Midterm Exam

(2017/4/13)

1) Consider the following linear first-order initial-value problem (IVP).

Ordinary differential equation (ODE): $y' + P(t) \times y = g(t) \cdots (1)$,

where $P(t) = -1$, $g(t) = e^{-t}$.

Initial condition (IC): $y(0) = y_0$, where y_0 is a real constant.

1A) (5%) Calculate the integrating factor $\rho(t) = e^{\int P(t)dt}$.

1B) (5%) Derive a new ODE [Equation (2)] by multiplying Equation (1) with $\rho(t)$. Show that the left hand side of the equality of Equation (2) becomes $\frac{d}{dt}[\rho(t) \times y(t)]$.

1C) (5%) Integrate Equation (2) in Problem 1B. Substitute the IC to get the solution $y(t)$.

1D) (5%) What is the value of y_0 such that the solution is bounded ($\left|\lim_{t\to\infty} y(t)\right| < \infty$)?

1E) (Bonus 10 points) Roughly sketch the direction field on the *yt*-plane (with $t > 0$) according to Equation (1): $y'(t) = y + e^{-t}$. Draw the solution curve $y(t)$ under the IC specified in Problem 1D.

1F) (5%) Perform Laplace transform for the ODE. Solve $Y(s)$.

1G) (5%) Perform partial fraction expansion for $Y(s)$. Solve $y(t)$ by $L^{-1}\{Y(s)\}$. Note: It has to be in agreement with that obtained in Problem 1C.

2) Consider the following linear second-order initial-value problem (IVP).

ODE: $x'' + 4x' + 4x = g(t) \cdots (3)$.

Initial conditions (ICs): $x(0) = x_0$, $x'(0) = v_0$, where x_0, v_0 are constants.

2A) (5%) Assume $g(t) = 0$ (source-free). Derive the auxiliary equation by substituting

$x(t) = e^{mt}$ into the ODE. Find the root(s).

2B) (10%) According to the result in Problem 2A, what is the homogeneous solution $x_c(t)$?

2C) (5%) Roughly sketch the solution curve $x_c(t)$ if $x_0 = 0$, $v_0 = 1$. Denote the time instant $t = t_0$ at which $x_c(t_0)$ is the maximum.

2D) (5%) Solve the particular solution $x_p(t)$ if $g(t) = \cos(2t)$.

2E) (10%) Solve the total solution $x(t) = x_c(t) + x_p(t)$ if { $g(t) = \cos(2t)$, $x_0 = 0$, $v_0 = 1$ }. (Note: Here the $x_c(t)$ is different from that obtained in Problem 2B.)

2F) (5%) Perform Laplace transform for the ODE with { $g(t) = \cos(2t)$, $x_0 = 0$, $v_0 = 1$ }. Solve $X(s)$.

2G) (5%) Perform partial fraction expansion for $X(s)$.

2H) (5%) Solve $x(t)$ by $L^{-1}\{X(s)\}$. Note: It has to be in agreement with that obtained in Problem 2E.

2I) (10%) Solve $x(t)$ for the ODE with { $g(t) = \delta(t)$, $x_0 = 0$, $v_0 = 1$ }. Denote the solution by $h(t)$ (impulse response).

2J) (10%) Whether the result obtained in Problem 2F is equal to $h(t) * \cos(2t)$?

2K) (10 bonus points) Solve $x(t)$ for the ODE with { $g(t) = \begin{cases} 1, \text{if } 0 < t < 1; \\ 0, \text{otherwise} \end{cases}$, $x_0 = 0$, $v_0 = 1$ }.