Chapter 4
Higher-order ODEs

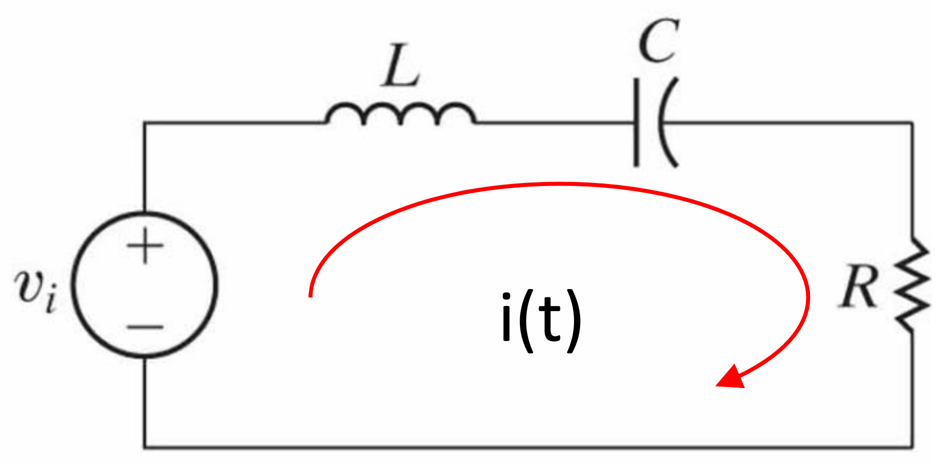
- Preliminary theory (4.1, 4.2)
- Complementary solutions (4.3)
- Particular solutions (4.4, 4.6)
- Systems of linear ODEs (4.8)

Ultrafast Photonics Lab

Edited by Shang-Da Yang (楊尚達)
National Tsing Hua University, Taiwan

Why higher-order ODEs?

- E.g. RLC circuit.



- By Kirchhoff voltage law, $v_L(t) = L\times i'(t)$, $i(t) = C\times v_c'(t)$:

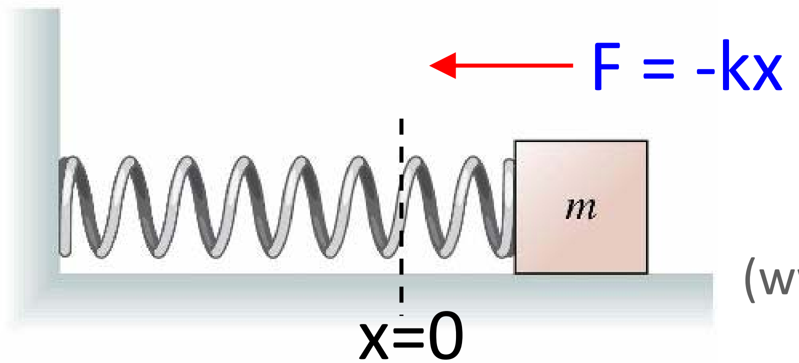
  $v_i(t) = L\times i'(t) + (1/C)\times\int i(t)dt + R\times i(t)$,

  $\Rightarrow$ $i''(t) + P(t)\times i'(t) + Q(t)\times i(t) = g(t)$, a 2nd-order ODE,

  where $P(t) = R/L$, $Q(t) = 1/(LC)$, $g(t) = v_i'(t)/L$.

Why higher-order ODEs?

- E.g. Horizontal mass-spring (harmonic oscillator).



(www.chegg.com)

- By Hook's and Newton's laws:

  $F = -k \times x(t) = m \times x''(t)$,

  $\Rightarrow x''(t) + P(t) \times x'(t) + Q(t) \times x(t) = g(t)$, 2nd-order ODE,

  where $P(t) = 0$, $Q(t) = k/m$, $g(t) = 0$.

❐ Preliminary theory

- Initial-value problems (IVPs)
- Boundary-value problems (BVPs)
- Solution to linear homogeneous ODEs
- Solution to linear nonhomogeneous ODEs
- Seek a 2nd solution by reduction of order

Linear IVPs

- A general linear, nth-order ODE is:

$$a_n(x)\frac{d^n y}{dx^n}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

- For a corresponding IVP, the function value and the values of all of its derivatives up to the (n-1)th order at some common position $x = x_0$ have to be specified:

$$y(x_0)=Y_0,\ \ y'(x_0)=Y_1,\ \ldots,\ y^{(n-1)}(x_0)=Y_{n-1}.$$

Existence of a unique solution

THEOREM 4.1.1 Existence of a Unique Solution

Let $a_n(x)$, $a_{n-1}(x)$, . . . , $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

2nd-order linear BVPs

- A general 2nd-order linear ODE is:

$$y''(x) + P(x) \times y'(x) + Q(x) \times y(x) = g(x)$$

- For a corresponding BVP, the linear combinations of function values and its 1st-order derivatives at different positions x = a, b have to be specified:

$$\begin{cases} \alpha_1 \times y(a) + \beta_1 \times y'(a) = \gamma_1 \\ \alpha_2 \times y(b) + \beta_2 \times y'(b) = \gamma_2 \end{cases}$$

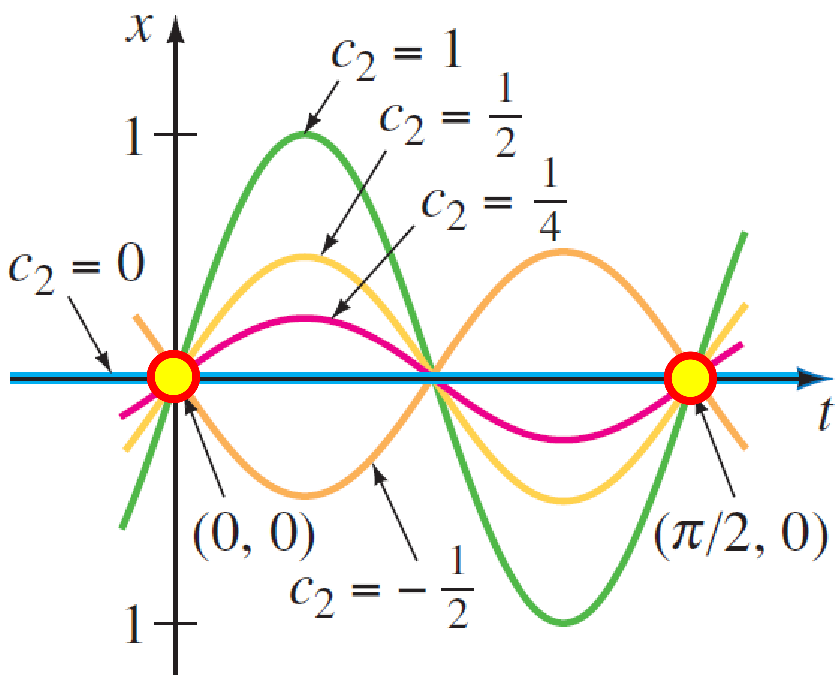
where $\alpha_{1,2}$, $\beta_{1,2}$, $\gamma_{1,2}$ are given constants (provided that $\alpha_i = \beta_i = 0$ is prohibited).

Ultrafast Photonics Lab

3 types of solutions (1)

- E.g. $x''(t) + 16x(t) = 0$
- You can verify that $x(t) = c_1\times\cos(4t) + c_2\times\sin(4t)$, for any combination of constant coefficients c_1, c_2.
- Case 1: Let the two BCs be $\{x(0) = 0, x(\pi/2) = 0\}$.
- Substituting $t = 0$ into the general solution: $c_1\times\cos(0) + c_2\times\sin(0) = c_1 = 0$, $\Rightarrow x(t) = c_2\times\sin(4t)$.
- Substituting $t = \pi/2$ into $x(t) = c_2\times\sin(4t)$, $\Rightarrow$ $c_2\times\sin(2\pi) = 0$, which is true for any choice of c_2.
- Infinitely many solutions: $x(t) = c_2\times\sin(4t)$.

3 types of solutions (2)



- $c_2 \times \sin(4t)$, no matter what the value of c_2 is, satisfies with the ODE $x''(t) + 16x(t) = 0$ and BCs $x(0) = 0$, $x(\pi/2) = 0$.

- Case 2: Let the two BCs be $\{x(0) = 0, x(\pi/8) = 0\}$.
- $x(0) = 0$ still leads to $c_1 = 0$. Substituting $t = \pi/8$ into $x(t) = c_2 \times \sin(4t)$, $\Rightarrow c_2 \times \sin(\pi/2) = c_2 = 0$.
- $x(t) = 0$ is the unique (but trivial) solution to the BVP.

3 types of solutions (3)

- Case 3: Let the two BCs be $\{x(0) = 0, x(\pi/2) = 1\}$.
- $x(0) = 0$ still leads to $c_1 = 0$.
- Substituting $t = \pi/2$ into $x(t) = c_2 \times \sin(4t)$, $\Rightarrow$ $c_2 \times \sin(2\pi) = 0 = 1$, $\Rightarrow$ a contradiction occurs.
- No solution to the BVP.
- Conclusion: BVPs essentially differ from IVPs. We will revisit BVPs in Ch11.

Linear homogeneous (LH) ODEs

- A general 2nd-order linear homogeneous ODE is:

$$y''(x)+P(x)\times y'(x)+Q(x)\times y(x)=0$$

where there is no excitation [g(x) = 0].

- Apparently, y(x) = 0 is always a (trivial) solution.
- For simplicity, we will assume P(x) and Q(x) are continuous within the interval of interest I.

Superposition principle (LH ODEs)

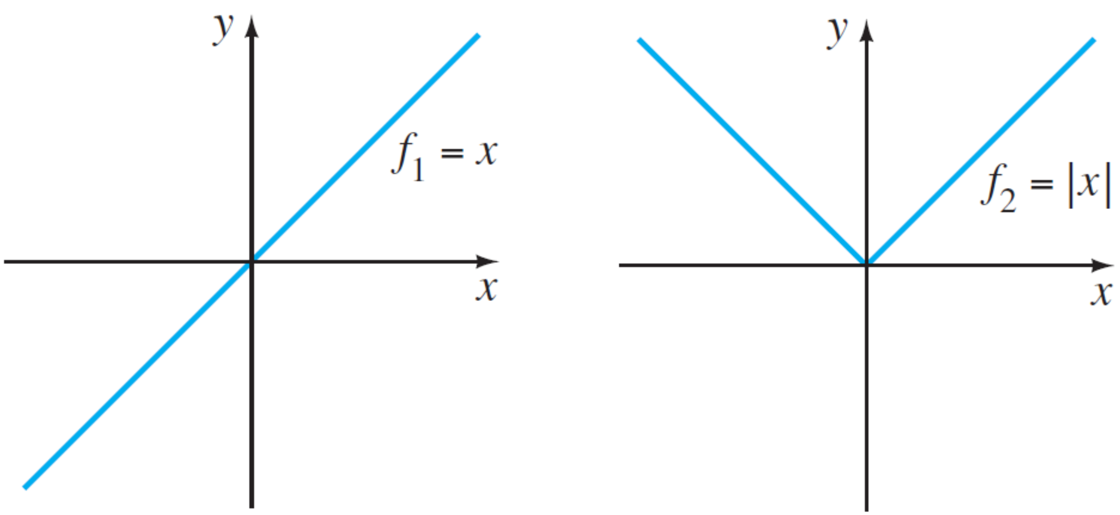
- Let $y_1(x)$, $y_2(x)$ be solutions to $y'' + P(x)y' + Q(x)y = 0$, i.e. $(y_{1,2})'' + P(x)(y_{1,2})' + Q(x)y_{1,2} = 0$. $\Rightarrow$ $y(x) = c_1y_1(x) + c_2y_2(x)$ (c_1, c_2 are arbitrary constants) is a solution.
- Proof: $y' = [c_1y_1(x) + c_2y_2(x)]' = c_1y_1' + c_2y_2'$; $y'' = [c_1y_1(x) + c_2y_2(x)]'' = c_1y_1'' + c_2y_2''$;

  $\Rightarrow y'' + P(x)y' + Q(x)y = [c_1y_1'' + c_2y_2''] + P(x)[c_1y_1' + c_2y_2'] + Q(x)[c_1y_1 + c_2y_2] = c_1[y_1'' + P(x)y_1' + Q(x)y_1] + c_2[y_2'' + P(x)y_2' + Q(x)y_2] = c_1\times 0 + c_2\times 0 = 0$,

  $\Rightarrow$ $y(x)$ is also a solution.

Linear (in)dependence

- Two functions $\{f_1(x), f_2(x)\}$ are linearly dependent on an interval I if they are in proportional to each other, i.e. $f_1(x) = c \times f_2(x)$, for every x in the interval I.
- $\{f_1(x), f_2(x)\}$ are linearly independent if $f_1(x) \neq c \times f_2(x)$.
- E.g. $\{x, |x|\}$ are linearly independent over $(-\infty,\infty)$.



Ultrafast Photonics Lab

If there are n functions…

- n functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are linearly dependent on an interval I if there exist n constants $\{c_1, c_2, \ldots, c_n$; not all zero$\}$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \text{ for all } x \text{ on } I.$$

- In other words, $f_i(x)$ $(i = 1, 2, \ldots, n)$ can be represented by the linear combination of the remaining n-1 functions in the set.
  $\Rightarrow f_i(x)$ is a “redundant” member of the set.

Ultrafast Photonics Lab

Example 5 (n = 4)

- Let $\{f_1 = \cos^2 x,\ f_2 = \sin^2 x,\ f_3 = \sec^2 x,\ f_4 = \tan^2 x\}$.
- $f_i(x) \neq c \times f_j(x)$, for any $i \neq j$.
- One can find a set of coefficients $\{c_1 = 1,\ c_2 = 1,\ c_3 = -1,\ c_4 = 1\}$ such that:

  $1 \times f_1(x) + 1 \times f_2(x) + (-1) \times f_3(x) + 1 \times f_4(x) = (\cos^2 x + \sin^2 x) - \sec^2 x + \tan^2 x = 1 + (-1) = 0$ for $x \in (-\infty, \infty)$
- In other words, $f_1(x) = f_3 - f_4 - f_2$, $f_2(x) = f_3 - f_4 - f_1$, ...
- $\{f_1, f_2, f_3, f_4\}$ are linearly dependent.

Ultrafast Photonics Lab

Linearly independent solutions

- Two solutions $\{y_1(x), y_2(x)\}$ of a 2nd-order linear homogeneous ODE

$$y''(x) + P(x) \times y'(x) + Q(x) \times y(x) = 0$$

are linearly independent if and only if their Wronskian $W(y_1, y_2) \neq 0$ for every x, where

$$W(y_1, y_2) \equiv \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- How to generalize W for a nth-order LH ODE?

Why is Wronskian useful?

- $\{y_1(x), y_2(x)\}$ are linearly independent if $c_1y_1 + c_2y_2 = 0$ is true only when $\{c_1 = 0, c_2 = 0\}$.
- Need two equations to solve c_1, c_2. Creating Eq. (2):

$$\begin{cases} c_1y_1(x)+c_2y_2(x)=0\cdots(1), \\ c_1y_1'(x)+c_2y_2'(x)=0\cdots(2), \end{cases} \Rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$c_{1,2}=\frac{W_{1,2}}{W},\ W_1=\begin{vmatrix} 0 & y_2 \\ 0 & y_2' \end{vmatrix}=0,\ W_2=\begin{vmatrix} y_1 & 0 \\ y_1' & 0 \end{vmatrix}=0.$$

- If $W \neq 0$, $\{c_1 = 0, c_2 = 0\}$ are unique (trivial) solution.
- If $W = 0$, $\{c_1, c_2\}$ have infinitely many solutions.

Ultrafast Photonics Lab

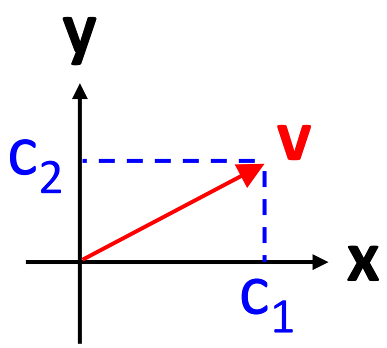
Fundamental set of solutions

- According to the superposition principle, the general (complementary) solution to a 2nd-order linear homogeneous ODE is the linear combination of two linearly independent solutions $\{y_1(x), y_2(x)\}$:

  $y_c(x) = c_1y_1(x) + c_2y_2(x)$, $\{c_1, c_2\}$ are arbitrary const.

- Analogy: A 2D vector can be represented by the linear combination of two orthogonal vectors $\{\mathbf{x},\mathbf{y}\}$:

$$\mathbf{V} = c_1\mathbf{x} + c_2\mathbf{y}.$$



Solution to LH IVP (1)

- Let $\{y_1(x), y_2(x)\}$ be linearly independent solutions to $y'' + P(x)y' + Q(x)y = 0$; $\Rightarrow$
  - $(y_{1,2})'' + P(x)(y_{1,2})' + Q(x)y_{1,2} = 0$.
  - $W(y_1,y_2) \neq 0$ for all x in the interested interval I.
- Note: $y_1(x)$, $y_2(x)$ do not have to satisfy the two ICs: $y(x_0) = Y_0$, $y'(x_0) = Y_1$, where $x_0 \in \mathrm{I}$ and $\{Y_0, Y_1\}$ are arbitrary constants.
- By the superposition principle, $G(x) \equiv c_1y_1(x) + c_2y_2(x)$ satisfies the ODE for any coefficients $\{c_1, c_2\}$.

Ultrafast Photonics Lab

Solution to LH IVP (2)

- Since $W(y_1,y_2) \neq 0$, the system of algebraic equations

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = Y_0, \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = Y_1, \end{cases} \Rightarrow \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

must give a unique combination of c_1, c_2, where

$$c_{1,2} = \frac{W_{1,2}}{W(y_1(x_0), y_2(x_0))}, \; W_1 = \begin{vmatrix} Y_0 & y_2(x_0) \\ Y_1 & y_2'(x_0) \end{vmatrix}, \; W_2 = \begin{vmatrix} y_1(x_0) & Y_0 \\ y_1'(x_0) & Y_1 \end{vmatrix}.$$

- G(x) satisfies the ICs, for $G(x_0) = c_1y_1(x_0) + c_2y_2(x_0) = Y_0$, $G'(x_0) = c_1y'_1(x_0) + c_2y'_2(x_0) = Y_1$.
- $\Rightarrow$ G(x) is the unique solution to the (ODE + IC).

Ultrafast Photonics Lab

Example 7

- ODE: y'' – 9y = 0.
- $y_1(x) = e^{3x}$, $y_2(x) = e^{-3x}$ satisfy the LH ODE (verify).
- Since

$$W(y_1, y_2) \equiv \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3-3 = -6 \neq 0$$

$\{y_1(x), y_2(x)\}$ are linearly independent for $x \in (-\infty,\infty)$.

- General solution must be: $y(x) = c_1e^{3x} + c_2e^{-3x}$.

Example 9

- ODE: y''' - 6y'' + 11y' - 6y = 0.
- $y_1(x) = e^x$, $y_2(x) = e^{2x}$, $y_3(x) = e^{3x}$ satisfy the 3rd-order LH ODE (verify).
- Since

$$W(y_1, y_2, y_3) \equiv \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

$\{y_1, y_2, y_3\}$ are linearly independent for $x \in (-\infty,\infty)$.

- General solution must be: $y(x) = c_1e^x + c_2e^{2x} + c_3e^{3x}$.

Linear nonhomogeneous (LN) ODEs

- A 2nd-order linear nonhomogeneous ODE is:

$$y''(x) + P(x) \times y'(x) + Q(x) \times y(x) = g(x)$$

- It can correspond to an IVP or BVP, depending the given conditions:
- IVP: if two ICs $y(x_0) = Y_0,\ y'(x_0) = Y_1$ are given;
- BVP: if two BCs

$$\begin{cases} \alpha_1 \times y(a) + \beta_1 \times y'(a) = \gamma_1 \\ \alpha_2 \times y(b) + \beta_2 \times y'(b) = \gamma_2 \end{cases}$$

are specified.

General solution to LN ODEs

- General solution is

$$y(x) = y_c(x) + y_p(x), \text{ where}$$

- $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ is the complementary solution to the corresponding LH ODE with $g(x) = 0$; $\{c_1, c_2\}$ are determined by ICs or BCs;
- $y_p(x)$ is a particular solution to the original LN ODE with $g(x) \neq 0$. It's independent of ICs or BCs.

- Note: In the absence of IC or BC, $y_p(x)$ alone is a solution to the LN ODE. It only refers to the "steady state" of the system (Example 5, Sec. 2.3).

Ultrafast Photonics Lab

Proof

- $y_c(x) = c_1y_1(x) + c_2y_2(x)$ satisfies the LH ODE: $(y_c)'' + P(x)\times(y_c)' + Q(x)\times y_c = 0$.
- $y_p(x)$ satisfies the LN ODE: $(y_p)'' + P(x)\times(y_p)' + Q(x)\times y_p = g(x)$.
- Substituting $y(x) = y_c(x) + y_p(x)$ into the LN ODE:
- $(y_c + y_p)'' + P(x)\times(y_c + y_p)' + Q(x)\times(y_c + y_p)$

  $= [(y_c)'' + P(x)\times(y_c)' + Q(x)\times(y_c)] + [(y_p)'' + P(x)\times(y_p)' + Q(x)\times y_p]$

  $= 0 + g(x) = g(x)$, $\Rightarrow$ $(y_c + y_p)$ is a solution to LN ODE.

Example 10

- ODE: $y''' - 6y'' + 11y' - 6y = 3x$.
- $y_1(x) = e^x$, $y_2(x) = e^{2x}$, $y_3(x) = e^{3x}$ satisfy the 3rd-order LH ODE (Example 9).
- $y_p(x) = -(11/12) - x/2$ is a particular solution. (verify)
- General solution must be:

$$y(x) = (c_1 e^x + c_2 e^{2x} + c_3 e^{3x}) - (11/12) - x/2.$$

Superposition principle (LN ODEs)

- Let $y_{p1}(x)$, $y_{p2}(x)$ be particular solutions to $y'' + P(x)y' + Q(x)y = g_{1,2}(x)$, i.e. $(y_{p1,p2})'' + P(x)(y_{p1,p2})' + Q(x)y_{p1,p2} = g_{1,2}(x)$. $\Rightarrow$ $y_p(x) = y_{p1}(x) + y_{p2}(x)$ is a particular solution to $y'' + P(x)y' + Q(x)y = g_1(x) + g_2(x)$.
- Proof: $(y_p)' = [y_{p1}(x) + y_{p2}(x)]' = (y_{p1})' + (y_{p2})'$; $(y_p)'' = [y_{p1}(x) + y_{p2}(x)]'' = (y_{p1})'' + (y_{p2})''$;

  $\Rightarrow$ $(y_p)'' + P(x)(y_p)' + Q(x)y_p = [(y_{p1})'' + (y_{p2})''] + P(x)[(y_{p1})' + (y_{p2})'] + Q(x)[y_{p1} + y_{p2}] = [(y_{p1})'' + P(x)(y_{p1})' + Q(x)y_{p1}] + [(y_{p2})'' + P(x)(y_{p2})' + Q(x)y_{p2}] = g_1(x) + g_2(x)$; $\Rightarrow$ $y_p(x)$ is a particular solution.

Motivation of reduction of order

- For a 2nd-order LH ODE: $y'' + P(x)y' + Q(x)y = 0$, we need two linearly independent solutions $y_1(x)$, $y_2(x)$ to expand the complementary solution $y_c(x)$.
- If we have $y_1(x)$ already, a linearly independent solution $y_2(x)$ can be found systematically by assuming $y(x) = u(x)\times y_1(x)$ and solving a linear separable 1st-order ODE of $w(x) \equiv u'(x)$.
- $y(x) \neq c\times y_1(x)$ (linearly independent) if $u(x) \neq \text{const.}$

Example 1 (Sec. 4.2)

- E.g. $y''(x) - y(x) = 0$ (2nd-order LH ODE)
- You can verify that $y_1(x) = e^x$ is a solution.
- Let $y(x) = u(x)\times e^x$, $\Rightarrow$ $y' = u'\times e^x + u\times e^x$, $y'' = u''\times e^x + 2u'\times e^x + u\times e^x$;
- $y'' - y = [(u'' + 2u' + u) - u]\times e^x = (u'' + 2u')\times e^x = 0$, $\Rightarrow$ $u'' + 2u' = 0$.
- Let $w = u'$, $\Rightarrow$ $w' + 2w = 0$ (1st-order LH ODE).
- It's separable: $-dw/(2w) = dx$, $w = c_1 e^{-2x}$.
- $u(x) = \int w dx = -(c_1/2)e^{-2x} + c_2$, $y = -(c_1/2)e^{-x} + c_2 e^x$, $\Rightarrow$ $y_2 = e^{-x}$.

- Complementary solutions
  - Auxiliary equation
  - Three types of solutions for 2nd-order LH ODEs
  - Two physical examples

Motivation

- For 1st-order LH ODEs $y'(x) + P(x)\times y = 0$, we have learnt two methods to solve them:
- separation of variables (Sec. 2.2)
- Integrating factor (Sec. 2.3)
- For constant coefficient $P(x) = -k$, ODE $y'(x) = k\times y$ has an intuitive solution:

$$y(x) = e^{mx}, \text{ for } y'(x) \propto y(x), \text{ self-consistent.}$$

- Substituting $y = e^{mx}$ into $y' = k\times y$ gives $me^{mx} = ke^{mx}$, $\Rightarrow m = k$, a 1st-order polynomial equation.
- $\Rightarrow$ Complementary solution is: $y_c(x) = c\times e^{kx}$.

Ultrafast Photonics Lab

2nd-order LH ODEs of constant coefficient

- ODE : $y'' + Py' + Qy = 0$, where P and Q are constant.
- Assume $y(x) = e^{mx}$, for $y''(x) \propto y'(x) \propto y(x)$.
- Substituting $y = e^{mx}$ into $y'' + Py' + Qy = 0$ gives a 2nd-order polynomial equation (auxiliary equation) $m^2 + Pm + Q = 0$.
- There are three types of solutions:
- Case 1: Two distinct real roots $m = \{m_1, m_2\}$
- Case 2: One repeated root $m = m_1$.
- Case 3: Two complex conjugate roots $m = \alpha \pm j\beta$.

Ultrafast Photonics Lab

Case 1: Two distinct real roots

- $m = \{m_1, m_2\}, \Rightarrow \{e^{m_1 x}, e^{m_2 x}\}$ are solutions.
- The Wronskian is nonzero for $x \in (-\infty,\infty)$ if $m_1 \neq m_2$:

$$W(e^{m_1 x}, e^{m_2 x}) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = (m_2 - m_1)e^{(m_1+m_2)x} \neq 0$$

- $\{e^{m_1 x}, e^{m_2 x}\}$ are linearly independent solutions, $\Rightarrow$ Complementary solution is their linear combination:

$$y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

- When accessing the ICs or BCs, derivative is needed:

$$y'_c(x) = c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x}$$

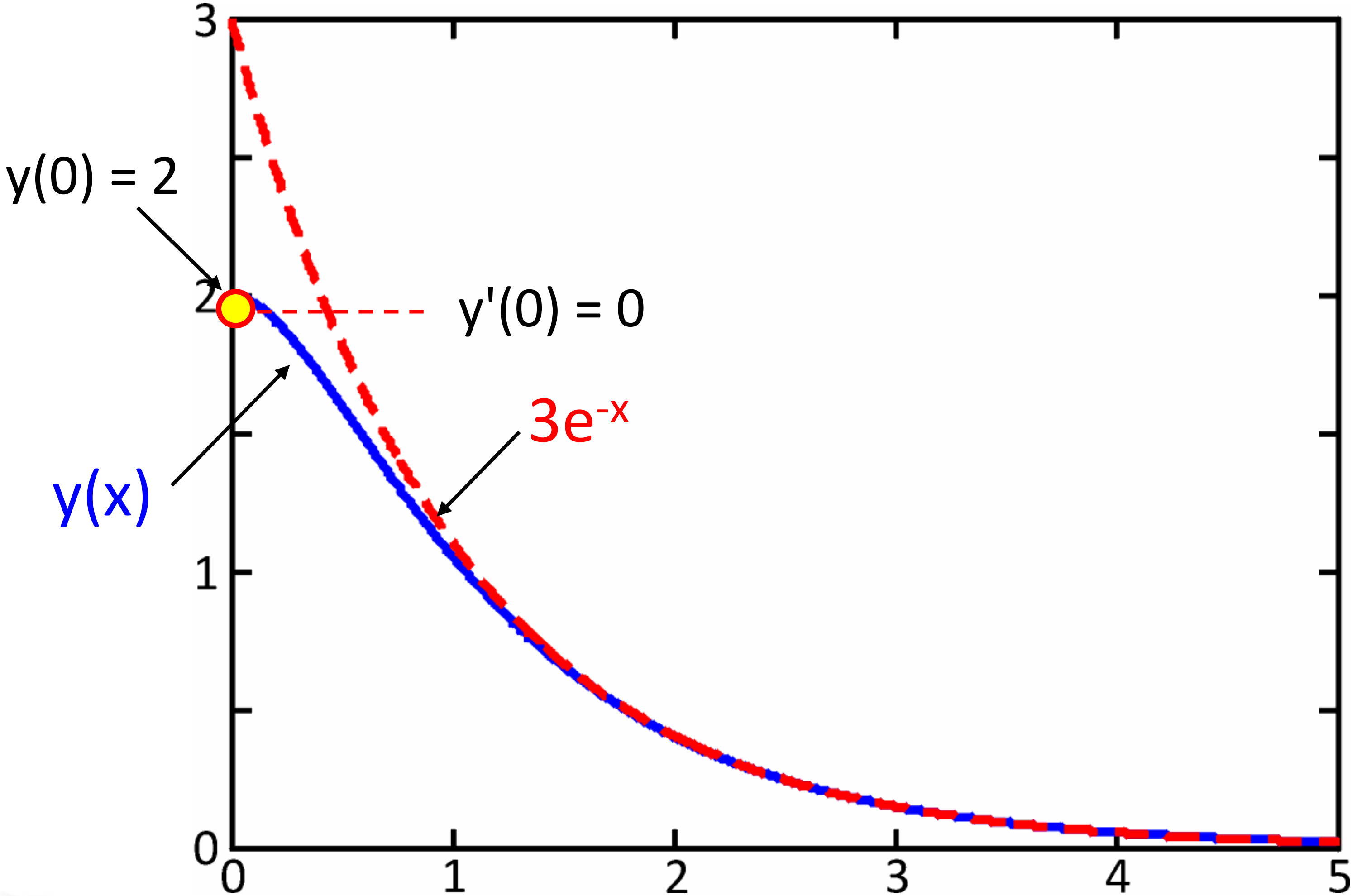
An IVP example of Case 1

- ODE: $y'' + 4y' + 3y = 0$
- ICs: $y(0) = 2$, $y'(0) = 0$
- Auxiliary equation: $m^2 + 4m + 3 = 0$, $\Rightarrow$ two distinct real roots $\{m_1 = -1, m_2 = -3\}$.
- $y_c(x) = c_1e^{-x} + c_2e^{-3x}$, $\Rightarrow y_c(0) = c_1 + c_2 = 2$.
- $y'_c(x) = -c_1e^{-x} - 3c_2e^{-3x}$, $\Rightarrow y'_c(0) = -c_1 - 3c_2 = 0$.
- Solving the system of algebraic equations gives $\{c_1 = 3, c_2 = -1\}$. $\Rightarrow$ The unique solution of the IVP is:

$$y(x) = 3e^{-x} - e^{-3x}$$

Ultrafast Photonics Lab

Unique solution plot



Case 2: One repeated real root (1)

- $m = m_1, \Rightarrow e^{m_1 x}$ is a solution.
- Reduction of order: given $y_1(x) = e^{mx}$ is one solution to the LH ODE, the 2nd solution linearly independent of $y_1(x)$ can be found by assuming $y(x) = u(x) \times y_1(x)$, … $\Rightarrow y_2(x) = xe^{mx}$.
- Verify the linear independence by Wroskian:

$$W(e^{m_1 x}, xe^{m_1 x}) = \begin{vmatrix} e^{m_1 x} & xe^{m_1 x} \\ m_1 e^{m_1 x} & (1+m_1 x)e^{m_1 x} \end{vmatrix} = e^{2m_1 x} \neq 0$$

Case 2: One repeated real root (2)

- $\{e^{m_1 x}, xe^{m_1 x}\}$ are linearly independent solutions, $\Rightarrow$ Complementary solution is their linear combination:

$$y_c(x) = (c_1 + c_2 x)\times e^{m_1 x}$$

- When accessing the ICs or BCs, derivative is needed:

$$y'_c(x) = [(c_1 m_1 + c_2) + (c_2 m_1)x]\times e^{m_1 x}$$

Ultrafast Photonics Lab

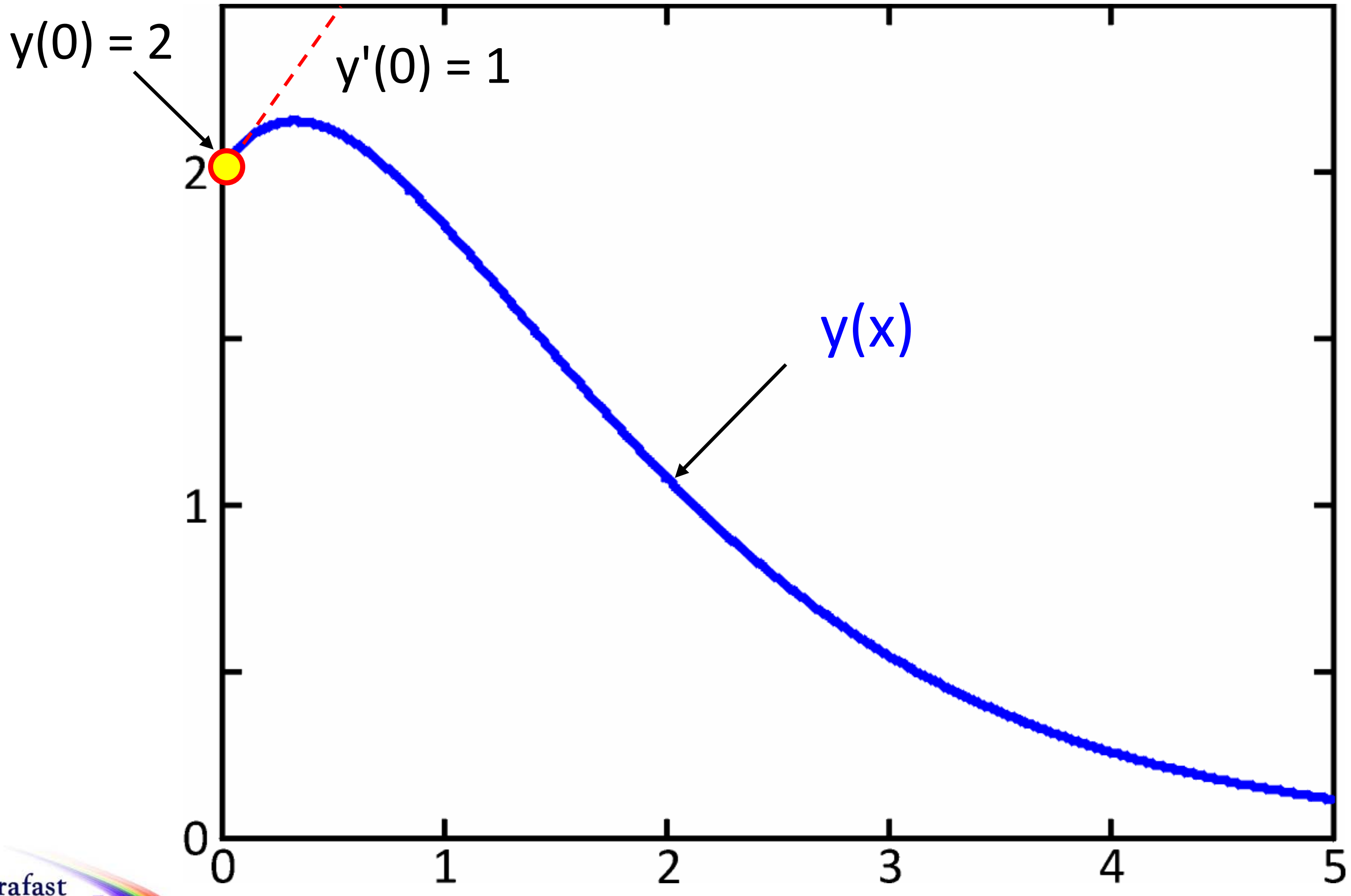
An IVP example of Case 2

- ODE: $y'' + 2y' + y = 0$
- ICs: $y(0) = 2$, $y'(0) = 1$
- Auxiliary equation: $m^2 + 2m + 1 = 0$, $\Rightarrow$ one repeated root $\{m_1 = -1\}$.
- $y_c(x) = (c_1 + c_2x)\times e^{-x}$, $\Rightarrow y_c(0) = c_1 = 2$.
- $y'_c(x) = [(-c_1+c_2)-(c_2)x]\times e^{-x}$, $\Rightarrow y'_c(0) = -c_1 + c_2 = 1$.
- Solving the system of algebraic equations gives $\{c_1 = 2, c_2 = 3\}$. $\Rightarrow$ The unique solution of the IVP is:

$$y(x) = (2 + 3x)\times e^{-x}$$

Ultrafast Photonics Lab

Unique solution plot



Case 3: Complex conjugate roots (1)

- For $m = \alpha \pm j\beta$, Case 1 formula $y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ still applies, $\Rightarrow y_1(x) = e^{\alpha x} \times e^{+j\beta x}$, $y_2(x) = e^{\alpha x} \times e^{-j\beta x}$ are two linearly independent "complex" solutions.
- Let $y_3 \equiv (y_1 + y_2)/2 = e^{\alpha x}(e^{j\beta x} + e^{-j\beta x})/2 = e^{\alpha x}\{[\cos(\beta x) + j\sin(\beta x)] + [\cos(\beta x) - j\sin(\beta x)]\}/2 = e^{\alpha x} \times \cos(\beta x)$, where Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$ is used.
- Similarly, $y_4 \equiv (y_1 - y_2)/(2j) = e^{\alpha x}(e^{j\beta x} - e^{-j\beta x})/(2j) = e^{\alpha x} \times \sin(\beta x)$.
- By the superposition principle (p12), $\{y_3, y_4\}$ remain solutions to the LH ODE.

Case 3: Complex conjugate roots (2)

- The Wronskian $W(y_3, y_4)$ is:

$$\begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ e^{\alpha x}(\alpha\cos\beta x-\beta\sin\beta x) & e^{\alpha x}(\alpha\sin\beta x+\beta\cos\beta x) \end{vmatrix} = \beta e^{2\alpha x} \neq 0$$

$\Rightarrow$ $\{y_3, y_4\}$ are linearly independent.

- Complementary solution to the 2nd-order LH ODE can also be the linear combination of $\{y_3, y_4\}$:

$$y_c(x) = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$$

- When accessing the ICs or BCs, derivative is needed:

$$y'_c(x) = e^{\alpha x}[(c_1\alpha+c_2\beta)\times\cos(\beta x) + (c_2\alpha-c_1\beta)\times\sin(\beta x)]$$

Ultrafast Photonics Lab

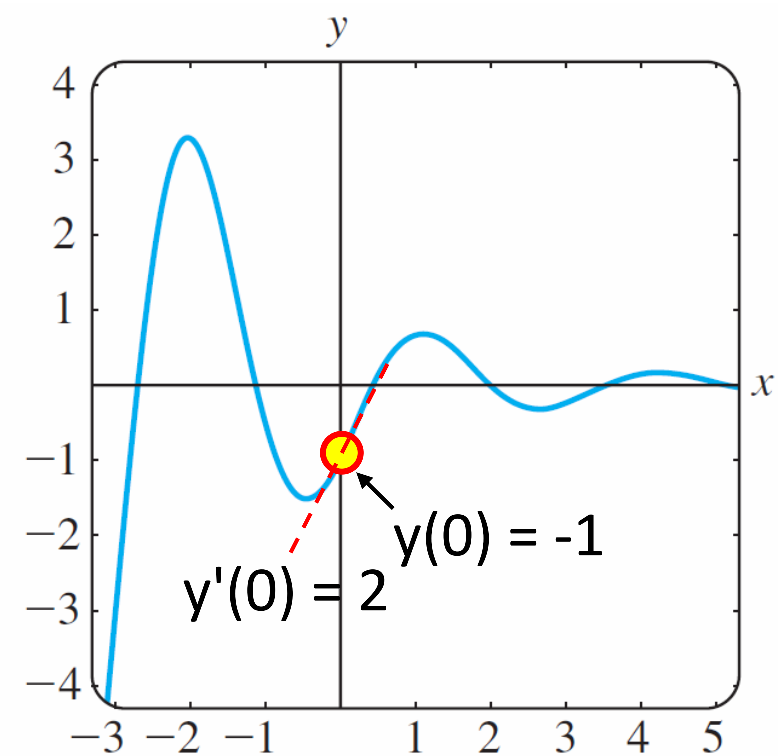
Example 2

- ODE: $4y'' + 4y' + 17y = 0$
- ICs: $y(0) = -1$, $y'(0) = 2$
- Auxiliary equation: $4m^2 + 4m + 17 = 0$, $\Rightarrow m = \alpha \pm j\beta = (-1/2) \pm j(2)$.
- $y_c(x) = e^{-x/2}[c_1\cos(2x) + c_2\sin(2x)]$, $\Rightarrow y_c(0) = 1\times[c_1\times 1 + c_2\times 0] = c_1 = -1$.
- $y'_c(x) = e^{-x/2}[(-c_1/2 + 2c_2)\times\cos(2x) + (-c_2/2 - 2c_1)\times\sin(2x)]$, $\Rightarrow y'_c(0) = 1\times[(½ + 2c_2)\times 1 + (-c_2/2 + 2)\times 0] = ½ + 2c_2 = 2$, $c_2 = ¾$.

Unique solution plot

- The IVP gives a unique solution:

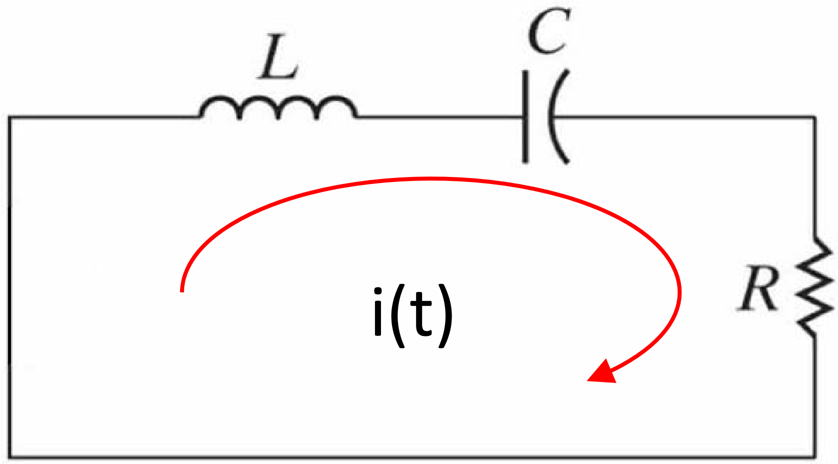
$$y(x) = e^{-x/2}[-1\cos(2x) + \tfrac{3}{4}\sin(2x)]$$



- y(x) behaves "damped oscillation", converging to zero as $x \to \infty$.
- Envelope decay depends on Re{m} = α (<0).
- Fringe oscillation depends on Im{m} = β.

Source-free RLC circuit

- ODE: $i''(t) + (R/L)\times i'(t) + 1/(LC)\times i(t) = 0$.



- Auxiliary equation: $m^2 + (R/L)m + 1/(LC) = 0, \Rightarrow$

$$m = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

Three types of natural response

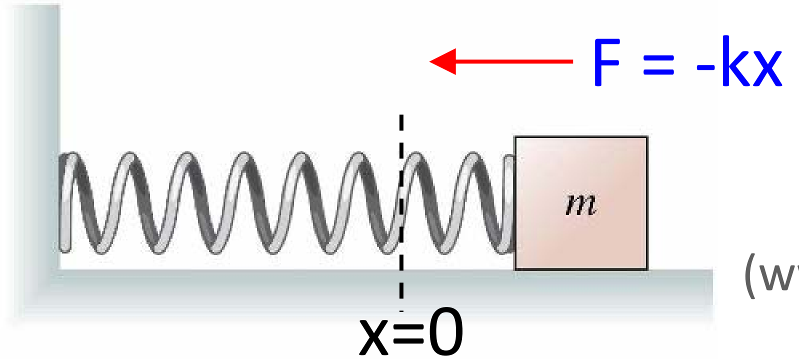
The Circuit is	When	Solutions
Over-damped	$R^2 > 4L/C$	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Critically-damped	$R^2 = 4L/C$	$(c_1 + c_2 x) \times e^{m_1 x}$
Under-damped	$R^2 < 4L/C$	$e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

- In Case 3 (under-damped), $\alpha = -R/(2L)$, $\Rightarrow$ envelope decays faster if R is larger.

Ultrafast Photonics Lab

Horizontal frictionless mass-spring

- ODE: $x''(t) + (k/m)x(t) = 0$.



(www.chegg.com)

- Auxiliary equation: $m^2 + k/m = 0$, $\Rightarrow m = \alpha \pm j\beta = \pm j\sqrt{\frac{k}{m}}$
- $x_c(t) = e^{\alpha x}[c_1\cos(\beta t) + c_2\sin(\beta t)] = c_1\cos[\sqrt{(k/m)}t] + c_2\sin[\sqrt{(k/m)}t] = A_0\times\cos[\sqrt{(k/m)}t+\phi]$, pure oscillation.
- Oscillation is faster if m is smaller or k is larger.

- Particular solutions
  - Method of undetermined coefficients
  - Method of variation of parameters

Method of undetermined coefficient

- A 2nd-order linear nonhomogeneous ODE of constant coefficients is:

$$y'' + P\times y' + Q\times y = g(x)$$

- If g(x) is one of the following forms:
  (1) polynomial, (2) $e^{\alpha x}$, (3) $\sin(\beta x)$, $\cos(\beta x)$, or (4) sums or products of (1-3), $\Rightarrow$ particular solution $y_p(x)$ and driving source g(x) should be of the same form.
- It does not apply if g(x) = ln(x), 1/x, tan(x), …

Ultrafast Photonics Lab

Example: g(x) = polynomial

- ODE: $y'' + y' + 2y = 4x^2 + 1$
- $g(x)$ is a 2nd-order polynomial, $\Rightarrow$ an educated guess is $y_p(x) = Ax^2 + Bx + C$. (Why?)
- Substituting $y'_p = (2Ax + B)$, $y''_p = (2A)$ into the ODE:
  $(2A) + (2Ax + B) + 2(Ax^2 + Bx + C)$
  $= (2A)x^2 + (2A+2B)x + (2A+B+2C) = 4x^2 + 1,$

$$\begin{cases} 2A = 4, \\ 2A + 2B = 0, \\ 2A + B + 2C = 1 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = -2 \\ C = -1/2 \end{cases}, \Rightarrow y_p(x) = 2x^2 - 2x - \frac{1}{2}$$

Example: $g(x) = e^{\alpha x}$

- ODE: $y'' - 5y' + 4y = 8e^{2x}$
- $g(x)$ is an exponential function, $\Rightarrow$ an educated guess is $y_p(x) = A\times e^{2x}$. (Why?)
- Substituting $y'_p = (2A\times e^{2x})$, $y''_p = (4A\times e^{2x})$ into the ODE:

  $(4A\times e^{x}) - 5(2A\times e^{2x}) + 4(A\times e^{2x}) = (4A-10A+4A)\times e^{2x} = -2A\times e^{2x} = 8e^{2x}$,

  $\Rightarrow -2A = 8,\ A = -4,\ y_p(x) = -4e^{2x}$.
- Note: $y_p(x)$ does not necessarily satisfy ICs or BCs. It's not the unique solution yet.

Ultrafast Photonics Lab

Example 4: A glitch of the method (1)

- By slightly changing the g(x): $y'' - 5y' + 4y = 8e^x$
- An educated guess is $y_p(x) = Ae^x$.
- Substituting $y'_p = (Ae^x)$, $y''_p = (Ae^x)$ into the ODE:
  $(Ae^x) - 5(Ae^x) + 4(Ae^x) = 0 = 8e^x$, $\Rightarrow$ no solution!
- Auxiliary equation to solve $y_c(x)$ is: $m^2 - 5m + 4 = 0$,
  $\Rightarrow m_1 = 1$, $m_2 = 4$; $\Rightarrow y_c(x) = c_1e^x + c_2e^{4x}$.
- Bug: Ae^x is already present in $y_c(x)$!

Example 4: A glitch of the method (2)

- Try another guess: $y_p(x) = Axe^x$. (Why?)
- Substituting $y'_p = [A(1+x)e^x]$, $y''_p = [A(2+x)e^x]$ into $y'' - 5y' + 4y = 8e^x$:

  $[A(2+x)e^x] - 5[A(1+x)e^x] + 4(Axe^x)$

  $= (2A-5A)e^x + (A-5A+4A)xe^x$

  $= (-3A)e^x + (0)xe^x = 8e^x$,

  $\Rightarrow -3A = 8$, $A = -8/3$, $y_p(x) = (-8/3)xe^x$

Ultrafast Photonics Lab

Example: $g(t) = \sin(\omega t)$

- ODE: $y'' + 2y' + 2y = -5\times\sin(2t)$
- Educated guess: $y_p(t) = A\times\cos(2t) + B\times\sin(2t)$.
- Substituting $y'_p = [-2A\times\sin(2t) + 2B\times\cos(2t)]$, $y''_p = [-4A\times\cos(2t) - 4B\times\sin(2t)] = -4y_p$ into the ODE:

  $[-4A\times\cos(2t) - 4B\times\sin(2t)] + 2[-2A\times\sin(2t) + 2B\times\cos(2t)] + 2[A\times\cos(2t) + B\times\sin(2t)]$

  $= (-4A+4B+2A)\times\cos(2t) + (-4B-4A+2B)\times\sin(2t)$

  $= (-2A+4B)\times\cos(2t) + (-4A-2B)\times\sin(2t) = -5\times\sin(2t)$,

$$\begin{cases} -2A+4B=0, \\ -4A-2B=-5, \end{cases} \Rightarrow \begin{cases} A=1 \\ B=\dfrac{1}{2}, \end{cases} \quad y_p(x) = \cos 2t + \frac{1}{2}\sin 2t$$

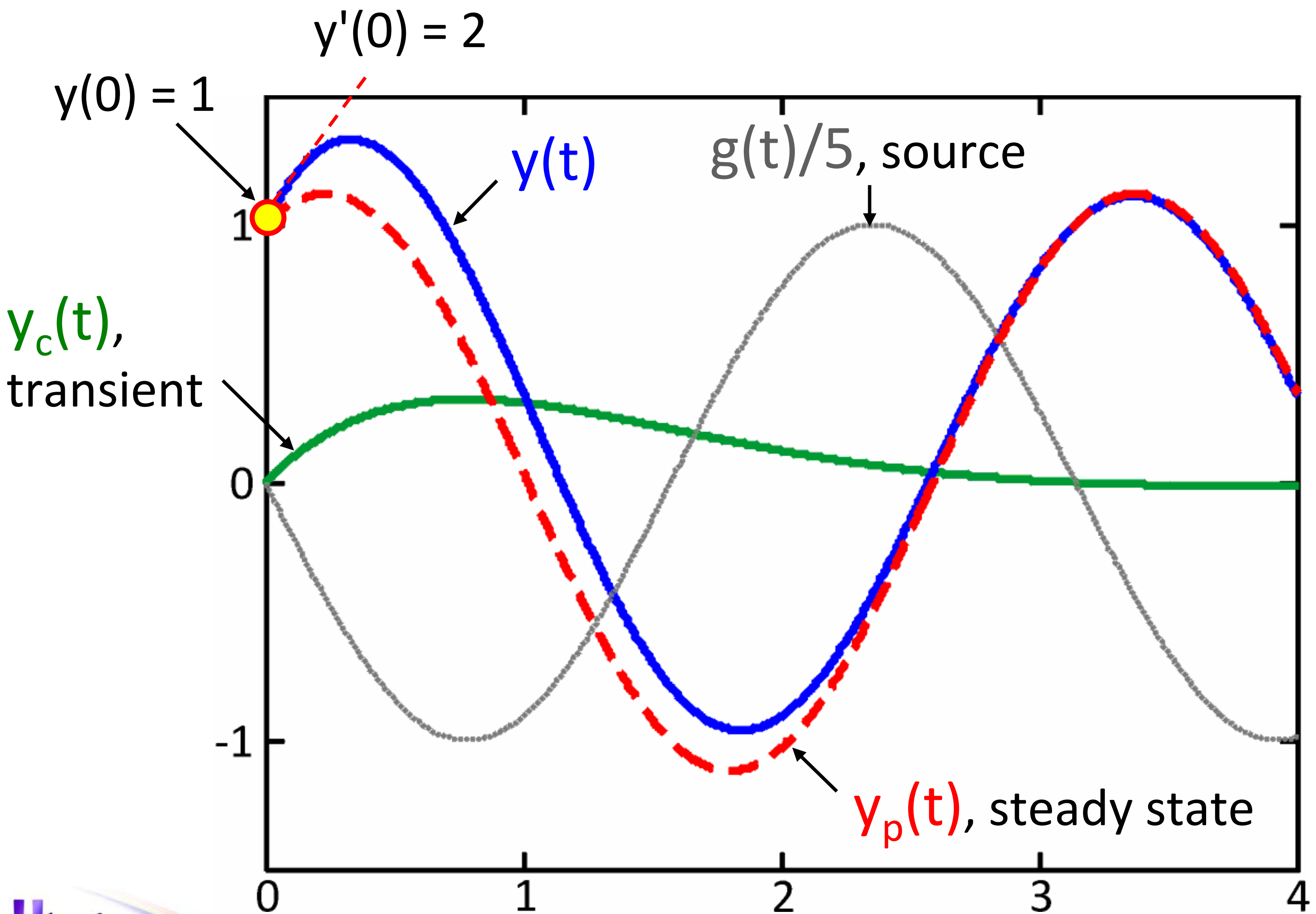
Ultrafast Photonics Lab

TSING HUA UNIVERSITY

IVP

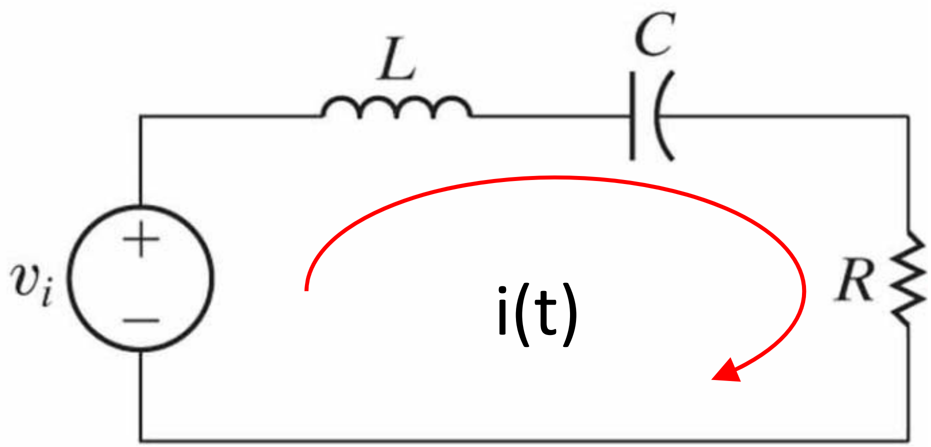
- Let the two associated ICs be: $y(0) = 1$, $y'(0) = 2$.
- Auxiliary equation to solve $y_c(t)$ is: $m^2 + 2m + 2 = 0$,
  $\Rightarrow m = -1 \pm j$ (Case 3), $\Rightarrow y_c(t) = e^{-t}[c_1\cos(t) + c_2\sin(t)]$.
- $y(t) = y_c + y_p = e^{-t}[c_1\cos(t) + c_2\sin(t)] + [\cos(2t) + \frac{1}{2}\times\sin(2t)]$
- $y(0) = c_1 + 1 = 1$, $\Rightarrow c_1 = 0$;
- $y(t) = c_2e^{-t}\sin(t) + [\cos(2t) + \frac{1}{2}\times\sin(2t)]$,
- $y'(t) = c_2e^{-t}[-\sin(t) + \cos(t)] + [-2\sin(2t) + \cos(2t)]$,
- $y'(0) = c_2 + 1 = 2$, $\Rightarrow c_2 = 1$;
- $y(t) = e^{-t}\sin(t) + [\cos(2t) + \frac{1}{2}\times\sin(2t)]$

Unique solution plot



Sinusoidally driven RLC circuit

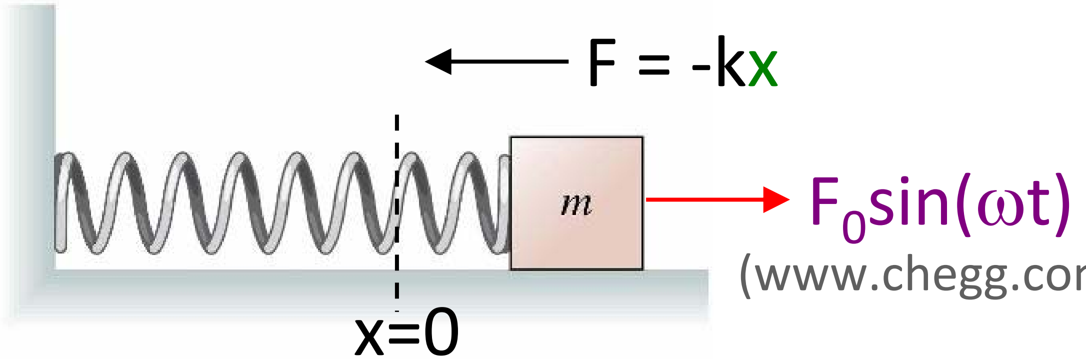
- ODE: $i''(t) + (R/L)\times i'(t) + 1/(LC)\times i(t) = v'(t)/L$.



- Element values R, L, C determines natural (transient) response $i_c(t)$.
- Driving source determines the "shape" of steady state $i_p(t)$, while its amplitude and timing depends on both the RLC circuit and the source.

Sinusoidally driven mass-spring

- ODE: $x''(t) + (k/m)\times x(t) = (F_0/m)\times\sin(\omega t)$.

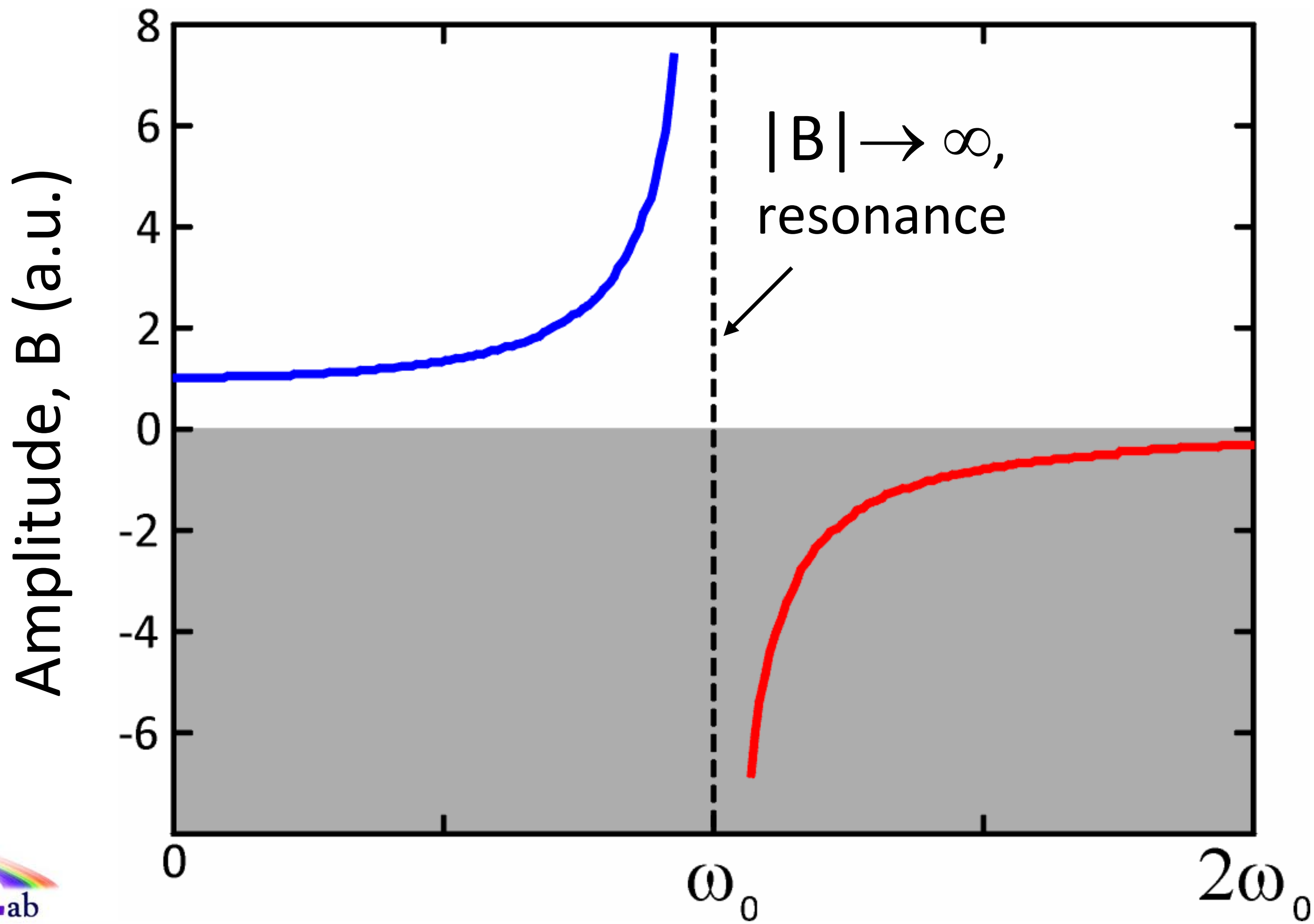


(www.chegg.com)

- $x_c(t) = A_0\cos(\omega_0 t+\phi)$, $\omega_0 = \sqrt{(k/m)}$ is the natural freq.
- Substituting $x_p(t) = A\times\cos(\omega t)+B\times\sin(\omega t)$, $x''_p = -\omega^2 x_p$ into the ODE: $-\omega^2 x_p + (\omega_0)^2 x_p = (F_0/m)\times\sin(\omega t)$,
  $[(\omega_0)^2-\omega^2]\times[A\times\cos(\omega t)+B\times\sin(\omega t)] = (F_0/m)\times\sin(\omega t)$,
  $\Rightarrow A = 0,\ \ B = F_0/\{m[(\omega_0)^2-\omega^2]\}$.

Resonance

- $x(t) = x_c+x_p = A_0\cos(\omega_0 t+\phi)+B\sin(\omega t)$, $B \propto [(\omega_0)^2-\omega^2]^{-1}$.



Method of variation of parameters (1)

- A 2nd-order LN ODE of varying coefficients is:

$$y'' + P(x)\times y' + Q(x)\times y = g(x)$$

- Let $y_c(x) = c_1y_1(x) + c_2y_2(x)$, where $W(y_1,y_2) \neq 0$.
- For any type of g(x), let $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$:

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2'$$

$$y_p'' = u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2'.$$

- Substituting y_p, y'_p, y''_p into the LN ODE:

$$\frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + [y_1'u_1' + y_2'u_2'] = g(x)\cdots(a)$$

Ultrafast Photonics Lab

TSING HUA UNIVERSITY

Method of variation of parameters (2)

- Two equations are needed to solve $u_1(x)$, $u_2(x)$.
- A convenient choice is having $y_1u_1' + y_2u_2' = 0$ such that Eq. (a) reduces to $y_1'u_1' + y_2'u_2' = g(x)$.
- $$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = g(x) \end{cases}, \Rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \times \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix},$$

$$\Rightarrow u_1' = \frac{W_1}{W} = -\frac{y_2 g(x)}{W},\ u_2' = \frac{W_2}{W} = \frac{y_1 g(x)}{W},$$

$$\text{where } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},\ W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix},\ W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}.$$

Example 1 (Sec. 4.6)

- ODE: y'' - 4y' + 4y = $(x+1)e^{2x}$
- Auxiliary equation: $m^2 - 2m + 4 = 0$, $\Rightarrow$ $m_1 = 2$ (Case 2), $\Rightarrow$ {$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$} are two linearly independent solutions to y'' - 4y' + 4y = 0.
- $W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{m_1 x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x},$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & (1+2x)e^{2x} \end{vmatrix} = -(x+1)xe^{4x}, \quad W_2 = (x+1)e^{4x},$$

$$\Rightarrow u_1' = \frac{W_1}{W} = -x^2 - x, \quad u_2' = \frac{W_2}{W} = x+1.$$

Example 1 (2)

- $u'_1 = -x^2 - x, \Rightarrow u_1 = -x^3/3 - x^2/2$;
- $u'_2 = x + 1, \Rightarrow u_2 = x^2/2 + x$;
- $y_p = u_1(x)\times y_1(x) + u_2(x)\times y_2(x)$
  $= (-x^3/3 - x^2/2)\times e^{2x} + (x^2/2 + x)\times(xe^{2x})$
  $= [(x^3/6)+(x^2/2)]e^{2x}$.
- Method of undetermined coefficients: guesses of $y_p(x) = Ae^{2x}$, $(Ax+B)e^{2x}$, $(Ax^2+Bx+C)e^{2x}$ give no solution.
- $y_p(x) = (Ax^3+Bx^2+Cx+D)e^{2x}$ leads to the same solution.

Example 3 (Sec. 4.6)

- ODE: y'' - y = 1/x
- Auxiliary equation: $m^2 - 1 = 0$, $\Rightarrow$ $\{m_1 = 1, m_2 = -1\}$ (Case 1), $\Rightarrow$ $\{y_1(x) = e^x, y_2(x) = e^{-x}\}$ are two linearly independent solutions to y'' - y = 0.
- $$W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2,$$

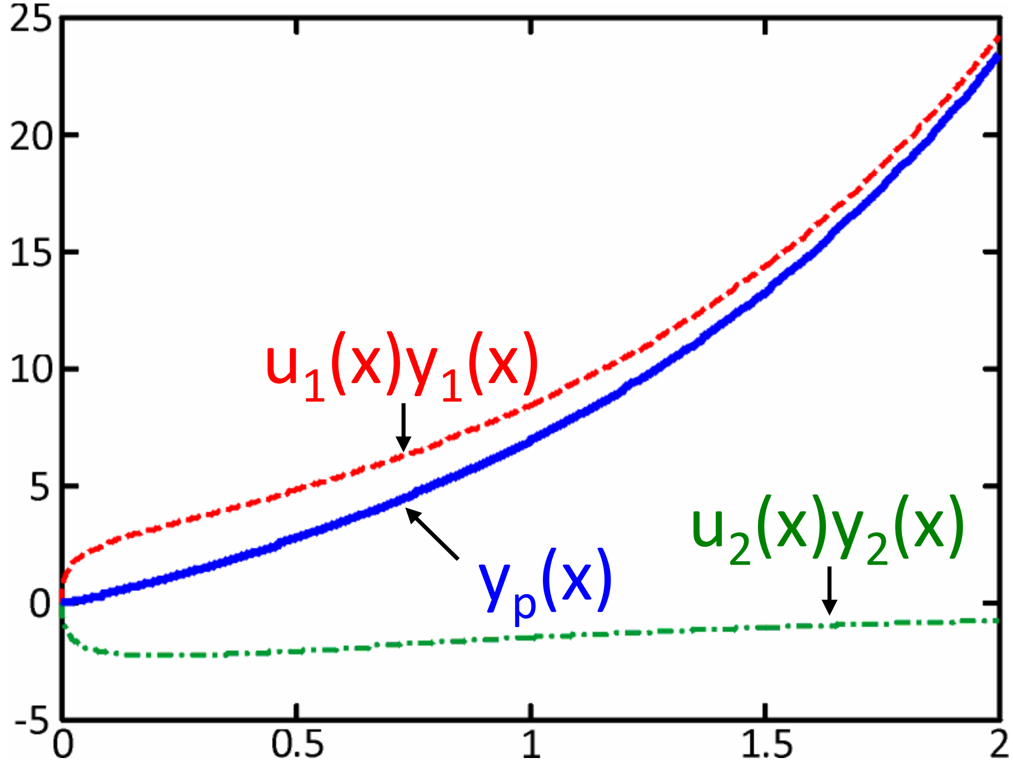
$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 1/x & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{x}, \quad W_2 = \begin{vmatrix} e^x & 0 \\ e^{-x} & 1/x \end{vmatrix} = \frac{e^x}{x},$$

$$\Rightarrow u_1' = \frac{W_1}{W} = \frac{e^{-x}}{2x}, \quad u_2' = \frac{W_2}{W} = -\frac{e^x}{2x}.$$

Example 3 (2)

- $u_1' = \dfrac{e^{-x}}{2x}, \quad \Rightarrow u_1(x) = \dfrac{1}{2}\displaystyle\int \dfrac{e^{-x}}{x}dx = \dfrac{1}{2}\int_{x_0}^{x} \dfrac{e^{-t}}{t}dt; \quad u_2(x) = -\dfrac{1}{2}\int_{x_0}^{x} \dfrac{e^{t}}{t}dt.$

$$y_p(x) = u_1 y_1 + u_2 y_2 = \frac{1}{2}\left[e^{x} \times \left(\int_{x_0}^{x} \frac{e^{-t}}{t}dt \right) - e^{-x} \times \left(\int_{x_0}^{x} \frac{e^{t}}{t}dt \right) \right]$$

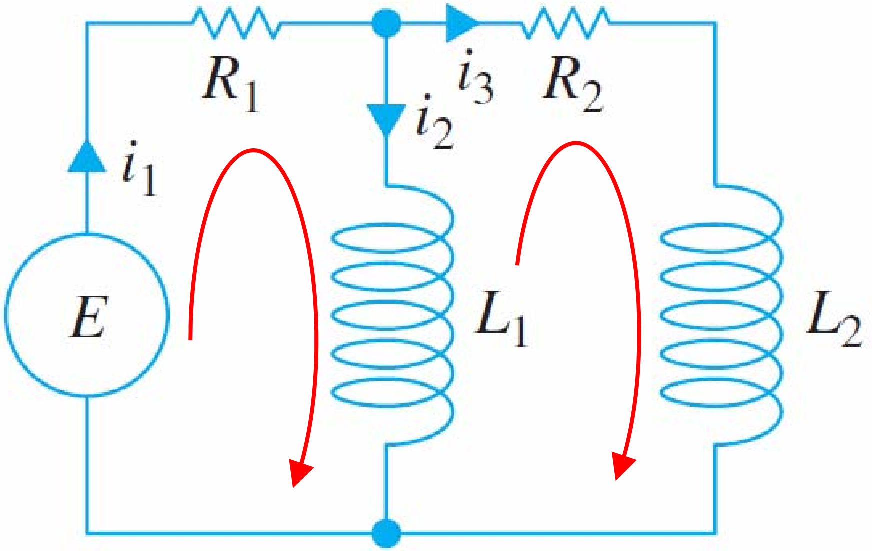


❐ Systems of linear ODEs

- Systems of LH ODEs
- Systems of LN ODEs

Why systems of ODEs?

- E.g. RL circuit. $v_L = L \times i'(t)$
  - $i_1 = i_2 + i_3$ …(1)
  - $E = R_1 i_1 + L_1 (i_2)'$ …(2)
  - $L_1 (i_2)' = R_2 i_3 + L_2 (i_3)'$ …(3)



- Substitute (1) into (2): $E = R_1(i_2+i_3) + L_1(i_2)'$, $\Rightarrow$

$$i_2' = -\frac{R_1}{L_1} i_2 - \frac{R_1}{L_1} i_3 + \frac{E}{L_1} \cdots (4)$$

- Substitute (4) into (3): $(-R_1 i_2 - R_1 i_3 + E) = R_2 i_3 + L_2 (i_3)'$,

$\Rightarrow$

$$i_3' = -\frac{R_1}{L_2} i_2 - \frac{R_1 + R_2}{L_2} i_3 + \frac{E}{L_2} \cdots (5)$$

Example: System of LH ODEs (1)

- $$\begin{cases} x'(t) - 3y(t) = 0 \cdots (1) \\ 2x(t) - y'(t) = 0 \cdots (2) \end{cases}$$

- Just like elimination of variables in solving systems of algebraic equations, trying to eliminate y(t):

$$\frac{d}{dt}(1) = x''(t) - 3y'(t) = 0 \cdots (3)$$

$$(3) - 3 \times (2) = x'' - 3y' - 3(2x - y') = x'' - 6x = 0 \cdots (4)$$

- Auxiliary equation: $m^2 - 6 = 0$, $\Rightarrow$ $m = \pm\sqrt{6}$ (Case 1),

$\Rightarrow$ $x_c(t) = c_1 e^{-(\sqrt{6})t} + c_2 e^{(\sqrt{6})t}$.

Example: System of LH ODEs (2)

- Eliminating x(t):

$$\frac{d}{dt}(2) = 2x'(t) - y''(t) = 0 \cdots (5)$$

$$2 \times (1) - (5) = 2(x' - 3y) - (2x' - y'') = y'' - 6y = 0 \cdots (6)$$

- Auxiliary equation: $m^2 - 6 = 0$, $\Rightarrow$ $m = \pm\sqrt{6}$ (Case 1), $\Rightarrow y_c(t) = c_3 e^{-(\sqrt{6})t} + c_4 e^{(\sqrt{6})t}$.

- $\{c_1, c_2, c_3, c_4\}$ are not arbitrary: Substitute x_c, y_c into Eq. (1): $x'_c - 3y_c = [-\sqrt{6}c_1 e^{-(\sqrt{6})t} + \sqrt{6}c_2 e^{(\sqrt{6})t}] - 3[c_3 e^{-(\sqrt{6})t} + c_4 e^{(\sqrt{6})t}] = (-\sqrt{6}c_1 - 3c_3)e^{-(\sqrt{6})t} + (\sqrt{6}c_2 - 3c_4)e^{(\sqrt{6})t} = 0$, $\Rightarrow \{c_3 = -(\sqrt{6}/3)c_1,\ c_4 = (\sqrt{6}/3)c_2\}$, where $\{c_1, c_2\}$ are determined by ICs or BCs.

Example 2: System of LN ODEs (1)

- $$\begin{cases} x' - 4x + y'' = t^2 \cdots (1) \\ x' + x + y' = 0 \cdots (2) \end{cases}$$
- For convenience, use operator D^n to symbolize nth-order derivative d^n/dt^n:

  $Dx - 4x + D^2y = (D-4)x + D^2y = t^2 \ldots(1)$

  $Dx + x + Dy = (D+1)x + Dy = 0 \ldots(2)$
- Eliminating x(t) by $(D+1)\times(1) - (D-4)\times(2)$:

  $(D+1)\times[(D-4)x + D^2y] - (D-4)\times[(D+1)x + Dy] = (D+1)t^2$,

  $(D^3+D^2-D^2+4D)y = (D^3+4D)y = 2t + t^2$,

  $\Rightarrow y''' + 4y' = t^2 + 2t \ldots(3)$.

Ultrafast Photonics Lab

Example 2 (2)

- Auxiliary equation of Eq. (3) is: $m^3 + 4m = 0$, $\Rightarrow$ m = 0, $\pm$j2, $\Rightarrow$ $y_c(t) = c_1 + c_2\cos(2t) + c_3\sin(2t)$ (why?)
- Since the driving source is $t^2 + 2t$, substituting $y_p(t) = At^3 + Bt^2 + Ct + D$ into Eq. (3):
  $y''' + 4y' = 6A + 4(3At^2+2Bt+C) = 12At^2 + 8Bt + (6A+4C) = t^2 + 2t$, $\Rightarrow$ {12A = 1, 8B = 2, 6A+4C = 0},
  $\Rightarrow$ {A = 1/12, B = ¼, C = -1/8, D is arbitrary}.
- $\Rightarrow y(t) = y_c + y_p = c_1 + c_2\cos(2t) + c_3\sin(2t) + t^3/12 + t^2/4 - t/8$ (why is D missing?)

Example 2 (3)

- Eliminating y(t) by (1)-D×(2):

  $[(D-4)x + D^2y] - D\times[(D+1)x + Dy] = t^2 - 0,$

  $(D-4-D^2-D)x = -(D^2+4)x = t^2, \Rightarrow x'' + 4x = -t^2$ …(4).

- Auxiliary equation of Eq. (4) is: $m^2 + 4m = 0, \Rightarrow m = \pm j2, \Rightarrow x_c(t) = c_4\cos(2t) + c_5\sin(2t)$

- Since the driving source is $-t^2$, substituting $x_p(t) = At^2 + Bt + C$ into Eq. (4):

  $x'' + 4x = 2A + 4(At^2 + Bt + C) = 4At^2 + 4Bt + (2A+4C)$

  $= -t^2, \Rightarrow \{4A = -1, 4B = 0, 2A+4C = 0\},$

  $\Rightarrow \{A = -¼, B = 0, C = 1/8\}, x_p(t) = -t^2/4 + 1/8.$

Example 2 (4)

- $x(t) = x_c + x_p = c_4\cos(2t) + c_5\sin(2t) - t^2/4 + 1/8$
- $\{c_1, c_2, c_3, c_4, c_5\}$ are not arbitrary: Substitute x(t), y(t) into Eq. (2):
  $x' + x + y' = [-2c_4\sin(2t) + 2c_5\cos(2t) - t/2] + [c_4\cos(2t) + c_5\sin(2t) - t^2/4 + 1/8] + [-2c_2\sin(2t) + 2c_3\cos(2t) + t^2/4 + t/2 - 1/8] = (2c_5+c_4+2c_3)\times\cos(2t) + (-2c_4+c_5-2c_2)\times\sin(2t) + [(-¼+¼)t^2 + (-½+½)t + (1/8-1/8)] = 0,$

$$\Rightarrow \begin{cases} 2c_5 + c_4 + 2c_3 = 0, \\ -2c_4 + c_5 - 2c_2 = 0, \end{cases} \begin{cases} c_4 = -(4c_2 + 2c_3)/5 \\ c_5 = (2c_2 - 4c_3)/5 \end{cases}$$

$$\Rightarrow x(t) = -\frac{4c_2 + 2c_3}{5}\cos 2t + \frac{2c_2 - 4c_3}{5}\sin 2t - \frac{t^2}{4} + \frac{1}{8}$$

Ultrafast Photonics Lab

Summary

- What're the differences between IVPs and BVPs?
- What are linearly independent functions $\{f_1, \ldots, f_n\}$? How to test the linear independence?
- For a LN ODE, $y(x) = y_c(x) + y_p(x)$. What are the meanings of y_c and y_p, respectively?
- For a 2nd-order LN ODE of constant coefficients, how to find y_c and y_p? What are the features of y_c and y_p?
- What does resonance mean in a mass-spring?