Chapter 6
Series Solutions

- Power series method (6.1)
- Legendre’s polynomials (6.3.2)
- Frobenius method (6.2)
- Bessel functions (6.3.1)

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Power series method

- Definition and convergence
- Operations
- Application to solving ODEs

What is a power series?

- An infinite series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots,$$

where a and $\{c_n\}$ are center and coefficients of the series, respectively.

- Note: All terms are in non-negative integral powers of (x-a).
- If a = 0, we got a Maclaurin series in powers of x:

$$\sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

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Examples

- Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ if } |x| < 1.$$

- Cosine and sine functions:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

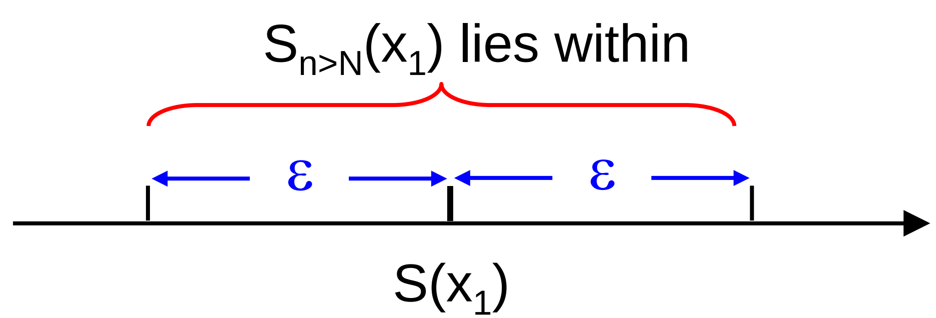
$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

Convergence of power series

- The Nth partial sum and the corresponding remainder of a power series S(x) are defines as:

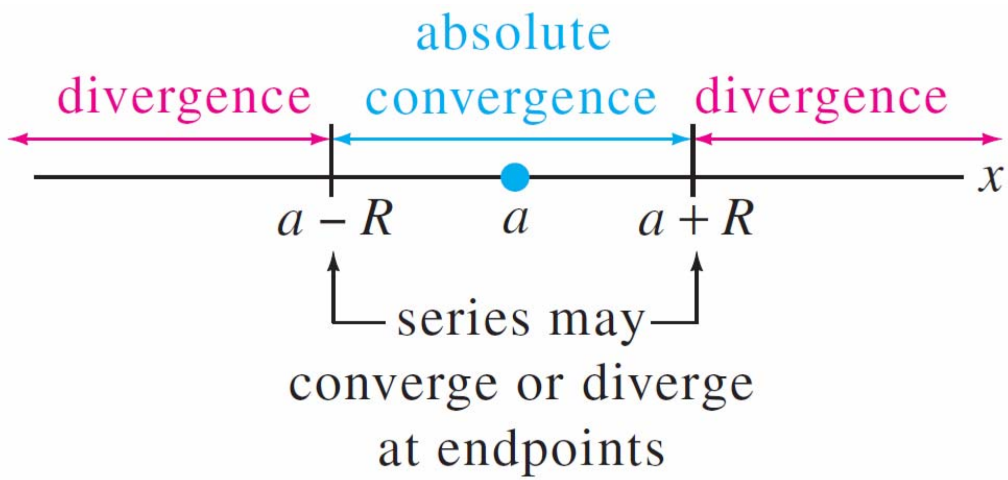
$$S_N(x) \equiv \sum_{n=0}^{N} c_n (x-a)^n, \;\; R_N(x) \equiv S(x) - S_N(x) = \sum_{n=N+1}^{\infty} c_n (x-a)^n.$$

- Series is convergent at x = x_1 if for any $\varepsilon > 0$, there is an N (depending on ε) such that $|R_n(x_1)| < \varepsilon$ for all n > N.



Radius of convergence R

- "R" is defined by: $\lim_{N\to\infty} S_N(x) = S(x)$ for all $|x-a| < R$.



- "R" can be evaluated by

$$R = \left(\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right)^{-1}, \text{ or } \left(\lim_{n\to\infty}\sqrt[n]{c_n}\right)^{-1}.$$

Example: 1/(1-x)

- Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \Rightarrow \{a = 0,\ c_n = 1\}.$$

$$\Rightarrow R = (\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|)^{-1} = (\lim_{n\to\infty}\left|\frac{1}{1}\right|)^{-1} = 1,$$

$\Rightarrow$ It is convergent for all $|x-0| < 1$, i.e. $-1 < x < 1$.

Example: e^x

- It's known that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \Rightarrow \left\{ a = 0,\ c_n = \frac{1}{n!} \right\}.$$

$$\Rightarrow R = (\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|)^{-1} = (\lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right|)^{-1} = (\lim_{n\to\infty} \left| \frac{1}{n+1} \right|)^{-1} = \infty,$$

$\Rightarrow$ It is convergent for all $|x-0| < \infty$, i.e. $-\infty < x < \infty$.

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Differentiation, addition

- Termwise differentiation:

  Let a = 0, $y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots;$

  $\Rightarrow y'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots$

  $\Rightarrow y''(x) = 2c_2 + 6c_3x + 12c_4x^2 + \ldots;$

- Termwise addition:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n,\ \ g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n,$$

$$\Rightarrow\ f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-a)^n.$$

Multiplication

- Termwise multiplication:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \;\; g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

$$\Rightarrow \; f(x) \times g(x) = \sum_{n=0}^{\infty} \boxed{(a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)} x^n$$

n+1 combinations

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

Shifting the summation index

- Example 1 of Sec. 6.1:

$$f(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \boxed{2c_2} + 6c_3 x + 12c_4 x^2 + \cdots,$$

Isolated x^0 term

$$g(x)=\sum_{n=0}^{\infty} c_n x^{n+1} = c_0 x + c_1 x^2 + c_2 x^3 + \cdots,$$

- Let n – 2 = k and n + 1 = k for the f(x) and g(x) series:

$$f(x)=\boxed{2c_2} + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} = 2c_2 + \sum_{k=1}^{\infty} \boxed{(k+2)(k+1)c_{k+2}}\, x^k,$$

n = 2

$$g(x)=\sum_{k=1}^{\infty} \boxed{c_{k-1}}\, x^k,$$

$$\Rightarrow f(x)+g(x) = 2c_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} + c_{k-1}\right]x^k.$$

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How to solve ODEs by power series?

- Consider a 2nd-order linear nonhomogeneous ODE:

$$y'' + p(x)y' + q(x)y = g(x)$$

- Step 1: Represent p(x), q(x), g(x) by power series. This step can be omitted if p, q, g are polynomials.
- Step 2: Assume the solution is also a power series:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

- Step 3: Substitute y, y', y'' into the ODE, collect like powers of x, $\{c_n\}$ can be solved by equating the coefficients of each power of x on both sides of "=".

Example: y' – y = 0

- Step 2: $y(x) = c_0 + c_1x + c_2x^2 + \ldots;$

  $\Rightarrow y'(x) = c_1 + 2c_2x + 3c_3x^2 + \ldots$

- Step 3: $y' - y = (c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + \ldots = 0;$

  $\Rightarrow c_1 - c_0 = 0,\ c_1 = c_0;$

  $\Rightarrow 2c_2 - c_1 = 0,\ c_2 = c_1/2 = c_0/(2!);$

  $\Rightarrow 3c_3 - c_2 = 0,\ c_3 = c_2/3 = c_0/(3!); \ldots$

  $\Rightarrow c_n = c_0/(n!);$

- The series solution is: $\sum_{n=1}^{\infty} \frac{c_0}{n!} x^n = c_0 e^x$

Example: y'' + y = 0

- Step 2: $y(x) = c_0 + c_1x + c_2x^2 + \dots; \Rightarrow y'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots; y''(x) = 2c_2 + 6c_3x + 12c_4x^2 + \dots;$
- Step 3: $y'' + y = (2c_2+c_0) + (6c_3+c_1)x + (12c_4+c_2)x^2 + \dots = 0;$

  $\Rightarrow 2c_2 + c_0 = 0,\ c_2 = -c_0/2 = -c_0/(2!);$

  $\Rightarrow 6c_3 + c_1 = 0,\ c_3 = -c_1/6 = -c_1/(3!);$

  $\Rightarrow 12c_4 + c_2 = 0,\ c_4 = -c_2/12 = +c_0/(4!); \dots$
- The series solution is:

$$y(x) = c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right),$$

Existence of power series solutions

- A real function f(x) is analytic at x = a if it can be represented by a power series in powers of (x-a):

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+\cdots,$$

- A 2nd-order linear nonhomogeneous ODE:

$$y'' + p(x)y' + q(x)y = g(x)$$

has power series solutions around x = a if p(x), q(x), g(x) are analytic at x = a.

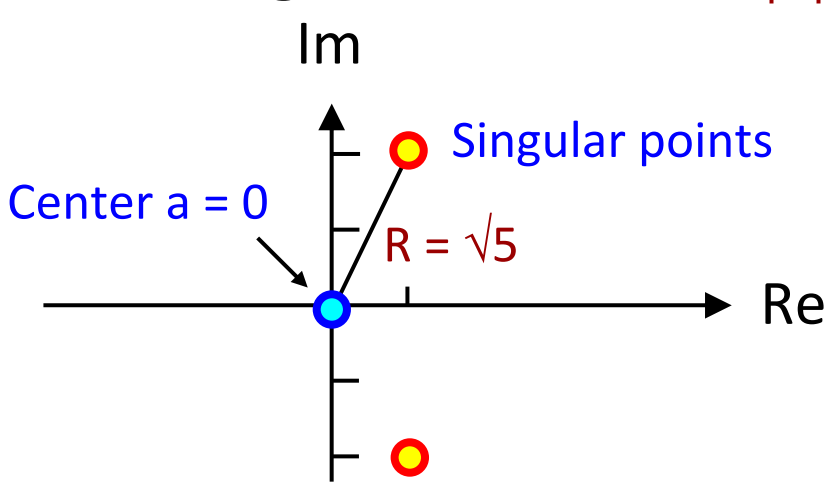
Example 2

- Consider $(x^2-2x+5)y'' + xy' - y = 0$,

  $\Rightarrow p(x) = x/(x^2-2x+5)$, $q(x) = -1/(x^2-2x+5)$, $g(x) = 0$.

- $p(x)$ and $q(x)$ are analytic except for $x^2-2x+5 = 0$, $\Rightarrow x = 1 \pm j2$ are two singular points.

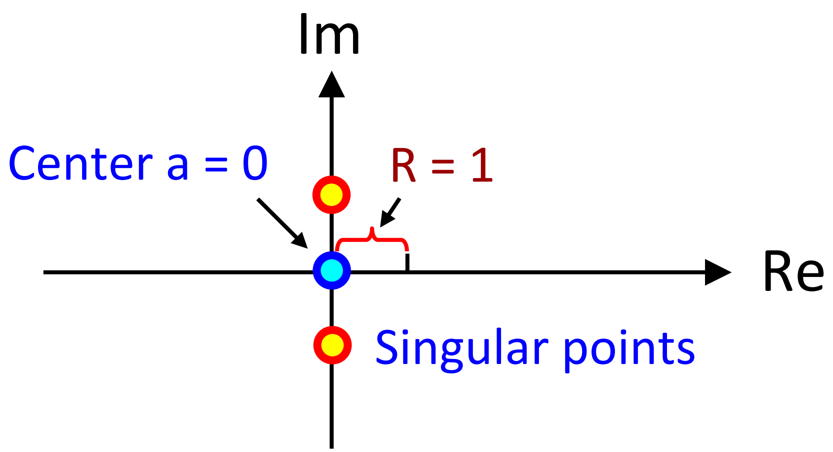
- We can find two power series solutions centered at $a = 0$, each is convergent at least for $|x| < \sqrt{5}$.



Example 4 (1)

- ODE: $(x^2 + 1)y'' + xy' - y = 0$,

  $\Rightarrow p(x) = x/(x^2 + 1)$, $q(x) = -1/(x^2 + 1)$, $g(x) = 0$.
- $x^2 + 1 = 0$, $\Rightarrow x = \pm j$ are two singular points.
- We can find two power series solutions centered at $a = 0$, each is convergent at least for $|x| < 1$.



Example 4 (2)

- Step 2: Assume $y(x) = \sum_{n=0}^{\infty} c_n x^n,$

$$\Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1},\ \ y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2},$$

- Step 3: $(x^2+1)y'' + xy' - y$

$$= \boxed{(x^2+1)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \boxed{\sum_{n=2}^{\infty} n(n-1) c_n x^n} + \boxed{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}} + \boxed{\sum_{n=1}^{\infty} n c_n x^n} - \boxed{\sum_{n=0}^{\infty} c_n x^n}$$

starting from x^2,
let k = n;

$2c_2 + 6c_3x + \sum_{n=4}(\ )$;
let k = n-2;

$c_1x + \sum_{n=2}(\ )$;
let k = n;

$c_0 + c_1x + \sum_{n=2}(\ )$;
let k = n;

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Example 4 (3)

$$(2c_2 - c_0) + 6c_3 x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0$$

$\Rightarrow 2c_2 - c_0 = 0$, $c_2 = c_0/2$;

$\Rightarrow 6c_3 = 0$, $c_3 = 0$;

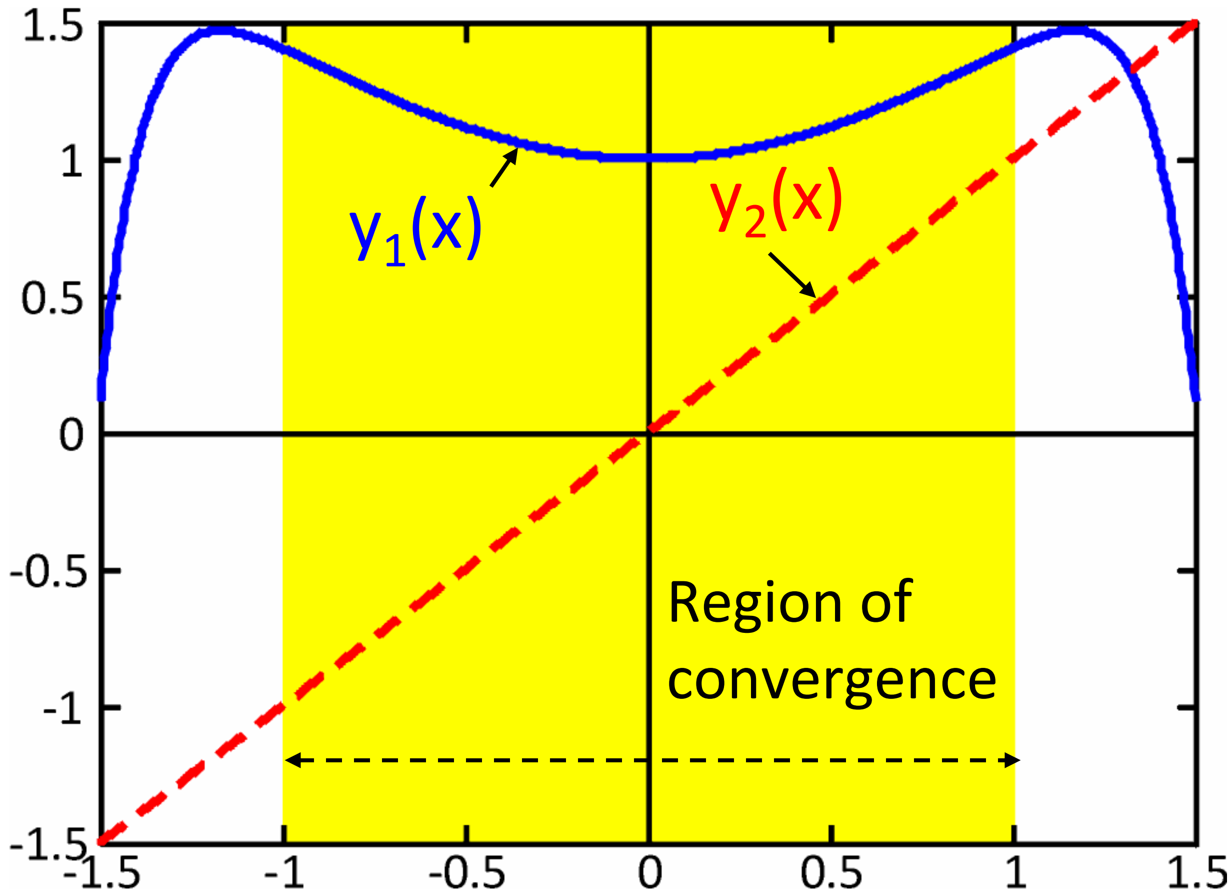
$\Rightarrow \cancel{(k+1)}(k-1)c_k + \cancel{(k+1)}(k+2)c_{k+2} = 0$;

$c_{k+2} = -[(k-1)/(k+2)]c_k$; $k = 2, 3, \ldots$ recursive relation

- For $k = 2$: $c_4 = -c_2/4 = -c_0/8$.
- For $k = 3$: $c_5 = -(2/5)c_3 = 0$. $\Rightarrow$ So are $c_7, c_9, \ldots = 0$.
- For $k = 4$: $c_6 = -c_4/2 = c_0/16$.

Example 4 (4)

- $y(x) = c_0\boxed{(1 + x^2/2 - x^4/8 + x^6/16 - \ldots)} + c_1\boxed{(x)}$, for $|x| < 1$.



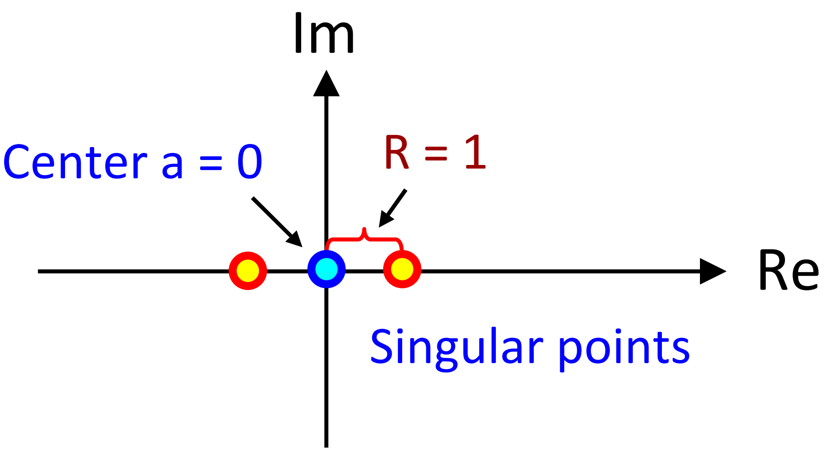
❑ Legendre polynomials

- Legendre Equation
- Linearly independent series solutions
- Legendre polynomials

Adrien-Marie Legendre (1752-1833)

Legendre's equation

- Some important ODEs cannot be solved by methods we've studied but series solutions.
- E.g. $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$, where $n \in R$ is a given constant (usually an integer).
- $1 - x^2 = 0, \Rightarrow x = \pm 1$ are two singular points. $\Rightarrow$ Maclaurin series solutions are convergent for $|x| < 1$.



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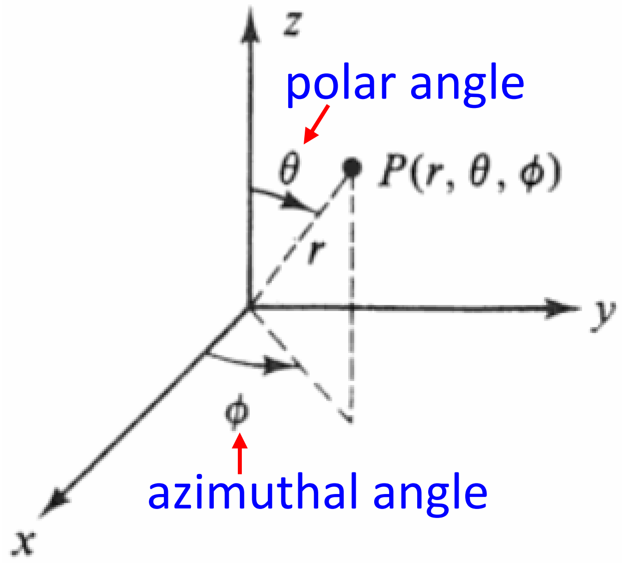
Why Legendre's equation?

- E.g. Find the electric potential $V(r,\phi,\theta)$ satisfying Laplace's equation $\nabla^2 V = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)V = 0$ and boundary condition $V(r_0,\phi,\theta) = g(\theta)$.
- Solution is $\sum_n A_n R_n(r) \times \Theta_n(\theta)$, where $\Theta_n(\theta) = P_n(x)$ ($x = \cos\theta$) satisfies the Legendre's equation.
- $0 < \theta < \pi, \Rightarrow -1 < x < 1$.



Power series solution (1)

- Step 2: Assume $y(x)=\sum_{m=0}^{\infty} c_m x^m$,

$$\Rightarrow y'(x)=\sum_{m=1}^{\infty} m c_m x^{m-1},\ \ y''(x)=\sum_{m=2}^{\infty} m(m-1) c_m x^{m-2},$$

- Step 3: $(1-x^2)y'' - 2xy' + n(n+1)y$

$$=\boxed{(1-x^2)\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2}}-2x\sum_{m=1}^{\infty} m c_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty} c_m x^m$$

$$=\boxed{\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2}-\sum_{m=2}^{\infty} m(m-1)c_m x^{m}}-2\boxed{\sum_{m=1}^{\infty} m c_m x^{m}}+n(n+1)\boxed{\sum_{m=0}^{\infty} c_m x^m}$$

$2c_2 + 6c_3x + \sum_{m=4}(\ )$; let k = m-2;

starting from x^2, let k = m;

$c_1x + \sum_{m=2}(\ )$; let k = m;

$c_0 + c_1x + \sum_{m=2}(\ )$; let k = m;

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Power series solution (2)

$$[2c_2+n(n+1)c_0]+[6c_3+\overbrace{(n^2+n-2)}^{(n+2)(n-1)}c_1]x+$$

$$\sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2}+\underbrace{(-k^2-k+n^2+n)}_{(n-k)(n+k+1)}c_k]x^k=0$$

$\Rightarrow$ $c_{k+2} = -c_k[(n-k)(n+k+1)/(k+2)(k+1)]$; k = 2, 3, ...

$\Rightarrow$ The recursive relation also applies to k = 0, 1.

- k = 0: $c_2 = -c_0[n(n+1)/2]$, same as from x^0 term;
- k = 1: $c_3 = -c_1[(n+2)(n+1)/6]$; same as from x^1 term;
- k = 2: $c_4 = -c_2(n-2)(n+3)/12 = -c_0(n-2)n(n+1)(n+3)/24$;
- k = 3: $c_5 = -c_1(n-3)(n-1)(n+2)(n+4)/120$; ...

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Power series solution (3)

- $y(x) = c_0\, y_1(x) + c_1\, y_2(x)$, where

$$
\begin{cases}
y_1(x) = 1 - \dfrac{\boxed{n(n+1)}}{2!}x^2 + \dfrac{(n-2)\boxed{n(n+1)}(n+3)}{4!}x^4 - \cdots, \\[2ex]
y_2(x) = x - \dfrac{\boxed{(n-1)(n+2)}}{3!}x^3 + \dfrac{(n-3)\boxed{(n-1)(n+2)}(n+4)}{5!}x^5 - \cdots,
\end{cases}
$$

are linearly independent (even & odd) solutions.

If "n" is a non-negative integer …

- Recursive relation

$$c_{k+2} = -\frac{\boxed{(n-k)}(n+k+1)}{(k+2)(k+1)} c_k, \quad k = 0, 1, 2, \cdots$$

implies that $c_{n+2} = 0$ (so are c_{n+4}, c_{n+6}, …) if "n" is a nonnegative integer.

- E.g. If $n = 2$, $c_2 = -c_0[n(n+1)/2] = -3c_0$, $c_4 = -c_2(n-2)(n+3)/12 = 0$, $c_6 = c_8 = \ldots = 0$,

  $\Rightarrow y_1(x) = 1 - 3x^2$, a 2nd-order polynomial.

- Note: $y_2(x) = x - (2/3)x^3 - (1/5)x^5 - \ldots$ remains an infinite series.

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Legendre polynomials (1)

- If n $\in$ even, properly choosing c_0 by

$$c_0 = (-1)^{n/2} \frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}, \text{ except that } c_0 = 1 \text{ if } n = 0.$$

makes $c_0y_1(x)$ (a particular solution) the nth-order Legendre polynomial, denoted by $P_n(x)$.

- If n $\in$ odd, $c_1y_2(x)$ = $P_n(x)$ is obtained by choosing

$$c_1 = (-1)^{(n-1)/2} \frac{1\cdot 3\cdots n}{2\cdot 4\cdots(n-1)}, \text{ except that } c_1 = 1 \text{ if } n = 1.$$

- Note: $\{P_n(x)\}$ are a set of orthogonal functions (Section 11.4 of the text).

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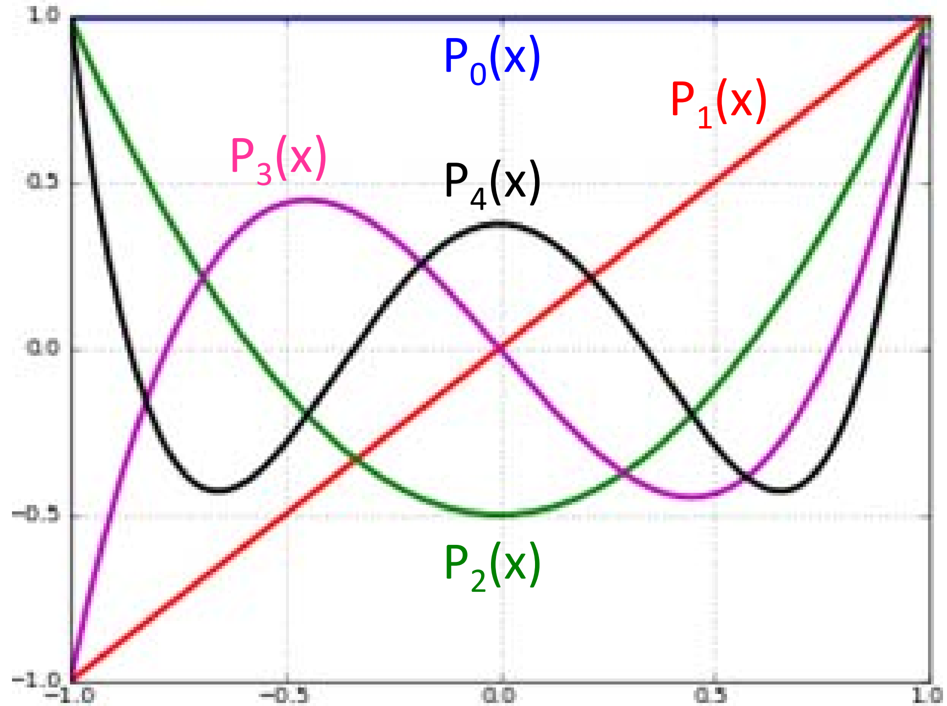
Legendre polynomials (2)

- E.g.

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



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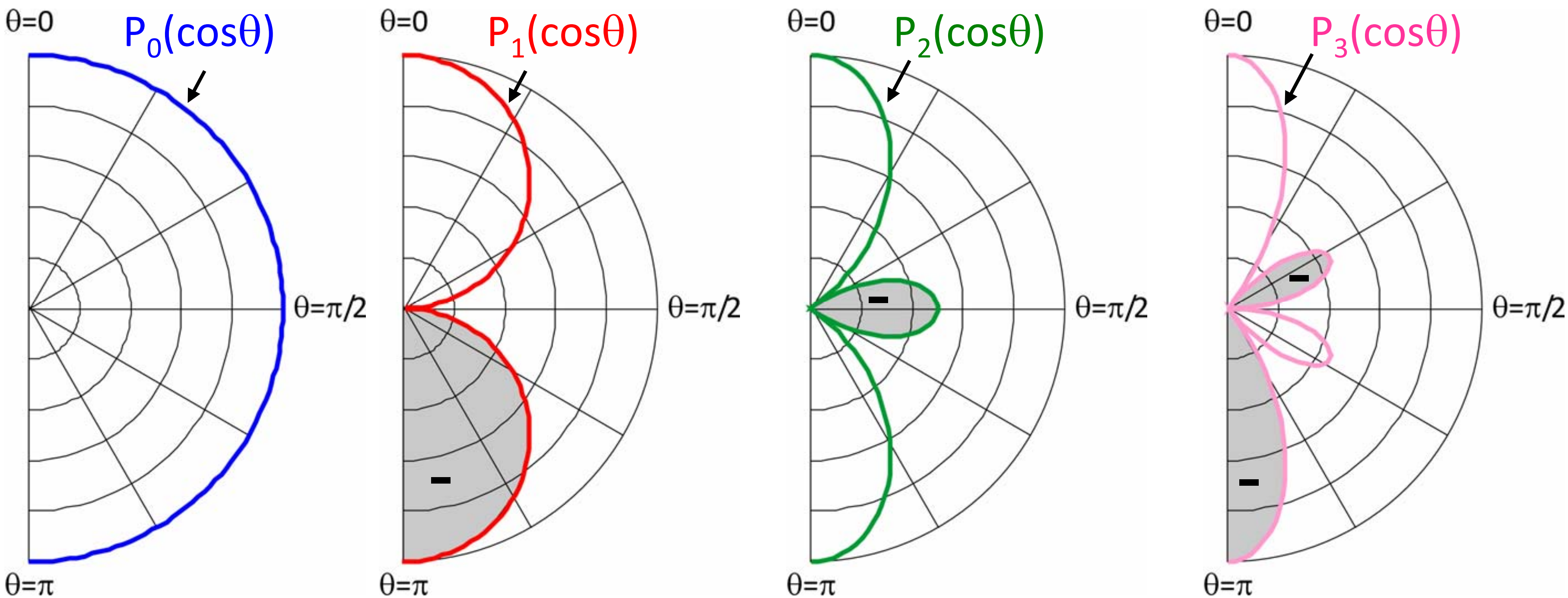
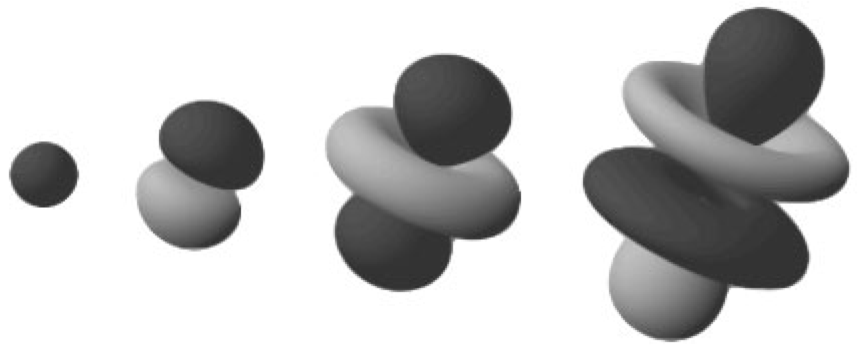
Plot of $P_n(\cos\theta)$





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Frobenius method

- Standard procedures
- E.g. Cauchy-Euler equation
- Linearly independent solutions in 3 cases

Overview

- A 2nd-order linear homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x) \equiv a(x)/x$, $q(x) \equiv b(x)/x^2$ at least has a series (but not power series) solution

$$y_1(x) = x^r(\sum c_n x^n),\ r \in C \text{ in general,}$$

if $a(x)$, $b(x)$ are analytic (differentiable) at $x = 0$.

- A 2nd linearly independent solution $y_2(x)$ can be found by reduction of order assuming $y_2 = u(x)y_1$ (Section 4.2 of the text).

Series solution (1)

- Step 1: Expand a(x), b(x) in power series:

$$a(x) = \sum_{n=0} a_n x^n, \; b(x) = \sum_{n=0} b_n x^n.$$

- Step 2: Assume $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{r+n}$,

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} (r+n) c_n x^{r+n-1} = x^{r-1} \sum_{n=0}^{\infty} (r+n) c_n x^n$$

not 1

$$= x^{r-1}\left[r c_0 + (r+1) c_1 x + \cdots \right],$$

$$\Rightarrow y''(x) = \sum_{n=0}^{\infty} (r+n)(r+n-1) c_n x^{r+n-2}$$

not 2

$$= x^{r-2}\left[r(r-1) c_0 + (r+1) r c_1 x + \cdots \right],$$

Series solution (2)

- Step 3: Substitute y, y', y" into the ODE:

$$x^2y'' + xa(x)y' + b(x)y = 0$$

$$\Rightarrow \cancel{x^2}\left[x^{r\cancel{-2}}\sum_{n=0}^{\infty}(r+n)(r+n-1)c_n x^n\right] \rightarrow x^r[r(r-1)c_0 + (r+1)rc_1x + \dots]$$

$$+\cancel{x}\left(\sum_{m=0}^{\infty}a_m x^m\right)\left[x^{r\cancel{-1}}\sum_{n=0}^{\infty}(r+n)c_n x^n\right] \rightarrow (a_0+a_1x+\dots)x^r[rc_0+(r+1)c_1x+\dots]$$

$$= x^r\{(ra_0c_0) + [ra_1c_0+(r+1)a_0c_1]x + \dots\}$$

$$+\left(\sum_{m=0}^{\infty}b_m x^m\right)\left(x^r\sum_{n=0}^{\infty}c_n x^n\right) \rightarrow (b_0+b_1x+\dots)x^r(c_0+c_1x+\dots)$$

$$= x^r\{(b_0c_0) + (b_0c_1+b_1c_0)x + \dots\}$$

Series solution (3)

- $x^r\{ [r(r-1) + ra_0 + b_0]c_0 + [(r+1)rc_1 + ra_1c_0 + (r+1)a_0c_1 + b_0c_1 + b_1c_0]x + [:]x^2 + \ldots\} = 0.$
- Step 4: x^{r+0} coefficient is $\propto r(r-1) + ra_0 + b_0 = r^2 + (a_0-1)r + b_0 = 0$ (indicial equation), $\Rightarrow r = \{r_1, r_2\}$.
- Step 5: x^{r+1} coefficient is $(r+1)rc_1 + ra_1c_0 + (r+1)a_0c_1 + b_0c_1 + b_1c_0 = [r^2 + (a_0+1)r + (a_0+b_0)]c_1 + (a_1r+b_1)c_0 = 0$,

$$\Rightarrow c_1 = -\frac{a_1 r + b_1}{r^2 + (a_0 + 1)r + (a_0 + b_0)} c_0$$

- x^{r+n} coefficient gives a recursive relation $c_n = (?)c_{n-1} = [?]c_0$, from which $y_1(x) = x^{r_1}\sum c_n x^n$ is determined.

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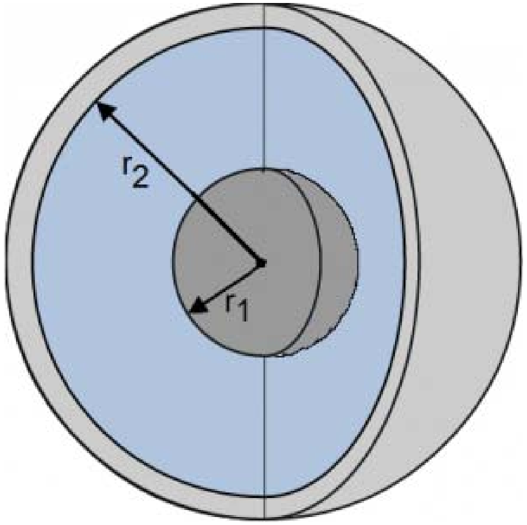
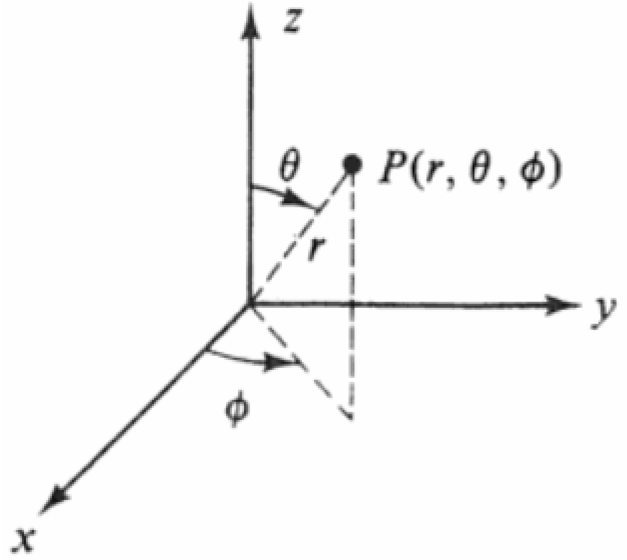
Example: Potential btw. spheres (1)

- Potential distribution $V(r,\phi,\theta)$ between two concentric conducting spheres satisfies:
- PDE: $\nabla^2 V = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)V = 0, \Rightarrow$
- ODE: $V''(r) + (2/r)V'(r) = 0$ if no dependence on ϕ, θ.
- BCs: $V(r_1=4) = 110$, $V(r_2=8) = 0$.





Example: Potential btw. spheres (2)

- Change of notations: $V(r) \rightarrow y(x)$
- ODE: $y'' + (2/x)y' = 0, \Rightarrow xy'' + 2y' = 0,$
- BCs: $y(4) = 110,\ y(8) = 0.$
- Step 2: Assume $y(x) = x^r\sum_{n=0}c_nx^n = \sum_{n=0}c_nx^{n+r}, \Rightarrow y' = \sum_{n=0}(n+r)c_nx^{n+r-1},\ y'' = \sum_{n=0}(n+r-1)(n+r)c_nx^{n+r-2}.$
- Step 3: Substitute y'', y' into the ODE: $x[\sum_{n=0}(n+r-1)(n+r)c_nx^{n+r-2}] + 2[\sum_{n=0}(n+r)c_nx^{n+r-1}] = 0,$

  $\Rightarrow x^{r-1}\{\sum_{n=0}[(n+r-1)(n+r)+2(n+r)]c_nx^n\} = 0,$

  $\Rightarrow (n+r)(n+r+1)c_n = 0,\ n = 0, 1, 2, \ldots$

Example: Potential btw. spheres (3)

- $(n+r)(n+r+1)c_n = 0$:
- Step 4: By For $n = 0$, we got the indicial equation

  $r(r+1) = 0$ (if $c_0 \neq 0$), $\Rightarrow r = \{0, -1\}$.
- Step 5: For $r = 0$, $(n+r)(n+r+1)c_n = n(n+1)c_n = 0$,

  $\Rightarrow \{c_n = 0\}$ for $n = 1, 2, \ldots$; because $n(n+1) \neq 0$.

  $\Rightarrow y_1(x) = x^0 c_0 = c_0$ (constant).
- For $r = -1$, $(n+r)(n+r+1)\mathbf{C_n} = (n-1)nC_n = 0$,

  $\Rightarrow \{C_n = 0\}$ for $n = 2, 3, \ldots$ because $(n-1)n \neq 0$.

  $\Rightarrow y_2(x) = x^{-1}(C_0 + C_1 x) = C_0/x + C_1$.

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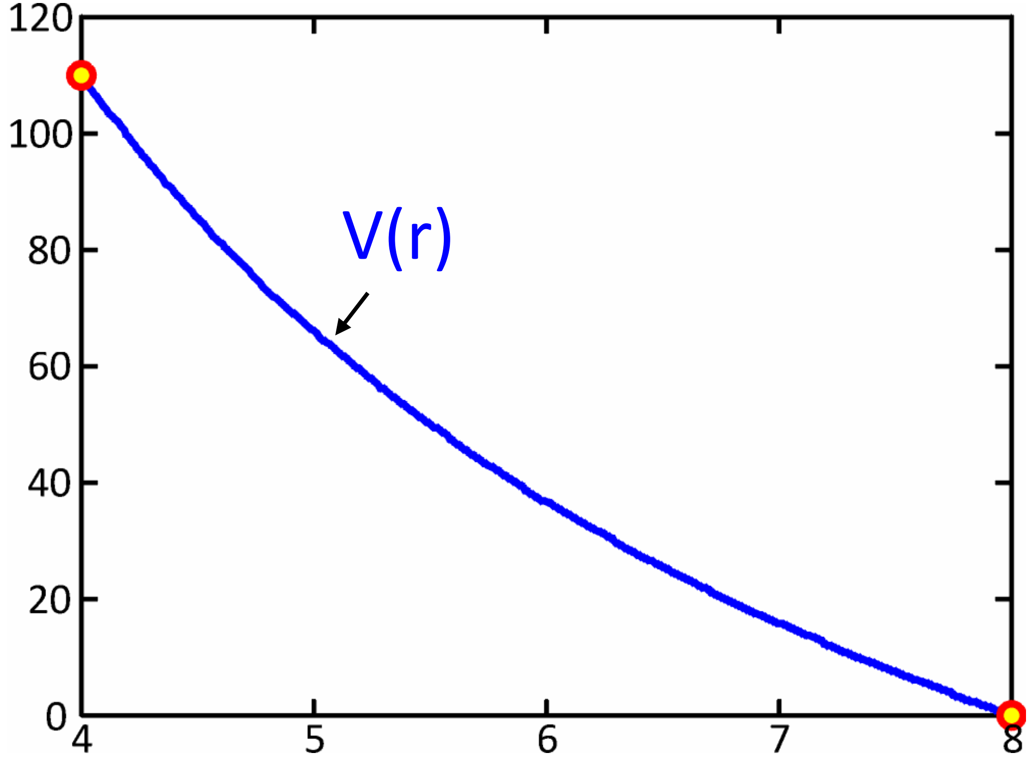
Example: Potential btw. spheres (4)

- $y(x) = Ay_1(x) + By_2(x) = Ac_0 + B(C_0/x + C_1) = a + b/x$.
- Two BCs $y(4) = 110$, $y(8) = 0$ gives $\{a = -110, b = 880\}$.
- $y(x) = -110 + 880/x$, $\Rightarrow V(r) = -110 + 880/r$.



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Example 3: Case 1 (1)

- ODE: $2xy'' + (1+x)y' + 1y = 0$.
- Step 2: Assume $y(x) = x^r\sum_{n=0}c_nx^n = \sum_{n=0}c_nx^{n+r}$,

  $\Rightarrow y'(x) = \sum_{n=0}(n+r)c_nx^{n+r-1}$,

  $\Rightarrow y''(x) = \sum_{n=0}(n+r)(n+r-1)c_nx^{n+r-2}$;
- Step 3: Substitute y'', y', y into $2xy'' + (1+x)y' + 1y = 0$:

  $[\sum_{n=0}2(n+r)(n+r-1)c_nx^{n+r-1}] + [\sum_{n=0}(n+r)c_nx^{n+r-1} + \sum_{n=0}(n+r)c_nx^{n+r}] + (\sum_{n=0}c_nx^{n+r}) = 0$;

  $\Rightarrow x^r\{ [2r(r-1)c_0x^{-1} + \sum_{n=1}2(n+r)(n+r-1)c_nx^{n-1}] + [rc_0x^{-1} + \sum_{n=1}(n+r)c_nx^{n-1} + \sum_{n=0}(n+r)c_nx^n] + (\sum_{n=0}c_nx^n) \} = 0$;

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Example 3: Case 1 (2)

- Step 4: x^{r-1} term: $2r(r-1)c_0 + rc_0 = (2r^2 - r)c_0 = 0$,

  $\Rightarrow$ the indicial equation gives $r = \{r_1, r_2\} = \{½, 0\}$.

- Case 1: When "r" has two distinct real roots whose difference $r_1-r_2 = ½$ is not an integer, $\Rightarrow$ the two series solutions $y_1(x)$, $y_2(x)$ are linearly independent:

- $y_1(x) = x^{r_1}\sum_{n=0} c_n x^n$
- $y_2(x) = x^{r_2}\sum_{n=0} C_n x^n$.

- General solution $y(x) = Ay_1(x) + By_2(x)$, where the two coefficients {A, B} are determined by ICs or BCs.

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Example 3: Case 1 (3)

- Step 5: By shifting the summation index:

  $[\sum_{n=1} 2(n+r)(n+r-1)c_n x^{n-1}] + [\sum_{n=1} (n+r)c_n x^{n-1} + \sum_{n=0} (n+r)c_n x^n] + (\sum_{n=0} c_n x^n) = 0;$

  $\Rightarrow [\sum_{k=0} 2(k+r+1)(k+r)c_{k+1} x^k] + [\sum_{k=0} (k+r+1)c_{k+1} x^k + \sum_{k=0} (k+r)c_k x^k] + (\sum_{k=0} c_k x^k) = 0;$

- x^{r+k} term: $\cancel{(k+r+1)}(2k+2r+1)c_{k+1} + \cancel{(k+r+1)}c_k = 0;$
- Recursive relation:

$$c_{k+1} = -\frac{1}{2k+2r+1} c_k, \quad k = 0, 1, 2, \cdots$$

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Example 3: Case 1 (4)

- For r = r_1 = ½: $c_{k+1} = -\dfrac{1}{2(k+1)} c_k, \;\; k = 0, 1, 2, \cdots$
- k = 0: $c_1 = -c_0/2$;
- k = 1: $c_2 = -c_1/4 = c_0/8$;
- k = 2: $c_3 = -c_2/6 = -c_0/48$; ...
- $c_n = c_0(-1)^n/(2^n n!)$.
- $y_1(x) = \cancel{c_0}\left(x^{1/2} - \dfrac{1}{2} x^{3/2} + \dfrac{1}{8} x^{5/2} - \cdots \right).$ Not a power series!
- $R = (\lim_{n\to\infty} |c_{n+1}/c_n|)^{-1} = (\lim_{n\to\infty} \dfrac{1}{2(n+1)})^{-1} = \infty,$ convergent for x > 0 ($x^{1/2}$ term forbids x < 0).

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Example 3: Case 1 (5)

- For r = $r_2 = 0$: $C_{k+1} = -\dfrac{1}{2k+1} C_k,\ k = 0, 1, 2, \cdots$
  - k = 0: $C_1 = -C_0$;
  - k = 1: $C_2 = -C_1/3 = C_0/3$;
  - k = 2: $C_3 = -C_1/5 = -C_0/15$; ...
  - $C_n = C_0(-1)^n/(1\cdot3\cdot5\cdots)$.
- $y_2(x) = \cancel{C_0}\left(1 - x + \dfrac{1}{3}x^2 - \dfrac{1}{15}x^3 - \cdots\right)$. Power series!
- $R = (\lim\limits_{n\to\infty} |c_{n+1}/c_n|)^{-1} = (\lim\limits_{n\to\infty} \dfrac{1}{2n+1})^{-1} = \infty,$ convergent for all $x \in R$.

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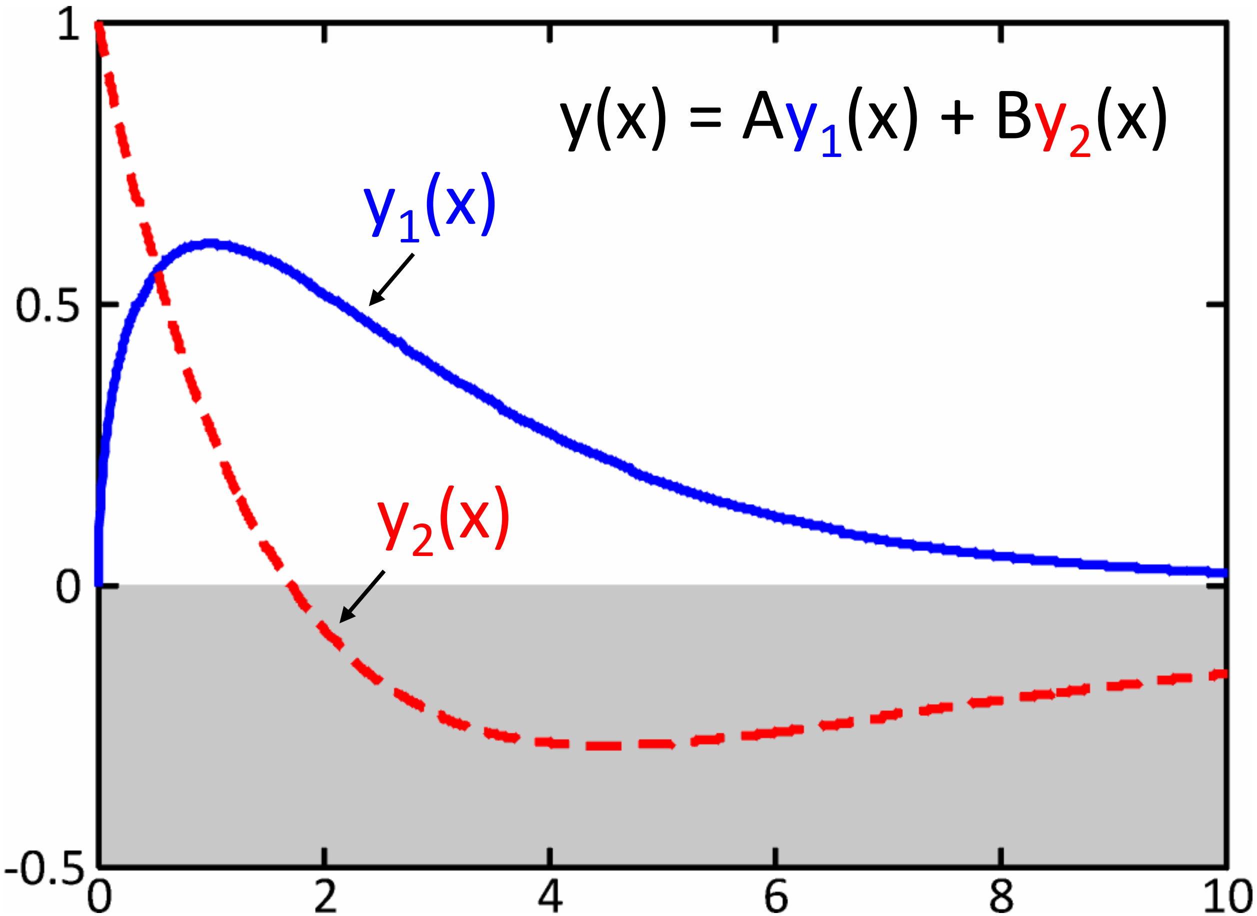
Example 3: Case 1 (6)

- $y_1(x) = x^{0.5} - x^{1.5}/2 + x^{2.5}/8 - \ldots$
- $y_2(x) = 1 - x + x^2/3 - x^3/15 + \ldots$



Examples 4-5: Case 2 (1)

- ODE: $xy'' + 1y = 0$.
- Step 2: Assume $y(x) = x^r\sum_{n=0}c_nx^n = \sum_{n=0}c_nx^{n+r}$,

  $\Rightarrow y'(x) = \sum_{n=0}(n+r)c_nx^{n+r-1}$,

  $\Rightarrow y''(x) = \sum_{n=0}(n+r)(n+r-1)c_nx^{n+r-2}$;
- Step 3: Substitute y'', y into $xy'' + 1y = 0$:

  $[\sum_{n=0}(n+r)(n+r-1)c_nx^{n+r-1}] + (\sum_{n=0}c_nx^{n+r}) = 0$;

  $\Rightarrow x^r\{ [r(r-1)c_0x^{-1} + \sum_{n=1}(n+r)(n+r-1)c_nx^{n-1}] +$

  $(\sum_{n=0}c_nx^n) \} = 0$.

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Examples 4-5: Case 2 (2)

- Step 4: x^{r-1} term: $r(r-1)c_0 = 0$, indicial equation gives r = $\{r_1, r_2\} = \{1, 0\}$.
- Case 2: Two distinct real roots whose difference $r_1 - r_2 = 1$ is an integer.
- Step 5: By shifting the summation index:

  $[\sum_{n=1}(n+r)(n+r-1)c_n x^{n-1}] + (\sum_{n=0} c_n x^n) = 0;$

  $\Rightarrow [\sum_{k=0}(k+r+1)(k+r)c_{k+1}x^k] + (\sum_{k=0} c_k x^k) = 0;$

- x^{r+k} term: $(k+r+1)(k+r)c_{k+1} + c_k = 0;$
- Recursive: $c_{k+1} = -\dfrac{1}{(k+r+1)(k+r)} c_k, \quad k = 0, 1, 2, \cdots$

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Examples 4-5: Case 2 (3)

- For r = r_1 = 1: $c_{k+1} = -\dfrac{1}{(k+2)(k+1)} c_k,\ k = 0, 1, 2, \cdots$
  - k = 0: $c_1 = -c_0/2$;
  - k = 1: $c_2 = -c_1/6 = c_0/12$;
  - k = 2: $c_3 = -c_2/12 = -c_0/144$; ...
- $y_1(x) = x^1 \sum_{n=0}^{\infty} c_n x^n = c_0\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{12} - \dfrac{x^4}{144} \cdots\right)$. Power series
- Radius of convergence:

$$R = (\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|)^{-1} = (\lim_{n\to\infty}\frac{1}{(n+2)(n+1)})^{-1} = \infty,$$

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$\Rightarrow$ the power series is convergent for $x \in R$.

Examples 4-5: Case 2 (4)

- For r = r_2 = 0: $C_{k+1} = -\frac{1}{(k+1)k} C_k$ would be troubled if k = 0 (why?)

- Recursive relation is from $(k+r+1)(k+r)c_{k+1} + c_k = 0$. For r = 0, k = 0, it becomes $1\times 0\times c_1 + c_0 = 0$, $\Rightarrow c_0 = 0$.

- k = 1: $c_2 = -c_1/2$;
- k = 2: $c_3 = -c_2/6 = c_1/12$;
- k = 3: $c_4 = -c_3/12 = -c_1/144$; ...

- $y_2(x) = x^0 \sum_{n=1}^{\infty} c_n x^n = c_1\left(x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} \cdots\right)$.

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- $y_2(x) \propto y_1(x)$, not linearly independent solutions!

Examples 4-5: Case 2 (5)

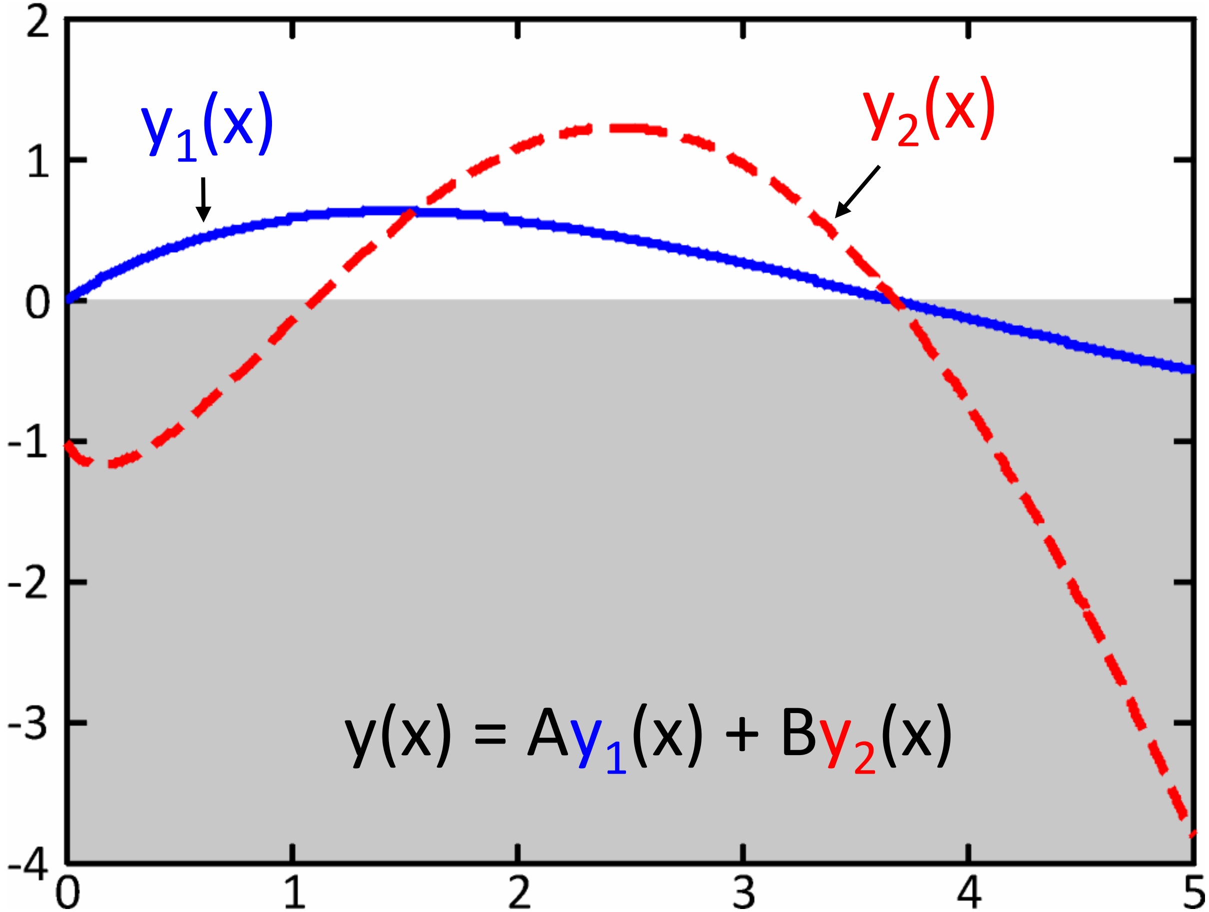
- To find a 2nd linearly independent solution, assume $y_2(x) = u(x) \times y_1(x)$ (reduction of order).
- It turns out that in Case 2, the two linearly independent solutions must be of the form:

  $y_1(x) = x^{r_1}\sum_{n=0} c_n x^n$,

  $y_2(x) = C y_1(x) \times \ln(x) + x^{r_2}\sum_{n=0} C_n x^n$; where $C = 0$ is possible (in that case, $y_2 = x^{r_2}\sum_{n=0} C_n x^n$).

Examples 4-5: Case 2 (6)

- $y_1(x) = x - x^2/2 + x^3/12 - x^4/144 + \ldots$
- $y_2(x) = y_1(x)\times\ln(x) + (-1 - x/2 + x^2/2 + \ldots)$



Example: Case 3 (1)

- ODE: $x(x-1)y'' + (3x-1)y' + 1y = 0$.
- Step 2: Assume $y(x) = x^r\sum_{n=0}c_nx^n = \sum_{n=0}c_nx^{n+r}$,

  $\Rightarrow y'(x) = \sum_{n=0}(n+r)c_nx^{n+r-1}$,

  $\Rightarrow y''(x) = \sum_{n=0}(n+r)(n+r-1)c_nx^{n+r-2}$;
- Step 3: Substitute y, y', y'' into the ODE:

  $[\sum_{n=0}(n+r)(n+r-1)c_nx^{n+r} - \sum_{n=0}(n+r)(n+r-1)c_nx^{n+r-1}] +$
  $[\sum_{n=0}3(n+r)c_nx^{n+r} - \sum_{n=0}(n+r)c_nx^{n+r-1}] + (\sum_{n=0}c_nx^{n+r}) = 0$;

  $\Rightarrow x^r\{ [\sum_{n=0}(n+r)(n+r-1)c_nx^n - r(r-1)c_0x^{-1} - \sum_{n=1}$
  $(n+r)(n+r-1)c_nx^{n-1}] + [\sum_{n=0}3(n+r)c_nx^n - rc_0x^{-1} -$
  $\sum_{n=1}(n+r)c_nx^{n-1}] + (\sum_{n=0}c_nx^n) \} = 0$;

Example: Case 3 (2)

- Step 4: x^{r-1} term: $-r(r-1)c_0 - rc_0 = -r^2c_0 = 0$, indicial equation gives $r = 0$. $\Rightarrow$ Case 3: Repeated real root.
- Step 5: By shifting the summation index:

$$[\textstyle\sum_{n=0}(n+r)(n+r-1)c_nx^n - \sum_{n=1}(n+r)(n+r-1)c_nx^{n-1}] + [\sum_{n=0}3(n+r)c_nx^n - \sum_{n=1}(n+r)c_nx^{n-1}] + (\sum_{n=0}c_nx^n)$$

$$\Rightarrow [\textstyle\sum_{k=0}(k+r)(k+r-1)c_kx^k - \sum_{k=0}(k+r+1)(k+r)c_{k+1}x^k] + [\sum_{k=0}3(k+r)c_kx^k - \sum_{k=0}(k+r+1)c_{k+1}x^k] + (\sum_{k=0}c_kx^k) = 0;$$

- x^{r+k} term: $-(k+r+1)^2c_{k+1} + [(k+r)(k+r+2)+1]c_k = 0$;
- Recursive: $c_{k+1} = \dfrac{(k+r)(k+r+2)+1}{(k+r+1)^2}c_k, \quad k = 0, 1, 2, \cdots$

Example: Case 3 (3)

- For r = 0: $c_{k+1} = \dfrac{k(k+2)+1}{(k+1)^2} c_k = c_k, \; k = 0, 1, 2, \cdots$

  $\Rightarrow \{c_0 = c_1 = c_2 = \ldots\}$

  $$\Rightarrow y_1(x) = c_0 x^0 \sum_{n=0}^{\infty} 1 \times x^n = c_0 \left(1 + x + x^2 + x^3 + \cdots\right) = \frac{1}{1-x}.$$

- Radius of convergence: $R = \left(\lim_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right|\right)^{-1} = \left(\lim_{n\to\infty} 1\right)^{-1} = 1,$

  $\Rightarrow$ the power series is convergent for $|x| < 1$.

Example: Case 3 (4)

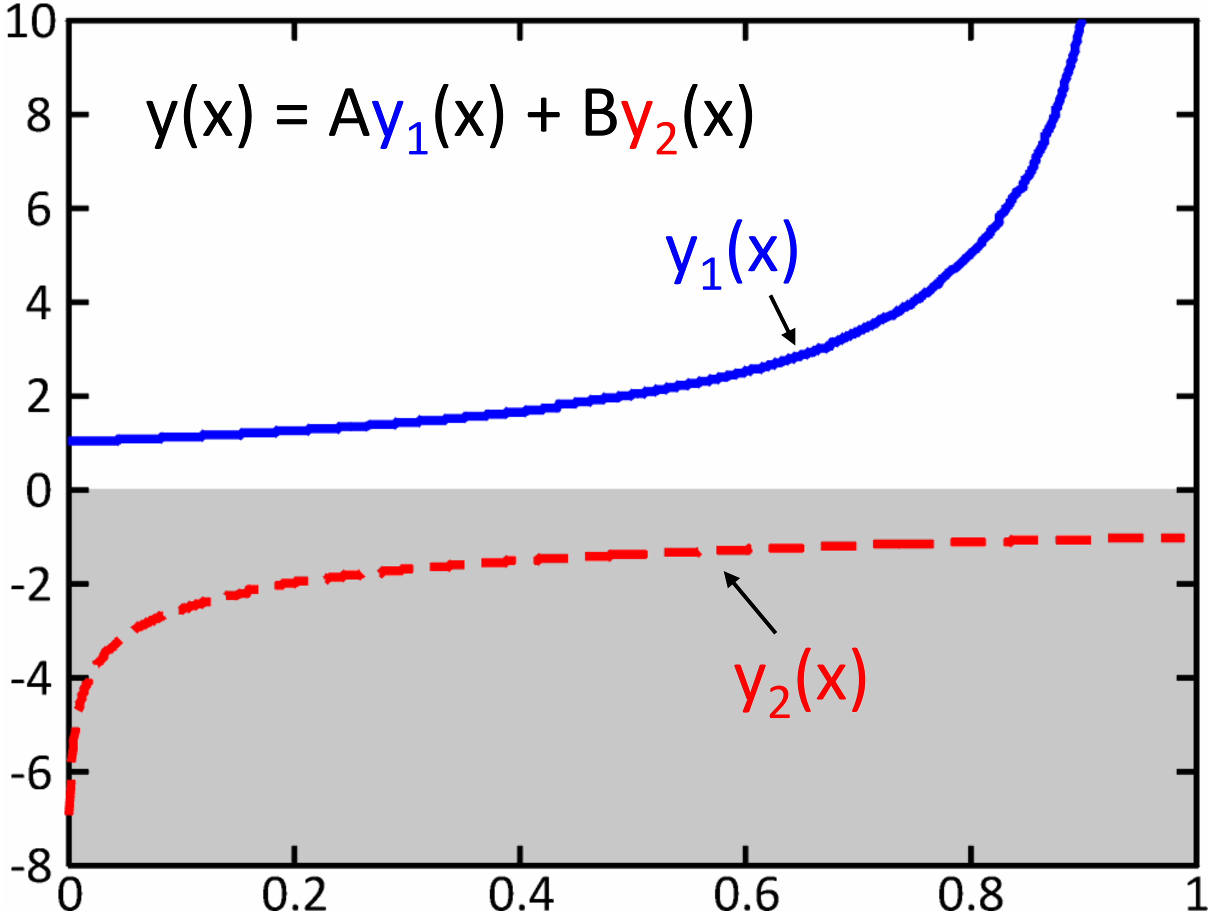
- For the lack of a distinct 2nd root, $\Rightarrow$ find the 2nd linearly independent solution by assuming $y_2(x) = u(x)\times y_1(x)$ (reduction of order).
- The governing 1st-order ODE of $u(x)$ gives $u(x) = \ln(x)$, $\Rightarrow y_2(x) = \ln(x)/(1-x)$.
- It turns out that in Case 3, the two linearly independent solutions must be of the form:

  $y_1(x) = x^r\sum_{n=0} c_n x^n$,

  $y_2(x) = y_1(x)\times\ln(x) + x^r\sum_{n=0} C_n x^n$

Example: Case 3 (5)

- $y_1(x) = 1/(1-x)$
- $y_2(x) = \ln(x)/(1-x)$



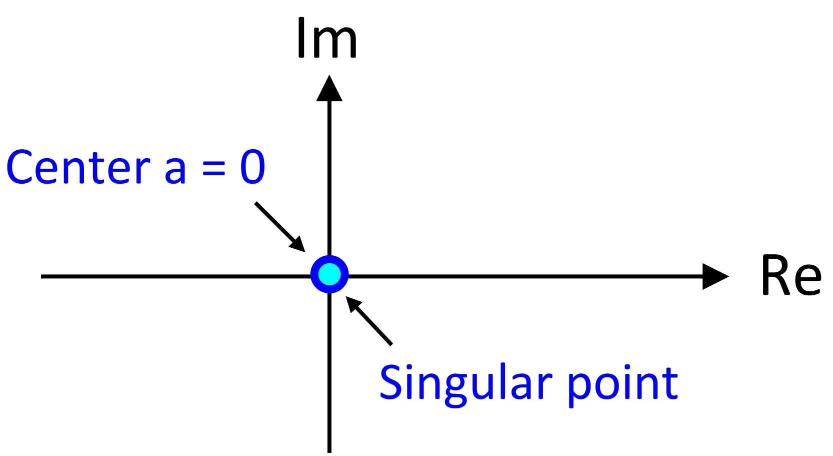
❐ Bessel functions

- Bessel's equation
- $\{J_\nu(x), Y_\nu(x)\}$
- Properties
- Varieties: $\{I_\nu(x), K_\nu(x)\}$, $\{J_{\pm 1/2}(x)\}$

Friedrich Wilhelm
Bessel
(1784-1846)

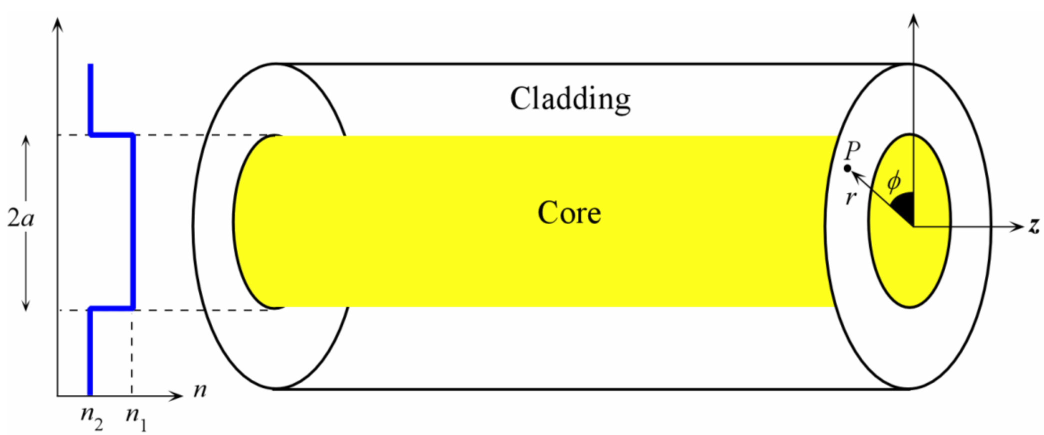
Bessel's equation

- $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, where $\nu > 0$ is a given constant (usually an integer).
- $x = 0$ is a singular point. $\Rightarrow$ no Maclaurin series solution, Frobenius method is needed.



Why Bessel's equation?

- E.g. The z-component of e-field $E_z(r,\phi,z) = R(r)\times\Phi(\phi)\times e^{-j\beta z}$ that can propagate in a cylindrical fiber without distortion satisfies:
- $\Phi(\phi) = e^{\pm j\nu\phi}$, $\nu = 0, 1, ..$; such that $\Phi(\phi+2n\pi) = \Phi(\phi)$.
- In the core ($r < a$, $n = n_1$), $R(r)$ satisfies $r^2R'' + rR' + (\kappa^2r^2-\nu^2)R = 0$, where $\kappa^2 \equiv (n_1k_0)^2 - \beta^2$ is unknown.



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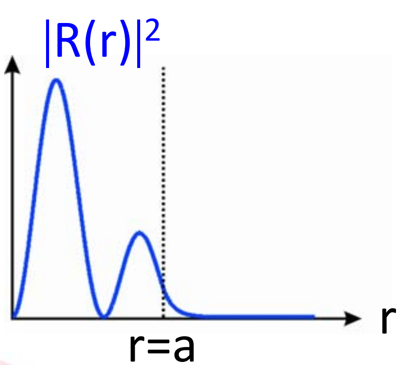
Change of variable

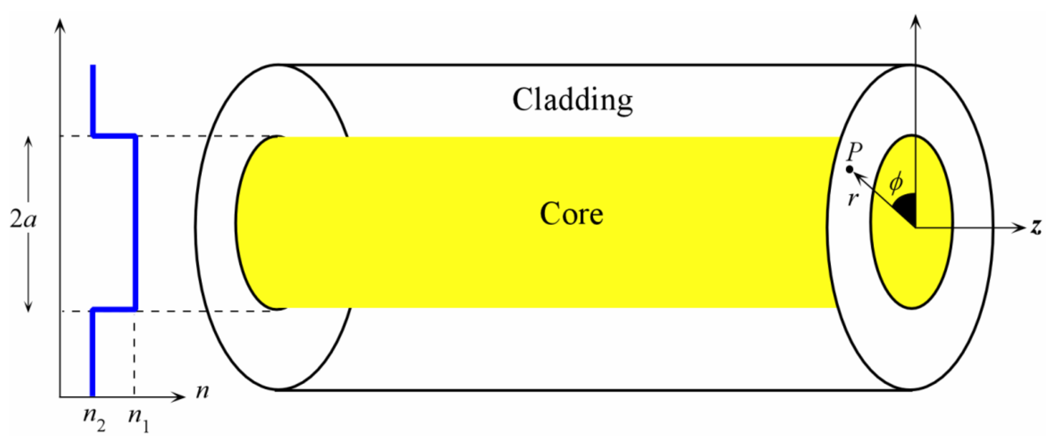
- By change of variable $R(r) = y(x/\kappa)$:

$\Rightarrow dR/dr = (dy/dx)\times(dx/dr) = \kappa y'(x)$,

$\Rightarrow d^2R/dr^2 = \kappa(dy'/dx)\times(dx/dr) = \kappa^2 y''(x)$,

$\Rightarrow r^2R'' + rR' + (\kappa^2 r^2 - \nu^2)R = 0$ becomes $(x/\kappa)^2\kappa^2 y''(x) + (x/\kappa)\kappa y'(x) + (x^2 - \nu^2)y = x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ ... Bessel's equation.





Series solution (1)

- Step 2: Assume $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{r+n}$,

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} (r+n) c_n x^{r+n-1}, \;\; y''(x) = \sum_{n=0}^{\infty} (r+n)(r+n-1) c_n x^{r+n-2}.$$

- Step 3: $x^2 y'' + x y' + (x^2 - \nu^2) y$

$$= \cancel{x^2} \sum_{n=0}^{\infty} (r+n)(r+n-1) c_n x^{r+n-\cancel{2}} + \cancel{x} \sum_{n=0}^{\infty} (r+n) c_n x^{r+n-\cancel{1}}$$

$$+ x^2 \sum_{n=0}^{\infty} c_n x^{r+n} - \nu^2 \sum_{n=0}^{\infty} c_n x^{r+n}$$

$$= x^r \left\{ \sum_{n=0}^{\infty} \overset{(r+n)^2-\nu^2}{\boxed{\left[(r+n)(r+n-1) + (r+n) - \nu^2\right]}} c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \right\}$$

$(r^2-\nu^2)c_0 + [(r+1)^2-\nu^2]c_1 x + \sum_{n=2}(\;)$; let k = n-2; let k = n;

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Series solution (2)

$$= x^r \begin{Bmatrix} (r^2 - \nu^2)c_0 + [(r+1)^2 - \nu^2]c_1 x \\ + \sum_{k=0}^{\infty} \{[(r+k+2)^2 - \nu^2]c_{k+2} + c_k\} x^{k+2} \end{Bmatrix} = 0,$$

- Step 4: x^{r+0} coefficient is $(r^2-\nu^2)c_0 = 0$, $\Rightarrow$ $r^2 - \nu^2 = 0$ (indicial equation), $\Rightarrow$ r = {ν, $-\nu$}.
- Step 5: x^{r+1} coefficient is $[(r+1)^2 - \nu^2]c_1$, for $r = \nu$, = $(2\nu+1)c_1 = 0$, $\Rightarrow$ $c_1 = 0$ as long as $\nu \neq -½$.
- For $r = \nu$, x^{r+n} coefficient gives a recursive relation:

$$c_{k+2} = \frac{-1}{(k+2)(k+2\nu+2)} c_k, \quad k = 0, 1, 2, \cdots$$

Series solution (3)

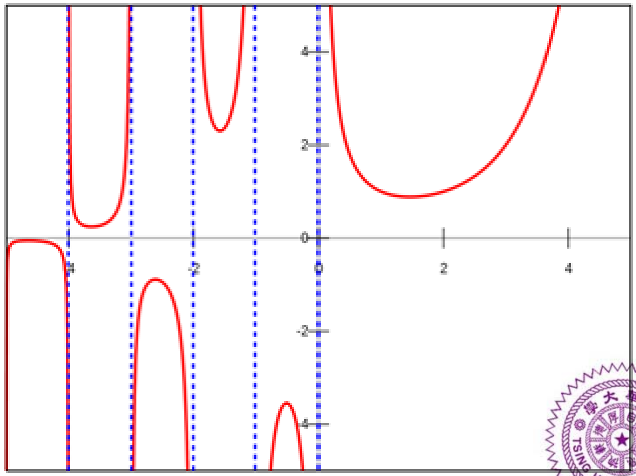
- k = 1: $c_3 = -c_1/[3(3+2\nu)] = 0$, so are $c_5 = c_7 = \ldots = 0$.
- k = 0: $c_2 = -c_0/[2^2(1+\nu)]$;
- k = 2: $c_4 = -c_2/[2^3(2+\nu)] = c_0/[2^4(2+\nu)]$; ...
- $c_{2n} = (-1)^n c_0/[2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)]$, n = 1, 2, 3, ...
- By choosing $c_0 = [2^\nu \Gamma(1+\nu)]^{-1}$, where

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} dx,$$

$$\Gamma(n) = (n-1)!, \text{ if } n \text{ is an integer;}$$

$c_{2n} = (-1)^n/[2^{2n+\nu} n!\Gamma(1+\nu+n)]$,

n = 0, 1, 2, ...



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Bessel function of the 1st kind $\{J_\nu(x)\}$

- For $r = \nu$ and the choice of $c_0 = [2^\nu \Gamma(1+\nu)]^{-1}$,

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

Bessel function of the 1st kind of order ν

- If ν is not an integral multiple of ½, $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is not an integer, $\Rightarrow$ Case 1 of Frobenius method, a 2nd linearly independent solution is obtained by:

$$y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} C_n x^n = J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

- General solution: $y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$, convergent over $x \in (0,\infty)$.

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Bessel function of the 2nd kind $\{Y_\nu(x)\}$

- If ν is an integer (as usual), so is $r_1 - r_2 = 2\nu$, $\Rightarrow$ Case 2, $J_{-n}(x)$ is linearly dependent on $J_n(x)$.
- Define $Y_\nu(x) \equiv \dfrac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$, Bessel function of the 2nd kind of order ν

  which is linearly independent of $J_\nu(x)$ if ν is not an integer.
- By L'Hôpital rule, $\lim_{\nu\to m} Y_\nu(x) = Y_m(x)$ exists and is linearly independent of $J_m(x)$ if $\nu = m$ is an integer.
- General solution: $y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$, for arbitrary ν (integer or non-integer) and $x \in (0,\infty)$.

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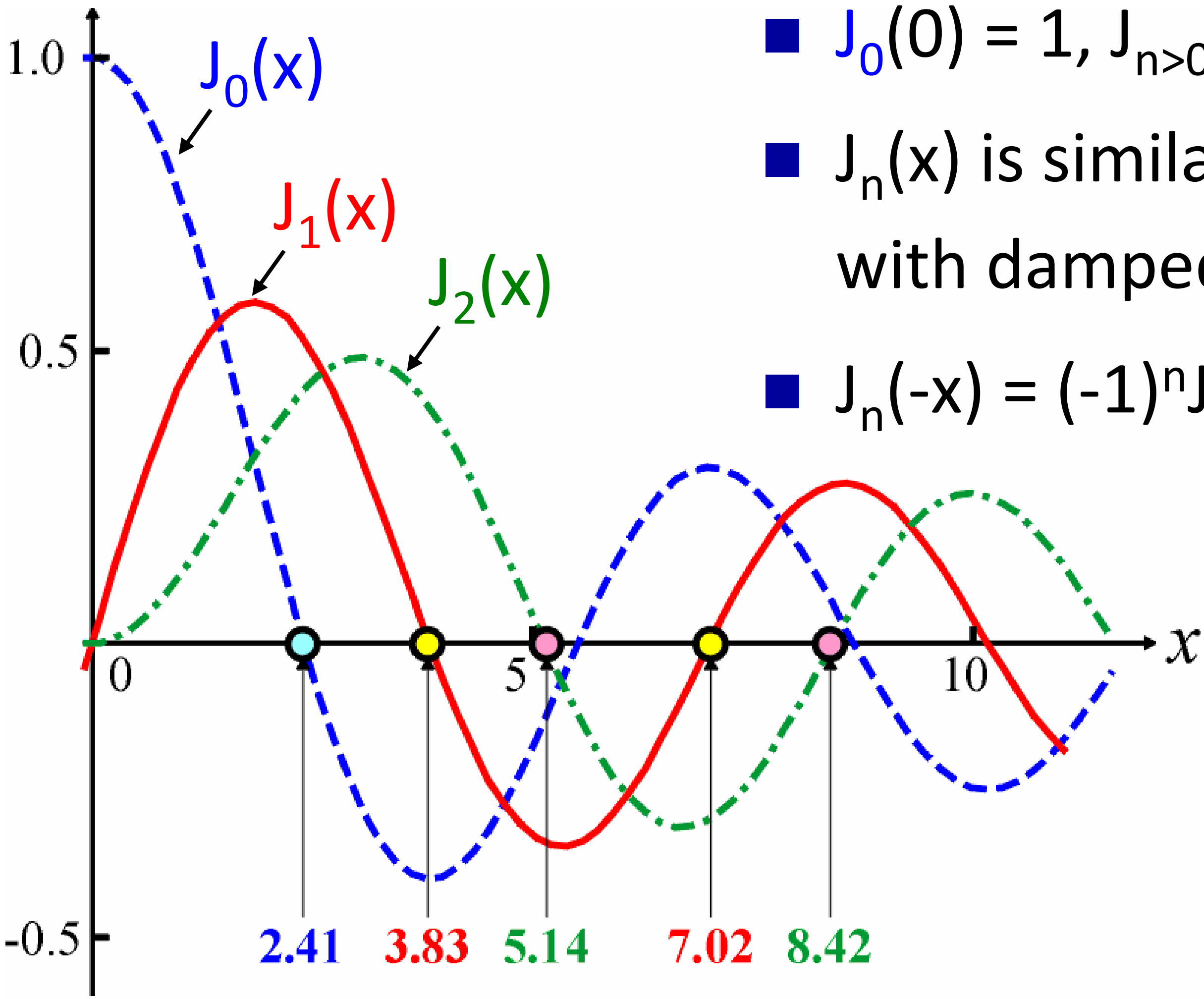
Plots of $\{J_n(x)\}$



- $J_0(0) = 1$, $J_{n>0}(0) = 0$.
- $J_n(x)$ is similar to cosine/sine with damped envelope.
- $J_n(-x) = (-1)^n J_n(x)$.

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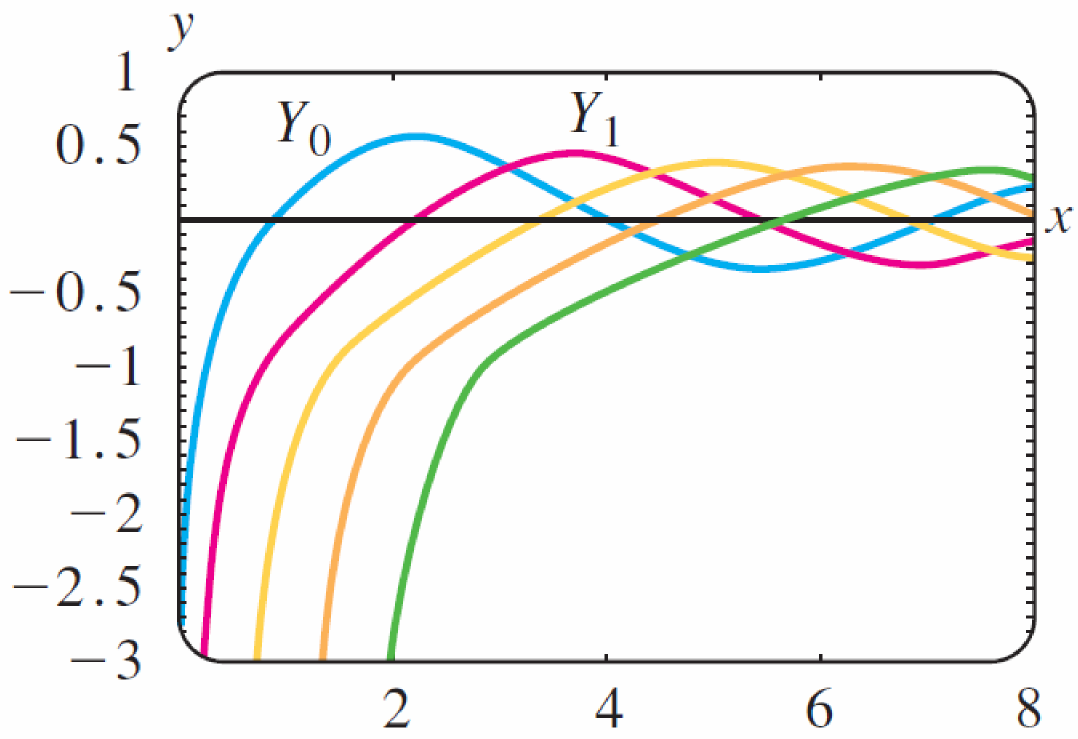
Plots of $\{Y_n(x)\}$

- $Y_n(0) \rightarrow -\infty$ (unbounded at x = 0).
- Not applicable to describing e-field's radial distribution R(r) in the fiber core region (r < a).



Differential recurrence relation (1)

- $xJ_\nu'(x) = -xJ_{\nu+1}(x) + \nu J_\nu(x)$ E.g. $\nu = 0$, $\Rightarrow J'_0(x) = -J_1(x)$.
- Proof:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu},$$

$$\Rightarrow xJ_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \boxed{2\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n}}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}} + \nu \overbrace{\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}}}^{J_\nu(x)}$$

$$= x\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} + \nu J_\nu(x)$$

Differential recurrence relation (2)

- Having n = k+1, the 1st term becomes

$$x\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$= x\boxed{\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!\,\Gamma(2+\nu+k)}\left(\frac{x}{2}\right)^{2k+\nu+1}},$$

$-J_{\nu+1}(x)$

because

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu},$$

$$\Rightarrow J_{\nu+1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(2+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu+1}.$$

Example 4 (1)

- The displacement x(t) of a frictionless mass on an aging spring is governed by ODE: $x'' + (ke^{-\alpha t}/m)x = 0$.
- Note: The ODE reduces to $x'' + (k/m)x = x'' + (\omega_0)^2 x = 0$ if $\alpha = 0$ (spring without aging).
- Change of variable: $s = \dfrac{2}{\alpha}\omega_0 e^{-\frac{\alpha}{2}t}$

$\Rightarrow$ $dx/dt = (dx/ds)\times(ds/dt) = -\omega_0 e^{-\alpha t/2} x'(s)$,

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega_0\left[-\frac{\alpha}{2}e^{-\frac{\alpha}{2}t}x'(s) + e^{-\frac{\alpha}{2}t}\frac{d}{dt}x'(s)\right] = \frac{\alpha s}{2}\left[\frac{\alpha}{2}x'(s) - \boxed{\frac{d}{dt}x'(s)}\right],$$

where $dx'/dt = (dx'/ds)\times(ds/dt) = -(\alpha s/2)x''(s)$.

Example 4 (2)

- $x''(t) + (\omega_0)^2 e^{-\alpha t} x(t) = x''(t) + (\alpha s/2)^2 x(t)$

$$= \frac{\alpha s}{2}\left[\frac{\alpha}{2}x'(s) + \frac{\alpha s}{2}x''(s)\right] + (\frac{\alpha s}{2})^2 x(s) = 0,$$

$\Rightarrow x'/s + x'' + x = 0,$

$\Rightarrow s^2 x'' + s x' + s^2 x = 0$... Bessel's equation with $\nu = 0$.

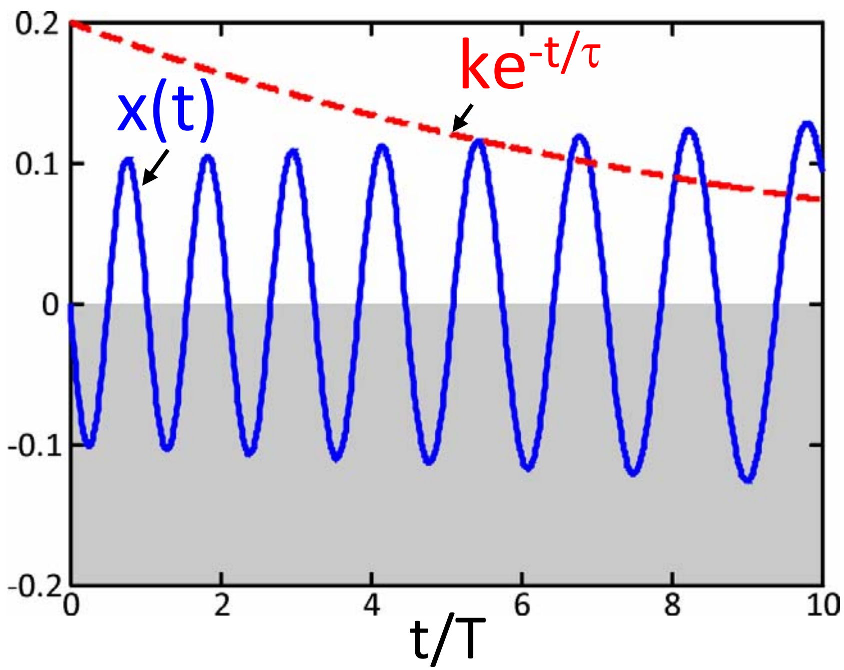
- General solution: $x(s) = c_1 J_0(s) + c_2 Y_0(s)$, i.e.

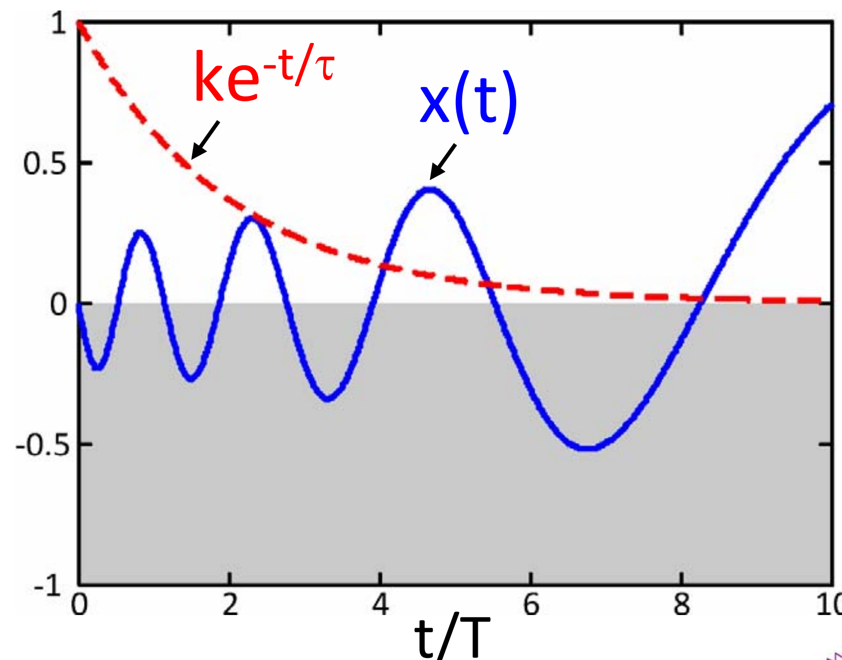
$$x(t) = c_1 J_0(2\omega_0 \tau e^{-\frac{t}{2\tau}}) + c_2 Y_0(2\omega_0 \tau e^{-\frac{t}{2\tau}}),\ \text{if}\ \ \tau \equiv \frac{1}{\alpha}.$$

Impact of aging

- Weak aging:
  - τ = 10T ($T = 2\pi/\omega_0$)
  - $c_1 = c_2 = 1$

- Strong aging:
  - τ = 2T ($T = 2\pi/\omega_0$)
  - $c_1 = c_2 = 1$





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Modified Bessel's equation

- The radial distribution R(r) of the e-field in the cladding region ($r > a$, $n = n_2$) satisfies

  $r^2R'' + rR' + (-\alpha^2r^2-\nu^2)R = 0$, $\alpha^2 \equiv \beta^2 - (n_2k_0)^2$.

- By change of variables $R(r) = y(x/\alpha)$, it gives $x^2y'' + xy' + (-x^2 - \nu^2)y = 0$ ... modified Bessel's equation.

- By another change of variable "$t = ix$" [$i = \sqrt{(-1)}$]:

  $\Rightarrow dy/dx = (dy/dt)\times(dt/dx) = iy'(t)$,

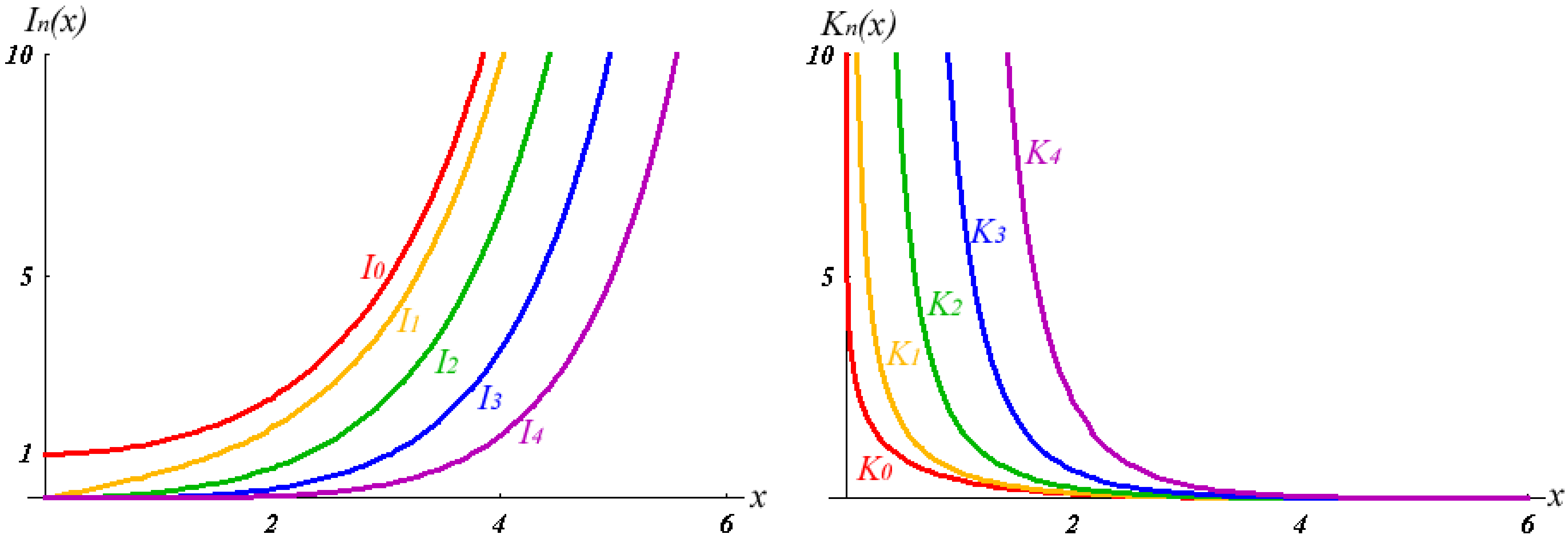
  $\Rightarrow d^2y/dx^2 = i(dy'/dt)\times(dt/dx) = -y''(t)$,

  $\Rightarrow (-it)^2(-y'') + (-it)(iy') + [-(-it)^2 - \nu^2]y = t^2y'' + ty' + (t^2 - \nu^2)y = 0$ ... standard Bessel's equation.

Modified Bessel functions $\{I_\nu, K_\nu\}$

- However, the solutions $\{J_\nu(t) = J_\nu(ix), Y_\nu(t)=Y_\nu(ix)\} \in C$.
- To get real-valued solutions, define:
- $I_\nu(x) = (i)^{-\nu} J_\nu(ix) \in R$, modified Bessel function of the 1st kind,
- $K_\nu(x) \equiv \dfrac{\pi}{2}\dfrac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$, … modified Bessel, 2nd kind
- By L'Hôpital rule, $\lim_{\nu\to m} K_\nu(x) = K_m(x)$ exists and is linearly independent of $I_m(x)$ if $\nu = m$ is an integer.
- General solution: $y(x) = c_1 I_\nu(x) + c_2 K_\nu(x)$, for arbitrary ν (integer or non-integer) and $x \in (0,\infty)$.

Plots of $\{I_n(x), K_n(x)\}$



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Spherical Bessel functions $\{J_{1/2}(x)\}$ (1)

- If $\nu = 1/2$, Bessel functions of the 1st kind

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+1/2+n)} \left(\frac{x}{2}\right)^{2n+1/2}$$

can be simplified by $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, $\Gamma(1/2) = \sqrt{\pi}$:

- n = 0: $\Gamma(1+1/2) = 1/2\times\Gamma(1/2) = \sqrt{\pi}/2$,
- n = 1: $\Gamma(1+3/2) = (3/2)\times\Gamma(3/2) = 3\sqrt{\pi}/2^2$,
- n = 2: $\Gamma(1+5/2) = (5/2)\times\Gamma(5/2) = (5!)\sqrt{\pi}/(2^5 2!)$, ...
- $\Gamma(1+1/2+n) = (2n+1)!\sqrt{\pi}/(2^{2n+1}n!)$.

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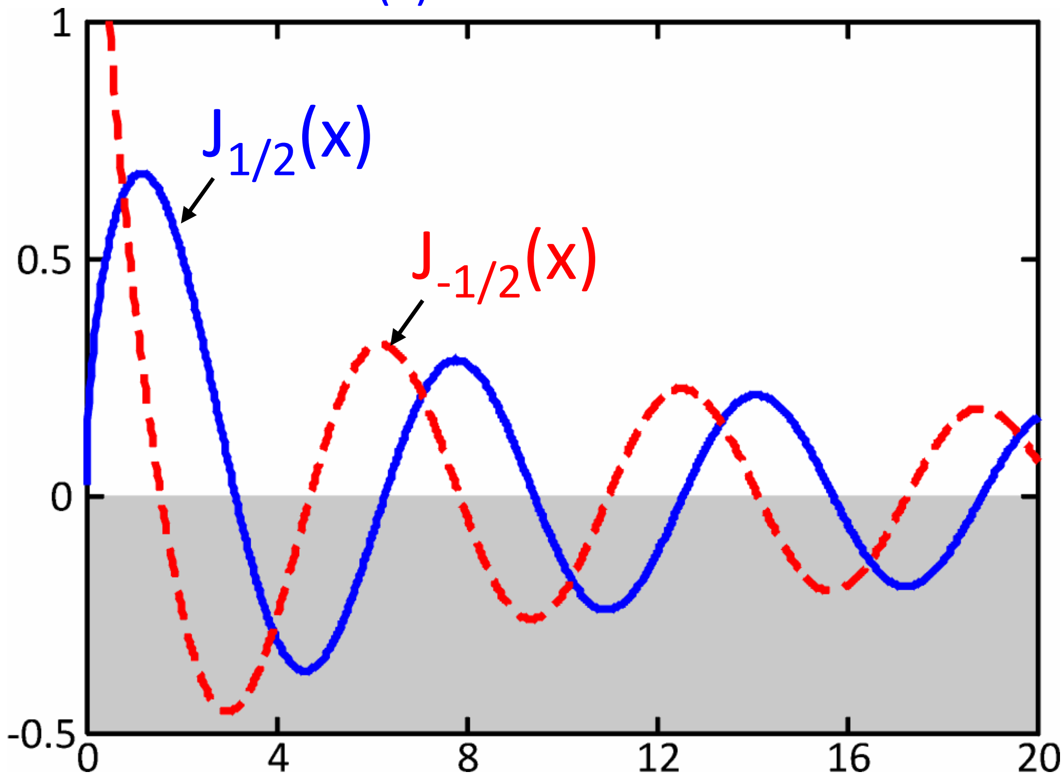
Spherical Bessel functions $\{J_{½}(x)\}$ (2)

- Hence,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

sin(x)



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Summary

- How to solve ODEs by power series method and Frobenius method, respectively?
- When is Frobenius method necessary?
- When are Legendre polynomials useful?
- When are Bessel functions useful?