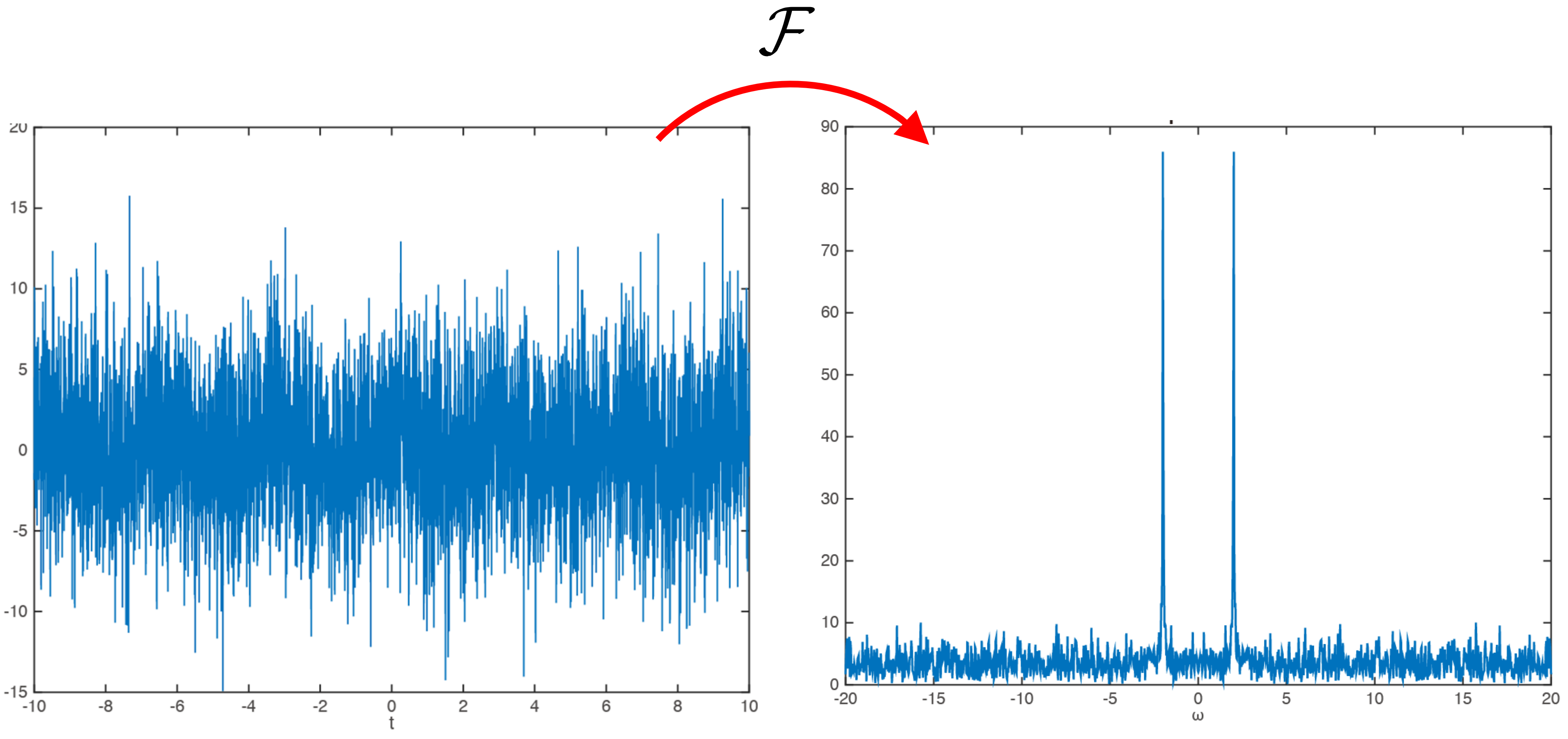
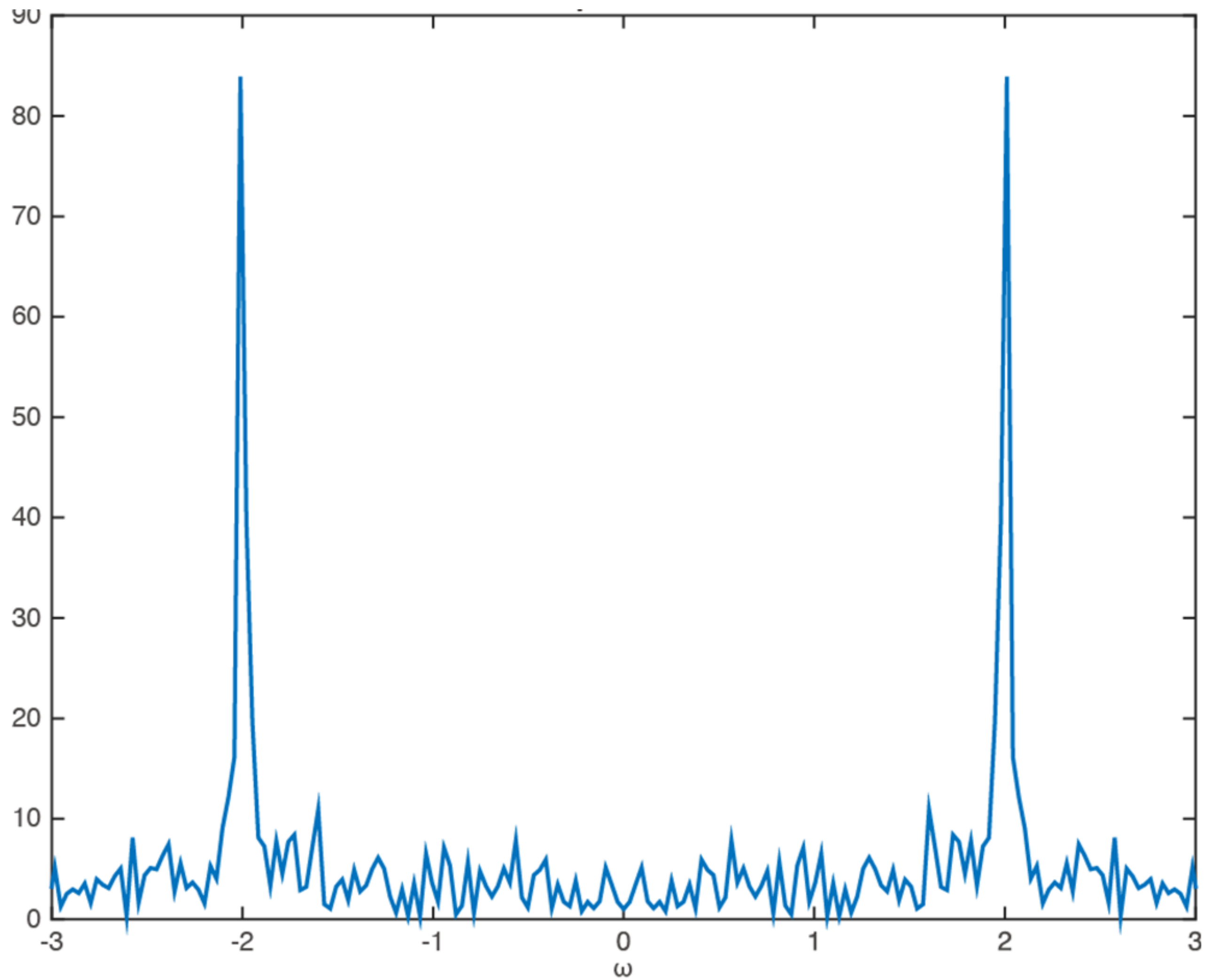
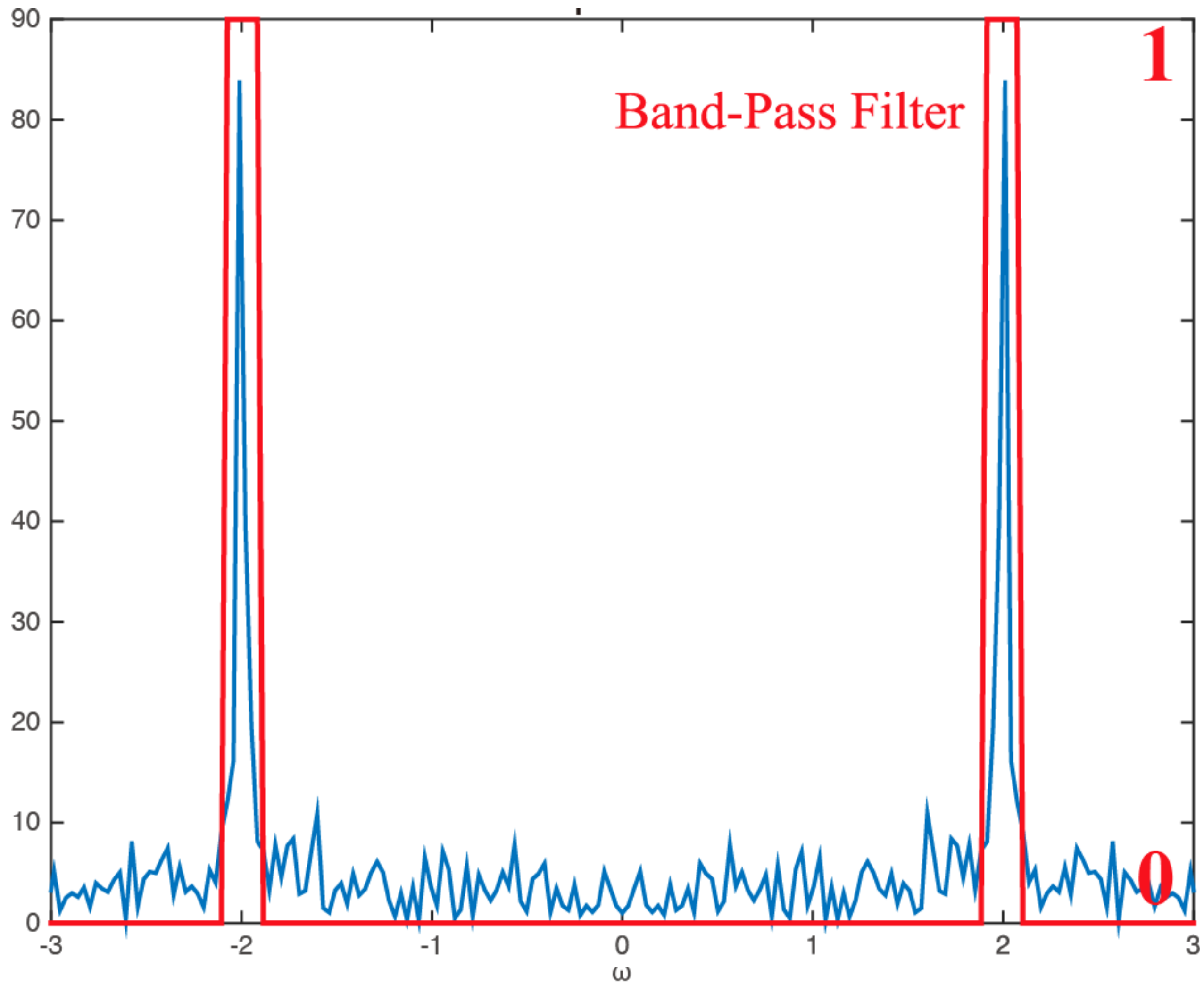
Try Fourier transform



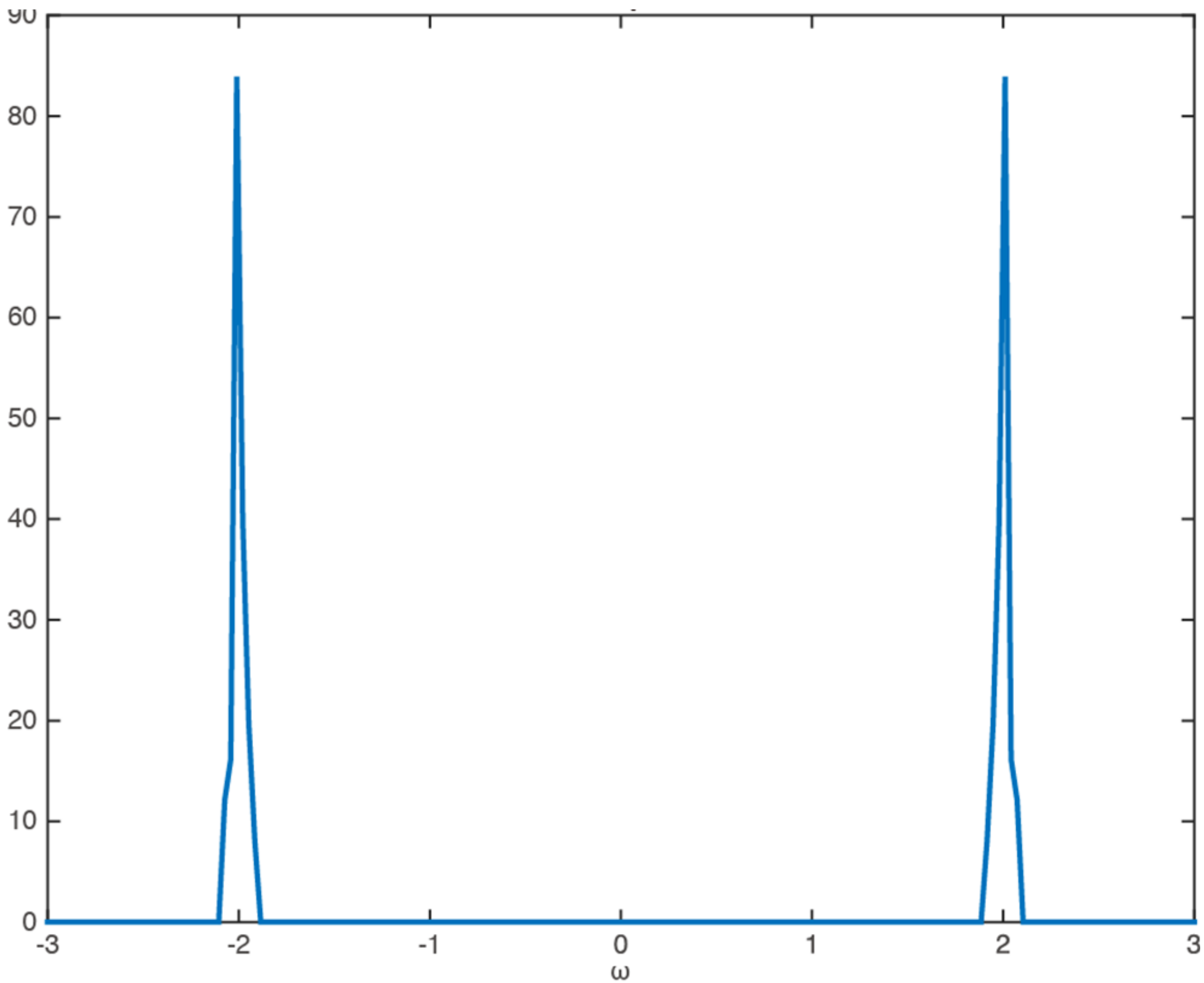
Zoom in



Band-Pass Filter



“Cleaned” spectrum



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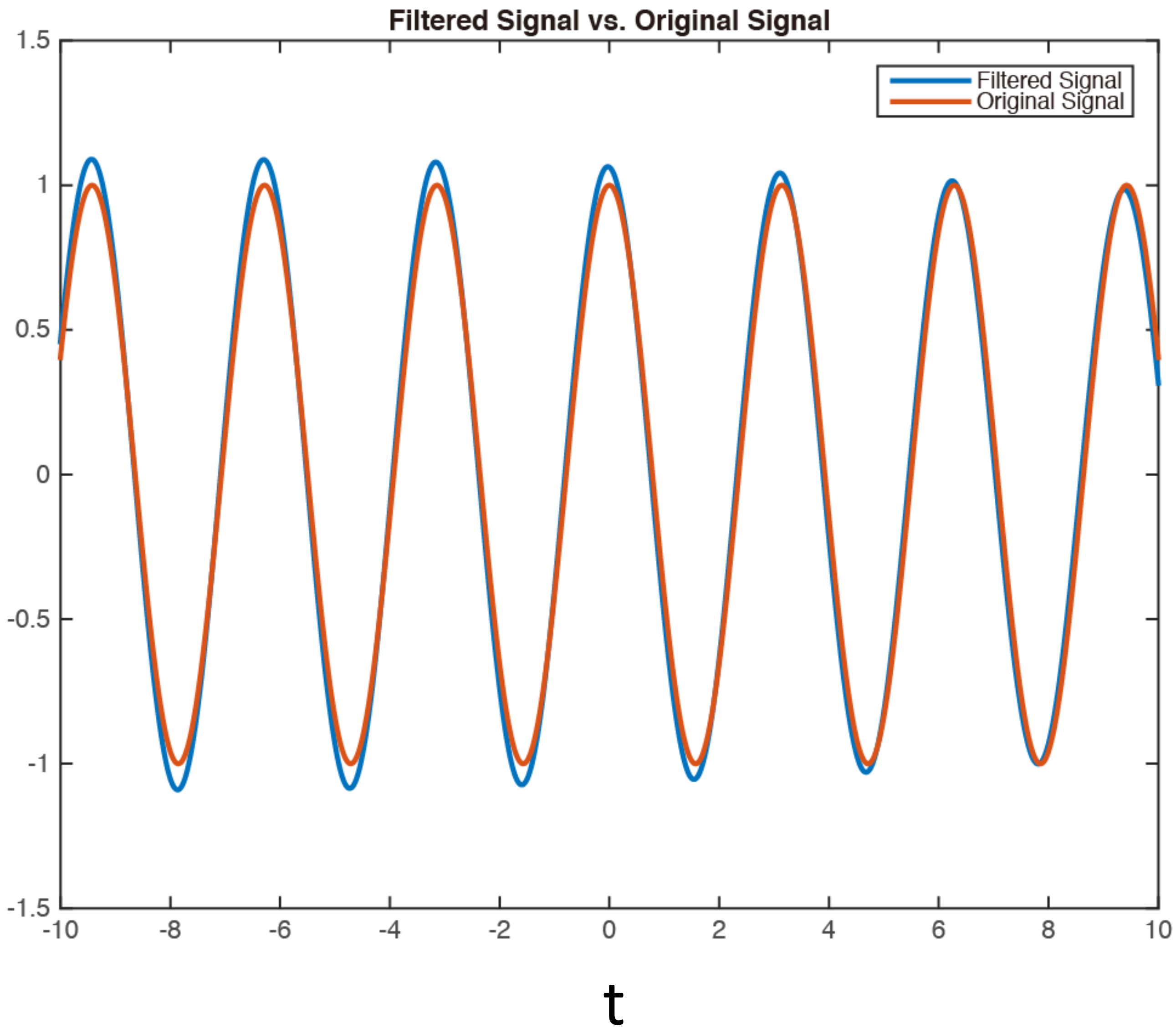
Inverse Fourier Transform



Flow chart