Chapter 11
Fourier Series

- Orthogonal functions (11.1)
- Fourier series (11.2)
- Fourier series of even & odd functions (11.3)

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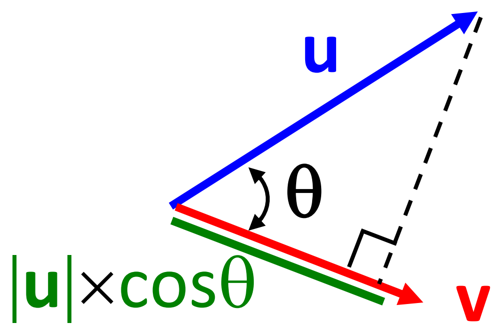
Orthogonal functions

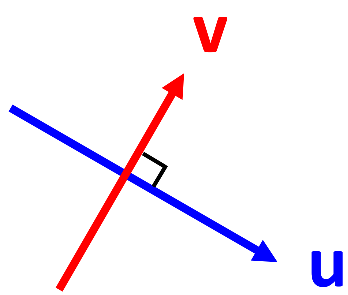
- Inner product
- Orthogonal & orthonormal sets
- Orthogonal series expansion

Vector algebra

- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\times|\mathbf{v}|\times\cos\theta =$ (the projection length of **u** on the direction of **v**) times (length of v).



- Two vectors are “orthogonal” (perpendicular) with each other if their “inner product” is zero:



$$\mathbf{u}\cdot\mathbf{v} = 0$$

Inner product of two functions

- The inner product of two complex-valued functions $f_1(x)$, $f_2(x)$ on an interval $x \in [a,b]$ is defined by their overlap integral:

$$(f_1, f_2) \equiv \int_a^b \left[f_1(x) \times f_2^*(x) \right] dx.$$

- If $f_{1,2}(x) \in R$, $\Rightarrow$ $(f_1, f_2) \equiv \int_a^b \left[f_1(x) \times f_2(x) \right] dx.$

Example

- Let f_1 = x, f_2 = sin(x). $\Rightarrow$ The inner product of f_1, f_2 ($\in$R) on the interval x $\in$ [-π,π] is:

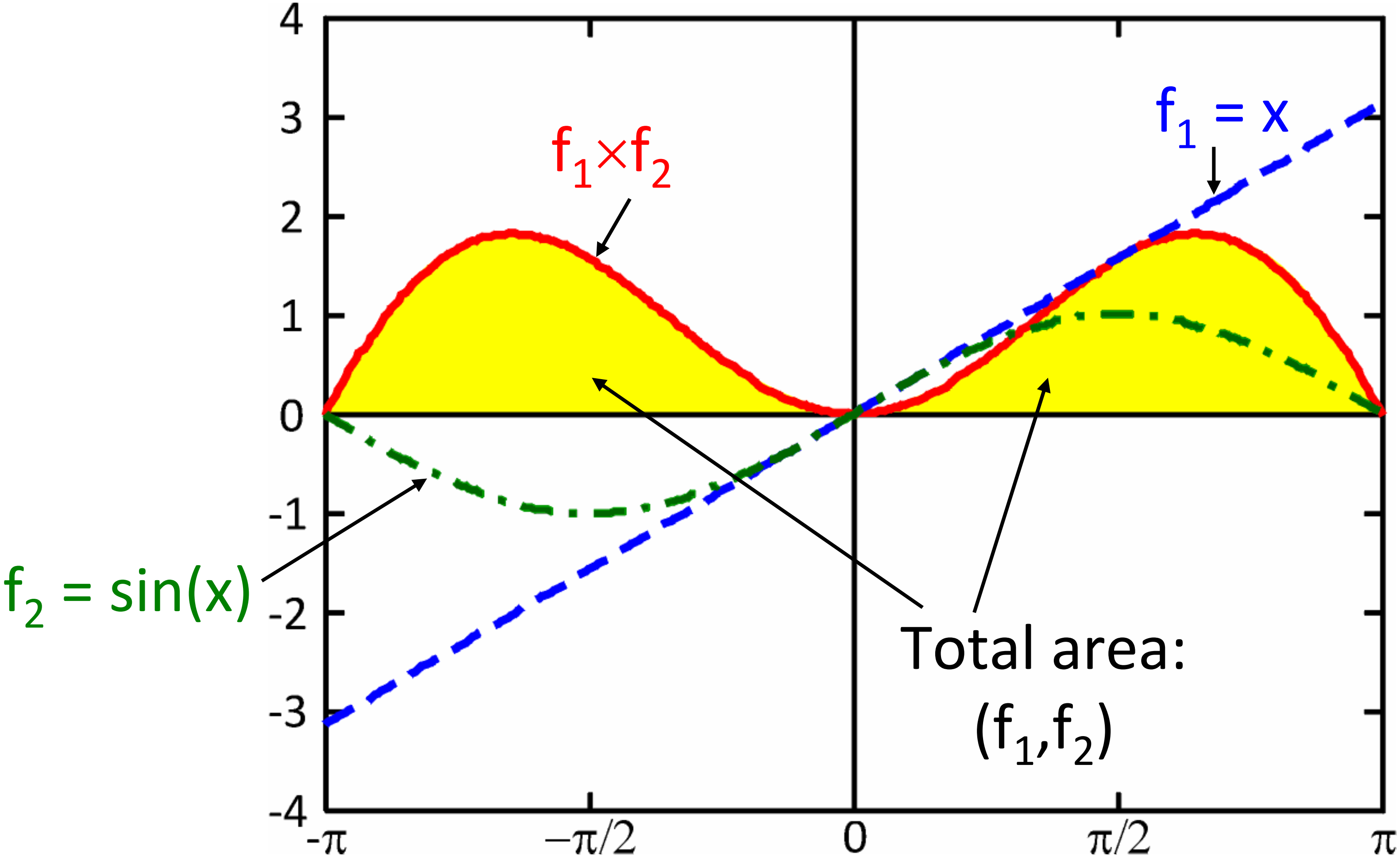
$$(f_1, f_2) \equiv \int_{-\pi}^{\pi} [x \times \sin(x)] dx = 2\int_0^{\pi} [x \times \sin(x)] dx.$$

- Integration by parts: {u = x, v' = sin(x)}, $\Rightarrow$ {u' = 1, v = -cos(x)}:

$$(f_1, f_2) = 2\left\{\left[-x\cos(x)\Big|_0^{\pi}\right] + \left[\int_0^{\pi}\cos(x)dx\right]\right\}$$

$$= 2\left\{\left[0 - \pi \underbrace{\boxed{\cos(\pi)}}_{-1}\right] + \left[\cancel{\sin(x)\Big|_0^{\pi}}\right]\right\} = 2\pi$$

Plot: (x,sinx) = 2π



Orthogonal functions

- Two functions $f_1(x)$ and $f_2(x)$ are orthogonal on an interval $x \in [a,b]$ if their inner product is zero:

$$(f_1, f_2) \equiv \int_a^b \left[f_1(x) \times f_2^*(x) \right] dx = 0.$$

- E.g. x and $\sin(x)$ are not orthogonal within $x \in [-\pi,\pi]$, for $(x, \sin x) = 2\pi \neq 0$.
- E.g. x and $\cos(x)$ are orthogonal within $x \in [-\pi,\pi]$, for $(x, \cos x) = 0$. (why?)
- E.g. x and $\cos(x)$ are not orthogonal within $x \in [0,\pi]$, for $(x, \cos x) = -2 \neq 0$.

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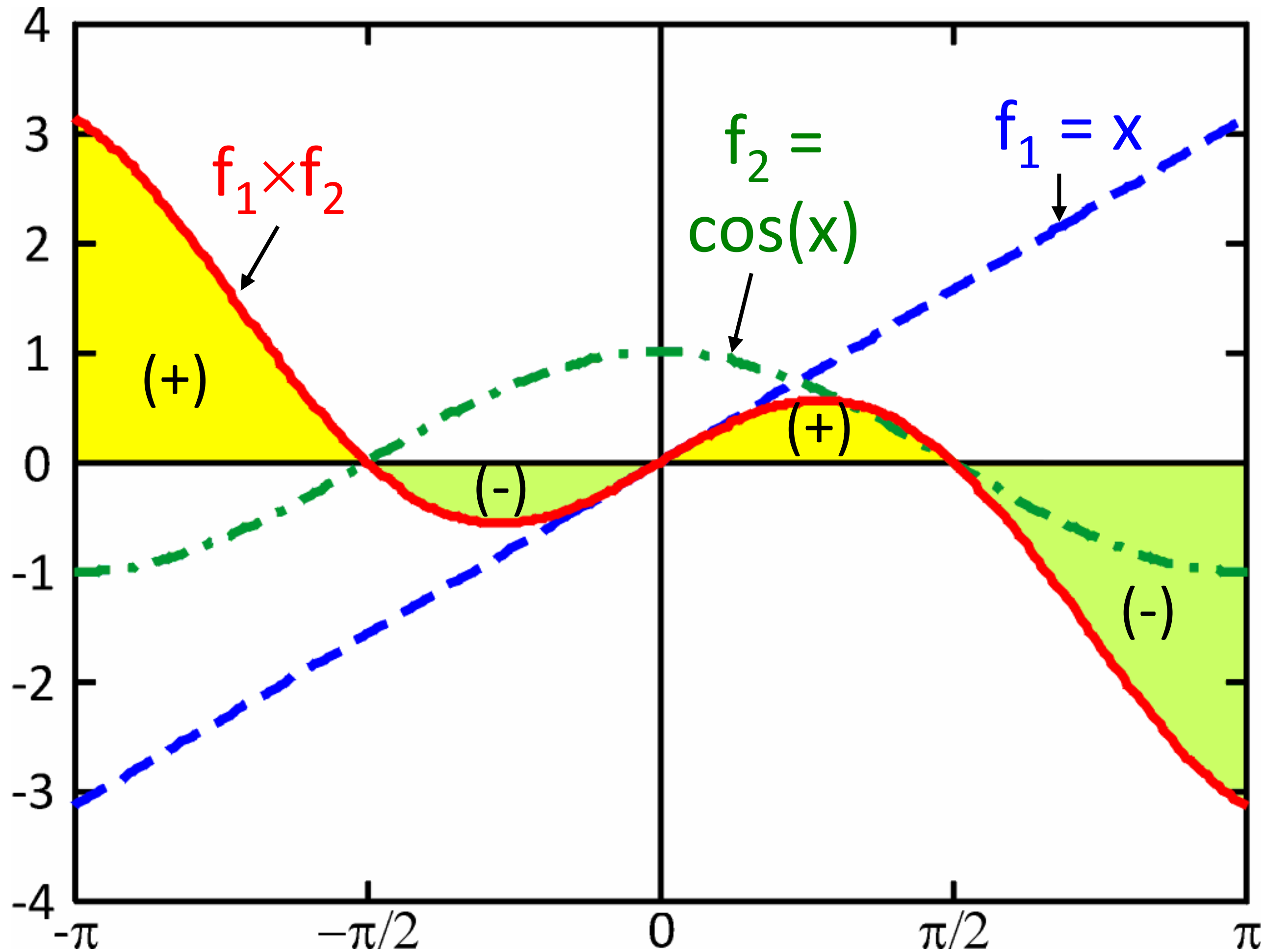
Plot: (x,cosx) = 0



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Norm of a function

- The length of a vector **v** is **|v| = √(v·v)**.
- The norm (generalized length) of a complex-valued function f(x) on an interval x∈[a,b] is:

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_a^b \left[f(x)\times f^*(x)\right]dx} = \sqrt{\int_a^b |f(x)|^2\,dx}.$$

- If f(x) ∈ R, $\|f\| = \sqrt{\int_a^b f^2(x)dx}.$

- E.g. The norm of function x on x ∈ [-π,π] is:

$$\|x\| = \sqrt{\int_{-\pi}^{\pi} x^2 dx} = \sqrt{\frac{x^3\big|_{-\pi}^{\pi}}{3}} = \pi\sqrt{\frac{2\pi}{3}}.$$

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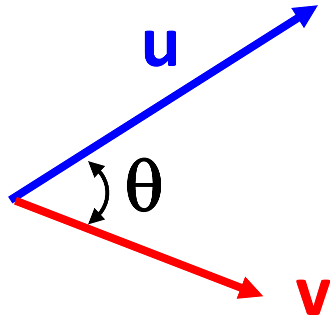
Generalized included angle θ

- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\times|\mathbf{v}|\times\cos\theta$, $\Rightarrow$ the included angle between vectors **u** and **v** is:

$$\theta = \cos^{-1}[(\mathbf{u}\cdot\mathbf{v})/(|\mathbf{u}|\times|\mathbf{v}|)].$$



- The generalized included angle between f(x) and g(x) is: $\theta = \cos^{-1}[(f,g)/(\|f\|\times\|g\|)]$.

- E.g. $\{f = x,\ g = \sin(x),\ x \in [-\pi,\pi]\}$,

  $\Rightarrow (f,g) = 2\pi,\ \|f\| = \pi\sqrt{2\pi/3},\ \|g\| = \sqrt{\pi}$,

  $\Rightarrow \theta = \cos^{-1}[(2\pi)/(\pi^2\sqrt{(2/3)})] = \cos^{-1}(\sqrt{6}/\pi) \approx 38.8^o$.

Orthogonal and orthonormal sets

- A set of real-valued functions $\{\phi_0(x), \phi_1(x), \ldots\}$ are orthogonal on an interval $x \in [a,b]$ if:

$$(\phi_m, \phi_n) \equiv \int_a^b [\phi_m(x) \times \phi_n(x)] dx = 0, \text{ whenever } m \neq n.$$

- An orthogonal set is orthonormal if $\|\phi_n\| = 1$, for n = 0, 1, 2, ….
- Orthonormal set is analogous to the two mutually perpendicular unit vectors **x** and **y** in a 2D plane.

Example 1: {cos(nx)}

- Are {cos(nx)} are orthogonal on $x \in [-\pi, \pi]$?

$$(\cos(mx), \cos(nx)) = \int_{-\pi}^{\pi} \cos(mx) \times \cos(nx)\, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \{\cos[(m+n)x] + \cos[(m-n)x]\} dx$$

$\sin[(m \pm n)\pi] = 0$

$$= \frac{1}{2} \left\{ \frac{\sin[(m+n)x]\Big|_{-\pi}^{\pi}}{m+n} + \frac{\sin[(m-n)x]\Big|_{-\pi}^{\pi}}{m-n} \right\}$$

$$= \frac{1}{2} \left\{ \frac{0-0}{m+n} + \frac{0-0}{m-n} \right\} = 0, \text{ if } \quad m \neq n.$$

$\{\cos[(n+\delta)x]\}$

- Are cos(1.1x) and cos(2.1x) orthogonal on $x \in [-\pi,\pi]$?

$$\int_{-\pi}^{\pi} \cos(1.1x)\times\cos(2.1x)dx = \frac{1}{2}\int_{-T/2}^{T/2}\left[\cos(3.2x)+\cos(-x)\right]dx$$

$$= \frac{1}{2}\left\{\frac{\sin(3.2x)\Big|_{-\pi}^{\pi}}{3.2} + \sin(x)\Big|_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{2}\left\{\frac{2\sin(3.2\pi)}{3.2} + (0-0)\right\} = \frac{-\sin(0.2\pi)}{3.2} \neq 0.$$

- A non-integer detuning δ will destroy the orthogonality.

{cos(nx+ϕ)}

- Are cos(x) and cos(2x+ϕ) orthogonal on x $\in$ [-π,π]?

$$\int_{-\pi}^{\pi}\cos(x)\times\cos(2x+\phi)dx=\frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(3x+\phi)+\boxed{\cos(-x-\phi)}\right]dx$$

cos(x+ϕ)

$$=\frac{1}{2}\left\{\frac{\sin(3x+\phi)\Big|_{-\pi}^{\pi}}{3}+\sin(x+\phi)\Big|_{-\pi}^{\pi}\right\}$$

$$=\frac{1}{2}\left\{\frac{\sin(3\pi+\phi)-\sin(-3\pi+\phi)}{3}+[\sin(\pi+\phi)+\sin(-\pi+\phi)]\right\}$$

$$=\frac{1}{2}\left\{\frac{\cancel{-\sin(\phi)+\sin(\phi)}}{3}+[\cancel{-\sin(\phi)+\sin(\phi)}]\right\}=0.$$

- Orthogonality is independent of phase shift ϕ.

Example 2: $\|\cos(nx)\|$

$$\|\cos(nx)\| \equiv \sqrt{\int_{-\pi}^{\pi} \cos^2(nx)\,dx} = \sqrt{\frac{1}{2}\int_{-\pi}^{\pi} \left[1+\cos(2nx)\right]dx}$$

- If $n \neq 0$:

$$\|\cos(nx)\| = \sqrt{\frac{1}{2}\left[2\pi + \cancel{\frac{\sin(2nx)\big|_{-\pi}^{\pi}}{2n}}\right]} = \sqrt{\pi}\,.$$

- If n = 0:

$$\|\cos(0x)\| \equiv \sqrt{\int_{-\pi}^{\pi} \cos^2(0)\,dx} = \sqrt{2\pi}\,.$$

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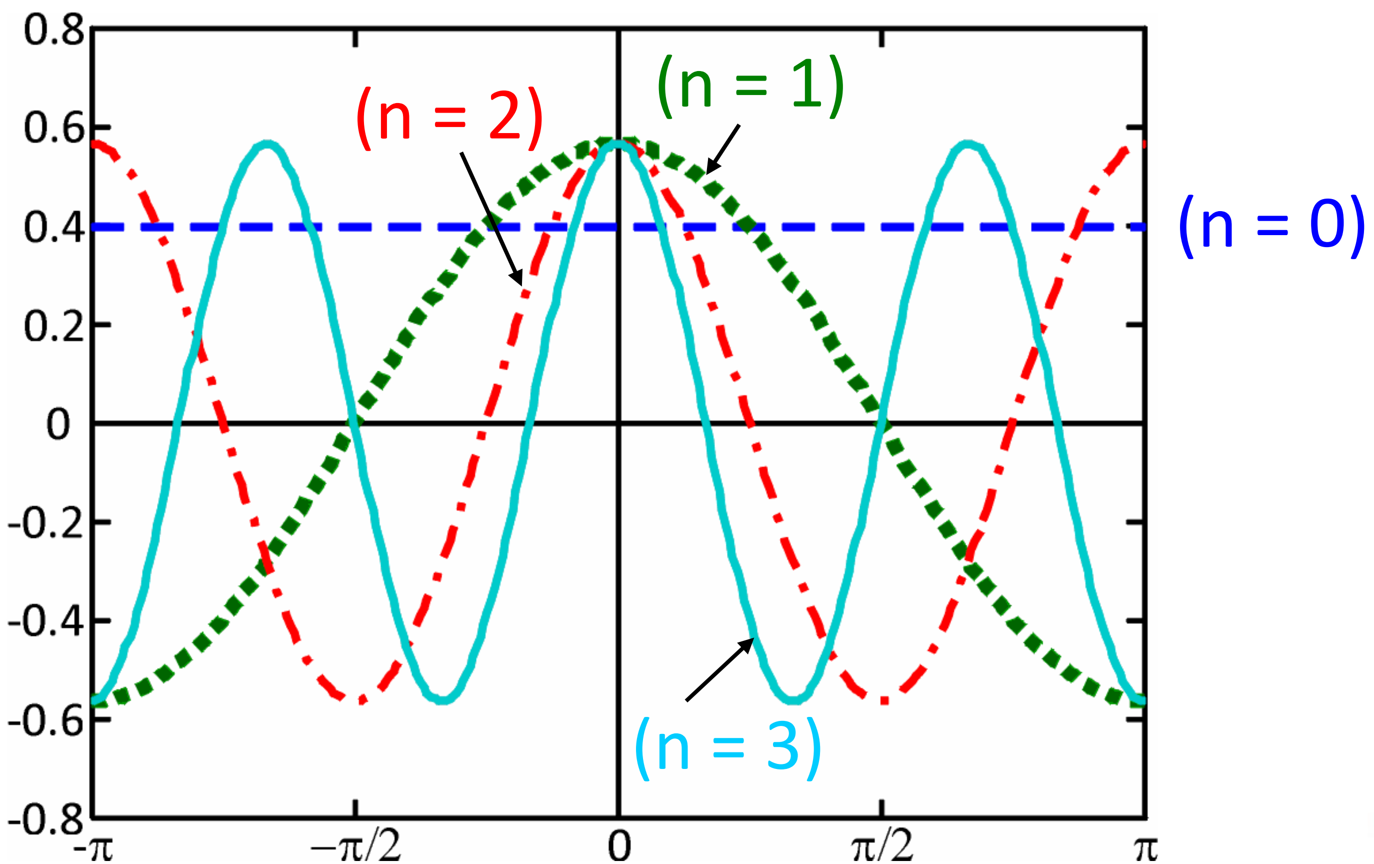
Normalize {cos(nx)}

- $\{\cos(nx)/\|\cos(nx)\|\} = \{1/\sqrt{2\pi}, \cos(x)/\sqrt{\pi}, \cos(2x)/\sqrt{\pi}, \ldots\}$ is an orthonormal set on $x \in [-\pi, \pi]$.



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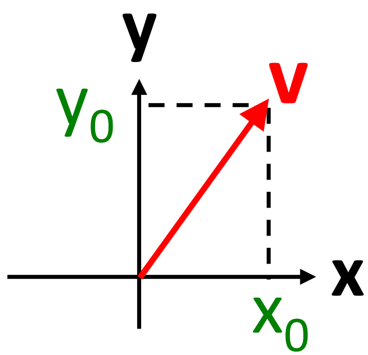
Vector expansion

- A 2D vector can be expanded in terms of a set of two orthogonal (orthonormal) vectors **x** and **y**:



$$\mathbf{v} = x_0\mathbf{x} + y_0\mathbf{y}$$

- $\mathbf{v}\cdot\mathbf{x} = (x_0\mathbf{x} + y_0\mathbf{y})\cdot\mathbf{x} = x_0(\mathbf{x}\cdot\mathbf{x}) + y_0(\mathbf{y}\cdot\mathbf{x}) = x_0|\mathbf{x}|^2 + y_0(0)$,
  $\Rightarrow x_0 = (\mathbf{v}\cdot\mathbf{x})/|\mathbf{x}|^2 = (\mathbf{v}\cdot\mathbf{x})$ if $|\mathbf{x}| = 1$.
- $\mathbf{v}\cdot\mathbf{y} = (x_0\mathbf{x} + y_0\mathbf{y})\cdot\mathbf{y} = x_0(\mathbf{x}\cdot\mathbf{y}) + y_0(\mathbf{y}\cdot\mathbf{y}) = x_0(0) + y_0|\mathbf{y}|^2$,
  $\Rightarrow y_0 = (\mathbf{v}\cdot\mathbf{y})/|\mathbf{y}|^2 = (\mathbf{v}\cdot\mathbf{y})$ if $|\mathbf{y}| = 1$.

Orthogonal series expansion

- A function f(x) on an interval $x \in [a,b]$ can be expanded in terms of a complete set of infinitely many orthogonal (orthonormal) functions $\{\phi_0(x), \phi_1(x), \ldots\}$:

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$$

where the expansion coefficient c_n is obtained by:

$$c_n = \frac{(f, \phi_n)}{\|\phi_n\|^2} = \frac{\int_a^b [f(x) \times \phi_n(x)] dx}{\int_a^b \phi_n^2(x) dx}.$$

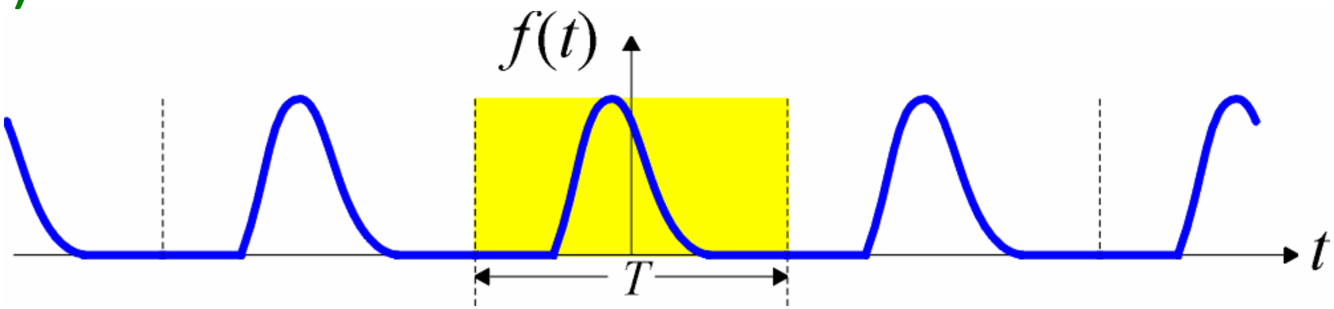
Complete set

- The only function that is orthogonal to each member of $\{\phi_0(x), \phi_1(x), \ldots\}$ is the zero function;
- i.e. for any function f(x), there must be some nonzero expansion coefficient ($c_n \neq 0$) unless f(x) = 0.
- E.g. {cos(nx)} is orthogonal but incomplete, for f(x) = x gives $\{c_n = 0\}$ for all n = 0, 1, 2, …

- Fourier series

Fourier series expansion

- A periodic function f(t) of period T satisfies f(t+n×T) = f(t).



- A periodic function f(t) of period T can be expanded in terms of sinusoidal functions whose angular frequencies are integral multiples of the fundamental frequency $\omega_0 = 2\pi/T$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right].$$

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Are {cos(mω_0t), cos(nω_0t)} orthogonal?

- Orthogonality of periodic functions is determined within one period: t $\in$ [-T/2,T/2]:

$$(\cos(m\omega_0 t),\cos(n\omega_0 t)) = \int_{-T/2}^{T/2}\left[\cos(m\omega_0 t)\times\cos(n\omega_0 t)\right]dt$$

$$=\frac{1}{2}\int_{-T/2}^{T/2}\left\{\cos[(m+n)\omega_0 t]+\cos[(m-n)\omega_0 t]\right\}dt$$

sin[(m$\pm$n)ω_0T/2] = sin[(m$\pm$n)π] = 0

$$=\frac{1}{2}\left\{\frac{\sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2}}{(m+n)\omega_0}+\frac{\sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}}{(m-n)\omega_0}\right\}$$

$$=\frac{1}{2}\left\{\frac{0-0}{(m+n)\omega_0}+\frac{0-0}{(m-n)\omega_0}\right\}=0,\ \text{if } m\neq n.$$

Are {cos(mω$_0$t), sin(nω$_0$t)} orthogonal?

$$(\cos(m\omega_0 t),\sin(n\omega_0 t)) = \int_{-T/2}^{T/2}\left[\cos(m\omega_0 t)\times\sin(n\omega_0 t)\right]dt$$

$$= \frac{1}{2}\int_{-T/2}^{T/2}\left\{\sin[(m+n)\omega_0 t] - \sin[(m-n)\omega_0 t]\right\}dt$$

$\cos[(m\pm n)\omega_0 T/2] = \cos[(m\pm n)\pi] = (-1)^{m\pm n}$

$$= \frac{1}{2}\left\{\frac{-\cos[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2}}{(m+n)\omega_0} + \frac{\cos[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}}{(m-n)\omega_0}\right\}$$

$$= \frac{1}{2}\left\{-\frac{\cancel{(-1)^{m+n} - (-1)^{m+n}}}{(m+n)\omega_0} + \frac{\cancel{(-1)^{m-n} - (-1)^{m-n}}}{(m-n)\omega_0}\right\} = 0,$$

for $m \neq n$ or $m = n$.

Fourier coefficients: a_0

$$\int_{-T/2}^{T/2} f(t)dt = \int_{-T/2}^{T/2}\left\{\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right]\right\}dt$$

$$=\frac{a_0}{2}\int_{-T/2}^{T/2}1dt+\sum_{n=1}^{\infty}\left[a_n\int_{-T/2}^{T/2}\cos(n\omega_0 t)dt+b_n\int_{-T/2}^{T/2}\sin(n\omega_0 t)dt\right]$$

$$=\frac{a_0}{2}T+\sum_{n=1}^{\infty}\left[a_n\frac{\sin(n\omega_0 t)\Big|_{-T/2}^{T/2}}{n\omega_0}-b_n\frac{\cos(n\omega_0 t)\Big|_{-T/2}^{T/2}}{n\omega_0}\right]$$

$$=\frac{a_0 T}{2}+\sum_{n=1}^{\infty}\left[a_n\frac{\cancel{0-0}}{n\omega_0}-b_n\frac{\cancel{(-1)^n-(-1)^n}}{n\omega_0}\right]=\frac{a_0 T}{2},$$

$$a_0=\frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = 2\times\underbrace{\boxed{\langle f(t)\rangle}}_{\text{average}}$$

Fourier coefficients: a_n, b_n

$$(f(t),\cos(n\omega_0 t)) = \int_{-T/2}^{T/2}\left[f(t)\times\cos(n\omega_0 t)\right]dt = (\cancel{\frac{a_0}{2},\cos(n\omega_0 t)})$$

$$+\sum_{m=1}^{\infty}\left[a_m(\cos(m\omega_0 t),\cos(n\omega_0 t)) + b_m\cancel{(\sin(m\omega_0 t),\cos(n\omega_0 t))}\right]$$

$$= \begin{cases} 0, \text{ if } m \neq n; \\ \int_{-T/2}^{T/2}\cos^2(n\omega_0 t)dt = \int_{-T/2}^{T/2}\frac{1+\cos(2n\omega_0 t)}{2}dt = \frac{T}{2}, \text{ if } m = n. \end{cases}$$

$$\Rightarrow (f(t),\cos(n\omega_0 t)) = a_n\frac{T}{2}, \quad a_n = \frac{2}{T}\int_{-T/2}^{T/2}\left[f(t)\times\cos(n\omega_0 t)\right]dt.$$

- Similarly, $b_n = \frac{2}{T}\int_{-T/2}^{T/2}\left[f(t)\times\sin(n\omega_0 t)\right]dt.$

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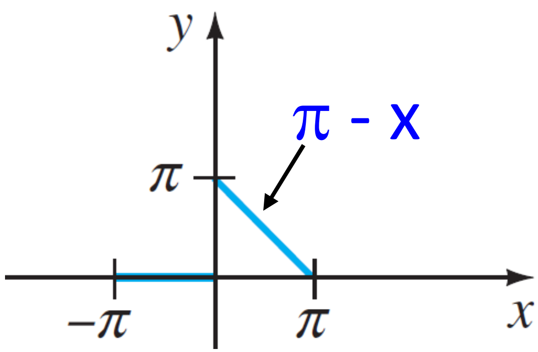
Example 1 (1)

- Expand a periodic function f(x) of period T = 2π, ⇒

ω_0 = 2π/T = 1:



$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx = \frac{1}{\pi}\left[\pi^2 - \left(\left.\frac{x^2}{2}\right|_0^{\pi}\right)\right] = \frac{\pi}{2}.$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2}\left[f(x)\times\cos(n\omega_0 x)\right]dx = \frac{1}{\pi}\int_0^{\pi}\left[(\pi - x)\times\cos(nx)\right]dx$$

$$= \frac{1}{\pi}\left[\pi\underbrace{\int_0^{\pi}\cos(nx)dx}_{0} - \underbrace{\int_0^{\pi}x\times\cos(nx)dx}_{I\,=\,?}\right].$$

Example 1 (2)

- Integration by parts: let u = x, v' = cos(nx), $\Rightarrow$ u' = 1, v = sin(nx)/n.

$$I=\int_0^{\pi} x\times\cos(nx)dx=\frac{x\times\sin(nx)\big|_0^{\pi}}{n}-\frac{1}{n}\int_0^{\pi}\sin(nx)dx$$

$$=\frac{0-0}{n}+\frac{1}{n^2}\left[\cos(nx)\big|_0^{\pi}\right]=\frac{(-1)^n-1}{n^2};\Rightarrow a_n=-\frac{I}{\pi}=\frac{1-(-1)^n}{n^2\pi}.$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}\left[f(x)\times\sin(n\omega_0 x)\right]dx=\frac{1}{\pi}\int_0^{\pi}\left[(\pi-x)\times\sin(nx)\right]dx$$

$$=\frac{1}{\pi}\left[\pi\underbrace{\int_0^{\pi}\sin(nx)dx}_{[1-(-1)^n]/n}-\underbrace{\int_0^{\pi}x\times\sin(nx)dx}_{(-1)^{n+1}\pi/n}\right]=\frac{1}{n}.$$

Partial sum

- The series of the first N sinusoidal pairs, counting up to cos($N\omega_0 t$) and sin($N\omega_0 t$):

$$S_N(t) \equiv \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right].$$

- Fourier series is useful if $\lim_{N\to\infty} S_N(t) = f(t)$.

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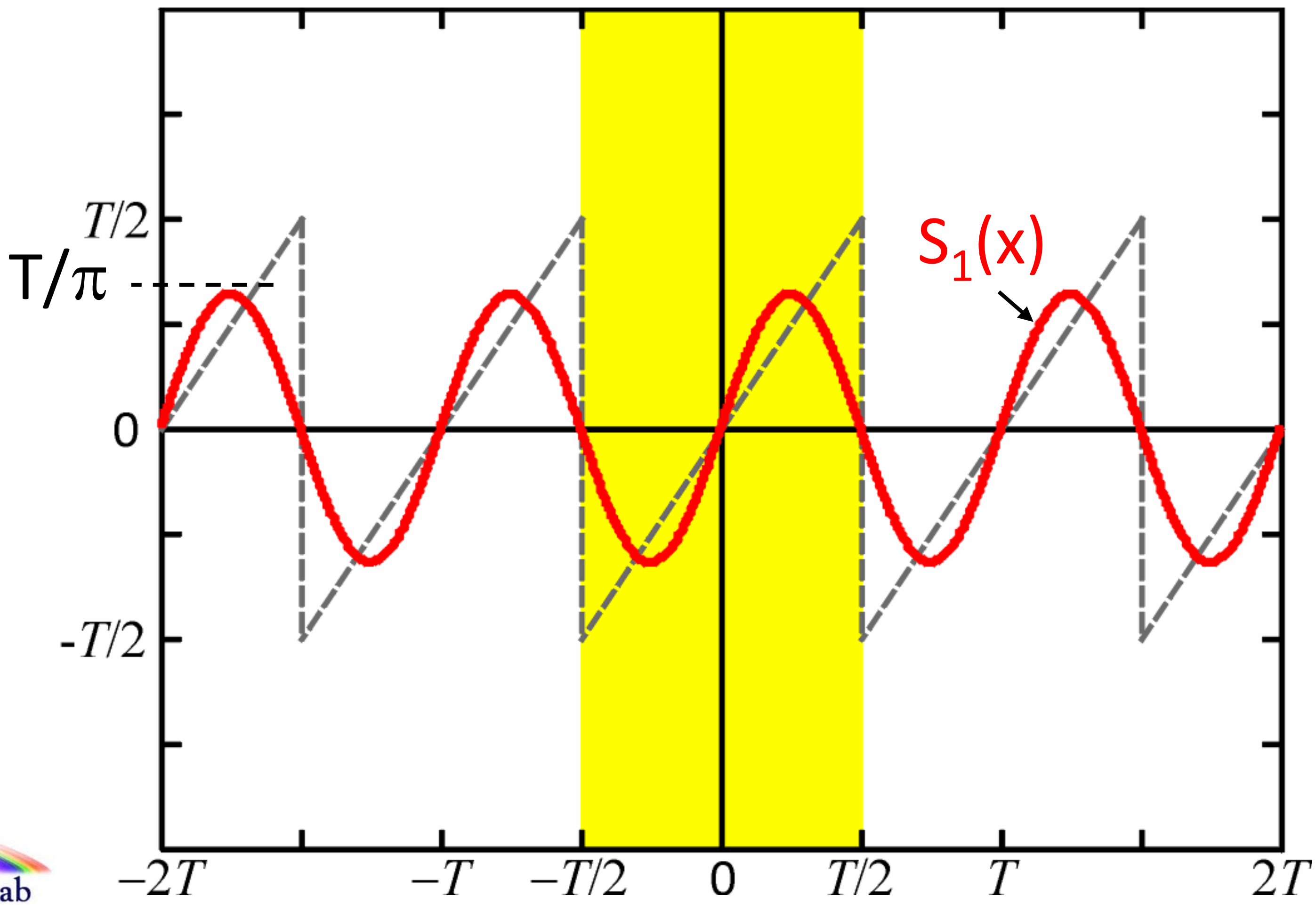
Partial sum of Fourier series

- $$S_N(x) = \frac{\pi}{4} + \sum_{n=1}^{N}\left[\frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{1}{n}\sin(nx)\right].$$



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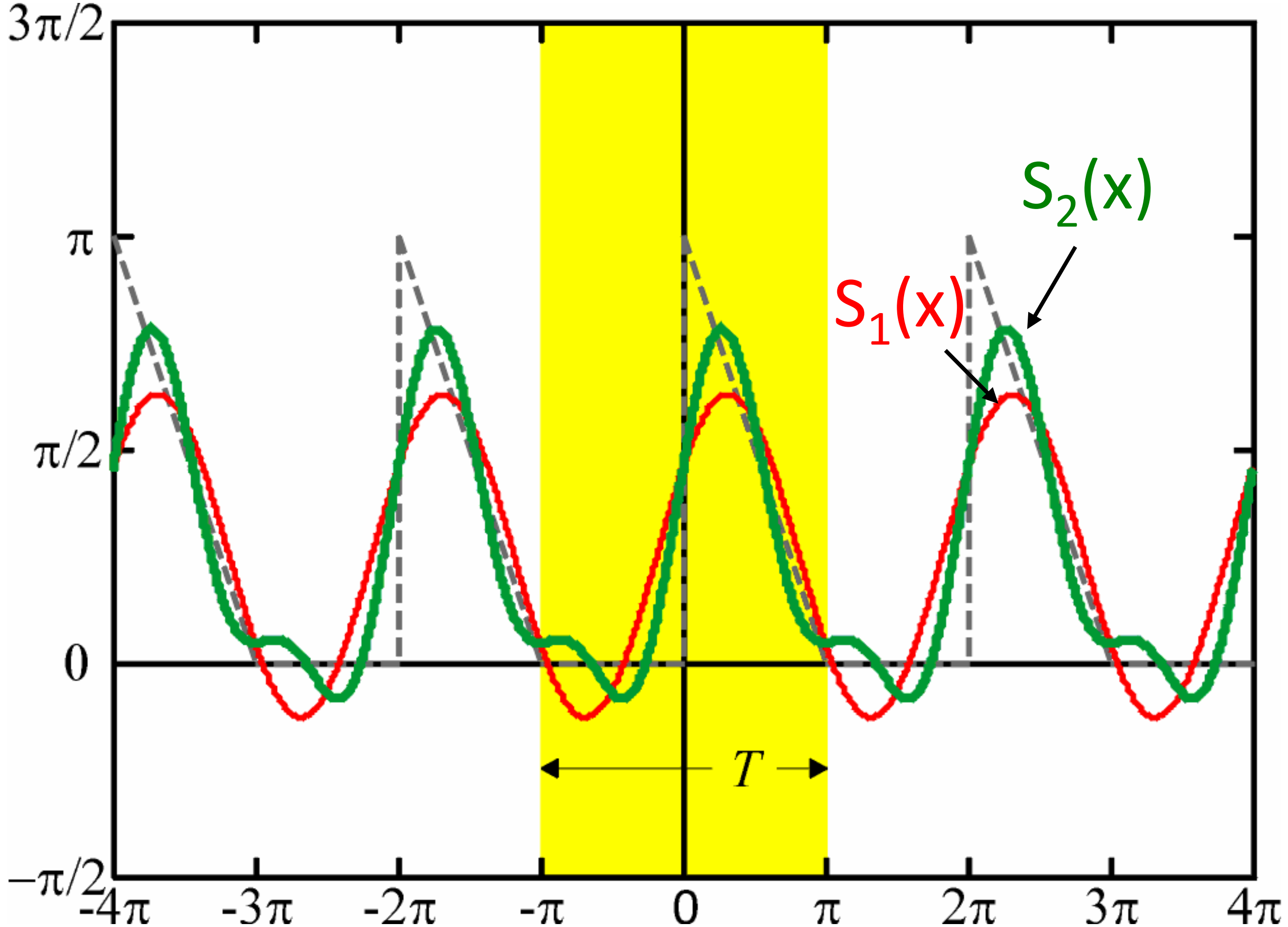
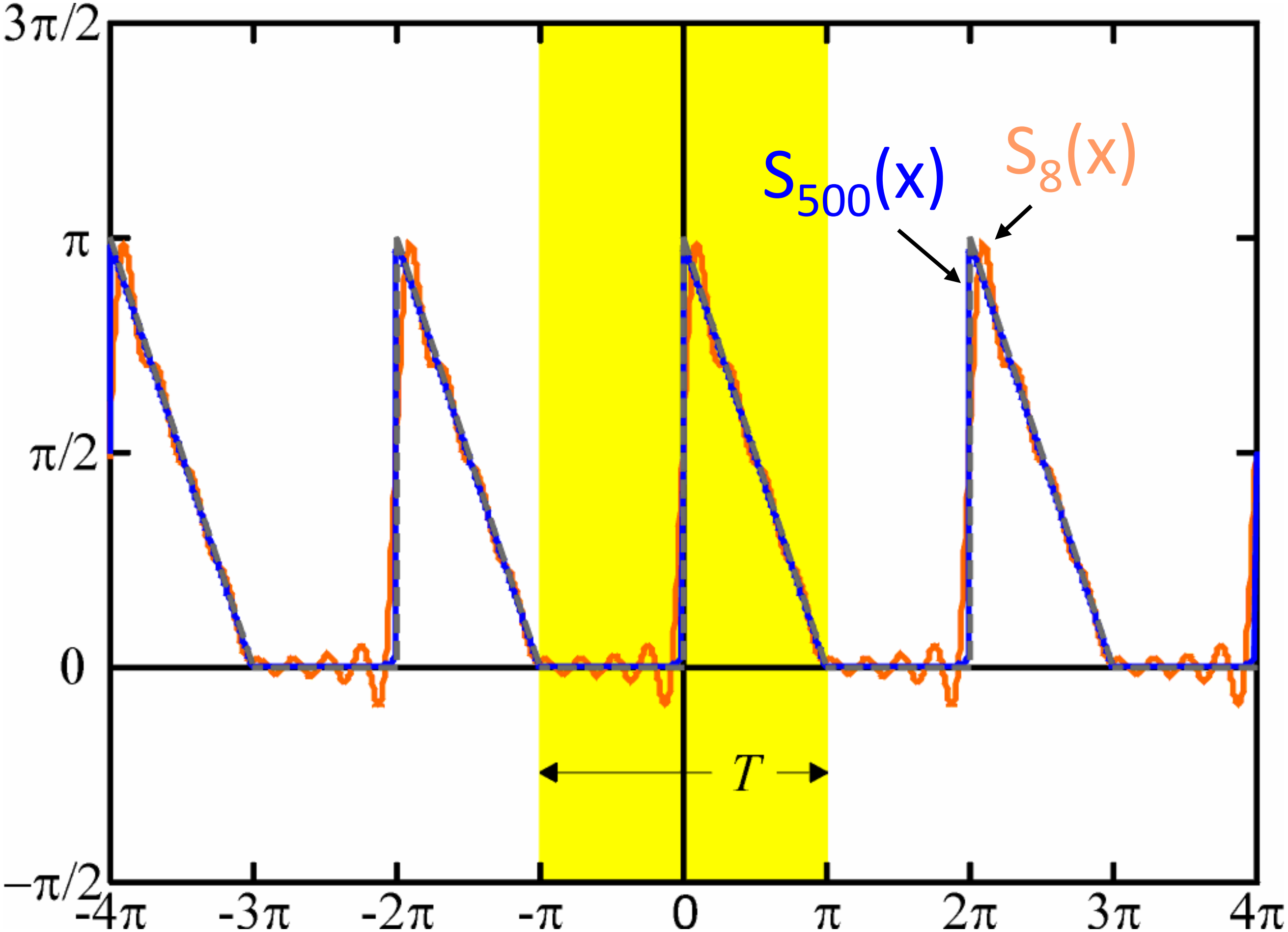
Partial sums (1)



Partial sums (2)



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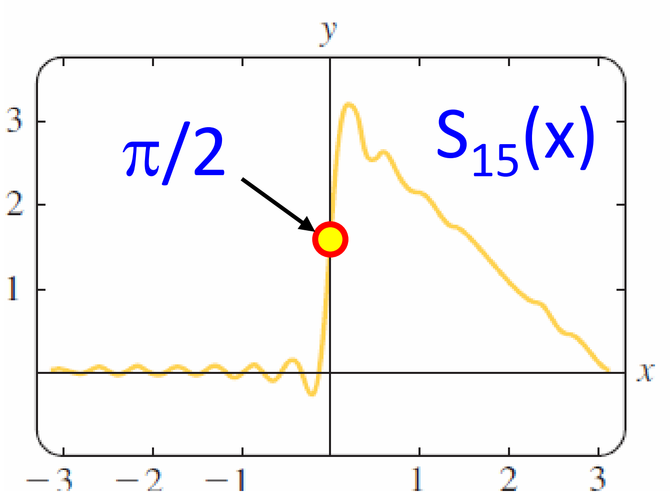
Convergence

- If f(t) and f'(t) are piecewise continuous within $t \in [-T/2, T/2]$, $\Rightarrow$ Fourier series expansion converges to

1) f(t) at any point of continuity.

2) $[f(t^+)+f(t^-)]/2$ (average) at a point of discontinuity.

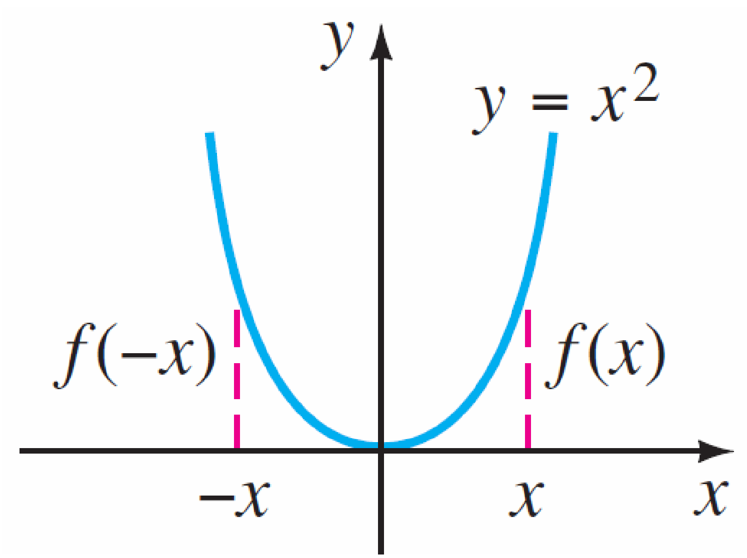
- In Example 1, $S_N(0) \rightarrow [f(0^-)+f(0^+)]/2 = \pi/2$.

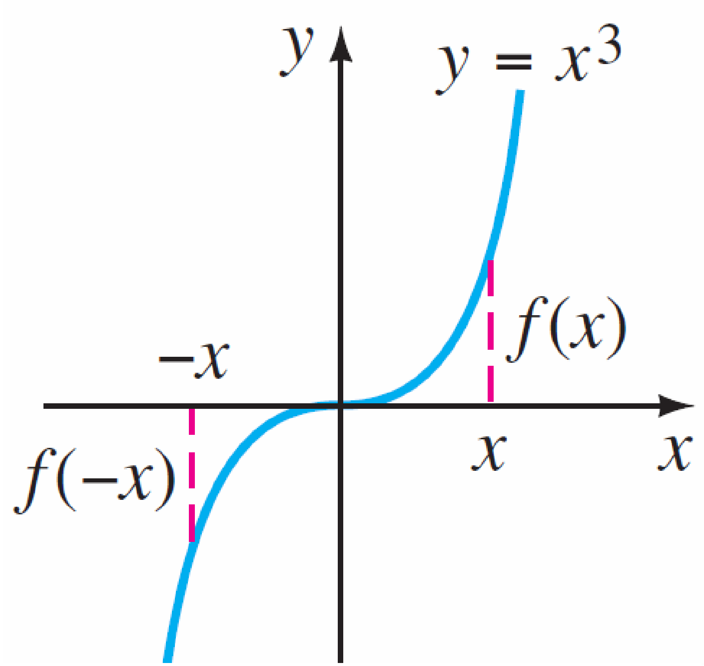


- Fourier series of even and odd functions

Definitions

- f(x) is even if f(-x) = f(x), $\Rightarrow$ symmetric with respect to the vertical axis (x = 0). E.g. f(x) = x^2.
- f(x) is odd if f(-x) = -f(x), $\Rightarrow$ anti-symmetric with respect to the vertical axis (x = 0). E.g. f(x) = x^3.





Properties

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

- Proof of (b): Let F(x) = f(x)×g(x), where f, g are odd.

  F(-x) = f(-x)×g(-x) = [-f(x)]×[-g(x)] = f(x)×g(x) = F(x), ⇒ F(x) is even.

Fourier series of even functions

- If f(t) is even, $\Rightarrow$ $f(t)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t)+\cancel{b_n\sin(n\omega_0 t)}\right].$

- Proof: $b_n=\dfrac{2}{T}\int_{-T/2}^{T/2}\underbrace{\boxed{[\overset{\text{even}}{f(t)}\times\overset{\text{odd}}{\sin(n\omega_0 t)}}}_{\text{odd}}]dt,$

  By Properties (c) and (g), $\Rightarrow$ $b_n = 0$.

- By properties (f) and (a),

$$\begin{cases} a_0=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} f(t)dt=\dfrac{4}{T}\int_{0}^{T/2} f(t)dt. \\ a_n=\dfrac{4}{T}\displaystyle\int_{0}^{T/2}[f(t)\times\cos(n\omega_0 t)]dt. \end{cases}$$

Fourier series of odd functions

- If f(t) is odd, $\Rightarrow$ $$f(t)=\cancel{\frac{a_0}{2}}+\sum_{n=1}^{\infty}\left[\cancel{a_n\cos(n\omega_0 t)}+b_n\sin(n\omega_0 t)\right].$$

- Proof: $$a_n=\frac{2}{T}\int_{-T/2}^{T/2}\underbrace{\boxed{f(t)\times\cos(n\omega_0 t)}}_{\text{odd}}\,dt,$$ where $f(t)$ is odd and $\cos(n\omega_0 t)$ is even.

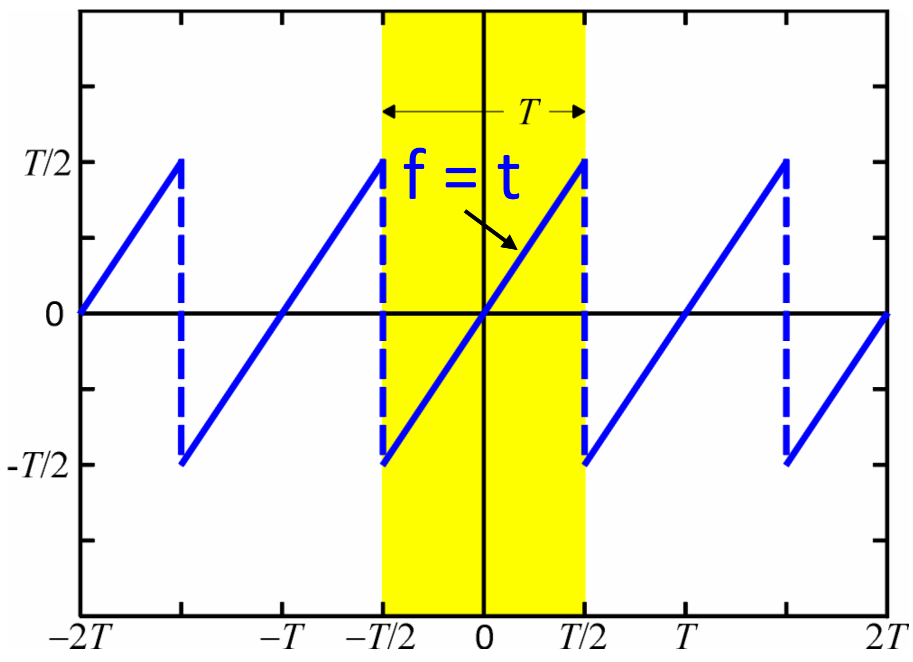
  By Properties (c) and (g), $\Rightarrow$ $a_n = 0$, for n = 0, 1, 2, …

- By properties (b) and (f),

$$b_n=\frac{4}{T}\int_{0}^{T/2}\underbrace{\boxed{f(t)\times\sin(n\omega_0 t)}}_{\text{even}}\,dt.$$ where $f(t)$ is odd and $\sin(n\omega_0 t)$ is odd.

Example 1: Sawtooth waveform

- Expand an odd function f(t) of period T ($\omega_0 = 2\pi/T$):



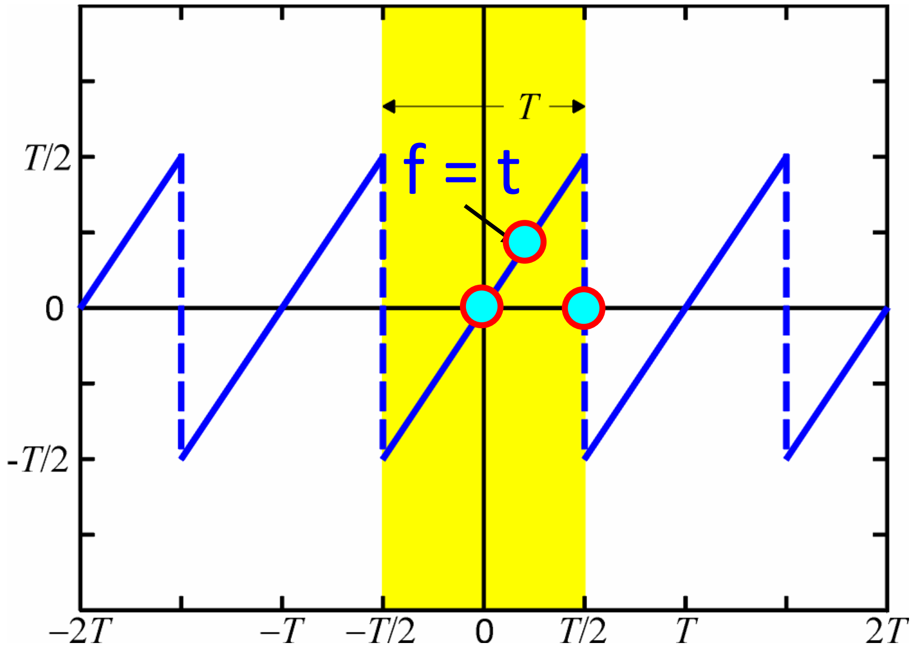
- $a_n = 0$.
- $b_n = \dfrac{4}{T}\displaystyle\int_0^{T/2}\left[t \times \sin(n\omega_0 t)\right]dt$

$$= \frac{4}{T}\frac{(-1)^{n+1}T}{2n\omega_0} = \frac{T}{\pi}\left[\frac{(-1)^{n+1}}{n}\right].$$

$$\Rightarrow f(t) = \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin(n\omega_0 t)\right], \text{where } \omega_0 = \frac{2\pi}{T}.$$

Verifications

$$\text{Given} \quad f(t) = \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin(n\omega_0 t)\right], \text{ where } \omega_0 = \frac{2\pi}{T}.$$



- For t = 0, $\Rightarrow \sin(n\omega_0 t) = 0$, $f(0) = (T/\pi)(\sum 0) = 0$.
- For t = T/2 (discontinuity), $\Rightarrow \sin(n\omega_0 t) = \sin(n\pi) = 0$, $f(T/2) = (T/\pi)(\sum 0) = 0$ (average value).
- For t = T/4, $\Rightarrow \sin(n\omega_0 t) = \sin(n\pi/2) = \{0, 1, -1\}$, $f(T/4) = (T/\pi)(1 - 3^{-1} + 5^{-1} - 7^{-1} + \ldots) \rightarrow T/4$.

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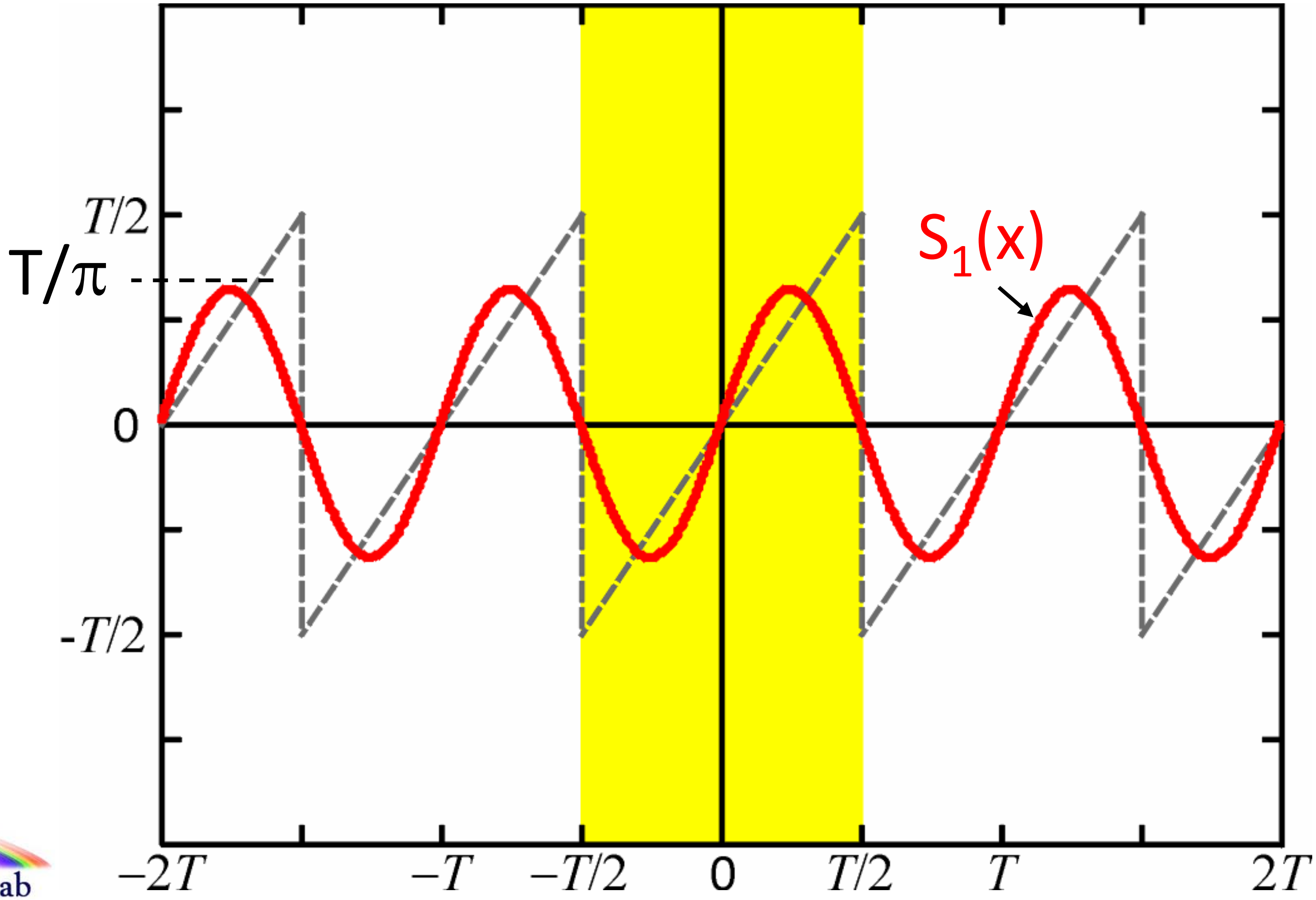
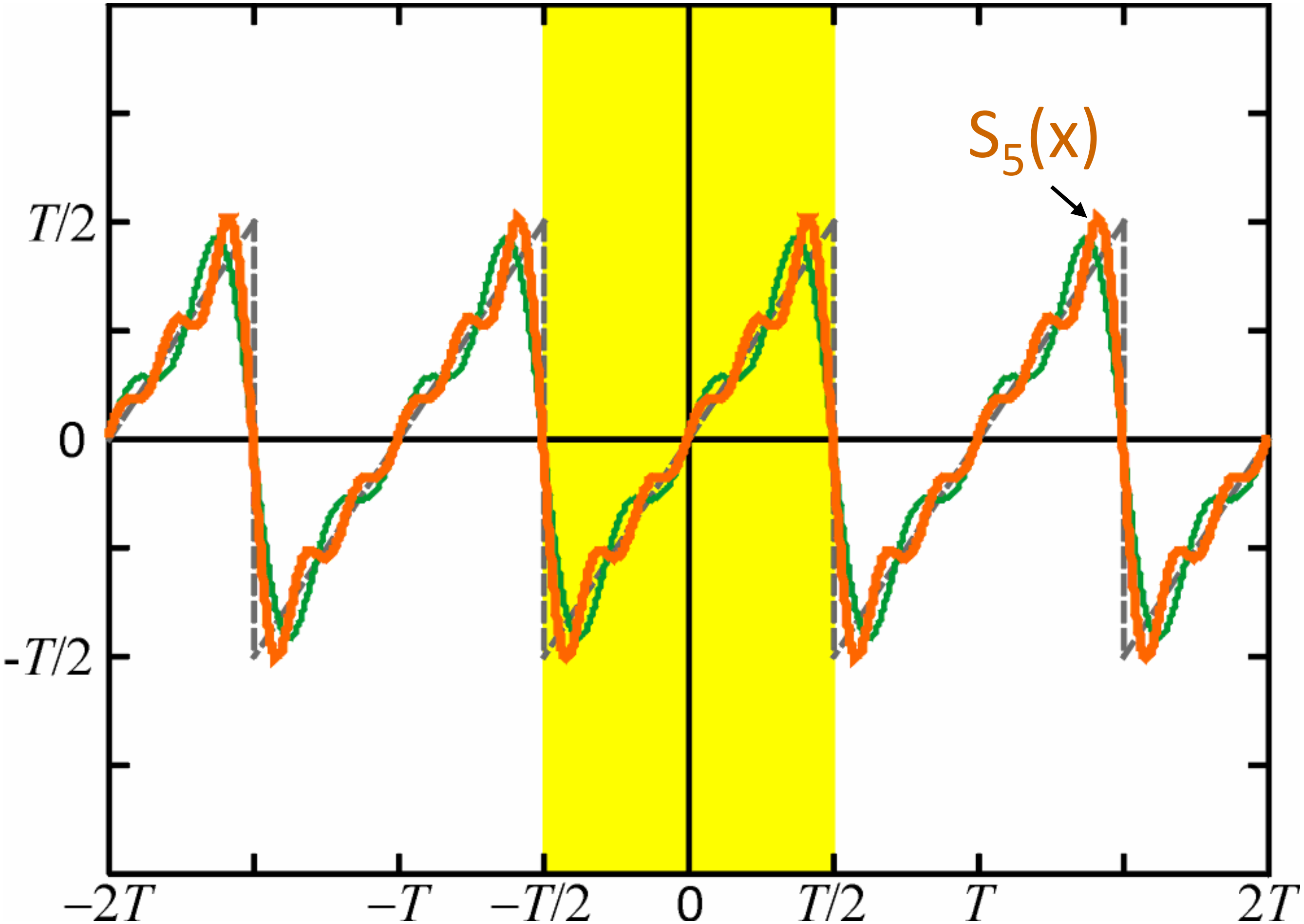
Partial sums (1)

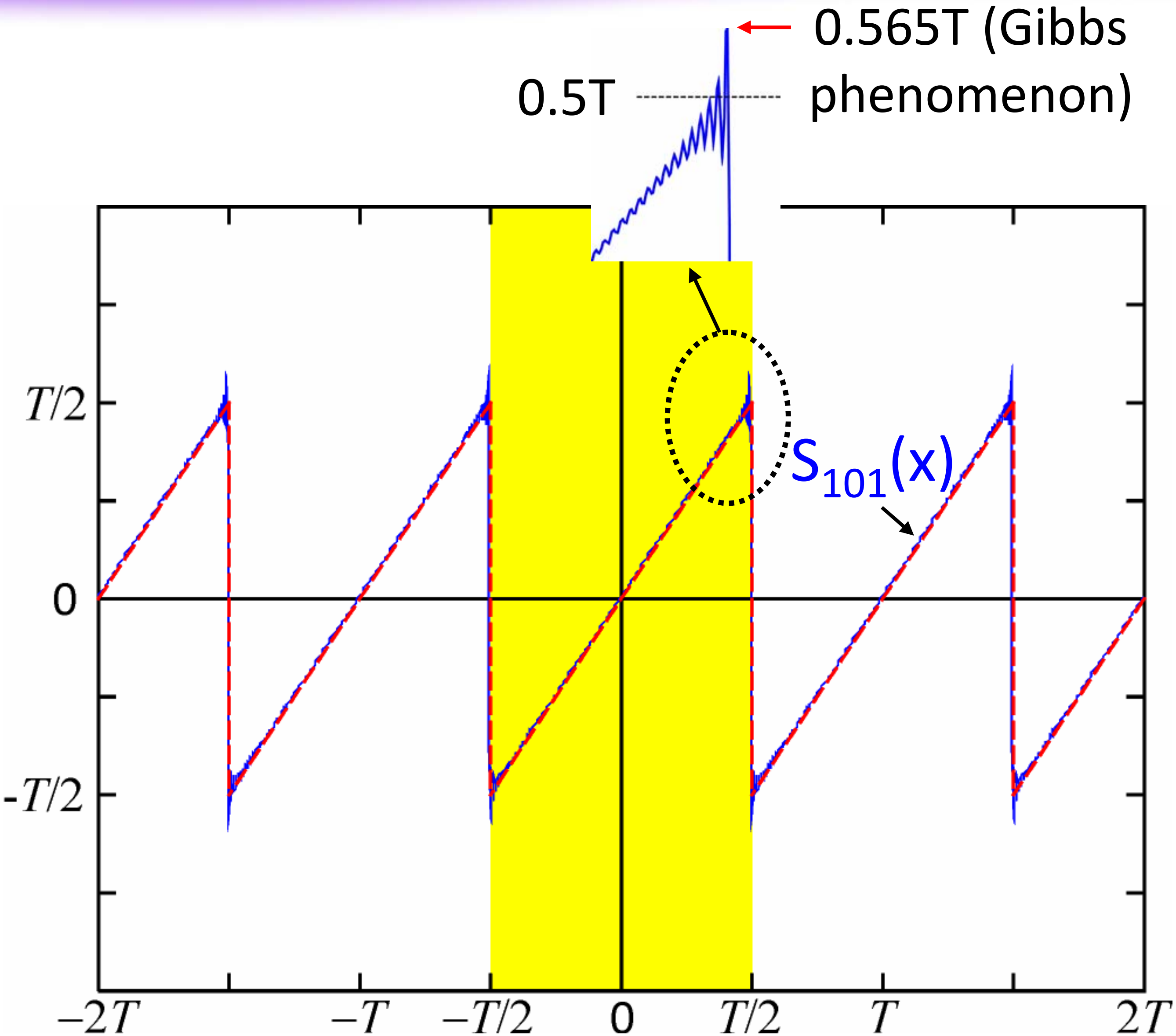
- $$S_N(t) \equiv \frac{T}{\pi}\sum_{n=1}^{N}\left[\frac{(-1)^{n+1}}{n}\sin\left(n\frac{2\pi}{T}t\right)\right].$$



Partial sums (2)

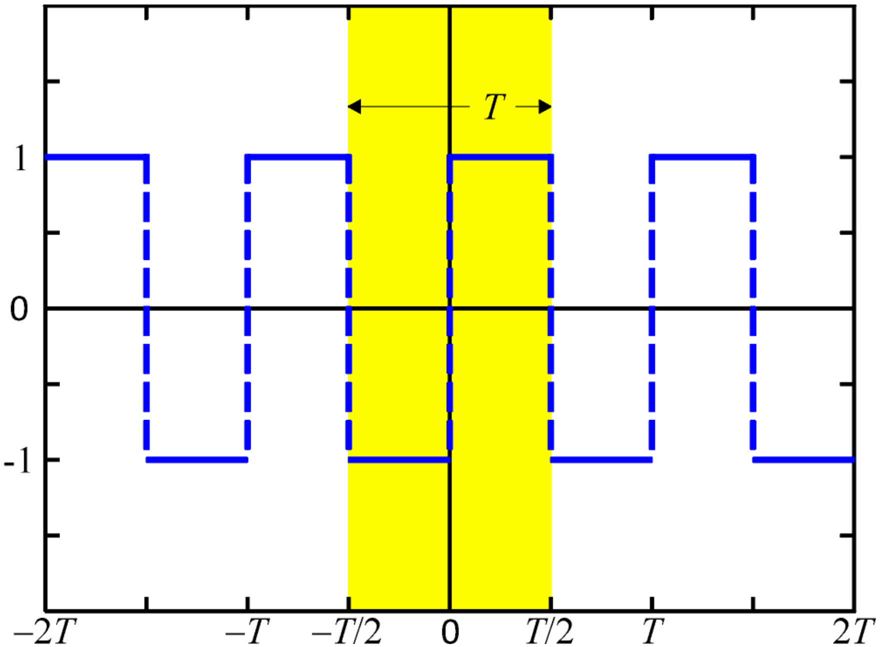


Partial sums (3)



Example 2: Square waveform

- Expand an odd function f(t) of period T ($\omega_0 = 2\pi/T$):



- $a_n = 0.$
- $b_n = \dfrac{4}{T}\int_0^{T/2}\left[1\times\sin(n\omega_0 t)\right]dt$

$$= \frac{4}{T}\frac{1-(-1)^n}{n\omega_0} = \frac{2}{\pi}\frac{1-(-1)^n}{n}.$$

0, if n ∈ even;
2, if n ∈ odd

$$\Rightarrow f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\boxed{1-(-1)^n}}{n}\sin(n\omega_0 t)\right], \text{ where } \omega_0 = \frac{2\pi}{T}.$$

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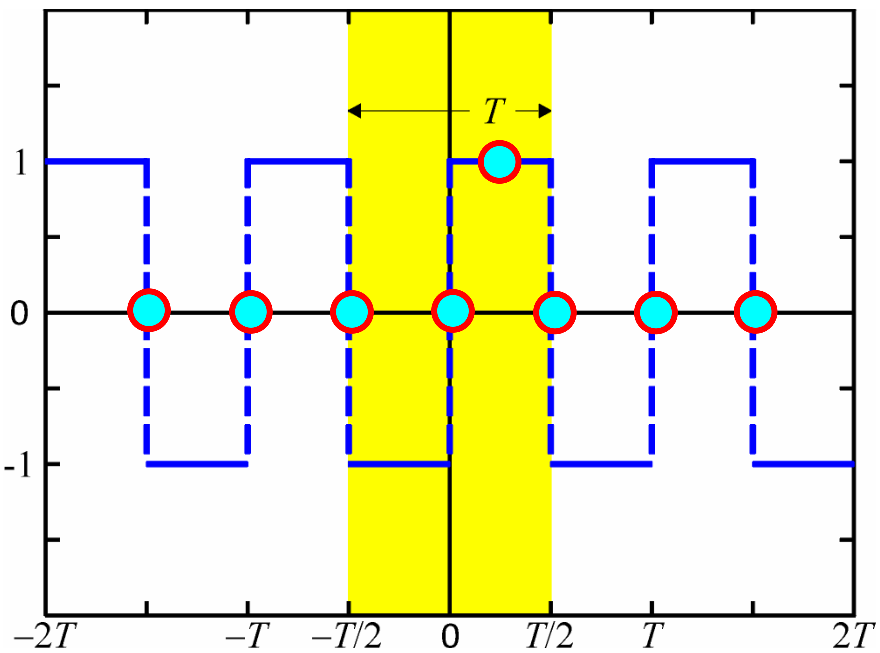
Verifications

$$\text{Given } f(t)=\frac{4}{\pi}\sum_{n\in\text{odd}}^{\infty}\left[\frac{1}{n}\sin(n\omega_0 t)\right],\ \text{where } \omega_0=\frac{2\pi}{T}.$$



- For t = m(T/2) (discontinuities), $\Rightarrow$ $\sin(n\omega_0 t) = \sin(mn\pi) = 0$, $f(mT/2) = (4/\pi)(\sum 0) = 0$ (average value).

- For t = T/4, $\Rightarrow$ $\sin(n\omega_0 t) = \sin(n\pi/2) = \{1, -1\}$ if $n \in$ odd, $f(T/4) = (4/\pi)(1 - 3^{-1} + 5^{-1} - 7^{-1} + \ldots) \to 1$.

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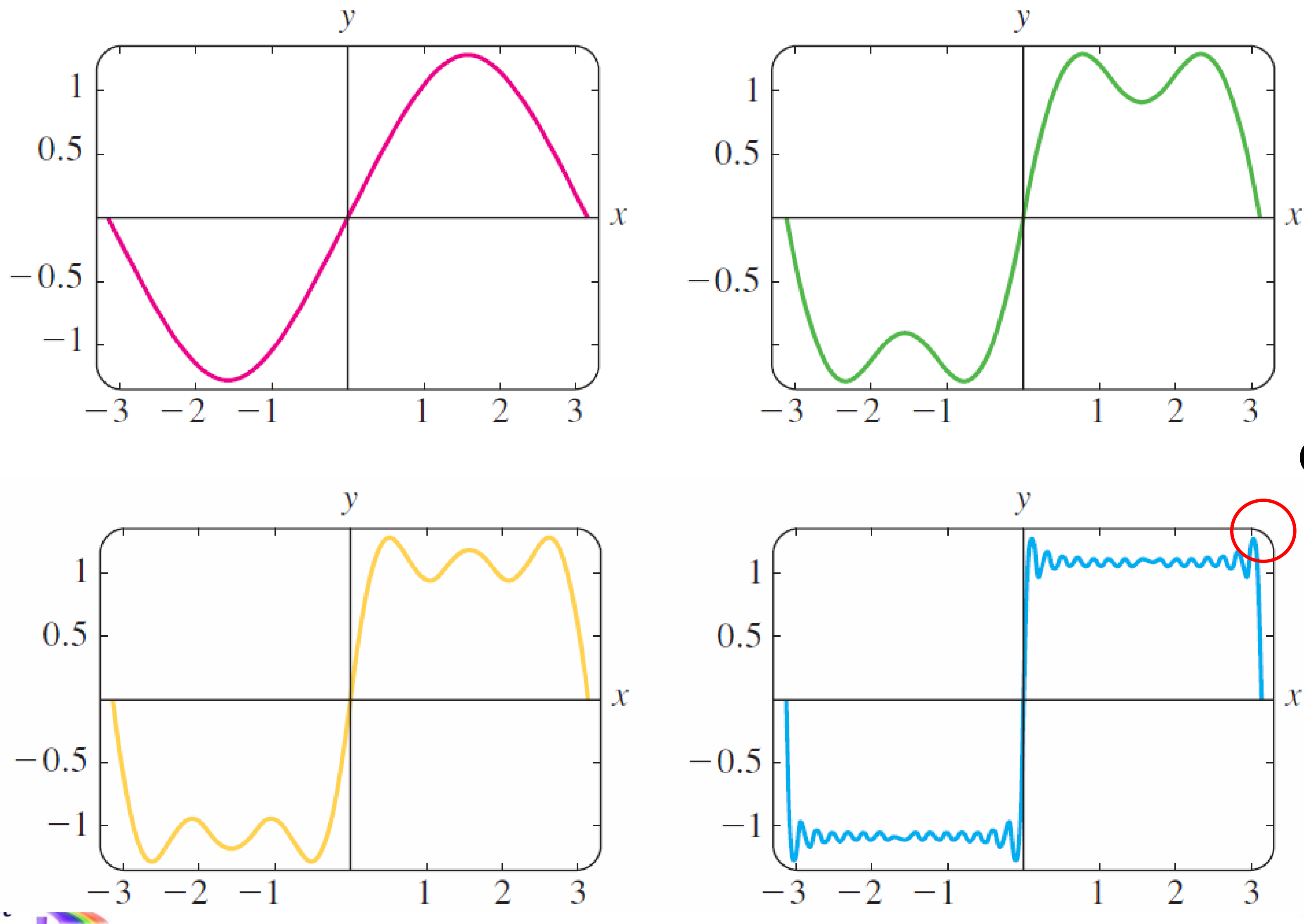
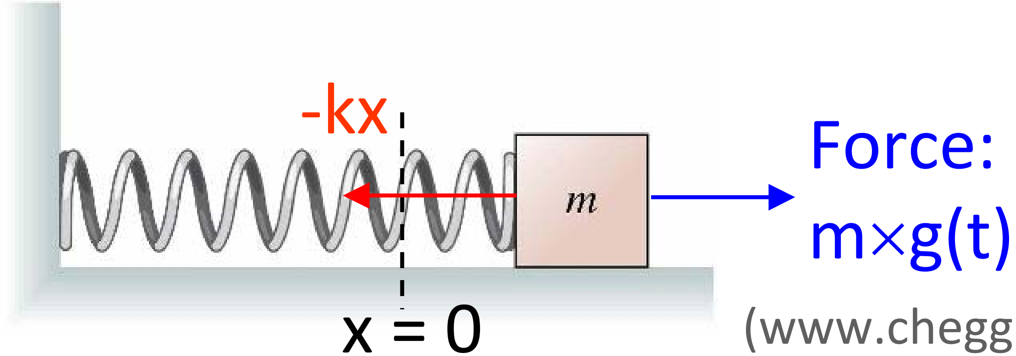
Partial sums



Gibbs

Periodically driven mass-spring

- ODE: $x''(t) + \Omega^2 x(t) = g(t)$, where the resonance frequency is $\Omega = \sqrt{(k/m)}$, $g(t)$ is an odd periodic driving function of period T (fundamental frequency $\omega_0 = 2\pi/T$).



(www.chegg.com)

- In Ch4, particular solution $x_p(t)$ can be determined only when $g(t)$ is of a limited number of forms.

Solving $x_p(t)$ by Fourier series (1)

- Expand the driving function by Fourier sine series with known $\{b_n\}$: $g(t)=\sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$.
- Expand $x_p(t)$ by another Fourier sine series with unknown $\{B_n\}$: $x_p(t)=\sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$.

$$\Rightarrow x_p' = \sum_{n=1}^{\infty} (n\omega_0 B_n) \times \cos(n\omega_0 t),$$

$$\Rightarrow x_p'' = \sum_{n=1}^{\infty} -(n\omega_0)^2 B_n \times \sin(n\omega_0 t),$$

Solving $x_p(t)$ by Fourier series (2)

- Substitute the series of x_p, x''_p into the ODE:

$$x''_p + \Omega^2 x_p = g(t),$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[-(n\omega_0)^2 + \Omega^2 \right] B_n \sin(n\omega_0 t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t),$$

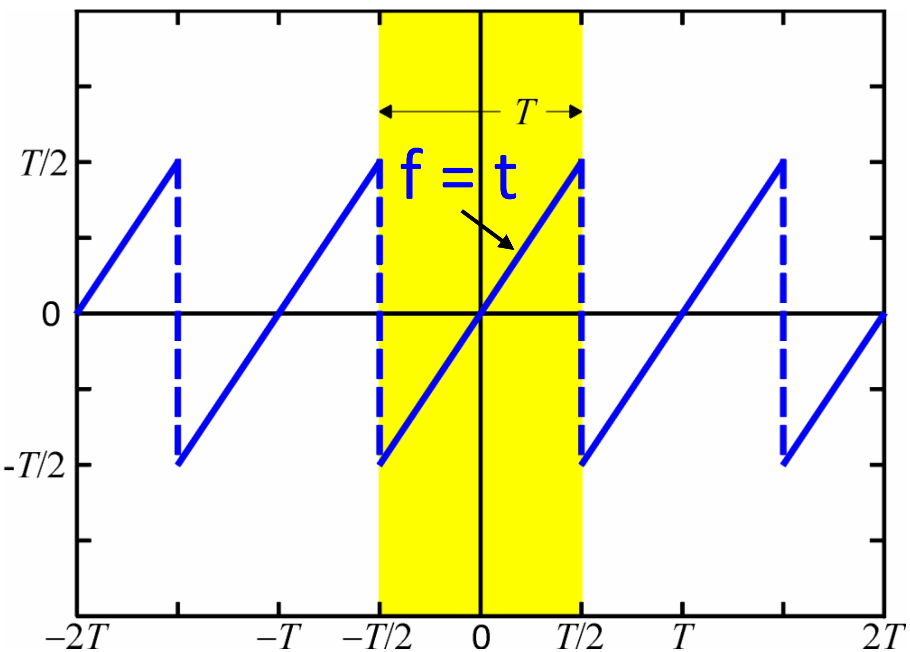
$$\Rightarrow \left[-(n\omega_0)^2 + \Omega^2 \right] B_n = b_n,$$

$$\Rightarrow B_n = \frac{b_n}{\Omega^2 - (n\omega_0)^2}.$$

$$\Rightarrow x_p(t) = \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t).$$ Fourier series solution

Example: Sawtooth driving

- If the mass-spring is driven by a sawtooth waveform:



$$g(t)=\sum_{n=1}^{\infty}\underbrace{\boxed{\frac{T}{\pi}\frac{(-1)^{n+1}}{n}}}_{}\sin(n\omega_0 t),$$

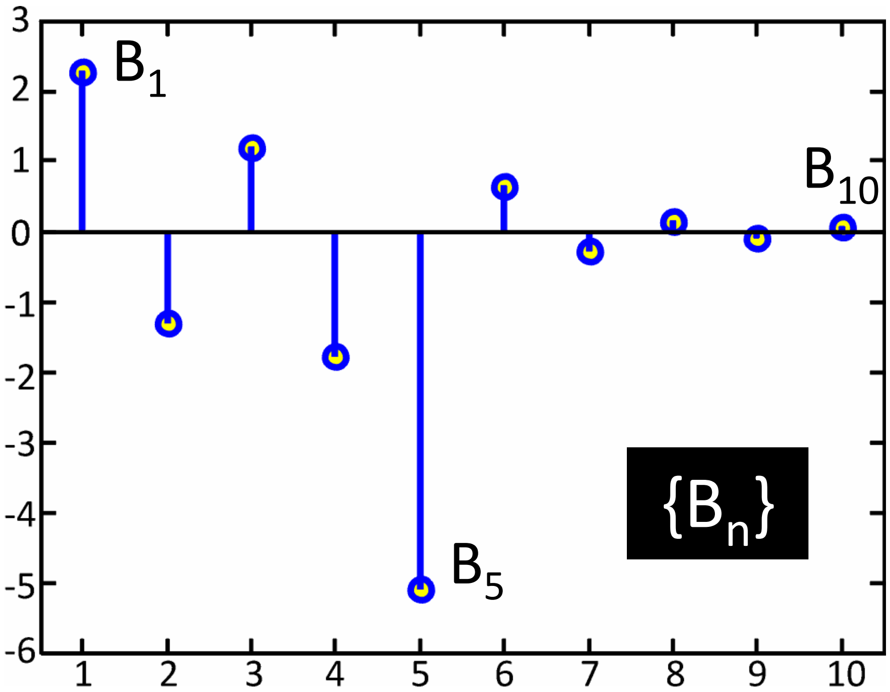
b_n

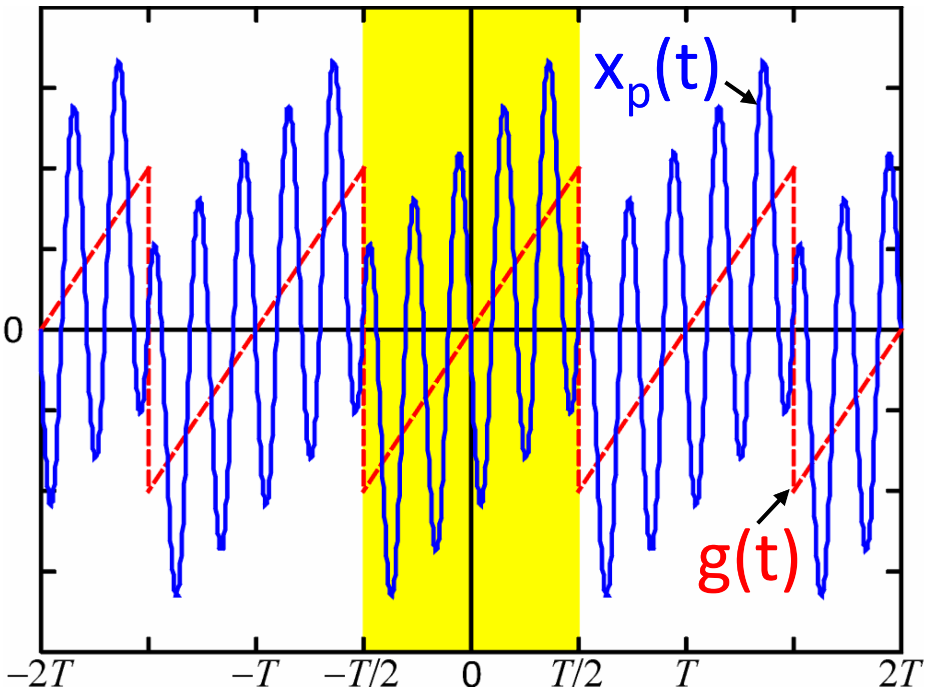
where $\omega_0=\dfrac{2\pi}{T}.$

$$\Rightarrow B_n=\frac{T}{\pi}\frac{(-1)^{n+1}}{n\left[\Omega^2-(n\omega_0)^2\right]}.$$

Solution plot

- E.g. $\Omega = 4.8\omega_0$, plot $x_p(t) \approx S_{101}(t)$.





- $x_p(t)$ can be obtained when $g(t)$ is of arbitrary form.

Summary

- What're inner product, norm, and orthogonality of functions f(x)?
- What's orthogonal series expansion of f(x)?
- What's the Fourier series expansion of a periodic function f(t) of period T?
- What happens to the Fourier series expansion of f(t) at a discontinuity? What's the Gibbs phenomenon?
- How to solve $x_p(t)$ by Fourier series method? What's its primary advantage?