# Chapter 2
# First-order ODEs

- Solution curves (2.1)
- Separable variables (2.2)
- Linear equations (2.3)
- A numerical method (2.6)

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## Solutions curves

- Direction field
- Autonomous ODEs

# Motivation



- For a general 1st-order ODE: $y'(x) = f(x,y)$, it's helpful to know what a solution curve $y(x)$ roughly looks like without actually solving it in detail.
- Visualization is particularly useful for nonlinear ODEs, for they are difficult to solve analytically.

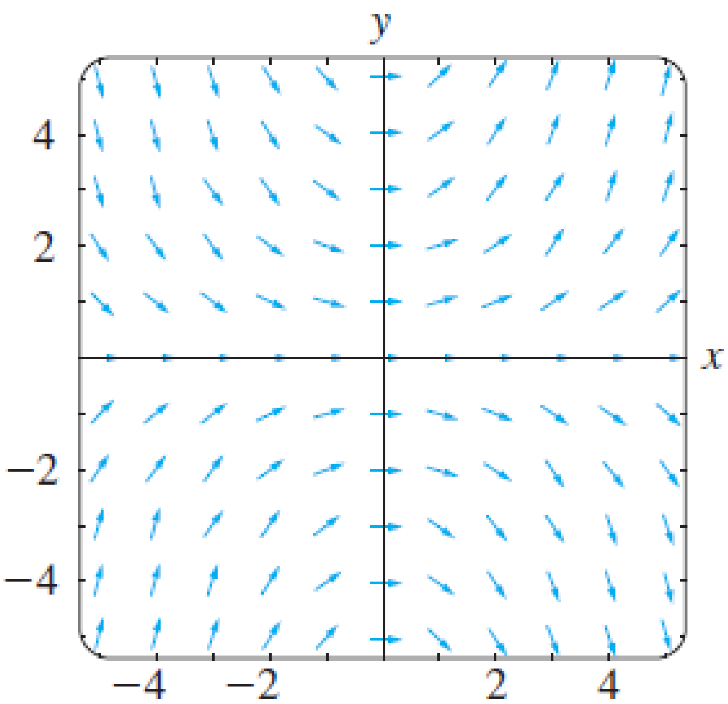
# Direction field



- $y'(x)$ means the slope of the solution curve $y(x)$ at position x.
- $y'(x) = f(x,y)$ means the slope of $y(x)$ at any position $(x,y)$ is specified by the given function $f(x,y)$.
- Direction field is established by drawing a grid of small arrows on the xy-plane, where the pointing direction of the arrow at position $(x_0,y_0)$ is determined by the value of $f(x_0,y_0)$.

# E.g. y' = 0.2xy (Example 1)



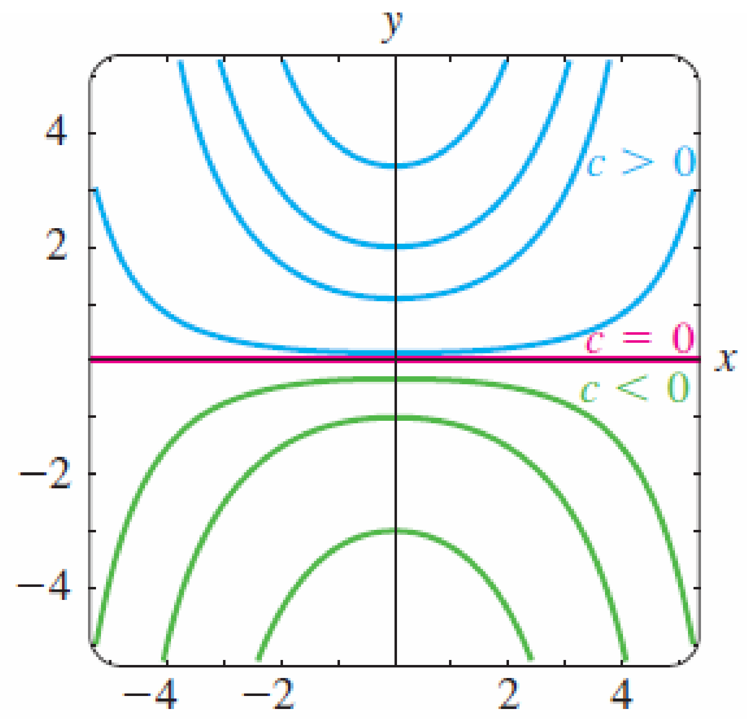



- For the 1st quadrant, slope function $f(x,y) = 0.2xy > 0$, all the arrows point upward.
- As x or y increases, magnitude of slope increases, arrows are increasingly steeper.
- For the 2nd quadrant, arrows point downward and are increasingly steeper as x decreases or y increases.

# Solution curves



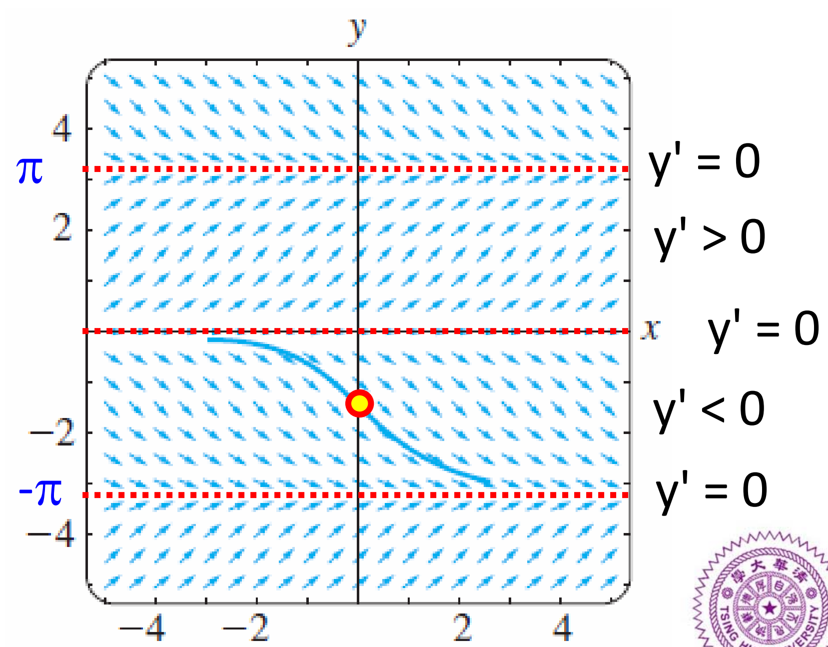
- A solution curve y(x) can be roughly sketched by
  - Starting from the point $(x_0, y_0)$ given by IC $y(x_0) = y_0$,
  - Extending y(x) by following the arrows of the direction field.
- E.g. The family of solution curves $y(x) = c\times\exp(0.1x^2)$ to the ODE $y' = 0.2xy$ agree with the direction field.



# E.g. y' = sin(y) (Example 2)



- Q: Is it linear or nonlinear?
- f(x,y) = sin(y) is independent of x, ⇒ all the points on the same row $(x,y_0)$ have a common slope.
- sin(y) is periodic, arrows repeat themselves in a y-interval of 2π.
- E.g. The solid curve represents the solution to y'(x) = sin(y) and IC y(0) = -1.5.



# What's autonomous ODE?



- An ODE whose slope function $y'(x) = f(y)$ does not vary with the independent variable x "explicitly" [i.e. after solving y(x), you can still see y'(x) generally varying with x].
- Models of physical laws that do not change with time are autonomous.
- E.g. RLC circuits with time-independent elements, Torricelli's law: $v = \sqrt{(2gh)}$, Newton's law: $F = ma$, ….

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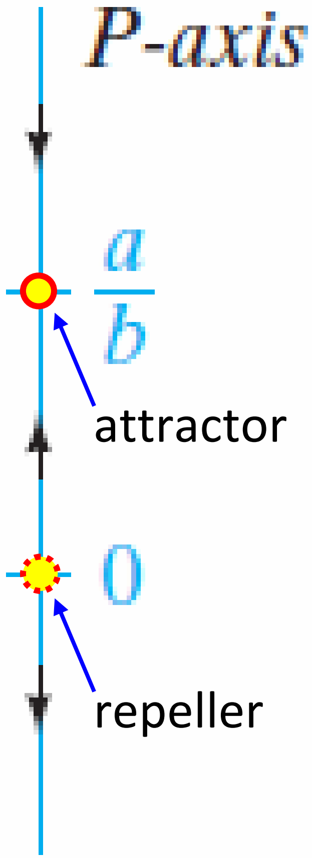
# E.g. P'(t) = P(a-b×P) (Example 3)




Phase portrait

- Slope function f(P) = P(a-b×P) = 0 has two solutions: P = 0 and P = a/b (assuming a, b are positive constant).
- P = 0 and P = a/b are equilibrium solution to the autonomous ODE P' = P(a-b×P) (verify it).
- For 0 < P < a/b, f(P) > 0, solution curve P(t) is monotonically increasing.
- For P < 0, or P > a/b, f(P) < 0, P(t) is monotonically decreasing.

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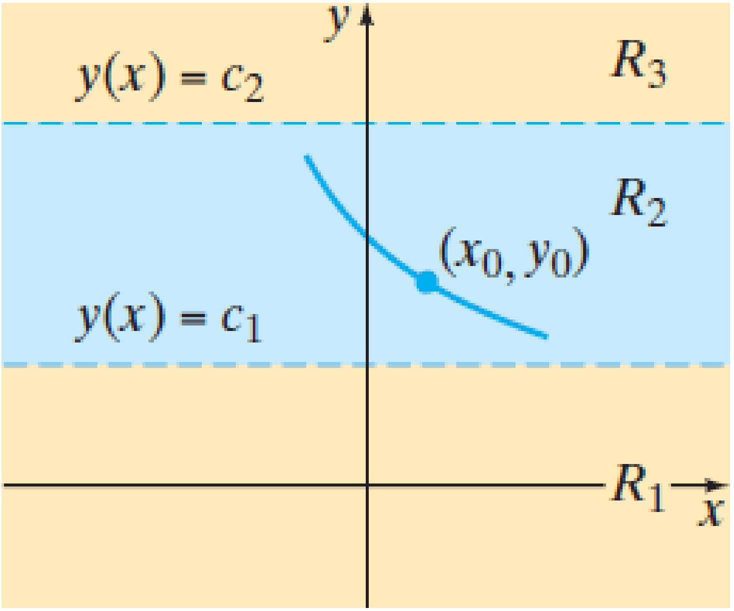
# Solution curves



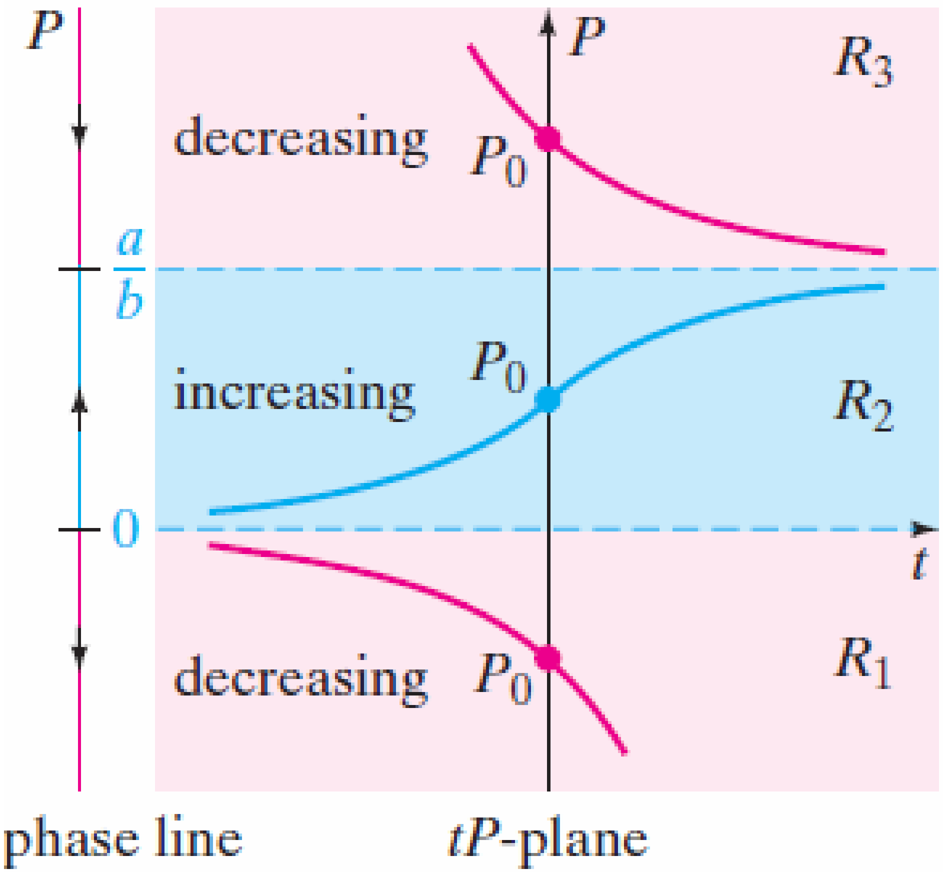
- A solution curve y(x) can be roughly sketched by
  - Starting from $(x_0,y_0)$ given by the IC: $y(x_0) = y_0$,
  - y(x) is increasing or decreasing within the sub-region $R_i$ bounded by equilibrium solutions if $f(y \in R_i) > 0$ or $f(y \in R_i) < 0$.
  - y(x) must approach but never cross an equilibrium solution (why?).



# E.g. Revisit P'(t) = P(a-b×P)



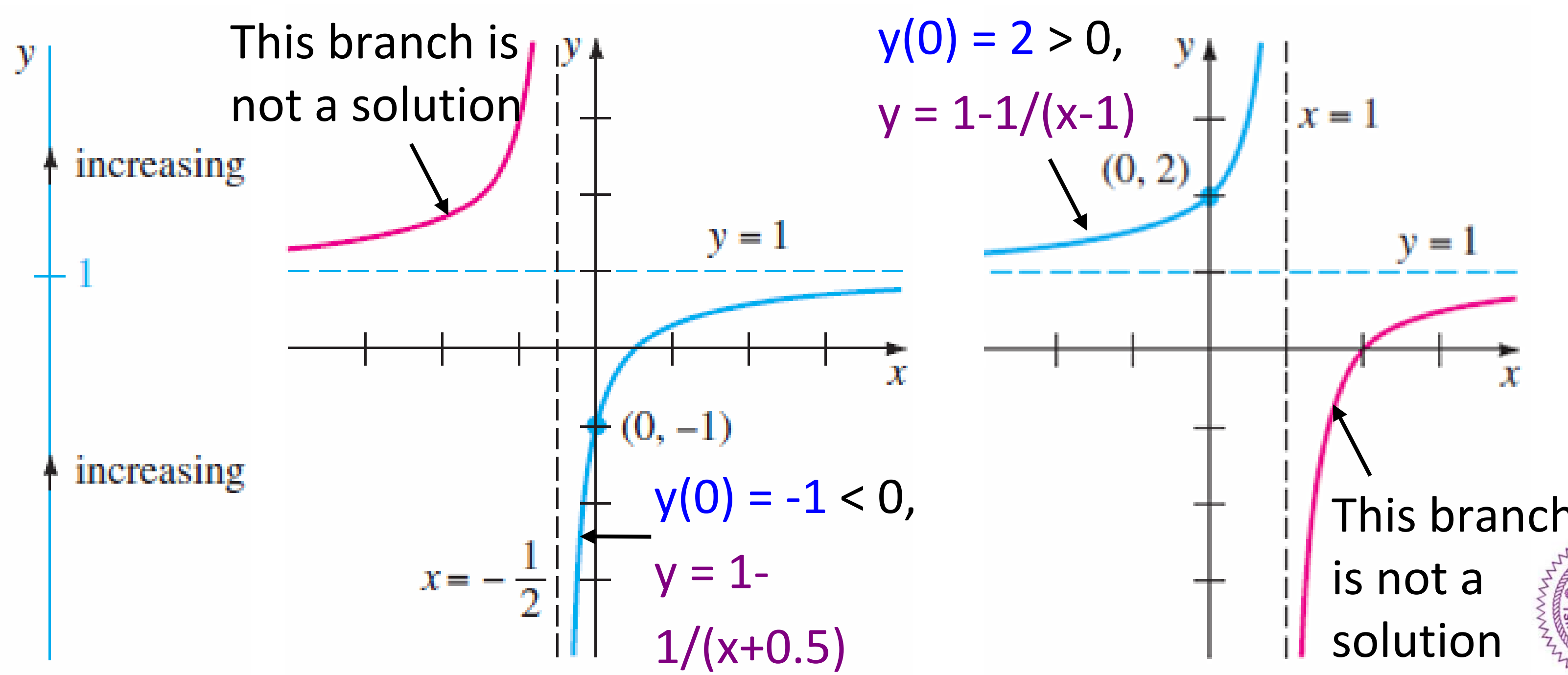
- Solution curve P(t) depends on the IC: $P(0) = P_0$:



# E.g. $y'(x) = (y-1)^2$ (Example 5)



- One equilibrium solution $y = 1$ is derived by solving $f(y) = (y-1)^2 = 0$.
- In either of the two sub-regions $R_1 = \{y<1\}$, $R_2 = \{y>1\}$, $y(x)$ must be increasing for $f(y) > 0$.

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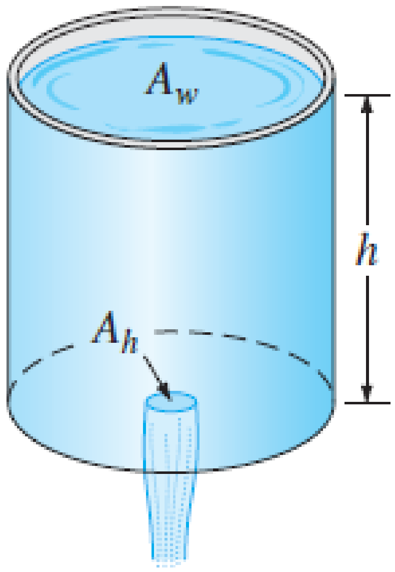
# E.g. Draining a tank



- ODE: $h'(t) = -(A_h/A_w)\times\sqrt{(2gh)}$
- Q: What does the direction field look like?
- Q: Is it autonomous?
- Q: What's the equilibrium solution?
- Q: What does a solution curve h(t) look like if the IC is h(0) = H?

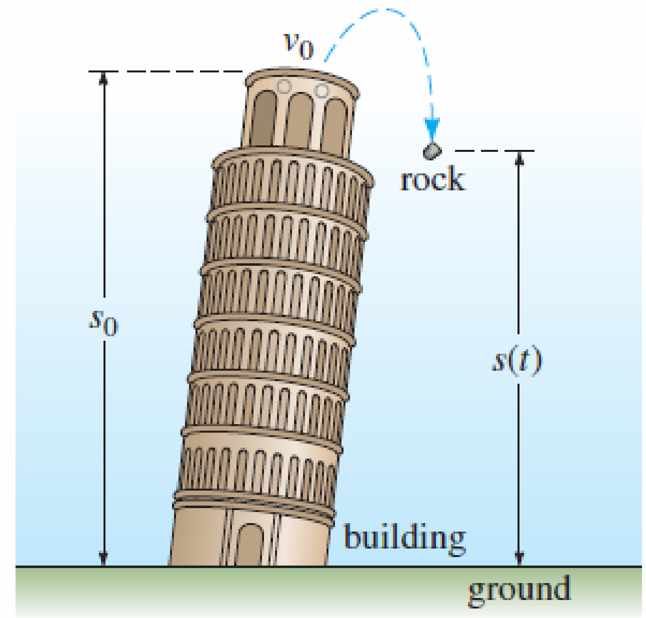



- Remark: Direction field and equilibrium solutions are particularly useful for nonlinear ODEs.

# E.g. Falling bodies



- ODE: $v'(t) = g - (k/m)v$
- Q: What does the direction field look like?
- Q: Is it autonomous?
- Q: What's the equilibrium solution?
- Q: What does a solution curve $v(t)$ look like if the IC is $v(0) = 0$?



- Separable variables

# What's separable ODE?



- $y'(x) = f(x,y) = g(x)\times h(y)$.

# E.g. y'(x) = -x/y (Example 2)



- Q: Is it autonomous or nonautonomous?
- Q: Is it linear or nonlinear?
- $dy/dx = -x/y, \Rightarrow y \times dy = -x \times dx;$
- Integration for both sides of equality:

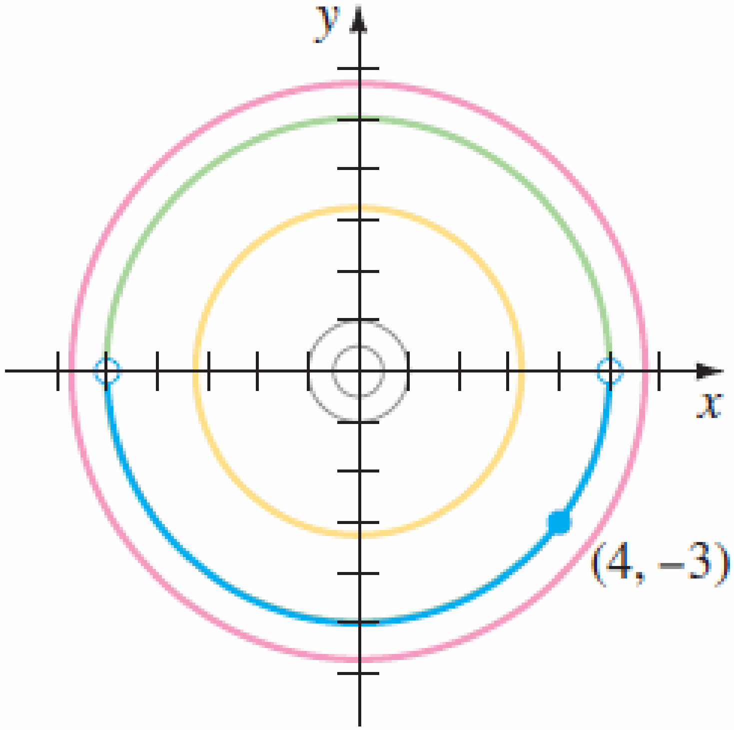
$$\int y\,dy = -\int x\,dx \longrightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

- Implicit form of the solutions: $x^2 + y^2 = c^2$ (concentric circles).

# E.g. Unique solution



- By imposing the IC: $y(4) = -3$, $\Rightarrow c^2 = 4^2 + (-3)^2 = 25$.
- However, the solution $y(x)$ is not a complete circle of radius 5 but its lower half $y = -\sqrt{(25-x^2)}$ (why?).



# E.g. An IVP (Example 4)



- ODE: $(e^{2y}-y)(\cos x)y' = e^{y}(\sin 2x)$.
- IC: $y(0) = 0$.
- Rearrangement:

$$\frac{e^{2y}-y}{e^{y}}\,dy = \frac{\sin 2x}{\cos x}\,dx, \Rightarrow (e^{y}-ye^{-y})dy = 2(\sin x)dx$$

- Integration for both sides of equality:

$$\int (e^{y} - ye^{-y})\,dy = 2\int \sin x\,dx$$

$$e^{y} + [ye^{-y} + e^{-y}] = -2\cos x + c.$$

- IC gives c = 4. But it's simply an implicit solution.

# Contour line



- A function of two variables G(x,y) can be illustrated in 3D like a topographic map (地形圖).
- G(x,y) = c is a contour line (等高線) obtained by sectioning the 3D map with a plane z = c.


(GinkgoMaps.com)

(yamol.tw)

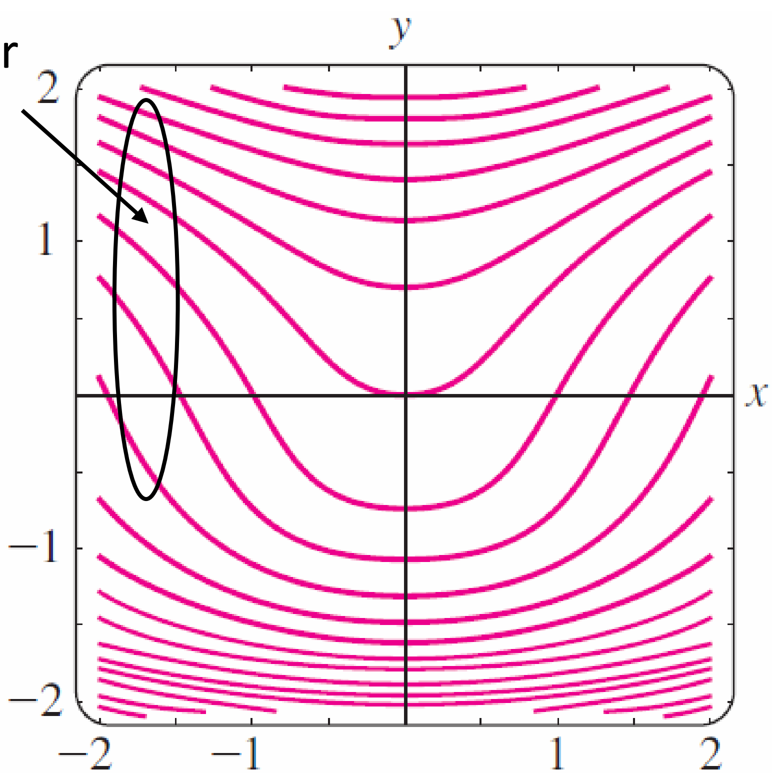
# Contour lines of the IVP

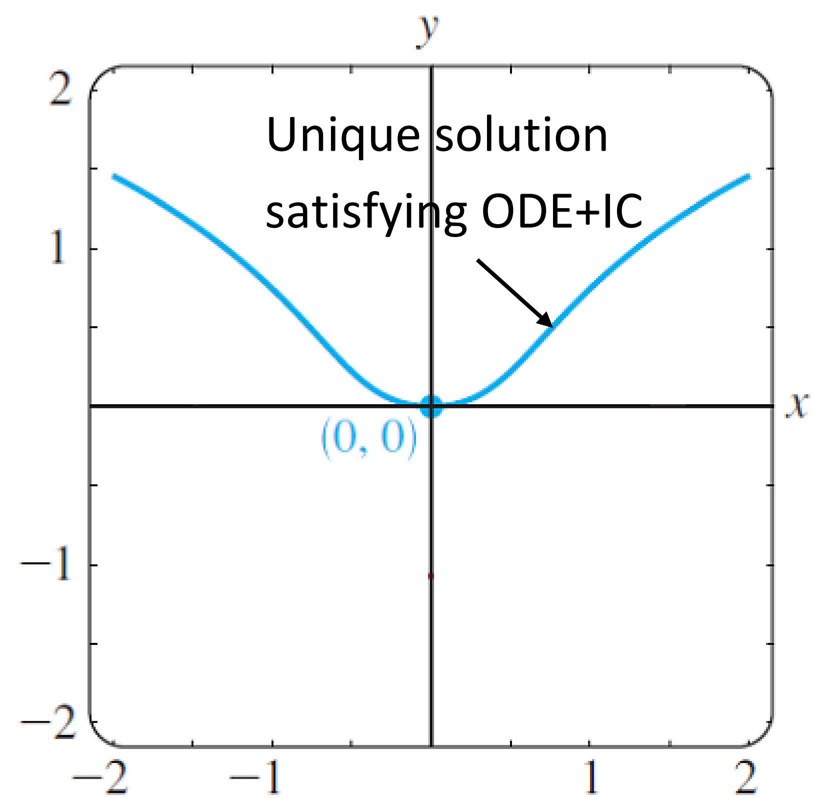


- Implicit solution $(e^{y}+ye^{-y}+e^{-y}) = -2(\cos x) + 4$ corresponds to a contour line

$$G(x,y) = e^{y} + ye^{-y} + e^{-y} + 2(\cos x) = 4$$

Family of contour lines



# E.g. An IVP (Example 5)



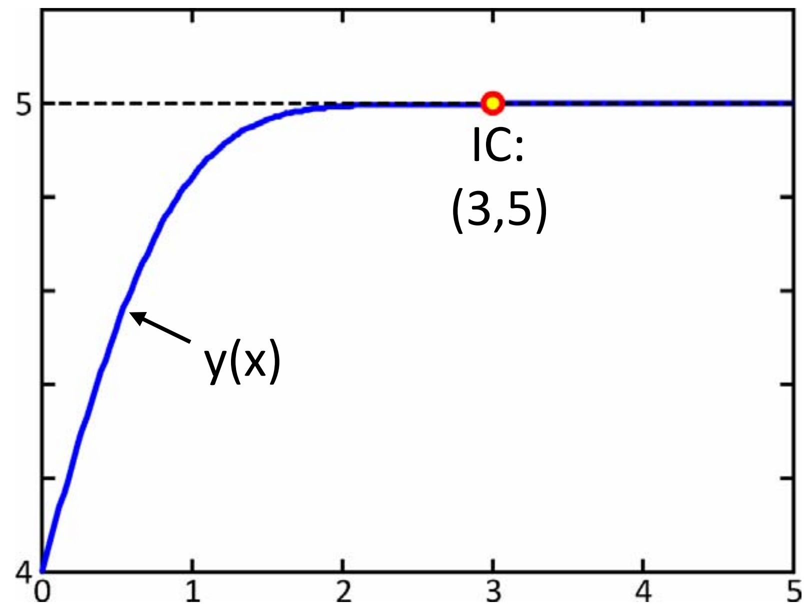
- ODE: $y' = \exp(-x^2)$.
- IC: $y(3) = 5$.

$$\int_3^x \frac{dy}{dt}\,dt = \int_3^x e^{-t^2}\,dt$$

$$y(t)\Big]_3^x = \int_3^x e^{-t^2}\,dt$$

$$y(x) - y(3) = \int_3^x e^{-t^2}\,dt$$

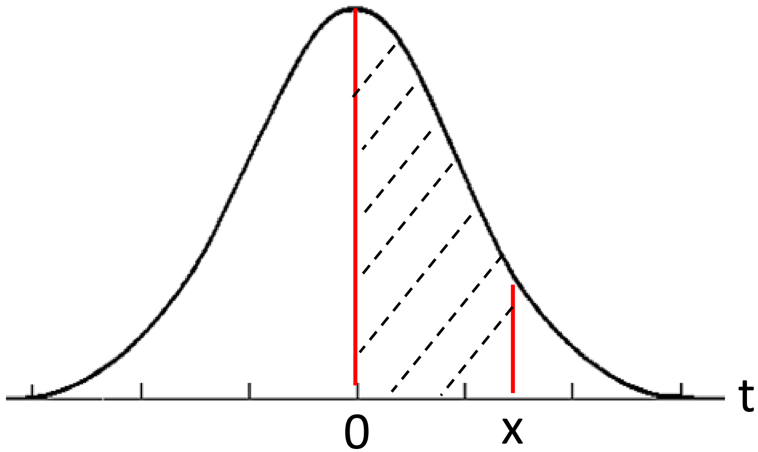
$$y(x) = 5 + \int_3^x e^{-t^2}\,dt.$$



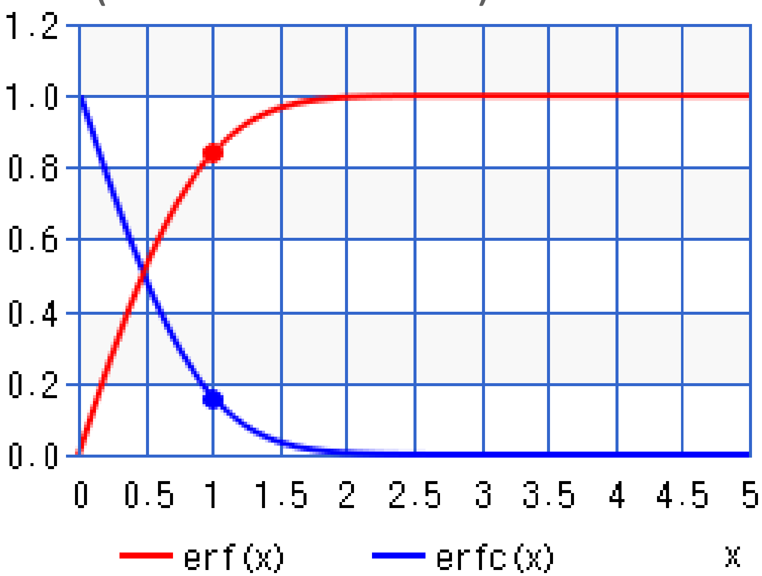
# Error function: erf(x)



- A non-elementary function useful in probability.
- Definition: $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$




(keisan.casio.com)




- Complementary error function: erfc(x) = 1 - erf(x)

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

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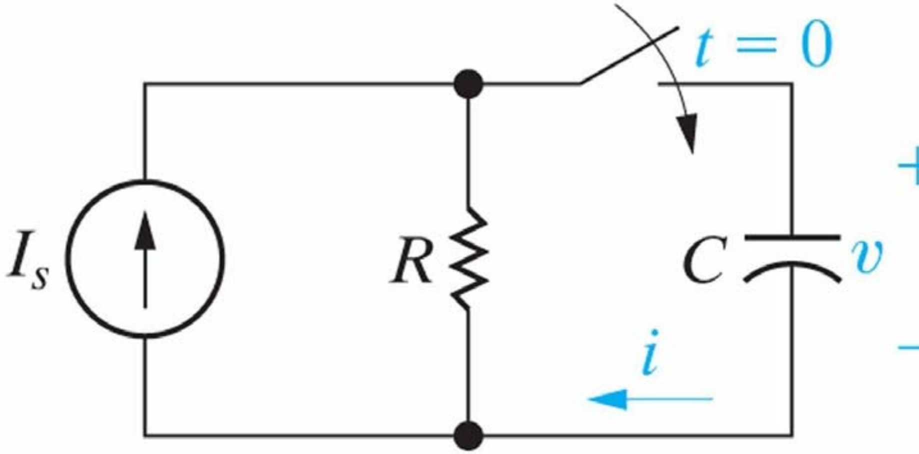
- Linear equations

# What's linear, 1st-order ODE?



- $y'(x) + P(x) \times y = f(x)$.
- It's an homogeneous ODE when $f(x) = 0$.
- In physical models, f(x) usually refers to an external driving force.
- E.g. The ODE of an RC circuit driven by a current source $I_s$ is

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{I_s}{C},$$



# How to solve a linear, 1st-order ODE?



- y'(x) + P(x)×y = f(x).
- Multiply both sides of the equality by the **integrating factor** $e^{\int P(x)dx}$ :

$$\left(e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y\right) = e^{\int P(x)dx}f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right]$$

$$\Rightarrow e^{\int P(x)dx} \times y(x) = c + \int e^{\int P(x)dx} \times f(x)dx$$

$$\Rightarrow y(x) = ce^{-\int P(x)dx} + e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} \times f(x)dx\right]$$

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# E.g. An IVP (Example 5)



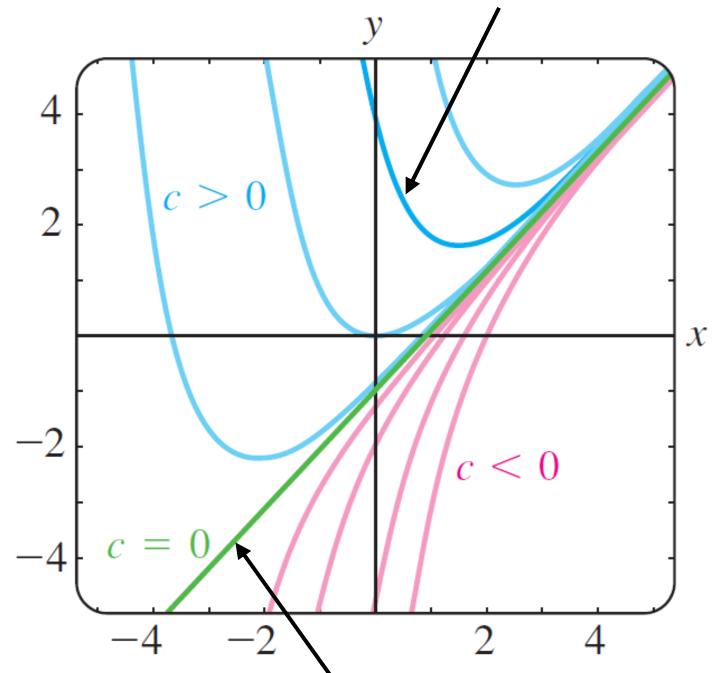
- ODE: y' + y = x; $\Rightarrow$ P(x) = 1, f(x) = x.
- IC: y(0) = 4.
- Integrating factor: $e^{\int P(x)dx} = e^x$

$$\Rightarrow \frac{d}{dx}\left[e^x y\right] = e^x x;\ e^x y = xe^x - e^x + c;$$

$$\Rightarrow y = (x-1) + (ce^{-x}).$$

- IC gives c = 5.
- All possible solutions to the ODE converges to x-1, $\Rightarrow$ $ce^{-x}$ is a transient term.

Unique solution to ODE+IC




Steady-state term: y = x-1

# E.g. An IVP (Example 2)



- ODE: y' - 3y = 6; $\Rightarrow$ P(x) = -3, f(x) = 6.
- IC: y(0) = 0.
- Integrating factor: $e^{\int P(x)dx} = e^{-3x}$

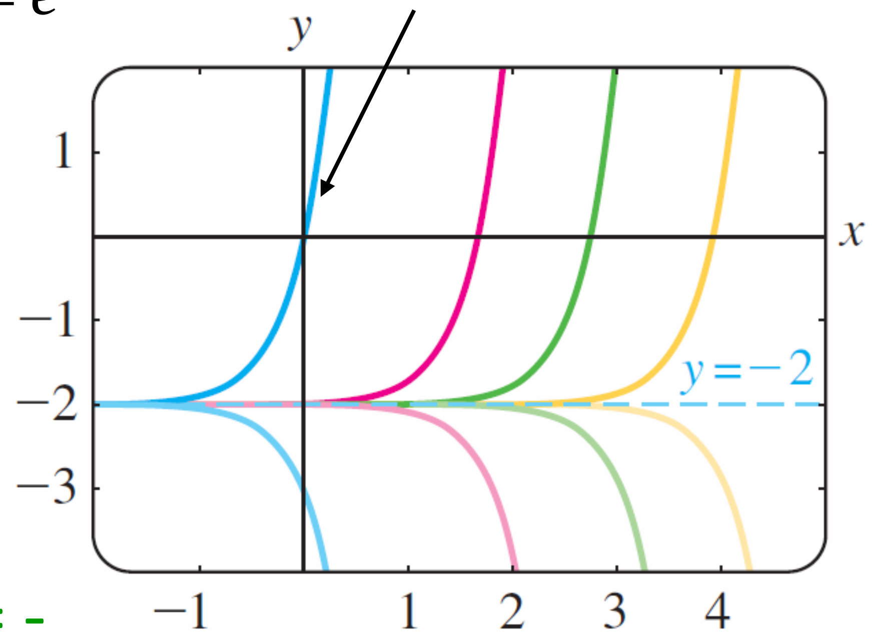
$$\Rightarrow \frac{d}{dx}\left[e^{-3x}y\right] = 6e^{-3x};$$

$$\Rightarrow e^{-3x}y = -2e^{-3x} + c;$$

$$\Rightarrow y = (-2) + (ce^{3x}).$$

- IC gives c = 2.
- Solutions diverge from y = -2, $\Rightarrow$ $ce^{3x}$ is NOT transient.

Unique solution to ODE+IC



- A numerical method
  - Euler's method

# Motivation



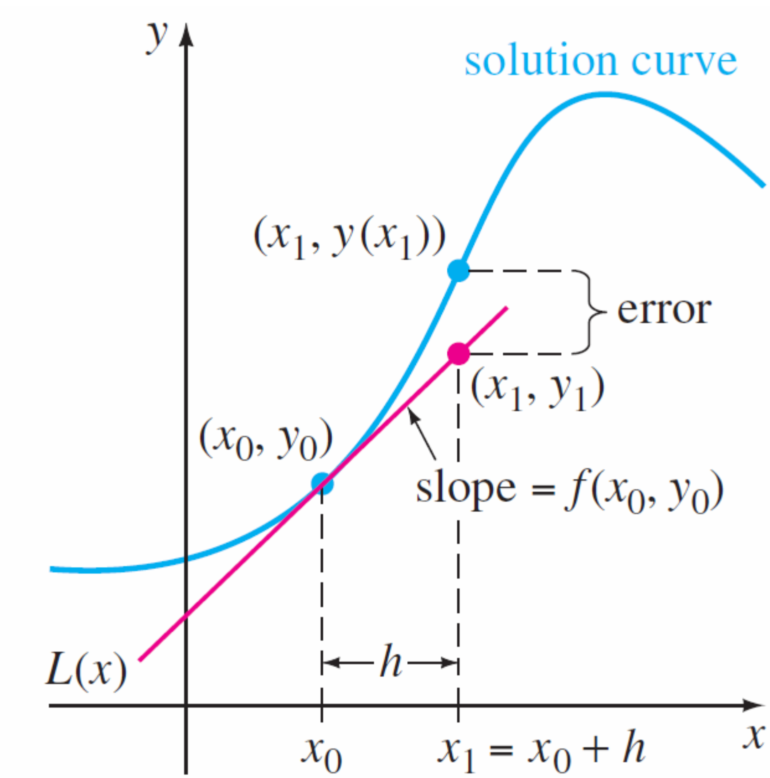
- Sometimes (quite often actually), the given IVP cannot be solved by any analytical method.
- E.g. ODE: $y' = 0.1\sqrt{y} + 0.4x^2$, IC: $y(2) = 4$.
- Q: Is the ODE linear or nonlinear?
- Q: Is it autonomous or not?

# Euler's method



- ODE: $y'(x) = f(x,y)$; IC: $y(x_0) = y_0$.
- Linearization of the solution curve $y(x)$ at $(x_0,y_0)$:

$$y(x) \approx L(x) = y_0 + f(x_0,y_0)\times(x-x_0)$$




- As a result, $y_1 \equiv y(x_1) \approx L(x_1) = y_0 + f(x_0,y_0)\times h$, where $h = x_1 - x_0$ is the step size.
- Recursive relation:

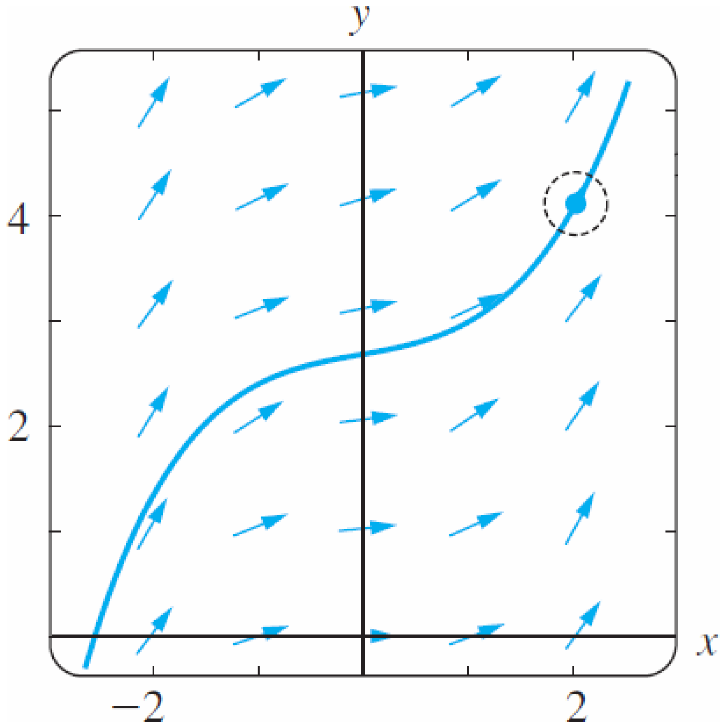
$$y_{n+1} \approx y_n + h\times f(x_n,y_n)$$

- It's quantitative realization of direction field (Sec. 2.1).

# E.g. An IVP (Example 1)



- ODE: $y' = 0.1\sqrt{y} + 0.4x^2$; IC: $(x_0,y_0) = (2,4)$
- Slope function $f(x,y) = 0.1\sqrt{y} + 0.4x^2$



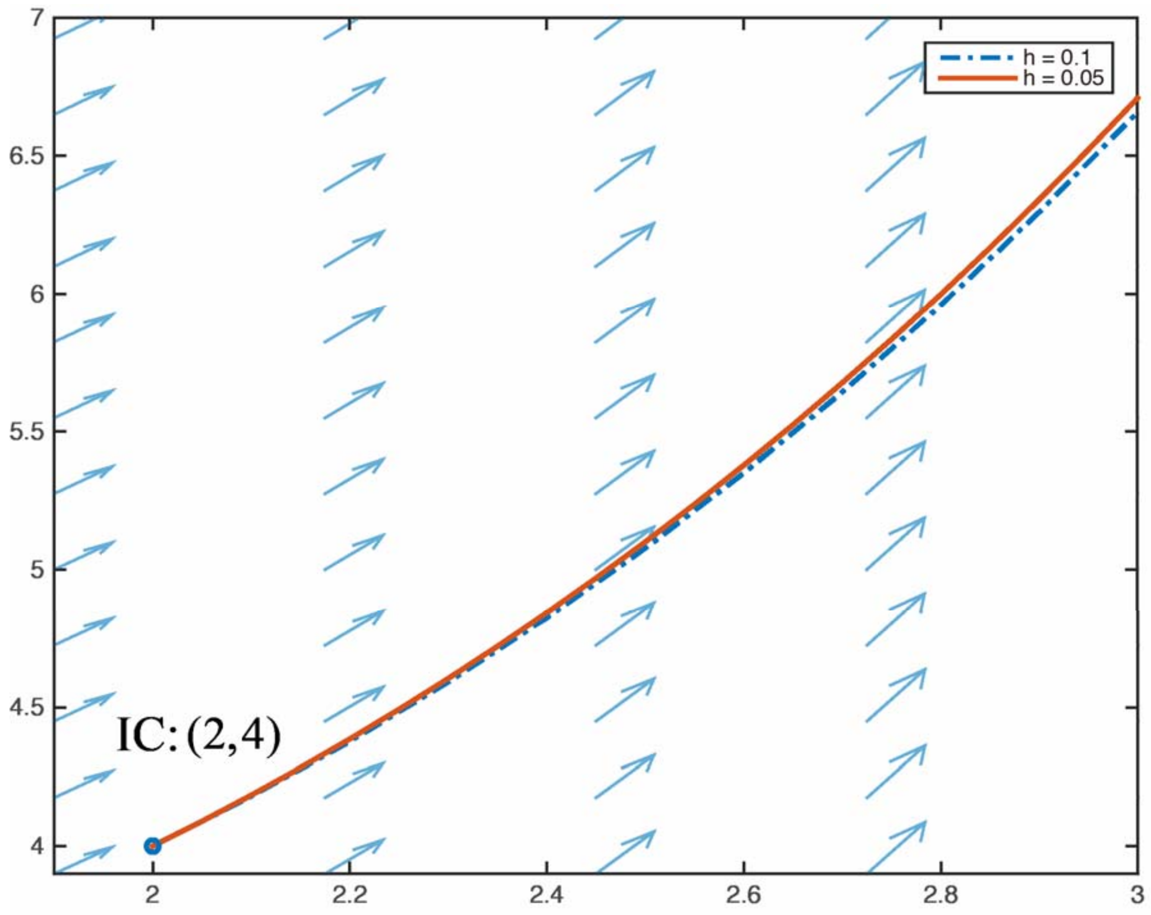

Direction field

- $y_{n+1} \approx y_n + h\times[0.1\sqrt{y_n} + 0.4(x_n)^2]$
- For $h = 0.1$:
  - $y_1 \approx 4 + 0.1\times[0.1\sqrt{4} + 0.4(2)^2] = 4.18$;
  - $y_2 \approx 4.18 + 0.1\times[0.1\sqrt{4.18} + 0.4(2.1)^2] = 4.3768, \ldots$
  - $y(2.5) = y_5 \approx 5.0768$.

# Impact of step size (Example 1)



- For h = 0.05, $y(2.5) = y_{10} \approx 5.0997$ [vs. $y(2.5) = y_5 \approx 5.0768$]



# E.g. An IVP (Example 2)



- ODE: $y' = 0.2xy$; IC: $(x_0,y_0) = (1,1)$
- Analytic solution is: $y(x) = \exp[0.1(x^2-1)]$
- Relative error $\varepsilon_r \equiv |[y_n-y(x_n)]/y(x_n)|$
- For h = 0.1:

| $x_n$ | $y_n$ | Actual value | Abs. error | % Rel. error |
|---|---|---|---|---|
| 1.00 | 1.0000 | 1.0000 | 0.0000 | 0.00 |
| 1.10 | 1.0200 | 1.0212 | 0.0012 | 0.12 |
| 1.20 | 1.0424 | 1.0450 | 0.0025 | 0.24 |
| 1.30 | 1.0675 | 1.0714 | 0.0040 | 0.37 |
| 1.40 | 1.0952 | 1.1008 | 0.0055 | 0.50 |
| 1.50 | 1.1259 | 1.1331 | 0.0073 | 0.64 |

- For h = 0.05, $\varepsilon_r = 0.32\%$ at $x_n = 1.5$.

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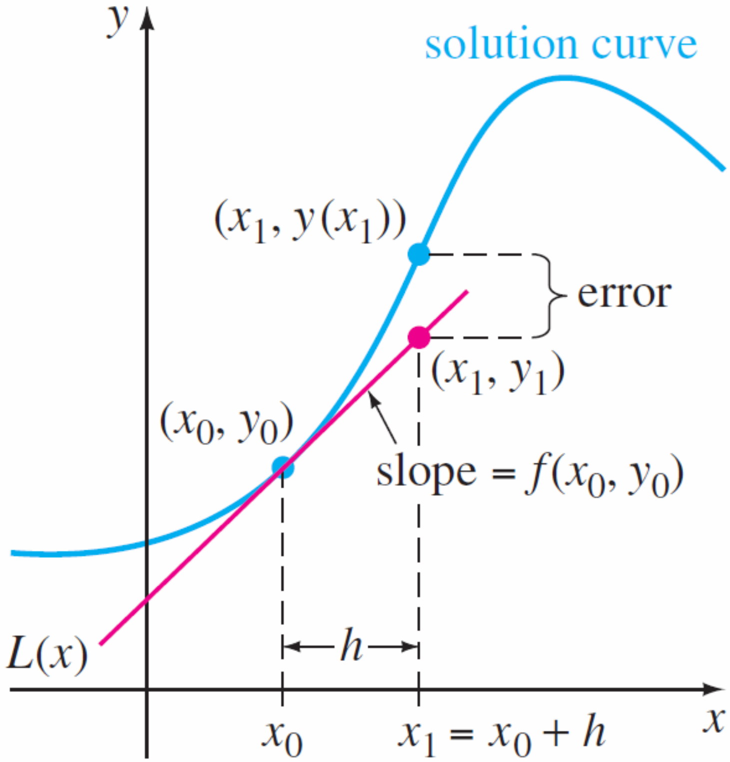
# Choice of step size



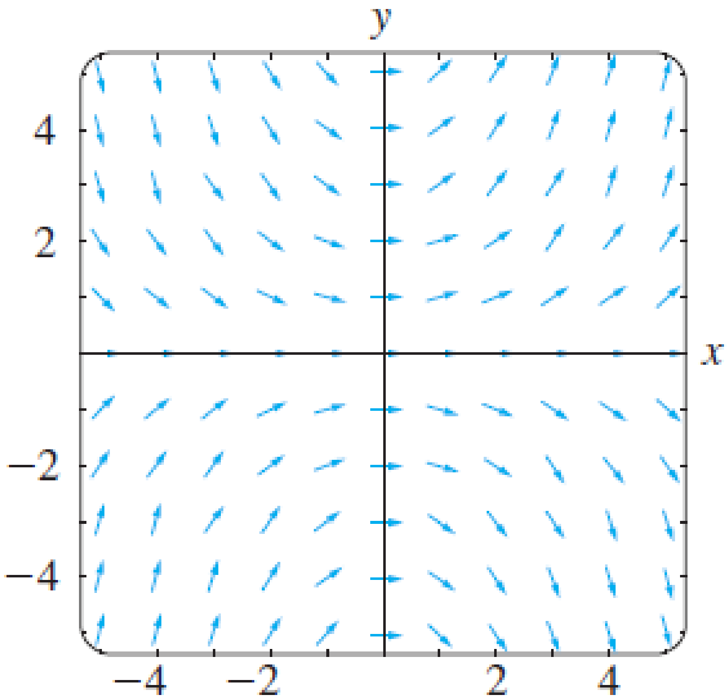
- Trade-off: A smaller h gives a more precise y(x) at the cost of heavier computation load.
- Q: How to choose h?

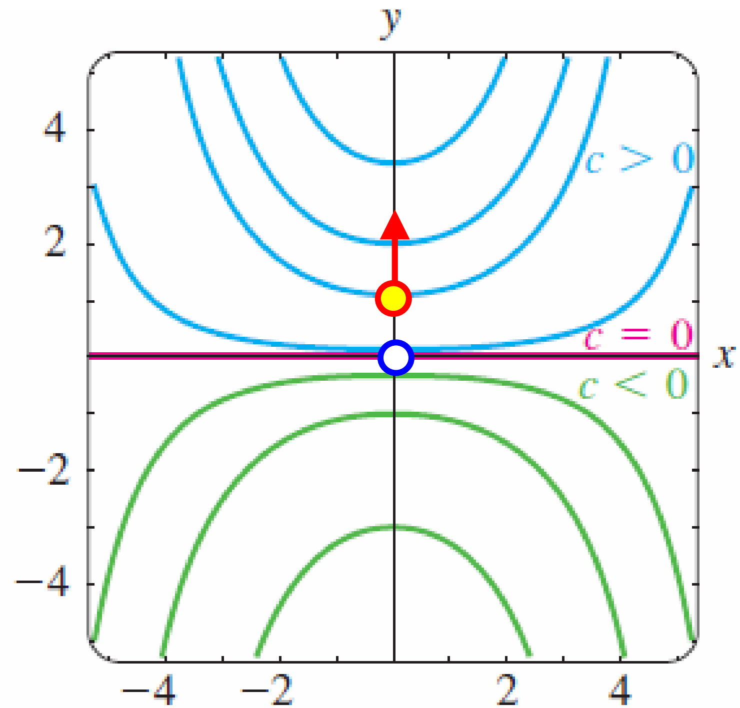


# E.g. $y' = 0.2xy$



- ODE: $y' = f(x,y) = 0.2xy$; $\Rightarrow$ $y(x) = c\times\exp(0.1x^2)$
- If $y(0) = 0$, Euler's method gives $y(x) = 0$, no error.
- If $y(0) = y_0$, error increases with $|y_0|$. So …



# Perspective



- Euler's method is seldom used because of the fast growth of the accumulated error.

- The forth-order Runge-Kutta (RK4) method (see Ch9) can give significantly greater accuracy even with a ridiculously large step size h.