Midterm Exam

(2017/6/15)

1) The inner product of two real functions $f_1(x)$, $f_2(x)$ over an interval $a < x < b$ is:

$$(f_1, f_2) \equiv \int_a^b [f_1(x) \times f_2(x)] dx. \tag{1}$$

The norm of a real function $f(x)$ is:

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_a^b |f(x)|^2 dx}. \tag{2}$$

The generalized included angle between two real functions $f_1(x)$, $f_2(x)$ satisfies:

$$\cos\theta = \frac{(f_1, f_2)}{\|f_1\| \times \|f_2\|}. \tag{3}$$

Let $f_1(x) = x^2$, $f_2(x) = x^4$, $a = -L/2$, $b = L/2$.

1A) (5%) Calculate (f_1, f_2).

1B) (5%) Calculate $\cos\theta$. Is it dependent on the interval length L?

1C) (5%) Does Taylor series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots$ belong to orthogonal series expansion?

2) A regular Sturm-Liouville problem is of the form:

Ordinary differential equation (ODE): $y'' + \lambda y = 0 \cdots (4)$,

Boundary conditions (BCs): $\begin{cases} y'(0) = 0 \cdots (5) \\ y(1) - y'(1) = 0 \cdots (6) \end{cases}$

2A) (5%) Show that there is only trivial solution $y(x) = 0$ if $\lambda \le 0$.

2B) (5%) If $\lambda = \beta^2 > 0$, derive the "eigenvalue equation" from which discrete eigenvalues $\{\beta_n\}$ can be determined.

2C) (5%) Roughly estimate the nth smallest eigenvalue λ_n if n is a large integer. (Hint: Use graphical method to solve the eigenvalue equation in Problem 2B.)

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3) The displacement $x(t)$ of a frictionless mass on an “aging” spring is governed by an ODE:

$$x'' + \omega_0^2 e^{-t/\tau} x = 0, \tag{7}$$

where $\omega_0 = \sqrt{k/m}$ and τ represent the resonance angular frequency and aging time constant of the spring, respectively.

3A) (10%) Describe how to solve Eq. (7) by the power series method without actually deriving the coefficients.

3B) (5%) What is the ratio between the 2nd and 0th order coefficients $\frac{c_2}{c_0}$?

Instead of using the power series method, Eq. (7) can be transformed into a Bessel’s equation of order $\nu = 0$

$$s^2 x'' + s x' + (s^2 - 0^2) x = 0 \tag{8}$$

by change of variable $s = 2\tau\omega_0 e^{-\frac{t}{2\tau}}$ (as performed in the lecture slides). This will quickly arrive at the general solution to Eq. (7):

$$x(t) = c_1 J_0(2\omega_0 \tau e^{-\frac{t}{2\tau}}) + c_2 Y_0(2\omega_0 \tau e^{-\frac{t}{2\tau}}), \tag{9}$$

where $J_0(x)$, $Y_0(x)$ are Bessel function of the 1st and 2nd kinds of order 0, and c_1, c_2 are two expansion coefficients to be determined by initial conditions.

3C) (5%) Roughly sketch $J_0(x)$ and $Y_0(x)$. [Note: Only the shapes and singular point (if any) matter. Never mind about the precise numbers.]

Figures 1(a) and 1(b) illustrate two $x(t)$ curves obtained by substituting $c_1 = c_2 = 1$ and (a) $\tau = 10T$, (b) $\tau = 2T$ into Eq. (9), where the resonance period $T \equiv \frac{2\pi}{\omega_0}$.

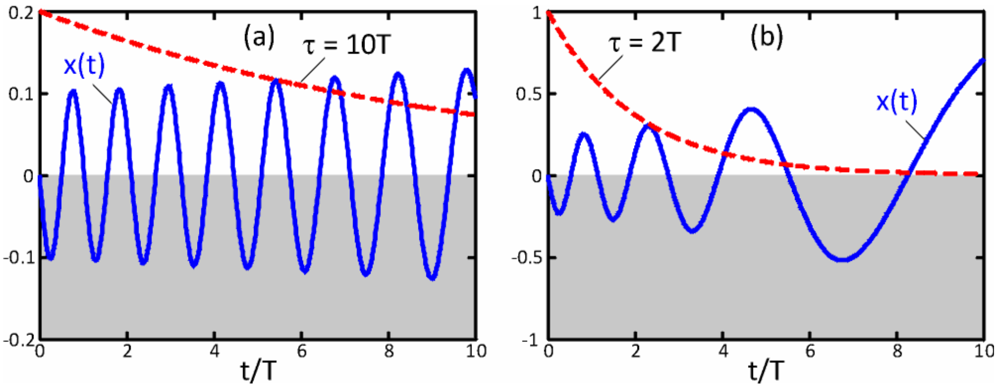

Fig. 1. Solution functions obtained by (a) slow aging $\tau = 10T$, and (b) fast aging $\tau = 2T$, respectively. The dashed curves show the decay of spring constant.

3D) (Bonus 5 points) Why does $x(t)$ roughly exhibit oscillation of period T in Fig. 1(a)?

3E) (Bonus 10 points) Why does $x(t)$ roughly exhibit decreasing oscillation period T in Fig. 1(b)? [Hint: Examine Eq. (7).]

4) The formulae of Fourier transform and inverse Fourier transform are

$$\begin{cases} F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \cdots (10) \\ f(t) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \cdots (11) \end{cases}$$

4A) (5%) Calculate the spectrum $F_1(\omega)$ if $f_1(t)$ is a square function of unit width [Fig. 2(a)].

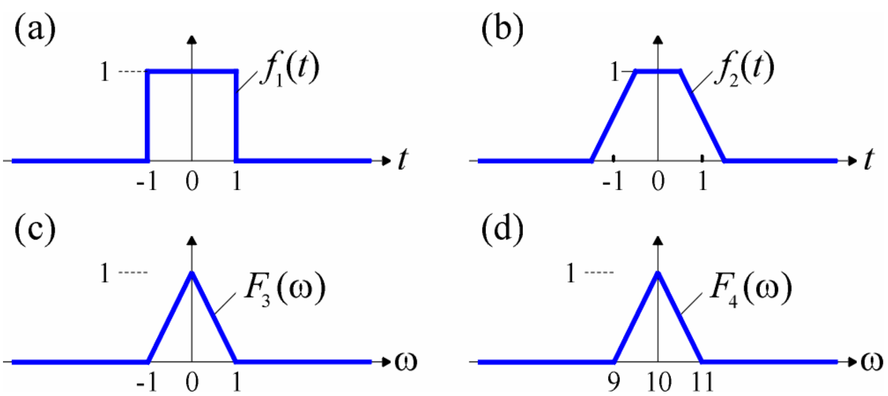

Fig. 2. Functions in (a-b) time and (c-d) frequency domains.

4B) (5%) Plot the spectral phase $\angle F_1(\omega)$ calculated in Problem 4A.

4C) (5%) What's the role of $\angle F_1(\omega)$ in synthesizing $f_1(t)$?

4D) (5%) Compare the difference(s) between spectra $F_1(\omega)$ and $F_2(\omega)$ if $f_2(t)$ is a trapezoidal function [Fig. 2(b)]. [Note: No need to calculate $F_2(\omega)$ precisely.]

4E) (5%) Compare the difference(s) between spectra $f_3(t)$ and $f_4(t)$ if $F_4(\omega)$ is a shifted version of $F_3(\omega)$ [Fig. 2(d)]. [Note: No need to calculate $f_{3,4}(t)$ precisely.]

5) The loop current $i(t)$ in a series-*RLC* circuit driven by a voltage source $v(t)$ is governed by an ODE:

$$i'' + \frac{1}{\tau} i' + \omega_0^2 i = g(t)\,, \tag{12}$$

where $\tau = \frac{L}{R}$, $\omega_0 = \frac{1}{\sqrt{LC}}$, $g(t) = \frac{v'(t)}{L}$, respectively. The Fourier transform of Eq. (12) must be of the form:

$$X(\omega) \times I(\omega) = G(\omega)\,. \tag{13}$$

5A) (5%) Solve $I(\omega)$, the spectrum of $i(t)$, by Eq. (13). [Note: $F\{i'(t)\} = j\omega \times I(\omega)$. $G(\omega) = F\{g(t)\}$ can be used in the expression of $I(\omega)$.]

5B) (10%) Roughly sketch $i(t)$ if $R = 0$, $L = 1$, $C = \frac{1}{\pi^2}$, and the driving voltage $v(t)$ behaves like Fig. 3. [Note: Pay attention to the characteristics of $X(\omega)$ when $R = 0$.]

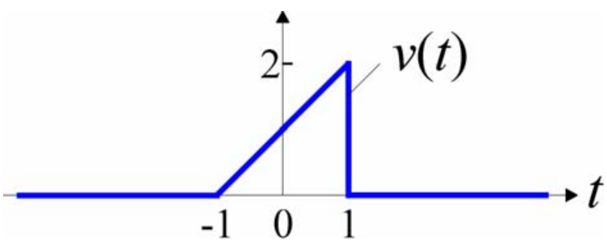

Fig. 3.

5C) (10%) What're the impacts of nonzero resistance $R > 0$ on the spectral magnitude response $|X(\omega)|$ of the circuit?