Chapter 7
Laplace Transform (LT)

- Definition, LT of basic functions (7.1)
- Inverse LT (7.2.1)
- Operational properties (7.2-7.4)
- Dirac delta function (7.5)

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Definition, LT of basic functions

- Definition
- LT of basic functions
- Existence of LT

What's "function"?

- A "relation" mapping input value(s) to output value(s), just like a "machine" mapping raw materials to products.
- E.g. $f(x) = \sin(x)$, mapping an input value x_0 to an output value $\sin(x_0)$.
- E.g. $f(x,y) = x + y$, mapping a pair of input values (x_0,y_0) to an output value $x_0 + y_0$.
- E.g. $f(z) = z^*$, mapping a complex number z_0 (a pair of real numbers) to a complex number $(z_0)^*$ (a pair of real numbers).

What's "transform"?

- A "relation" mapping an input function to an output, function.
- E.g. differentiation $D\{f(x)\} = f'(x)$, mapping an input function $f(x)$ to an output function $f'(x)$.
- E.g. partial differentiation $D_x\{f(x,y)\} = \partial f(x,y)/\partial x$, mapping a 2-variable function $f(x,y)$ to another 2-variable function $\partial f(x,y)/\partial x$.
- It's linear if $T\{a\times f_1(x) + b\times f_2(x)\} = a\times T\{f_1(x)\} + b\times T\{f_2(x)\}$.

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What's "integral transform"?

- The output function F(s) is obtained by integrating the product of the input function f(t) and a kernal function K(s,t):

$$F(s) \equiv T\{f(t)\} = \int_0^\infty f(t) \times K(s,t)\,dt$$

- Note: Output function has a new independent variable s.
- All integral transforms are linear. (Why?)

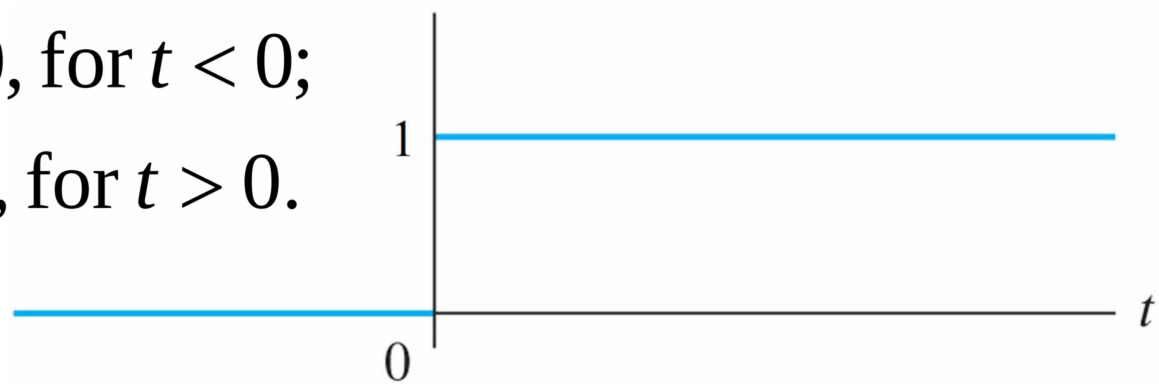
What's Laplace transform (LT)?

- A special case of integral transform where the kernal is an exponential function K(s,t) = e^{-st}:

$$F(s) \equiv L\{f(t)\} = \int_0^\infty f(t) \times e^{-st} dt.$$

- Integration is carried out over $0 < t < \infty$, $\Rightarrow$ L{f(t)} = L{f(t)×u(t)}, where the unit-step function is:

$$u(t) \equiv \begin{cases} 0, \text{ for } t < 0; \\ 1, \text{ for } t > 0. \end{cases}$$



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Example 1: L{1}

- By definition,

$$L\{1\} = L\{u(t)\} = \int_0^{\infty} u(t) \times e^{-st} dt = \int_0^{\infty} 1 e^{-st} dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{0-1}{-s} = \boxed{\frac{1}{s}}, \quad \text{if } s > 0.$$

- L{1} exists (integrable) only in a finite region of convergence ROC: {s > 0}.

Example 2: L{t}

- By definition,

$$L\{t\} = L\{t \times u(t)\} = \int_0^\infty t \times u(t) \times e^{-st} dt = \int_0^\infty t e^{-st} dt;$$

- Integration by parts: u = t, v' = e^{-st}, $\Rightarrow$ u'=1, v = -e^{-st}/s

$$L\{t\} = uv\Big|_0^\infty - \int_0^\infty u'v dt = -\frac{te^{-st}}{s}\Bigg|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} dt$$

$$= -\left(\frac{0-0}{s}\right) - \left(\frac{e^{-st}}{s^2}\Bigg|_0^\infty\right) = -\frac{0-1}{s^2} = \boxed{\frac{1}{s^2}}, \quad \text{if } s > 0$$

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Complex nature

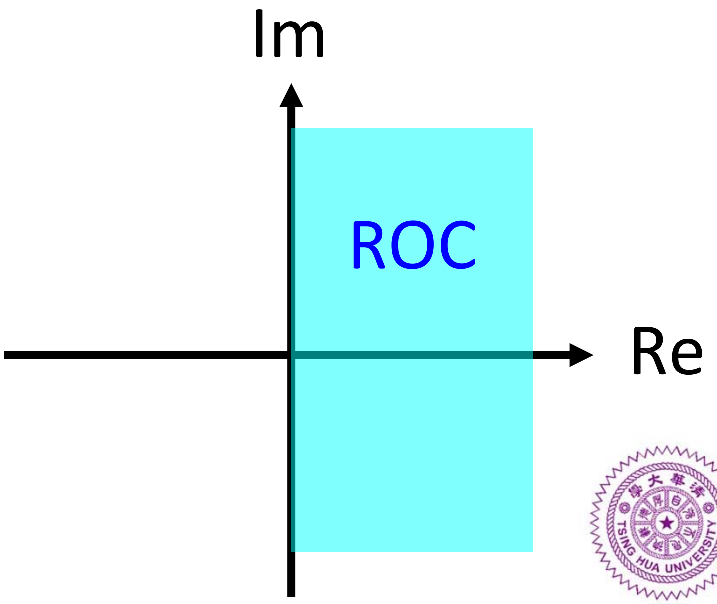
- The new variable s is actually complex ($s = \alpha + j\beta \in C$),

  $\Rightarrow K(s,t) = e^{-st} = e^{-(\alpha+j\beta)t} = e^{-\alpha t}\times[\cos(\beta t) + j\sin(\beta t)] \in C$,

  $$\Rightarrow \quad F(s) = \int_0^\infty f(t)\times e^{-st}dt \in C$$

- Input f(t): a real function of real variable t.
- Output F(s): a complex function of complex variable s,

  $\Rightarrow$ Re{F}, Im{F} are functions of 2 real variables.



- ROC: {s>0} is actually Re{s} > 0.

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Example: L{t²}

- By definition, $L\{t^2\} = \int_0^\infty t^2 e^{-st} dt;$
- Let {u = t^2, v' = e^{-st}}, ⇒ {u' = 2t, v = $-e^{-st}$/s}

$$L\{t^2\} = uv\Big|_0^\infty - \int_0^\infty u'v\,dt = -\frac{t^2 e^{-st}}{s}\Bigg|_0^\infty + \frac{2}{s}\int_0^\infty t e^{-st} dt$$

$$= 0 + \frac{2}{s}L\{t\} = \frac{2}{s}\times\frac{1}{s^2} = \boxed{\frac{2}{s^3}} = \frac{2!}{s^{2+1}}, \text{ if } \operatorname{Re}\{s\} > 0$$

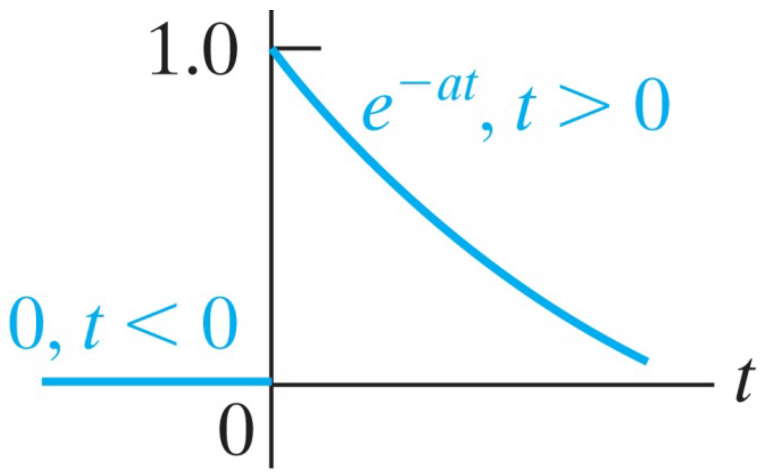
- Recursive relation gives L{t^n} = $n!/s^{n+1}$.

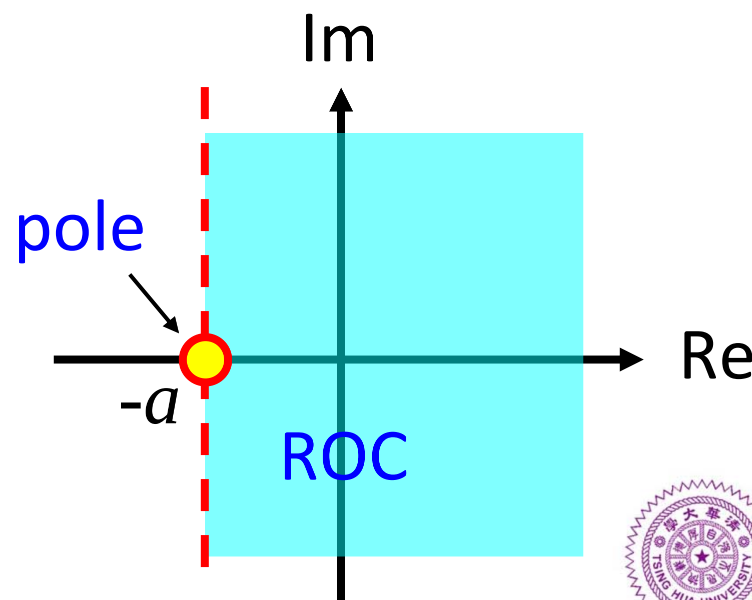
Example 3: L{e^{-at}}

- By definition,

$$L\{e^{-at}\} = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(a+s)t} dt$$

$$= \left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_{0^-}^{\infty} = \boxed{\frac{1}{s+a}}, \text{ if } \mathrm{Re}\{s\} > -a.$$





Example 4: L{sin(ωt)}

- By definition,

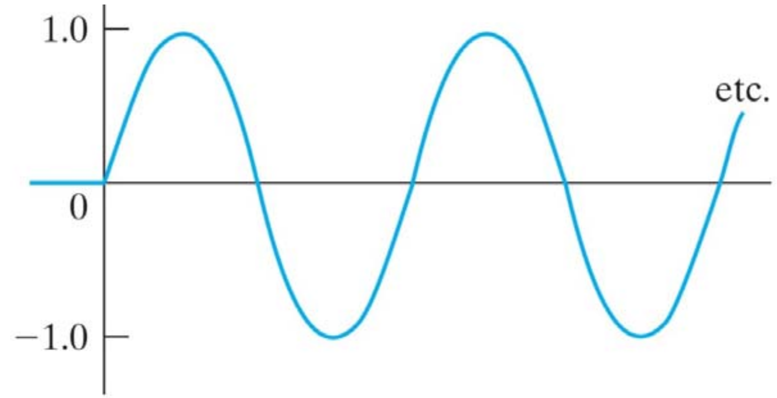
$$L\{\sin\omega t\} = \int_0^\infty (\sin\omega t) e^{-st} dt$$

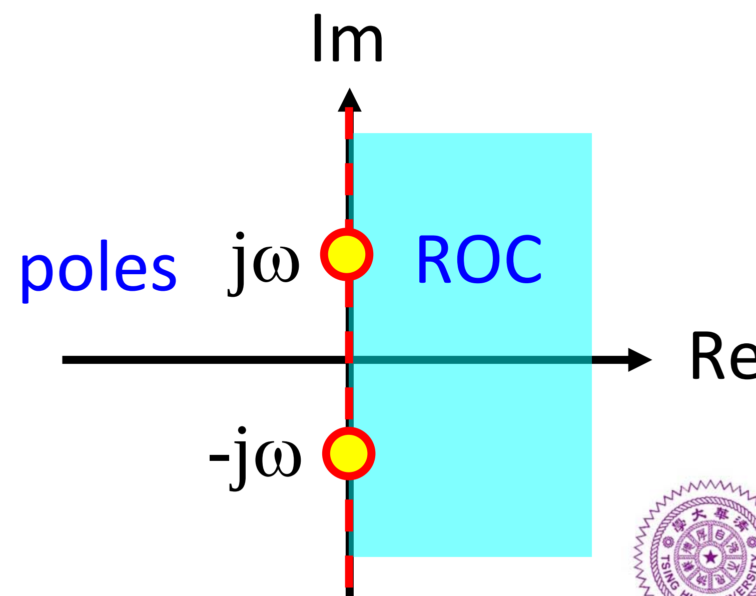
$$= \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st} dt$$

$$= \frac{1}{2j} \int_0^\infty \left[e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right] dt$$

$$= \frac{1}{2j} \left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right)$$

$$= \boxed{\frac{\omega}{s^2+\omega^2}}, \text{ if } \mathrm{Re}\{s\} > 0.$$





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LT of basic functions (1)

Type	$f(t)$	$F(s)$
unit-step	$u(t)$	$\frac{1}{s}$
ramp	$t \times u(t)$	$\frac{1}{s^2}$
polynomial	$t^n \times u(t)$	$\frac{n!}{s^{n+1}}$
exponential	$e^{-at} \times u(t)$	$\frac{1}{s+a}$

LT of basic functions (2)

Type	$f(t)$	$F(s)$
sine	$\sin \omega t \times u(t)$	$\dfrac{\omega}{s^2+\omega^2}$
cosine	$\cos \omega t \times u(t)$	$\dfrac{s}{s^2+\omega^2}$
damped ramp	$t\boxed{e^{-at}}\times u(t)$	$\dfrac{1}{(s\underline{+a})^2}$
damped sine	$\boxed{e^{-at}}\sin \omega t \times u(t)$	$\dfrac{\omega}{(s\underline{+a})^2+\omega^2}$

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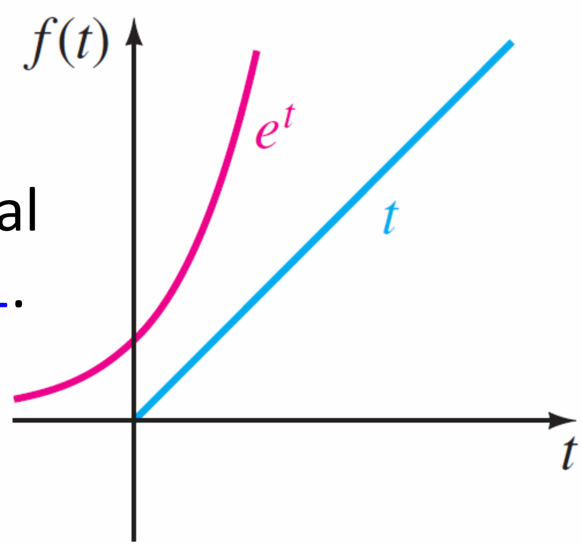
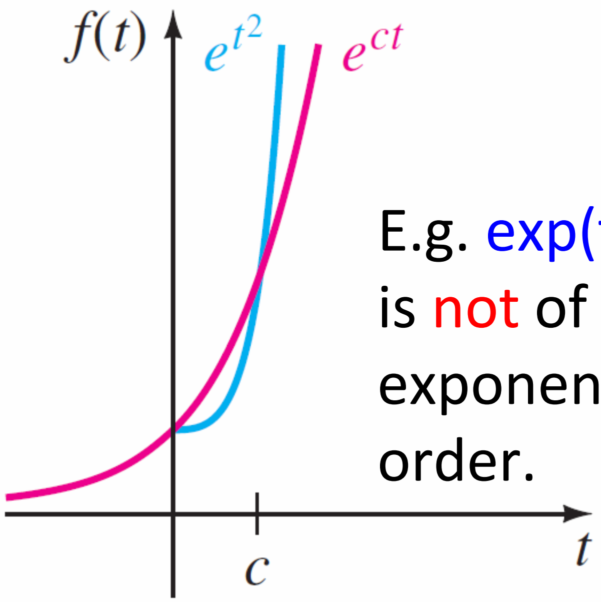
Exponential order

- A function f(t) is of exponential order c if there are constants c, M(>0), T(>0) such that f(t) is eventually bounded by an exponential function, i.e.

$$|f(t)| \le Me^{ct}, \text{ for } t > T.$$

E.g. t is of exponential order c = 1.





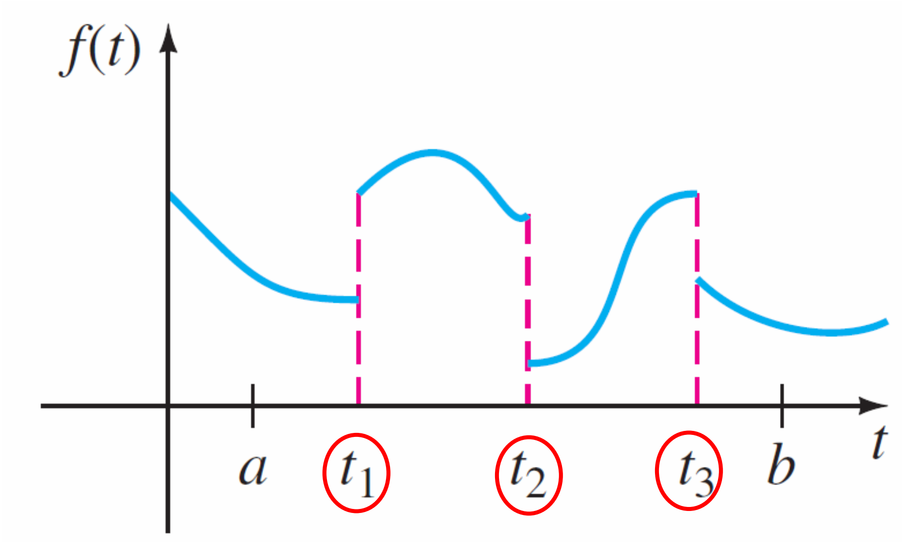
E.g. $\exp(t^2)$ is not of exponential order.

Existence of LT

- L{t^{-1}} and L{t^t} do not exist (not integrable).
- The sufficient (but not necessary) conditions that the L{f(t)} exist for Re{s} > c are:

  (1) f(t) is "piecewise continuous" for $t \in [0,\infty)$

  (2) f(t) is of "exponential order c" for t > T.



Piecewise continuous: only finite number of discontinuities t = {t_1, t_2, …, t_n}

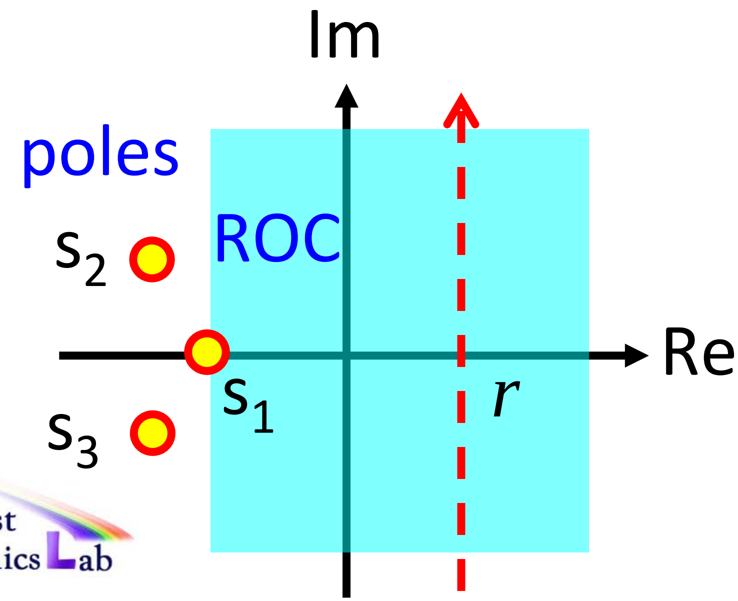
❐ Inverse LT

- Formula of complex integral
- Practical treatment

Inverse LT

- Notation: $L^{-1}\{F(s)\} = f(t)$
- The formula involves with complex integral and residue theorem (to be studied in EE 202):

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} F(s)\times e^{st}\,ds = \sum_{\text{all poles}} \operatorname*{Res}_{s_k}\left[e^{st}\times F(s)\right]$$



- If G(s) has a pole at s_i, $\Rightarrow$

$$G(s) = \sum_{n=1}^{\infty} a_n (s-s_i)^n + \sum_{n=1}^{\infty} \frac{b_n}{(s-s_i)^n},$$

$$\Rightarrow \operatorname*{Res}_{s_i}[G(s)] = b_1,$$

residue of G(s) at s_i.

Practical treatment of inverse LT

- Inverse LT is usually performed by using the table of LT of basic functions.
- Table (p13-4):

  (1) $L^{-1}\{s^{-n}\} = t^{n-1}/n!$,

  (2) $L^{-1}\{1/(s+a)\} = e^{-at}$,

  (3) $L^{-1}\{1/(s^2+\omega^2)\} = \sin(\omega t)/\omega$,

  (4) $L^{-1}\{s/(s^2+\omega^2)\} = \cos(\omega t)$, …
- For F(s) other than these basic forms, trying to simplify F(s) by linearity and partial fractions.

Linearity of ILT

- $L^{-1}\{a\times F_1(s) + b\times F_2(s)\} = a\times L^{-1}\{F_1(s)\} + b\times L^{-1}\{F_2(s)\} = a\times f_1(t) + b\times f_2(t)$.
- E.g. What's f(t) if $F(s) = (-2s+6)/(s^2+4)$?

$$f(t) = L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$$

$$= -2L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 3L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= -2\cos(2t) + 3\sin(2t)$$

Partial fractions, Example 3 (1)

$$F(s)=\frac{s^2+6s+9}{(s^3+s^2-10s+8)=(s-1)(s-2)(s+4)}$$

$$=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}=\frac{N(s)}{(s-1)(s-2)(s+4)}$$

$$N(s)=A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

$$=s^2+6s+9$$

- Method 1: Expanding N(s) as a 2nd-order polynomial, comparing the coefficients, solving a system of 3 algebraic equations.

Example 3 (2)

- Method 2: Substituting s = 1, 2, -4 into N(s).
- N(1) = A(1-2)(1+4) = -5A = $1^2 + 6\times 1 + 9 = 16$,

  $\Rightarrow$ A = -16/5;
- N(2) = B(2-1)(2+4) = 6B = $2^2 + 6\times 2 + 9 = 25$,

  $\Rightarrow$ B = 25/6;
- N(-4) = C(-4-1)(-4-2) = 30C = $(-4)^2 + 6\times(-4) + 9 = 1$,

  $\Rightarrow$ C = 1/30;

$$\Rightarrow f(t) = L^{-1}\left\{\frac{A}{s-1}\right\} + L^{-1}\left\{\frac{B}{s-2}\right\} + L^{-1}\left\{\frac{C}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Operational properties

- Differentiation
- Shift
- Convolution
- Integral
- Periodic functions

L{f'(t)}

- By definition, $L\{f'(t)\} = \int_0^\infty f'(t) \times e^{-st} dt;$

- Integration by parts: u = e^{-st}, v' = f', $\Rightarrow$ u' = -se^{-st}, v = f:

$$\int_0^\infty f'(t) \times e^{-st} dt = \left[f(t) \times e^{-st} \Big|_0^\infty \right] + s\left[\int_0^\infty f(t) \times e^{-st} dt \right]$$

$$= [0 - f(0)] + sF(s) = sF(s) - f(0);$$

if Re{s} > 0 and F(s) ≡ L{f(t)}.

- L{f'} = sF(s) – f(0): derivative in the time domain is equivalent to multiplication in the s-domain.

L{f"(t)}

- By definition, $L\{f''(t)\} = \int_0^{\infty} f''(t) \times e^{-st} dt;$
- Integration by parts: u = e^{-st}, v' = f", $\Rightarrow$ u' = -se^{-st}, v = f':

$$\int_0^{\infty} f''(t) \times e^{-st} dt = \left[f'(t) \times e^{-st} \Big|_0^{\infty} \right] + s\left[\int_0^{\infty} f'(t) \times e^{-st} dt \right]$$

$$= [0 - f'(0)] + s[sF(s) - f(0)] = s^2 F(s) - sf(0) - f'(0);$$

if Re{s} > 0.

- L{f"} = s^{2}F(s) – sf(0) – f'(0): derivative in the time domain is equivalent to multiplication (plus a polynomial) in the s-domain.

Example 5 (Sec. 7.2)

- ODE: $y'' - 3y' + 2y = e^{-4t}$; ICs: $\{y(0) = 1, y'(0) = 5\}$.
- Step 1: $L\{y'' - 3y' + 2y\} = [s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = (s^2-3s+2)Y(s) - (s+2) = L\{e^{-4t}\} = (s+4)^{-1}$.
- Step 2: Solve Y(s)

$$(s^2 - 3s + 2)Y(s) = (s+2) + \frac{1}{s+4};$$

$$\Rightarrow Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

- Step 3: $y(t) = L^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$ (p21).

Advantages of LT approach

- Unique solution y(t) is directly obtained without by way of $y_c(t)$ and $y_p(t)$.
- ICs have been taken into account by LT.
- Y(s) is obtained by solving an algebraic equation.

L{e^{at}×f(t)}

- By definition,

$$L\{e^{at}\times f(t)\}=\int_0^\infty e^{at}\times f(t)\times e^{-st}dt$$
$$=\int_0^\infty f(t)\times e^{-(s-a)t}dt$$
$$=F(s)\big|_{s\to s-a}=F(s-a).$$

- L{e^{at}×f(t)} = F(s-a): exponential modulation in the time domain is equivalent to translation (shift) in the s-domain.

Examples

- $L\{e^{at}\} = L\{e^{at} \times u(t)\} = L\{u(t)\}\big|_{s \to s+a} = \left.\frac{1}{s}\right|_{s \to s+a} = \frac{1}{s+a},$

  same as that obtained by definition (p11).

- $L\{t \times e^{-at}\} = L\{t\}\big|_{s \to s+a} = \left.\frac{1}{s^2}\right|_{s \to s+a} = \frac{1}{(s+a)^2},$

- $L\{e^{-at} \times \sin \omega t\} = L\{\sin \omega t\}\big|_{s \to s+a} = \frac{\omega}{s^2 + \omega^2}_{s \to s+a}$

$$= \frac{\omega}{(s+a)^2 + \omega^2},$$

as shown in the last two rows of the LT table (p14).

Example 3 (Sec. 7.3) (1)

- ODE: $y'' - 6y' + 9y = t^2e^{3t}$; ICs: $\{y(0) = 2, y'(0) = 17\}$.
- Step 1: $L\{y'' - 6y' + 9y\} = [s^2Y(s) - 2s - 17] - 6[sY(s) - 2] + 9Y(s) = (s^2-6s+9)Y(s) - (2s+5) = L\{t^2e^{3t}\} = 2/(s-3)^3$.
- Step 2: Solve Y(s). $(s-3)^2Y(s) = (2s+5) + 2/(s-3)^3$, $\Rightarrow$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5} = \left[\frac{A}{s-3} + \frac{B}{(s-3)^2}\right] + \frac{2}{(s-3)^5};$$

$$[\ ] = \frac{A(s-3)+B}{(s-3)^2} = \frac{As+(B-3A)}{(s-3)^2} = \frac{2s+5}{(s-3)^2}, \Rightarrow \begin{cases} A = 2, \\ B = 11 \end{cases}$$

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Example 3 (Sec. 7.3) (2)

- Step 3: $y(t) = L^{-1}\{Y(s)\}$

$$y(t) = L^{-1}\left\{\frac{2}{s-3}\right\} + L^{-1}\left\{\frac{11}{(s-3)^2}\right\} + L^{-1}\left\{\frac{2}{(s-3)^5}\right\}$$

$$= \left[2L^{-1}\left\{s^{-1}\right\} + 11L^{-1}\left\{s^{-2}\right\} + 2L^{-1}\left\{s^{-5}\right\}\right] \times e^{3t}$$

$$= \left[2u(t) + 11tu(t) + 2\frac{t^4 u(t)}{4!}\right] e^{3t}$$

$$= \left(2 + 11t + \frac{t^4}{12}\right) e^{3t} u(t)$$

Example 4 (Sec. 7.3) (1)

- ODE: y'' + 4y' + 6y = $1+e^{-t}$; ICs: {y(0) = 0, y'(0) = 0}.
- Step 1: L{y'' + 4y' + 6y} = [s^2Y(s) - 0s - 0] +4[sY(s) - 0] + 6Y(s) = (s^2+4s+6)Y(s) = L{$1+e^{-t}$} = s^{-1} + $(s+1)^{-1}$ = (2s+1)/[s(s+1)].
- Step 2: Solve Y(s), (s^2+4s+6)Y(s) = (2s+1)/[s(s+1)],

$$Y(s)=\frac{2s+1}{s(s+1)(s^2+4s+6)}=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2+4s+6};$$

$$A(s+1)(s^2+4s+6)+Bs(s^2+4s+6)+(Cs+D)s(s+1)$$

$$=2s+1,\ \Rightarrow\left\{A=\frac{1}{6},\ B=\frac{1}{3},\ C=-\frac{1}{2},\ D=-\frac{5}{3}\right\}$$

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Example 3 (Sec. 7.3) (2)

- Since $s^2 + 4s + 6 = 0$ has no real root, regrouping it as $(s - \alpha)^2 + \omega^2 = (s+2)^2 + (\sqrt{2})^2$,

$$F(s) \equiv \frac{Cs+D}{s^2+4s+6} = \frac{C(s+2)+(D-2C)}{(s+2)^2+(\sqrt{2})^2}$$

- Step 3: $y(t) = L^{-1}\{Y(s)\} = L^{-1}\{A/s\} + L^{-1}\{B/(s+1)\} + f(t)$,

$$f(t) = L^{-1}\left\{\frac{C(s+2)+(D-2C)}{(s+2)^2+(\sqrt{2})^2}\right\} = L^{-1}\left\{\frac{Cs+(D-2C)}{s^2+(\sqrt{2})^2}\right\}e^{-2t}$$

$$= \left[C \times L^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} + (D-2C) \times L^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\right\}\right]e^{-2t};$$

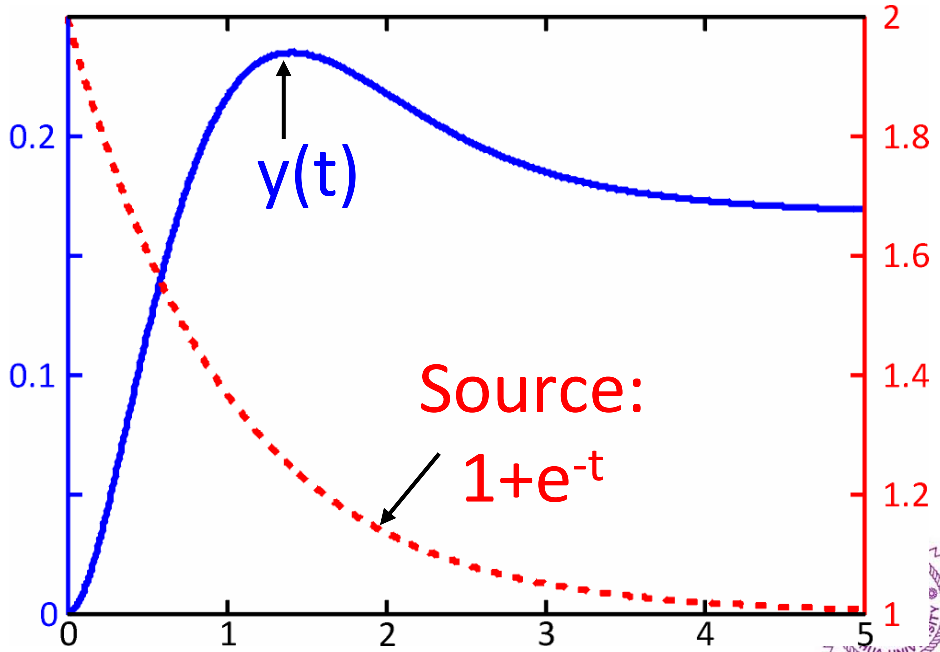
Example 3 (Sec. 7.3) (3)

$$f(t) = \left[C \times \cos(\sqrt{2}t) + (D - 2C) \times \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \right] e^{-2t} \Bigg|_{C=-\frac{1}{2}, D=-\frac{5}{3}}$$

$$= -\left[\frac{1}{2}\cos(\sqrt{2}t) + \frac{\sqrt{2}}{3}\sin(\sqrt{2}t) \right] e^{-2t}$$

$$\Rightarrow y(t) = \frac{1}{6} + \frac{1}{3}e^{-t}$$

$$-\left[\frac{1}{2}\cos(\sqrt{2}t) + \frac{\sqrt{2}}{3}\sin(\sqrt{2}t) \right] e^{-2t}$$



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L{f(t-a)×u(t-a)}

- By definition,

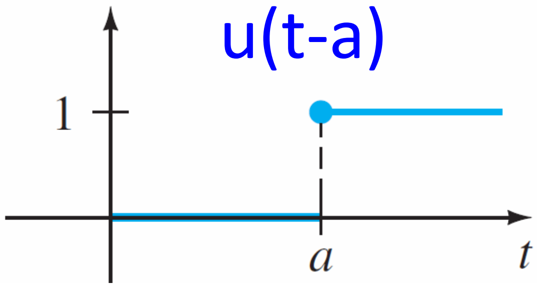
$$g(t) \equiv L\{f(t-a)\times u(t-a)\}$$

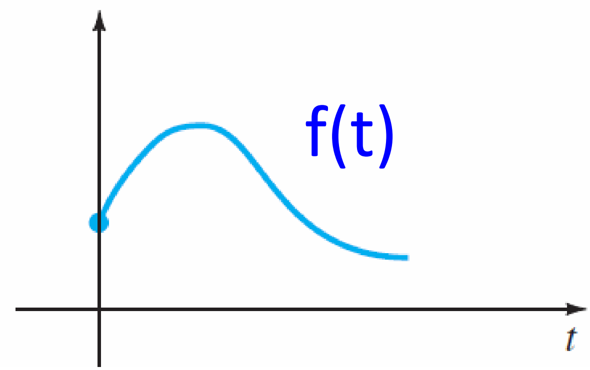
$$= \int_a^{\infty} f(t-a)\times e^{-st}dt;$$

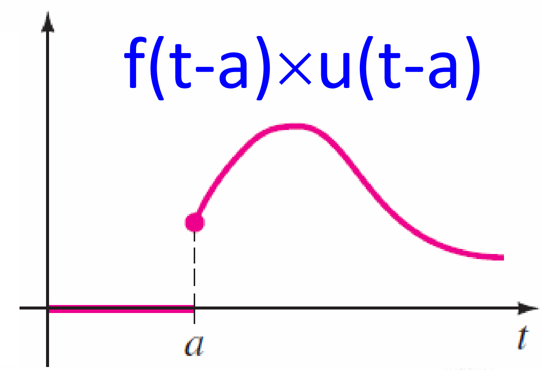
Let t-a = v, $\Rightarrow$ dt = dv;

$$g(t) = \int_0^{\infty} f(v)\times e^{-s(v+a)}dv$$

$$= e^{-sa}\left[\int_0^{\infty} f(v)\times e^{-sv}dv\right] = e^{-as}\times F(s).$$







- Shift in the t-domain, $\rightarrow$ modulation in the s-domain. Complementary relation of L{e^{at}×f(t)} = F(s-a):

L{f(t)×u(t-a)}

- By definition, $g(t) \equiv L\left\{ f(t) \times u(t-a) \right\} = \int_a^{\infty} f(t) \times e^{-st} dt;$

  Let t-a = v, $\Rightarrow$ dt = dv;

$$g(t) = \int_0^{\infty} f(v+a) \times e^{-s(v+a)} dv$$

$$= e^{-sa}\left[\int_0^{\infty} f(v+a) \times e^{-sv} dv \right] = e^{-as} \times L\left\{ f(t+a) \right\}.$$

- Activating f(t) at "t = a" is equivalent to activating f(t+a) at t = 0.

Example 8 (Sec. 7.3) (1)

- ODE: $y' + y = 3\cos(t)\times u(t-\pi)$; IC: $y(0) = 5$.
- Step 1: $L\{y' + y\} = [sY(s) - 5] + Y(s) = (s+1)Y(s) - 5 = L\{3\cos(t)\times u(t-\pi)\} = 3e^{-\pi s}\times L\{\cos(t+\pi) = -\cos(1t)\} = -3(e^{-\pi s})\times[s/(s^2+1)]$.
- Step 2: Solve $Y(s)$, $(s+1)Y(s) = 5 - 3e^{-\pi s}s/(s^2+1)$,

$$Y(s) = \frac{5}{s+1} - F(s)\times e^{-\pi s},\ \text{where}\ F(s) = \frac{3s}{(s+1)(s^2+1)}$$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+1},\ \Rightarrow A(s^2+1) + (Bs+C)(s+1) = 3s,$$

$$\Rightarrow \left\{A = -\frac{3}{2},\ B = \frac{3}{2},\ C = \frac{3}{2}\right\}$$

Example 8 (Sec. 7.3) (2)

$$\Rightarrow Y(s) = \frac{5}{s+1} + \frac{3}{2} e^{-\pi s} \left(\frac{1}{s+1} - \frac{1}{s^2+1} - \frac{s}{s^2+1} \right).$$

- Step 3: y(t) = L^{-1}{Y(s)}

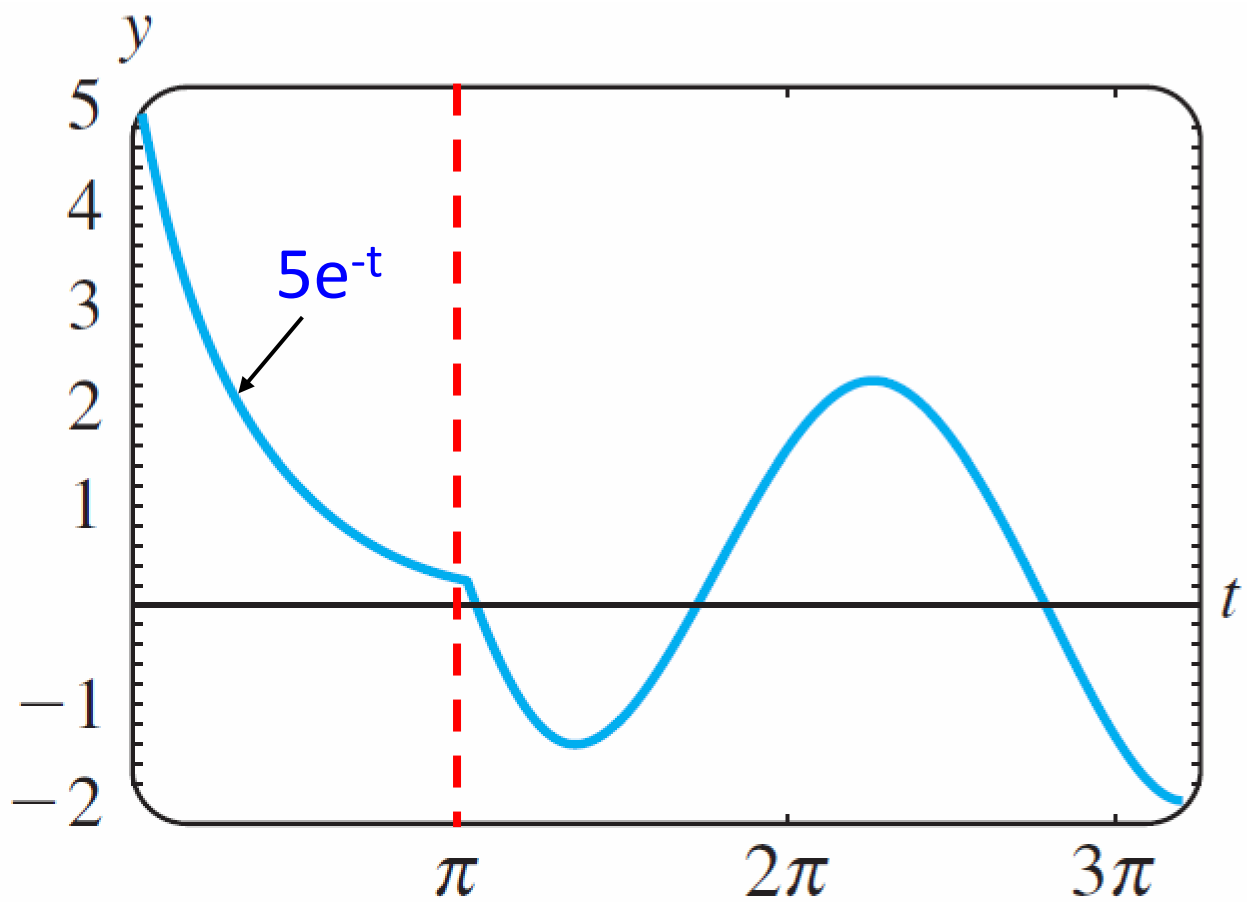
$$y(t) = L^{-1}\left\{ \frac{5}{s+1} \right\} + \frac{3}{2}\left[L^{-1}\left\{ \frac{1}{s+1} - \frac{1}{s^2+1} - \frac{s}{s^2+1} \right\} \right]_{t \to t-\pi}$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-t} - \sin(t) - \cos(t) \right]\Big|_{t \to t-\pi}$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} - \sin(t-\pi) - \cos(t-\pi) \right] \times u(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin(t) + \cos(t) \right] \times u(t-\pi)$$

Solution plot



$L^{-1}\{F'(s)\}$

- By definition, $\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^{\infty} f(t)\times e^{-st}dt;$
- If the order of differentiation and integration is interchangeable,

$$F'(s) = \int_0^{\infty} \frac{\partial}{\partial s}\left[f(t)\times e^{-st}\right]dt = \int_0^{\infty} f(t)\times\left(\frac{\partial}{\partial s}e^{-st}\right)dt$$

$$= \int_0^{\infty} f(t)\times\left(-te^{-st}\right)dt = \int_0^{\infty} -tf(t)\times e^{-st}dt = L^{-1}\{-tf(t)\}$$

- $L^{-1}\{F'\} = -t\times f(t)$: derivative in the s-domain is equivalent to multiplication in the time domain.

$L^{-1}\{F''(s)\}$

- Without using the inverse LT formula, let's guess $L^{-1}\{F''(s)\} = t^2 f(t)$ given $L^{-1}\{F'(s)\} = -t\times f(t)$ is true.
- Proof:

$$L\{t^2 f(t)\} = \int_0^\infty t^2 f(t)\times e^{-st}dt = \int_0^\infty t\times g(t)\times e^{-st}dt$$

$$= L\{t\times g(t)\} = -G'(s), \text{ where } g(t) \equiv t\times f(t).$$

Since $G(s) = -F'(s)$, $\Rightarrow L\{t^2 f(t)\} = F''(s)$.

- $L^{-1}\{F''\} = t^2 f(t)$: derivative in the s-domain is equivalent to multiplication in the time domain.

Example: $L\{t^2\}$

- Previously, we got $L\{t^2\} = 2/s^3$ by definition (p10). Here is a faster way:
- Let $f(t) = 1$, $\Rightarrow F(s) = 1/s$.
- $g(t) \equiv t^2 = t^2 \times f(t)$, $\Rightarrow G(s) = F''(s) = (s^{-1})'' = -(s^{-2}) = 2s^{-3}$.
- For $h(t) \equiv t^n = t^n \times f(t)$, $\Rightarrow$

$$H(s) = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n \frac{d^n}{ds^n} s^{-1}$$

$$= (-1)^{n+1} \frac{d^{n-1}}{ds^{n-1}} s^{-2} = (-1)^{n+2} (2) \frac{d^{n-2}}{ds^{n-2}} s^{-3} = \ldots$$

$$= (-1)^{n+n} n! s^{-(n+1)} = n! s^{-(n+1)}.$$

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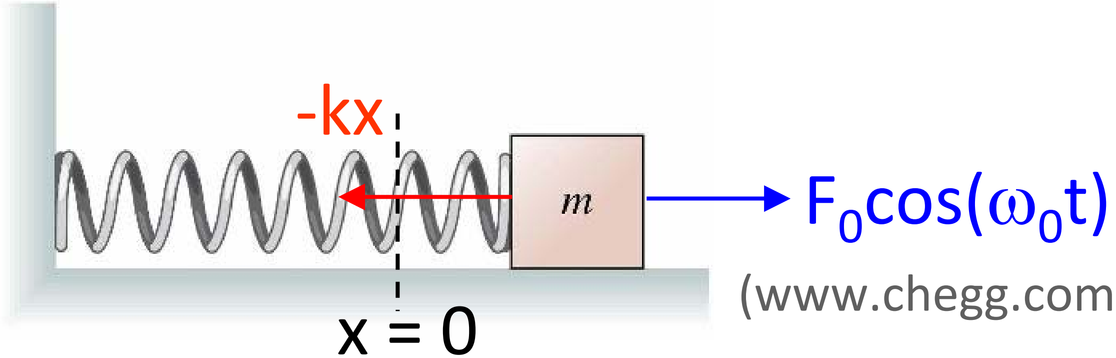
Resonantly driven mass-spring (1)

- ODE: $x''(t) + (\omega_0)^2 x(t) = A\times\cos(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$, $A = F_0/m$.
- ICs: $x(0) = X_0$, $x'(0) = V_0$.



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- Step 1: L{ODE}

$$L\{x'' + \omega_0^2 x\} = (s^2 X - X_0 s - V_0) + \omega_0^2 X$$

$$= (s^2 + \omega_0^2)X - (X_0 s + V_0);$$

$$L\{A\times\cos\omega_0 t\} = \frac{As}{s^2 + \omega_0^2}$$

Resonantly driven mass-spring (2)

- Step 2: Solve X(s)

$$(s^2+\omega_0^2)X-(X_0s+V_0)=\frac{As}{s^2+\omega_0^2}$$

$$\Rightarrow X(s)=\frac{X_0s+V_0}{s^2+\omega_0^2}+\frac{As}{(s^2+\omega_0^2)^2}\equiv X_1(s)+X_2(s)$$

- Step 3: $x(t) = L^{-1}\{X(s)\}$

$$x_1(t)=L^{-1}\left\{\frac{X_0s+V_0}{s^2+\omega_0^2}\right\}=X_0\cos\omega_0 t+\frac{V_0}{\omega_0}\sin\omega_0 t$$

- Initial position X_0 and initial velocity V_0 contribute to cos(ω_0t) and sin(ω_0t) terms, respectively.

Resonantly driven mass-spring (3)

- To calculate $L^{-1}\{X_2(s)\}$, let's look for $G(s) \equiv \int X_2(s)ds$ such that $x_2(t) = -t\times g(t)$.

$$G(s) = \int \frac{As}{(s^2+\omega_0^2)^2} ds = -\frac{A}{2}\frac{1}{s^2+\omega_0^2};$$

$$\Rightarrow g(t) = -\frac{A}{2\omega_0} L^{-1}\left\{\frac{\omega_0}{s^2+\omega_0^2}\right\} = -\frac{A}{2\omega_0}\sin\omega_0 t;$$

$$\Rightarrow x_2(t) = \frac{A}{2\omega_0} t \sin\omega_0 t;$$

- Resonant driving force causes linearly growing oscillation, unavailable by using Ch4's approach.

Example 2 (Sec. 7.4)

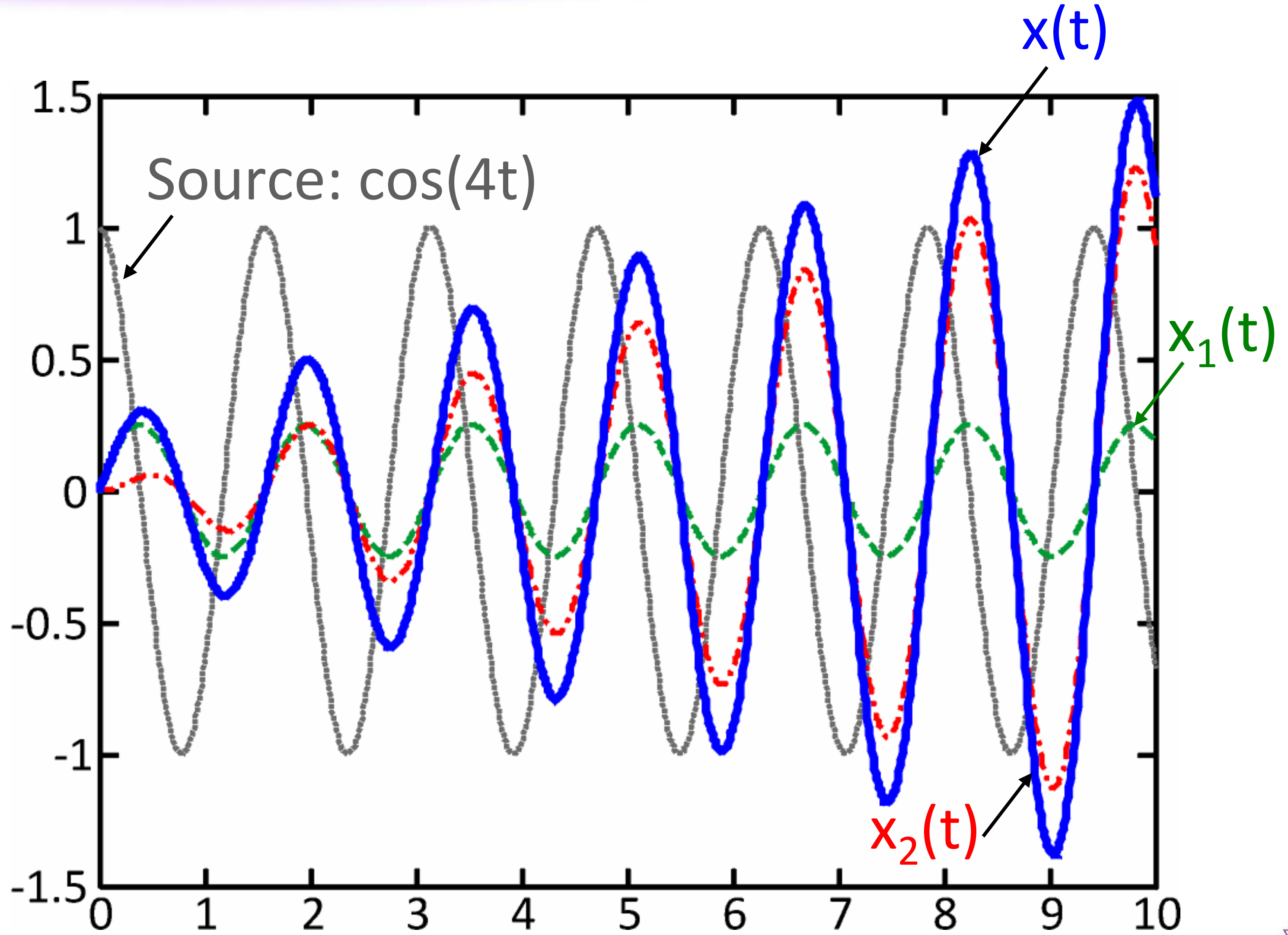
- ODE: $x''(t) + 16x(t) = \cos(4t)$; $\Rightarrow$ $\omega_0 = 4$, $A = 1$.
- ICs: $x(0) = X_0 = 0$, $x'(0) = V_0 = 1$.
- By the general analysis, $x(t) = x_1(t) + x_2(t)$

$$= X_0 \cos\omega_0 t + \frac{V_0}{\omega_0}\sin\omega_0 t + \frac{A}{2\omega_0} t\sin\omega_0 t$$

$$= 0\times\cos 4t + \frac{1}{4}\sin 4t + \frac{1}{2\times 4} t\sin 4t$$

$$= \frac{1}{4}\sin 4t + \frac{1}{8} t\sin 4t$$

Solution plot

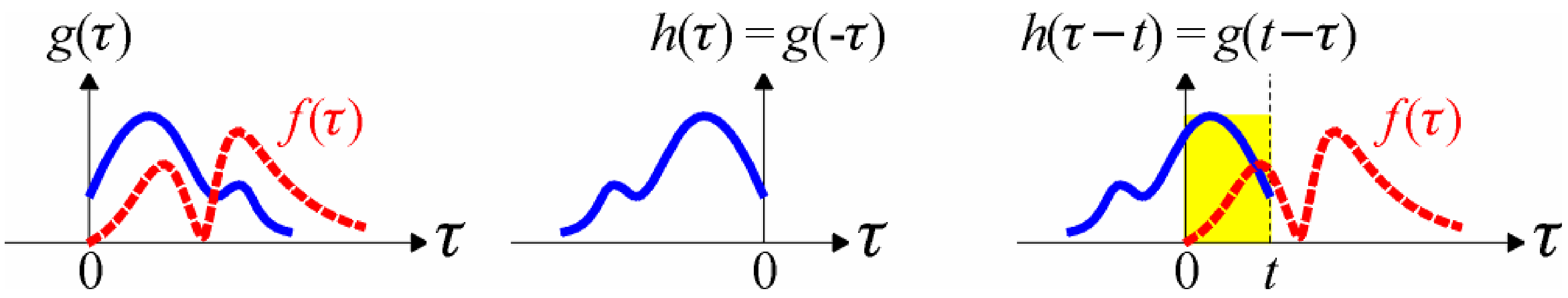


Convolution f*g

- If f(τ) and g(τ) are piecewise continuous over τ = [0,∞), the convolution of f and g is defined as

$$f*g=\int_0^t f(\tau)\times g(t-\tau)d\tau=\int_0^t f(t-\tau)\times g(\tau)d\tau$$

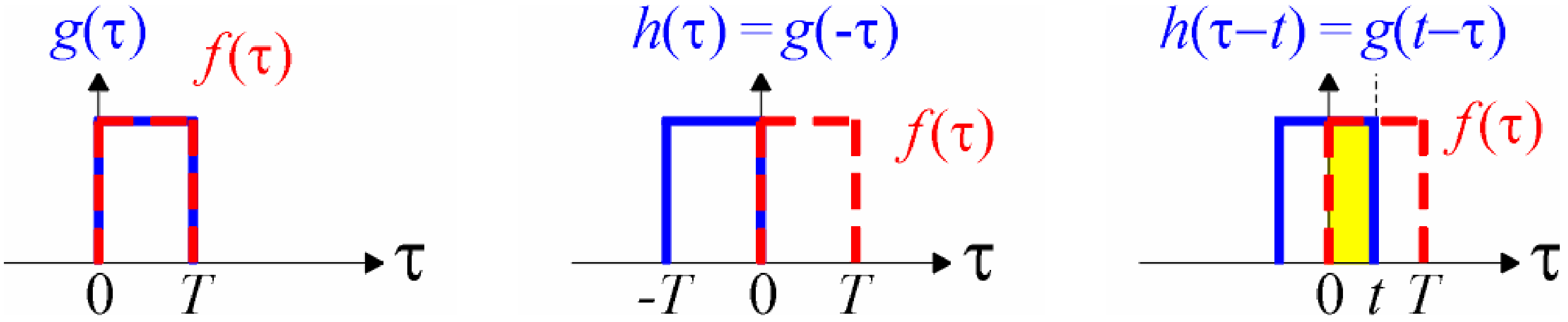
- Why is the integral range τ = (0,t)?



Example: ∏ * ∏

- Let ∏ (t) = {1, for 0 < t < T; 0, otherwise} is a rectangular function of width T.

$$\Pi * \Pi = \int_0^t \Pi(\tau) \times \Pi(t-\tau) d\tau$$



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L{f*g} = F×G (1)

- Proof:

$$F(s) \equiv L\{f(t)\} = \int_0^\infty f(\tau) \times e^{-s\tau} d\tau,$$

$$G(s) \equiv L\{g(t)\} = \int_0^\infty g(\beta) \times e^{-s\beta} d\beta.$$

$$\Rightarrow F(s) \times G(s) = \left[\int_0^\infty f(\tau) \times e^{-s\tau} d\tau\right] \times \left[\int_0^\infty g(\beta) \times e^{-s\beta} d\beta\right]$$

$$= \int_0^\infty \int_0^\infty f(\tau) g(\beta) \times e^{-s(\tau+\beta)} d\tau d\beta$$

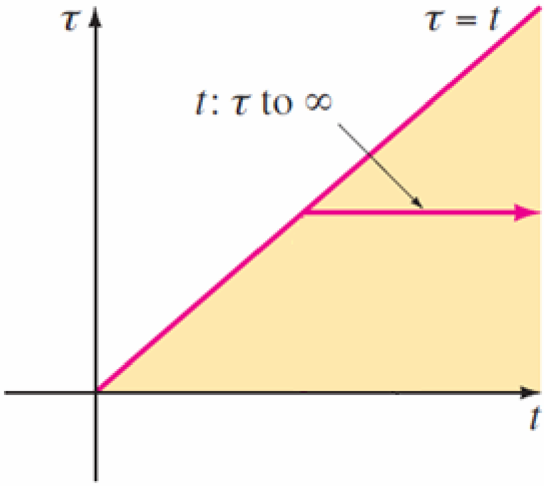
$$= \int_0^\infty f(\tau) \times \left[\int_0^\infty g(\beta) \times e^{-s(\tau+\beta)} d\beta\right] d\tau$$

L{f*g} = F×G (2)

- Let t = τ + β, ⇒ dβ = dt,

$$F(s)\times G(s)$$

$$=\int_0^\infty f(\tau)\times\left[\int_\tau^\infty g(t-\tau)\times e^{-st}dt\right]d\tau$$

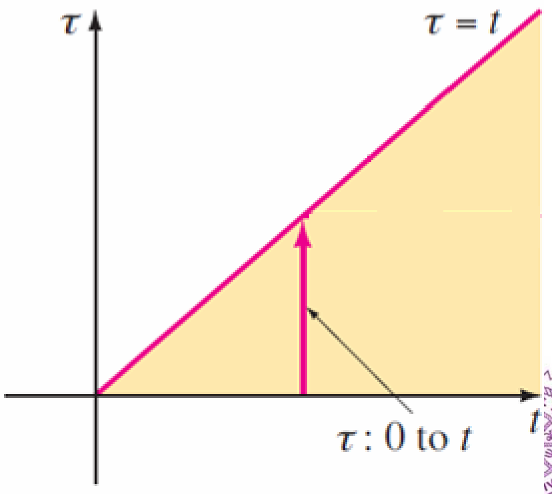


- Interchanging the integration order:

$$F(s)\times G(s)$$

$$=\int_0^\infty\left[\int_0^t f(\tau)\times g(t-\tau)d\tau\right]\times e^{-st}dt$$

$$=\int_0^\infty (f*g)\times e^{-st}dt = L\{f*g\}.$$



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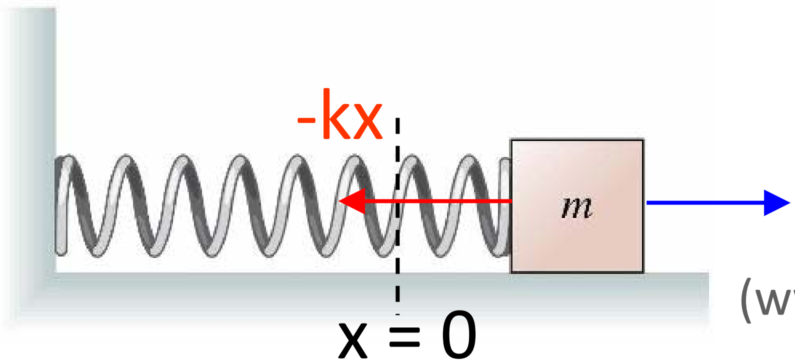
Revisit: Resonantly driven mass-spring (1)

- ODE: $x''(t) + (\omega_0)^2 x(t) = A\times\sin(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$, $A = F_0/m$.
- ICs: $x(0) = X_0$, $x'(0) = V_0$.



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- Step 1: L{ODE}

$$L\{x'' + \omega_0^2 x\} = (s^2 + \omega_0^2)X - (X_0 s + V_0);$$ ... same

$$L\{A\times\sin\omega_0 t\} = \frac{A\omega_0}{s^2+\omega_0^2}$$ v.s. $$L\{A\times\cos\omega_0 t\} = \frac{As}{s^2+\omega_0^2}$$

Revisit: Resonantly driven mass-spring (2)

- Step 2: Solve X(s)

$$(s^2+\omega_0^2)X-(X_0 s+V_0)=\frac{A\omega_0}{s^2+\omega_0^2}$$

$$\Rightarrow X(s)=\frac{X_0 s+V_0}{s^2+\omega_0^2}+\frac{A\omega_0}{(s^2+\omega_0^2)^2}\equiv X_1(s)+X_2(s)$$

- Step 3: $x(t) = L^{-1}\{X(s)\} = x_1(t) + x_2(t)$, where $x_1(t) = X_0\cos(\omega_0 t) + (V_0/\omega_0)\sin(\omega_0 t)$ is the same as before.
- However, no easy anti-derivative $G(s) \equiv \int X_2(s)ds$ exists, and $x_2(t) = -t\times g(t)$ does not apply.

Revisit: Resonantly driven mass-spring (3)

- Instead, $X_2(s) = \dfrac{A\omega_0}{(s^2+\omega_0^2)^2} \equiv A\omega_0 \times F(s) \times G(s),$

  where F(s) = G(s) = 1/[s^2 + $(\omega_0)^2$],

  $\Rightarrow$ f(t) = g(t) = sin(ω_0t)/ω_0; x_2(t) = Aω_0(f*g) = (A/ω_0)I,

  $$I \equiv \sin(\omega_0 t) * \sin(\omega_0 t) = \int_0^t \sin(\omega_0 \tau) \times \sin[\omega_0 (t-\tau)] d\tau;$$

- By the "product to sum" formula:

  $$\sin\alpha \times \sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2};$$

  the integrand is $\dfrac{\cos(2\omega_0\tau - \omega_0 t) - \cos(\omega_0 t)}{2};$

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Revisit: Resonantly driven mass-spring (4)

- $$I = \frac{1}{2}\int_0^t \left[\cos(2\omega_0 \tau - \omega_0 t) - \cos(\omega_0 t)\right] d\tau$$

variable

$$= \frac{1}{2}\left\{\left[\left.\frac{\sin(2\omega_0 \tau - \omega_0 t)}{2\omega_0}\right|_0^t\right] - \cos(\omega_0 t) \times \left(\left.\tau\right|_0^t\right)\right\}$$

$$= \frac{1}{2}\left[\frac{\sin(\omega_0 t) + \sin(\omega_0 t)}{2\omega_0} - t \times \cos(\omega_0 t)\right];$$

$$\Rightarrow x_2(t) = \frac{A}{2\omega_0}\left[\frac{\sin(\omega_0 t)}{\omega_0} - t \times \cos(\omega_0 t)\right]$$ v.s. $$\frac{A}{2\omega_0} t \sin \omega_0 t$$

L{ ∫f }

- Let g(t) = 1, $\Rightarrow$ G(s) = 1/s;

$$\Rightarrow f * g = \int_0^t [f(\tau) \times g(t-\tau)] d\tau = \int_0^t f(\tau) d\tau .$$

- By L{f*g} = F×G, $\Rightarrow L\left\{\int_0^t f(\tau) d\tau\right\} = \dfrac{F(s)}{s}$.

- L{∫f} = F(s)/s: integral in the time domain is equivalent to division in the s-domain.

- It's the complementary relation of L{f'} = sF(s) – f(0).

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Example: $L^{-1}\{F(s)/s\}$

- $G(s) \equiv \dfrac{1}{s(s^2+1)}, \quad \Rightarrow g(t) \equiv L^{-1}\{G(s)\} = ?$
- Let F(s) ≡ $1/(s^2+1)$, ⇒ f(t) = sin(t);
- G(s) = F(s)/s,

$$\Rightarrow g(t) = \int_0^t f(\tau)d\tau = \int_0^t \sin(\tau)d\tau = -\cos(\tau)\Big|_0^t = 1-\cos(t).$$

- Check: $G(s) = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1} = \dfrac{(A+B)s^2 + Cs + A = 1}{s(s^2+1)};$

$$\Rightarrow \{A=1,\ B=-1,\ C=0\};$$

$$\Rightarrow g(t) = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{s}{s^2+1}\right\} = 1-\cos(t).$$ (same)

Example: $L^{-1}\{F(s)/s^2\}$

- $H(s) \equiv \dfrac{1}{s^2(s^2+1)}, \quad \Rightarrow h(t) \equiv L^{-1}\{H(s)\} = ?$
- Let $G(s) \equiv 1/[s(s^2+1)]$, $\Rightarrow$ $g(t) = 1-\cos(t)$;
- $H(s) = G(s)/s$,

$$\Rightarrow h(t) = \int_0^t g(\tau)d\tau = \int_0^t [1-\cos(\tau)]d\tau$$
$$= t - \left[\sin(\tau)\Big|_0^t\right] = t - \sin(t).$$

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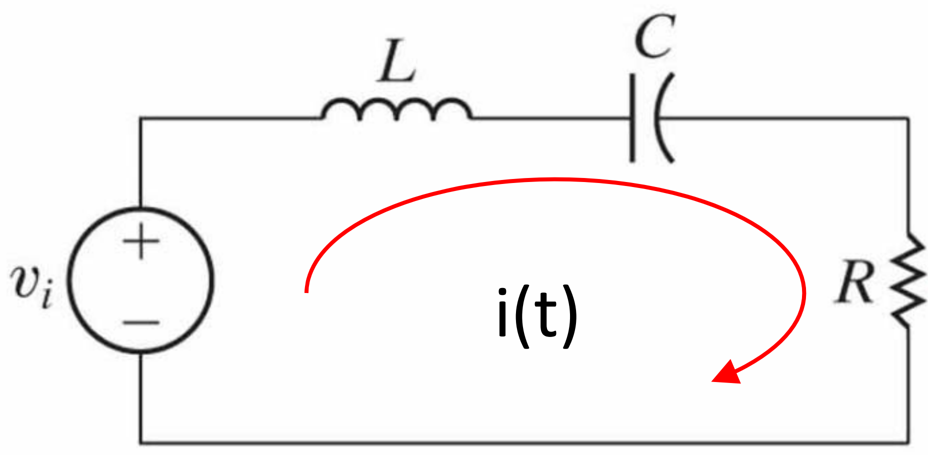
Integrodifferntial equations

- E.g. Series RLC circuit.



- By Kirchhoff voltage law, $v_L(t) = L \times i'(t)$, $i(t) = C \times v_c'(t)$:

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v_i(t)$$

Example 6 (1)

- Let $v_i(t) = V_0t[1-u(t-1)]$ (curve?), $i(0) = I_0$.
- Step 1: $L\left\{Li'(t)+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau\right\}$

$$= L[sI(s)-I_0]+R\times I(s)+\frac{1}{C}\times\frac{I(s)}{s};$$

By $L\{f(t)\times u(t-a)\}=e^{-as}\times L\{f(t+a)\times u(t)\}$,

$$L\{v_i(t)\}=V_0\big[L\{t\}-L\{t\times u(t-1)\}\big]$$

$$=V_0\left[\frac{1}{s^2}-e^{-s}\times L\{(t+1)\times u(t)\}\right]=V_0\left(\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}\right);$$

Example 6 (2)

- Step 2: Solve I(s)

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) - LI_0 = V_0\left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right)$$

$$\Rightarrow L\left[\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)I(s) - I_0 s\right] = V_0\left(\frac{1}{s} - \frac{e^{-s}}{s} - e^{-s}\right)$$

$$\Rightarrow I(s) = I_0\frac{s}{D(s)} + I_m\frac{\left(s^{-1} - s^{-1}e^{-s} - e^{-s}\right)}{D(s)},$$

$$\text{where } I_m = \frac{V_0}{L},\ D(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}.$$

Example 6 (3)

- Step 3: $L^{-1}\{I(s)\}$.
- Let L = 0.1 H, C = 0.1 F, R = 2 Ω, V_0 = 120, I_0 = 0, ⇒ I_m = 1200 A, D(s) = s² + 20s + 100 = (s+10)² … Case 2.

$$I(s) = \frac{\cancel{I_0} s}{(s+10)^2} + I_m \frac{\left(s^{-1} - s^{-1}e^{-s} - e^{-s}\right)}{(s+10)^2} = I_m\left(I_1 - I_2 - I_3\right),$$

Partial fractions

$$i_1(t) = L^{-1}\left\{\frac{1}{s(s+10)^2}\right\} = L^{-1}\left\{\frac{0.01}{s} - \frac{0.01}{s+10} - \frac{0.1}{(s+10)^2}\right\}$$

$$= \left[0.01 - 0.01e^{-10t} - 0.1te^{-10t}\right] \times u(t);$$

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Example 6 (4)

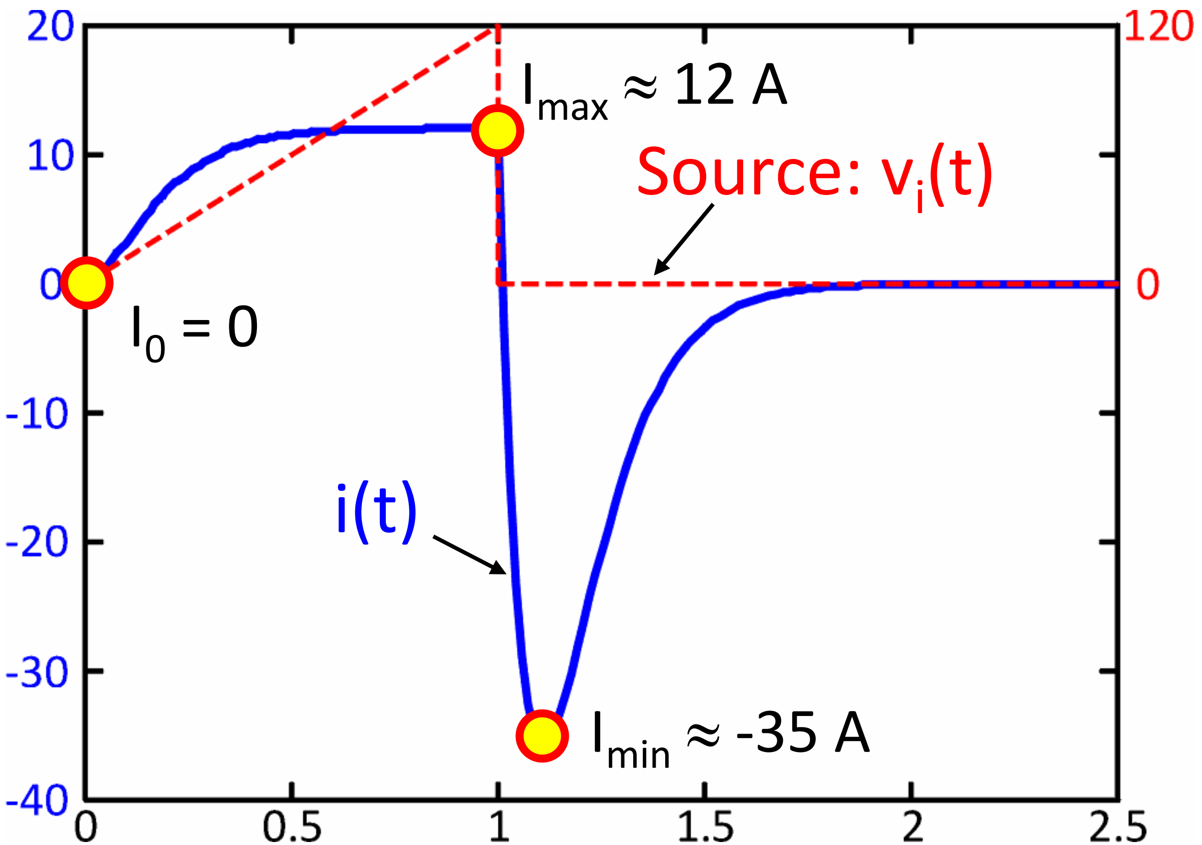
$$i_2(t) = L^{-1}\left\{\frac{e^{-s}}{s(s+10)^2}\right\} = L^{-1}\left\{e^{-s} \times I_1(s)\right\} = i_1(t-1);$$

$$i_3(t) = L^{-1}\left\{\frac{e^{-s}}{(s+10)^2}\right\} = L^{-1}\left\{\frac{1}{(s+10)^2}\right\}\Bigg|_{t\to t-1} = (t-1)e^{-10(t-1)};$$

$$\begin{aligned} i(t) &= I_m\left\{i_1(t) - \left[i_2(t) + i_3(t)\right]\right\} \\ &= 1200\left\{\left(0.01 - 0.01e^{-10t} - 0.1te^{-10t}\right)\times u(t) \right. \\ &\quad \left. - \left[0.01 - 0.01e^{-10(t-1)} + 0.9(t-1)e^{-10(t-1)}\right]\times u(t-1)\right\}, \end{aligned}$$

Solution plot

$$i(t)=\begin{cases}12\left[1-e^{-10t}-10te^{-10t}\right], \text{ if } 0<t<1.\\ 12\left[-e^{-10t}+e^{-10(t-1)}-10te^{-10t}-90(t-1)e^{-10(t-1)}\right], \text{ if } t>1.\end{cases}$$



L{f(t)}, with f(t+T) = f(t)

- $F(s) = L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t) \times e^{-st} dt.$
- Proof:

$$F(s) = \boxed{\int_0^T f(t) \times e^{-st} dt} + \int_T^\infty f(t) \times e^{-st} dt \equiv I_1 + I_2.$$

Let u = t-T, ⇒ dt = du: $I_2 = \int_0^\infty \underline{f(u+T)} \times e^{-s(u+T)} du$

$$= e^{-sT} \left[\int_0^\infty \underline{f(u)} \times e^{-su} du \right] = e^{-sT} \times F(s) = e^{-sT} (I_1 + I_2);$$

$$\Rightarrow (1-e^{-sT}) I_2 = e^{-sT} I_1, \quad I_2 = \frac{e^{-sT}}{1-e^{-sT}} I_1;$$

$$\Rightarrow F(s) = I_1 + \frac{e^{-sT}}{1-e^{-sT}} I_1 = \frac{1}{1-e^{-sT}} I_1.$$

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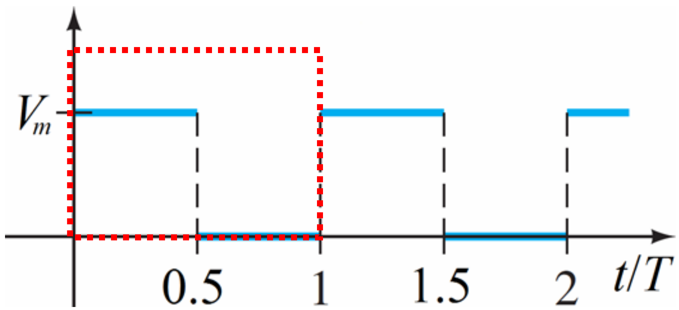
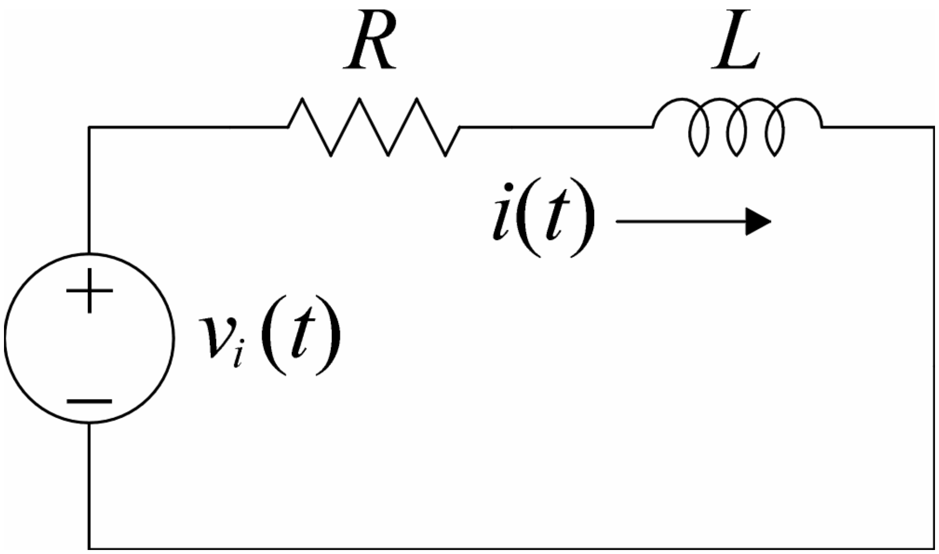
Example 8 (1)

- An RL circuit driven by a periodic square voltage $v_i(t)$:

$$\begin{cases} Li'(t) + Ri(t) = v_i(t); \\ i(0) = I_0 \end{cases}$$





$$V_i(s) = \frac{1}{1-e^{-Ts}} \int_0^{T/2} V_m \times e^{-st} dt = \frac{V_m}{1-e^{-Ts}} \times \frac{1-e^{-Ts/2}}{s} = \frac{V_m}{s(1+e^{-Ts/2})}.$$

Example 8 (2)

- Step 1: $\mathcal{L}\{Li'(t)+Ri(t)\}=\mathcal{L}[sI(s)-I_0]+RI(s)$

$$=(Ls+R)I(s)-LI_0=\mathcal{L}\{v_i(t)\}=V_i(s)=\frac{V_m}{s(1+e^{-Ts/2})}.$$

- Step 2: $I(s)=\dfrac{I_0}{s+R/L}+\dfrac{V_m/L}{s(s+R/L)(1+e^{-Ts/2})}\equiv I_1+I_2.$

- Step 3: $i_1(t)=\mathcal{L}^{-1}\left\{\dfrac{I_0}{s+R/L}\right\}=I_0e^{-t/\tau}$, where $\boxed{\tau=\dfrac{L}{R}}.$

$$I_2(s)=\frac{V_m}{L}\times\frac{1}{s(s+\tau^{-1})}\times\frac{1}{1+e^{-Ts/2}}\equiv\frac{V_m}{L}\times G(s)\times H(s),$$

Example 8 (3)

$$G(s)=\frac{1}{s(s+\tau^{-1})}=\frac{\tau}{s}-\frac{\tau}{s+\tau^{-1}},$$

$$H(s)=\frac{1}{1+e^{-Ts/2}}=1-e^{-\frac{T}{2}s}+e^{-Ts}-\ldots,$$

$$I_2(s)=\frac{V_m}{L}\times\tau\left(\frac{1}{s}-\frac{1}{s+\tau^{-1}}\right)\times(1-e^{-\frac{T}{2}s}+e^{-Ts}-\ldots)$$

$$=I_m\left[(\frac{1}{s}-\frac{e^{-\frac{T}{2}s}}{s}+\frac{e^{-Ts}}{s}-\ldots)-(\frac{1}{s+\tau^{-1}}-\frac{e^{-\frac{T}{2}s}}{s+\tau^{-1}}+\frac{e^{-Ts}}{s+\tau^{-1}}-\ldots)\right],\ I_m=\frac{V_m}{R}.$$

$$i_2(t)=I_m\{[u(t)-u(t-T/2)+u(t-T)-\ldots]$$
$$-[e^{-t/\tau}u(t)-e^{-(t-1)/\tau}u(t-T/2)+e^{-(t-2)/\tau}u(t-T)-\ldots]\}$$

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Example 8 (4)

$$i_2(t) = I_m\{(1-e^{-t/\tau})\times u(t) - [1-e^{-(t-T/2)/\tau}]\times u(t-T/2)$$
$$+[1-e^{-(t-T)/\tau}]\times u(t-T) - \ldots\}$$

$$\Rightarrow i_2(t) = I_m\sum_{q=0}^{n}(-1)^q[1-e^{-(t-qT/2)/\tau}], \text{ if } \frac{n}{2}T < t < \frac{n+1}{2}T.$$

- For t < T/2 (n = 0): $i_2(t) = I_m(1-e^{-t/\tau})$;

$$\Rightarrow i(t) = i_1(t) + i_2(t) = I_0e^{-t/\tau} + I_m(1-e^{-t/\tau}).$$

- For T/2 < t < T (n = 1): $i_2(t) = I_m\{(\cancel{1}-e^{-t/\tau}) - [\cancel{1}-e^{-(t-T/2)/\tau}]\}$

$$= I_m[-e^{-t/\tau} + e^{-(t-T/2)/\tau}];$$

$$\Rightarrow i(t) = I_0e^{-t/\tau} + I_m[-e^{-t/\tau} + e^{-(t-T/2)/\tau}].$$

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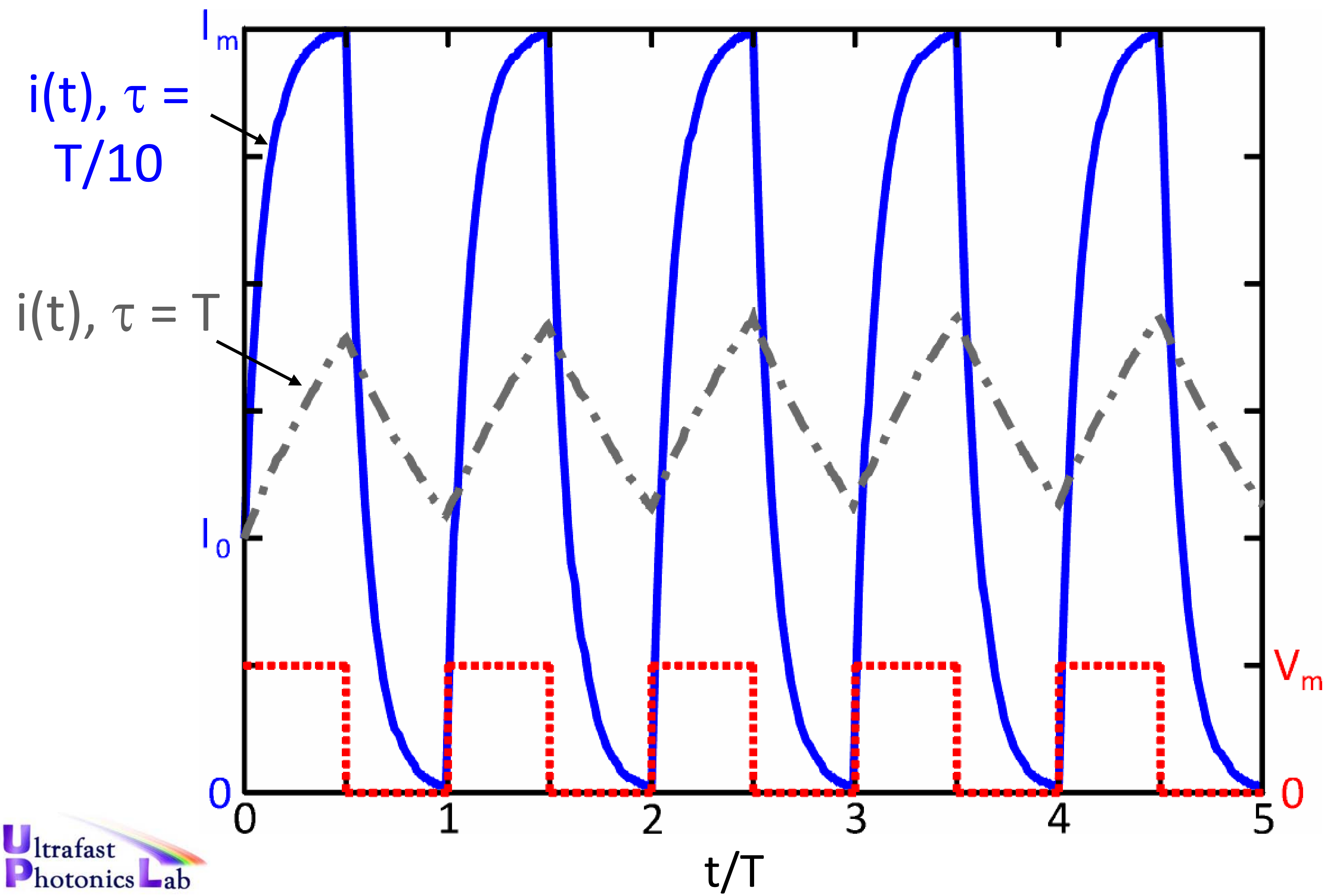
Solution plot

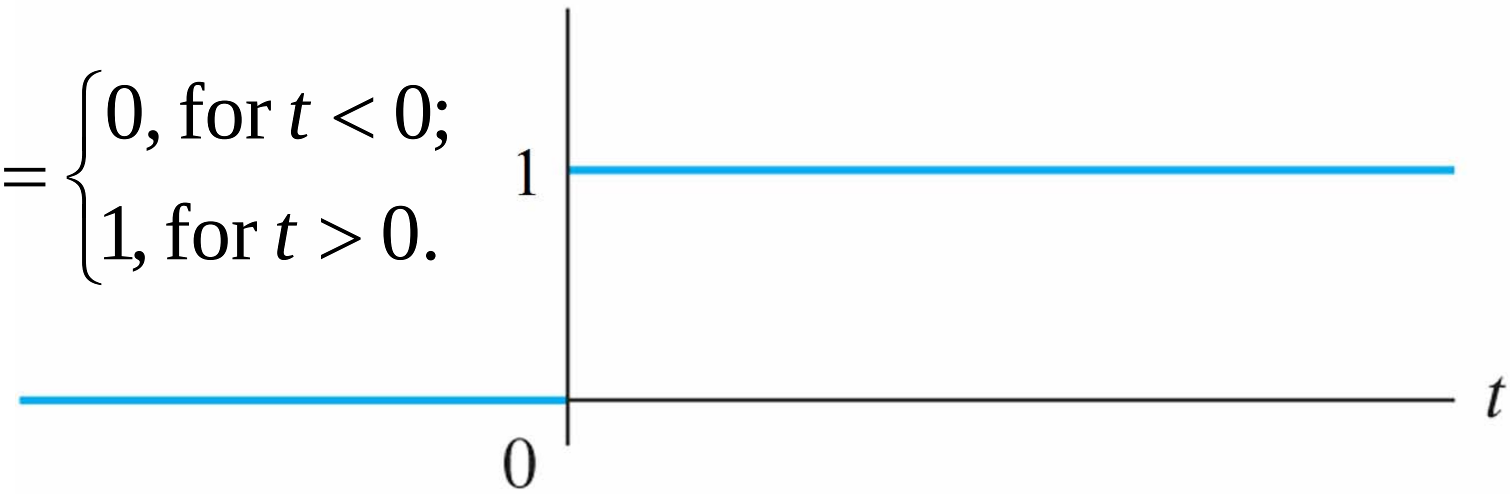


❑ Dirac delta function $\delta(t)$

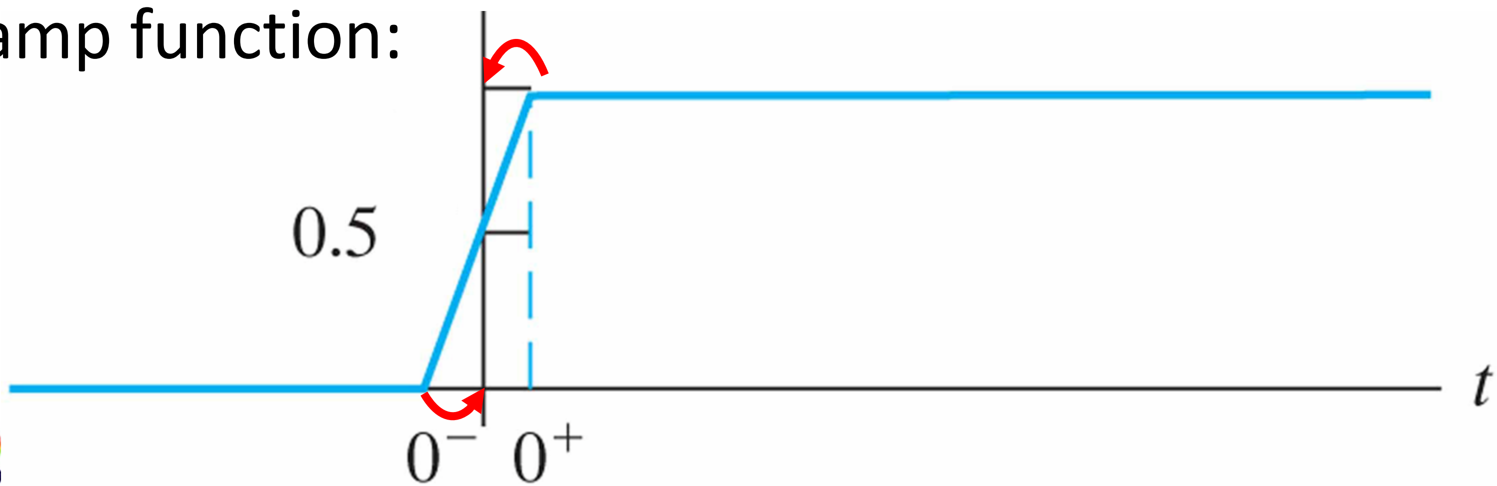
- Definition of $\delta(t)$
- $\delta'(t)$
- Impulse response $h(t)$

Revisit the unit-step function u(t)

$$u(t) = \begin{cases} 0, \text{ for } t < 0; \\ 1, \text{ for } t > 0. \end{cases}$$



- u(t) can be approximated by the limit of a linear ramp function:



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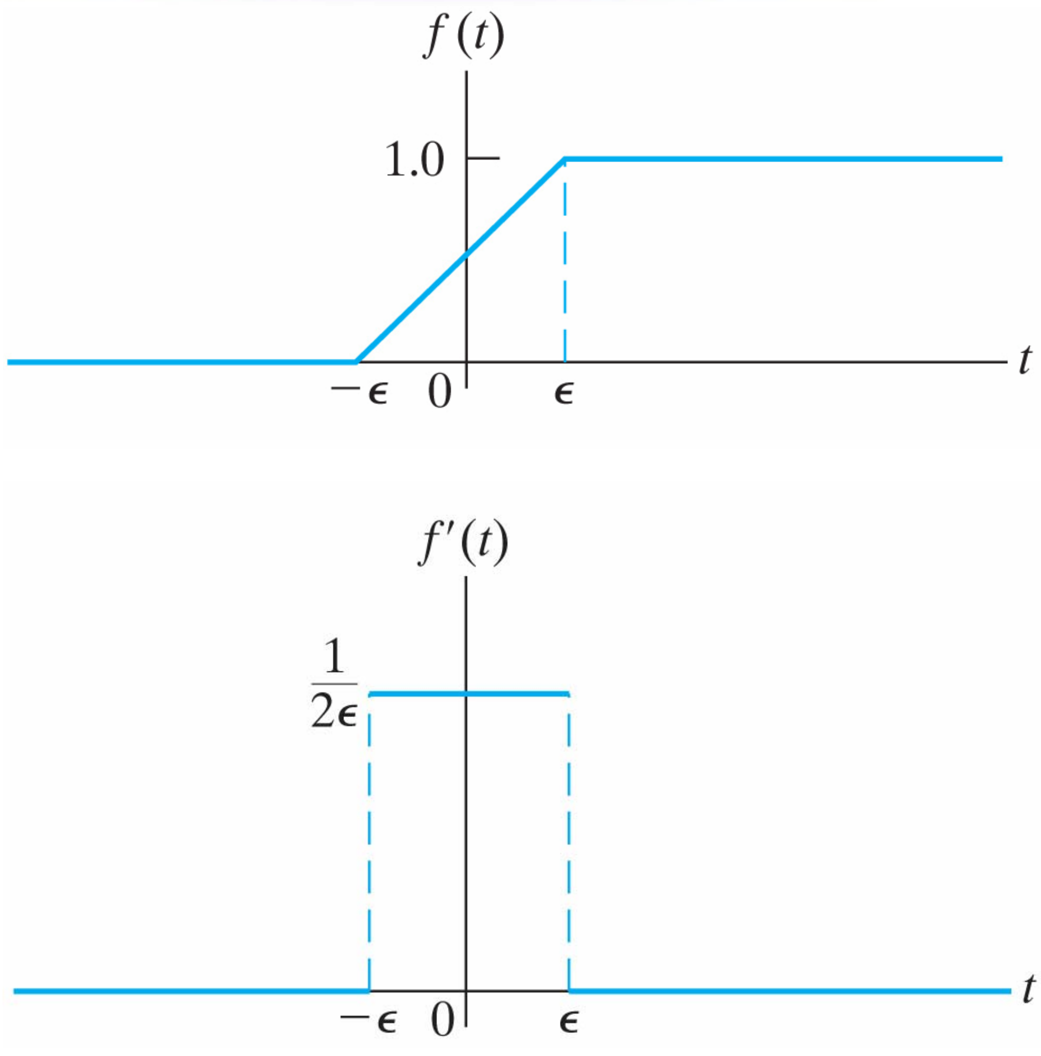
Impulse function δ(t)



$$u(t) = \lim_{\varepsilon \to 0} f(t)$$

$$\delta(t) = \lim_{\varepsilon \to 0} f'(t) = u'(t)$$

$$\Rightarrow \begin{cases} \delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & \text{otherwise;} \end{cases} \\ \int_{-\infty}^{\infty} \delta(t)dt = 1. \end{cases}$$

L{δ(t)}

- Sampling of f(t) at t = a can be formulated by integral of the product of f(t) and δ(t-a):

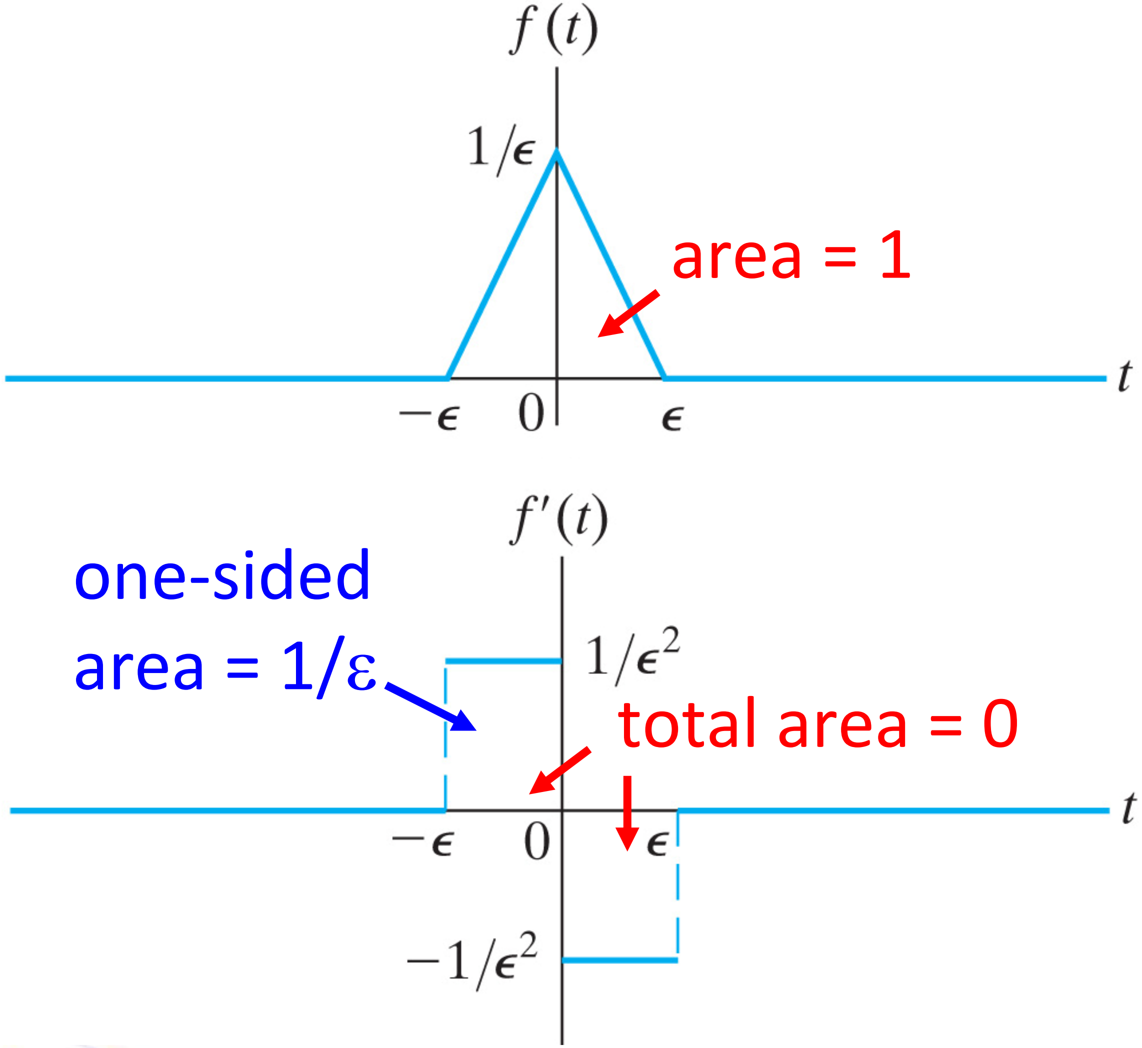
$$\int_0^\infty f(t)\times\delta(t-a)dt=\lim_{\varepsilon\to 0}\int_{a-\varepsilon}^{a+\varepsilon} f(t)\times\delta(t-a)dt$$

$$= f(a)\left[\lim_{\varepsilon\to 0}\int_{a-\varepsilon}^{a+\varepsilon}\delta(t-a)dt\right]= f(a)\times 1= f(a).$$

- It can be used in deriving L{δ(t)} = 1:

$$L\{\delta(t)\}=\int_0^\infty \delta(t)\times e^{-st}dt=e^{-s(0)}=1,\ \text{for any } s\in C.$$

δ'(t) = ?



$$\delta(t) = \lim_{\varepsilon \to 0} f(t)$$

$$\delta'(t) = \lim_{\varepsilon \to 0} f'(t)$$

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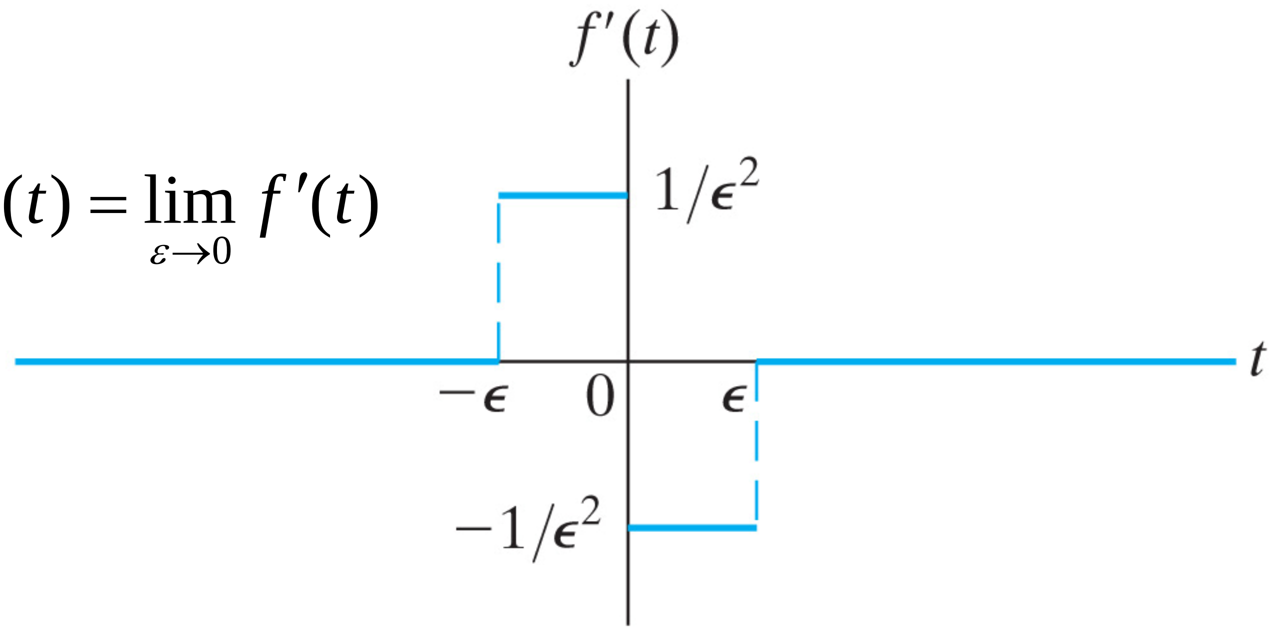
L{δ'(t)}

$$\delta'(t) = \lim_{\varepsilon \to 0} f'(t)$$



$$L\{\delta'(t)\} = \lim_{\varepsilon \to 0} \int_{0^-}^{\infty} f'(t) e^{-st} dt = \lim_{\varepsilon \to 0} \left[\int_{-\varepsilon}^{0^-} \frac{1}{\varepsilon^2} e^{-st} dt + \int_{0^+}^{\varepsilon} \left(\frac{-1}{\varepsilon^2} \right) e^{-st} dt \right]$$

$$= \lim_{\varepsilon \to 0} \frac{e^{s\varepsilon} + e^{-s\varepsilon} - 2}{s\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{(e^{s\varepsilon} - e^{-s\varepsilon})s}{2s\varepsilon} = \lim_{\varepsilon \to 0} \frac{(e^{s\varepsilon} + e^{-s\varepsilon})s^2}{2s} = s.$$

- Even δ'(t) has well defined Laplace transform!

L^{-1}\{improper rational function\}

$$F(s)=\frac{s^4+13s^3+66s^2+200s+300}{s^2+4s+20}$$

$$=\left(s^2+4s+10\right)-\frac{20}{s+4}+\frac{50}{s+5}.$$

$$f(t)=\delta''(t)+4\delta'(t)+10\delta(t)$$

$$+\left[-20e^{-4t}+50e^{-5t}\right]u(t).$$

Example 1 (1)

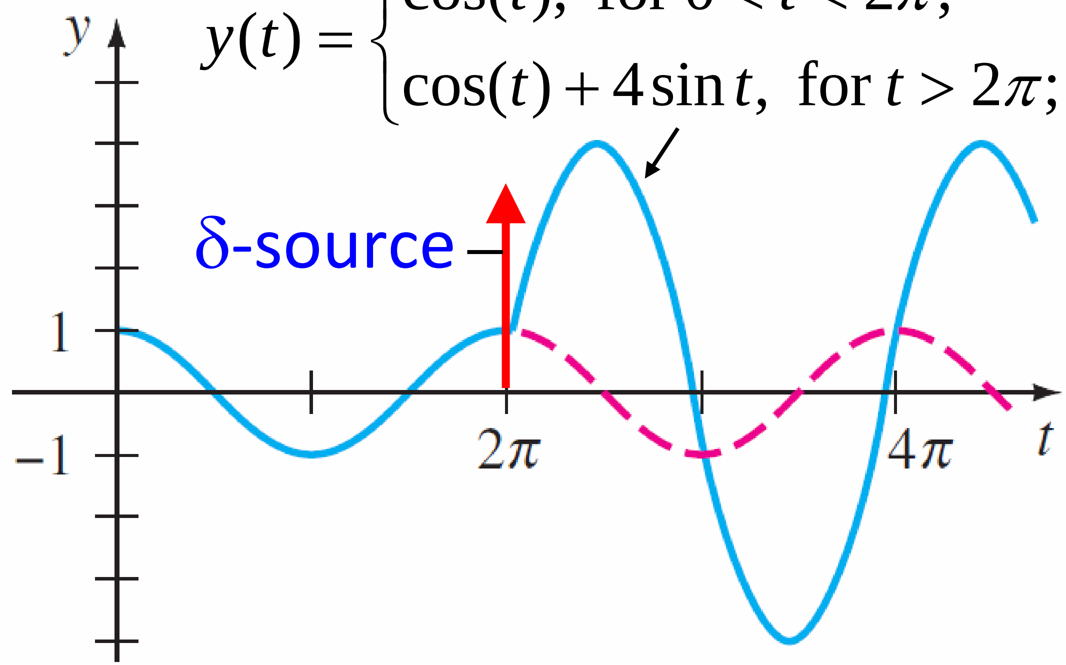
- ODE: $y'' + y = 4\times\delta(t-2\pi)$
- ICs: $y(0) = 1$, $y'(0) = 0$.
- Step 1: $L\{y'' + y\} = [s^2Y(s) - s\times 1 - 0] + Y(s) = (s^2 + 1)Y(s) - s = L\{4\times\delta(t-2\pi)\} = 4\times L\{\delta(t)\}\times e^{-2\pi s} = 4e^{-2\pi s}$.
- Step 2: $(s^2 + 1)Y(s) - s = 4e^{-2\pi s}$, $\Rightarrow$

$$Y(s) = \frac{s}{s^2+1^2} + \frac{4e^{-2\pi s}}{s^2+1^2} \equiv Y_1 + Y_2.$$

- Step 3: $y_1(t) = \cos(t)$; $y_2(t) = 4L^{-1}\{1/(s^2+1^2)\}|_{t\to t-2\pi} = 4\sin(t-2\pi)\times u(t-2\pi) = 4\sin(t)\times u(t-2\pi)$.

Example 1 (2)

$$y(t) = \begin{cases} \cos(t), & \text{for } 0 < t < 2\pi; \\ \cos(t) + 4\sin t, & \text{for } t > 2\pi; \end{cases}$$



- An impulse source to an LC circuit or mass-spring system could cause discontinuous slope but not necessarily discontinuous function value.

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Impulse response

- ODE: $y'' + Py' + Qy = \delta(t)$
- ICs: $y(0) = 0$, $y'(0) = 0$.
- Step 1: $L\{y'' + Py' + Qy\} = [s^2Y(s) - s\times 0 - 0] + P[sY(s) - 0] + Q\times Y(s) = (s^2 + Ps + Q)Y(s) = L\{\delta(t)\} = 1$.
- Step 2: $(s^2 + Ps + Q)Y(s) \equiv D(s)\times Y(s) = 1$, $\Rightarrow Y(s) \equiv H(s) = 1/D(s)$.
- Step 3: $y(t) \equiv h(t) = L^{-1}\{1/D(s)\}$, which is named the "impulse response" (the output as a result of impulse source).

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Why is h(t) important ?

- For an arbitrary source $f(t)$, the output function $y(t)$ can be calculated by $h*f$:
- Step 1: $L\{y'' + Py' + Qy\} = D(s)\times Y(s) = L\{f(t)\} = F(s)$.
- Step 2: $Y(s) = F(s)/D(s) = H(s)\times F(s)$.
- Step 3: $y(t) = L^{-1}\{H(s)\times F(s)\} = h(t)*f(t)$, which is the convolution between the impulse response and the source function.
- A linear system is fully characterized by $h(t)$ or $H(s)$.

Summary

- How to calculate L{f(t)} and L^{-1}{F(s)}?
- How to solve ODEs by Laplace transform?
- What're the complementary operations between time- and s-domains?
- What really happens in a resonantly driven system?
- What's $\delta(t)$? What's $F\{\delta(t)\}$?
- What's impulse response h(t)? Why is it useful?