Lesson 16 Plane Waves in Homogeneous Media

■ Introduction

By eq's (15.5-7), $\vec{E}$ and $\vec{H}$ in a charge-free ($\rho = 0$), nonconducting ($\vec{J} = 0$), simple (linear, homogeneous, isotropic) medium are governed by the vector wave equations:

$$\nabla^2 \vec{E} - \frac{1}{u_p^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0,\quad \nabla^2 \vec{H} - \frac{1}{u_p^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0, \text{ where } u_p = \frac{1}{\sqrt{\mu\varepsilon}}.$$

The general solutions are waves propagating with phase velocity u_p.

A plane wave is a particular solution to the vector wave equation, where every point on an infinite plane perpendicular to the direction of propagation has the same electric field $\vec{E}$ and magnetic field $\vec{H}$ (in terms of magnitude and direction, see Fig. 16-1).

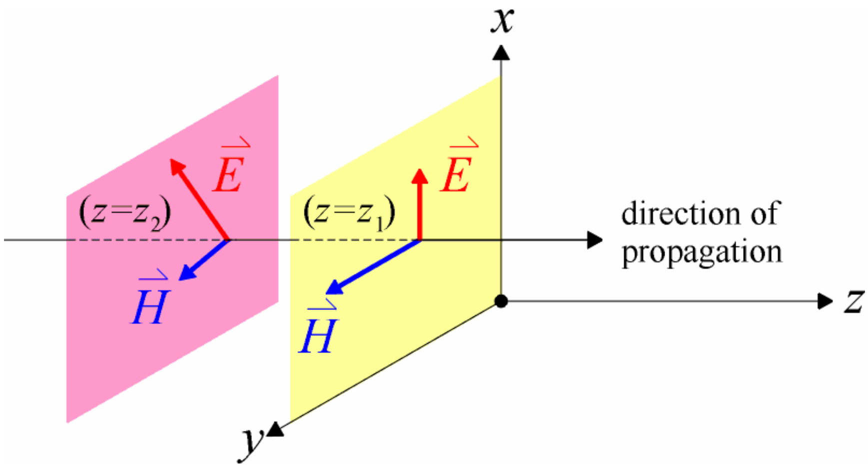

Fig. 16-1. A general plane wave propagates in +z-direction illustrated at some time instant. The fields $\vec{E}$ and $\vec{H}$ are constant throughout the plane $z = z_1$, but may differ from those on another plane $z = z_2$.

Time-harmonic (sinusoidal) plane waves are of particular importance because:

1) The mathematical treatments are greatly simplified.
2) Superposition of time-harmonic plane waves remains a solution to the (linear) vector wave equation. They can be used as a basis to describe general electromagnetic waves.

E.g. Superposition of time-harmonic plane waves of different frequencies and common direction of propagation can describe plane wave packets. Superposition of time-harmonic plane waves of the same frequency and different directions of propagation can describe Gaussian beam.

3) It is a good approximation for far-field EM waves (**e.g.** sunlight received on the earth), and waves whose cross-sections have a linear dimension much larger than the wavelength (**e.g.** typical laser beam).

16.1 Plane Waves in Vacuum

■ Most simplified time-harmonic (sinusoidal) plane waves

The vector phasor $\vec{E}$ of a time-harmonic E-field is satisfied with:

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \tag{15.13}$$

where $k = \omega / u_p$ [eq. (15.15)] denotes the wavenumber. For simplicity, consider a time-harmonic plane wave propagating in the z-direction and the electric field is polarized in the x-direction, ⇒

$$\vec{E} = \vec{a}_x E_x(z) . \tag{16.1}$$

Note that eq. (16.1) does represent a plane wave, for any point on an infinite plane $z = z_0$ (perpendicular to the direction of propagation $\vec{a}_z$) must have the same electric field $\vec{a}_x E_x(z_0)$. In this particular case, eq. (15.13) reduces to a scalar ordinary differential equation:

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \tag{16.2}$$

In the absence of physical boundary, the general solution to eq. (16.2) consists of two counter- propagating waves:

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{+jkz} , \tag{16.3}$$

where $E_0^+ = |E_0^+|e^{j\phi^+}$ and $E_0^- = |E_0^-|e^{j\phi^-}$ are complex amplitudes corresponding to time-harmonic waves propagating in the $+z$ and $-z$ directions, respectively.

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1) As in transmission lines [eq's (4.7), (4.8)], phasor $E_x^+(z) = E_0^+ e^{-jkz}$ describes a time-harmonic wave with velocity $u_p = \omega/k$ because:

$$E_x^+(z,t) = \mathrm{Re}\{E_x^+(z)e^{j\omega t}\} = |E_0^+|\cos(\omega t - kz + \phi^+)$$

is a function of variable $\tau = t - \dfrac{z}{(\omega/k)}$. By eq. (15.15), $u_p = \dfrac{\omega}{\omega\sqrt{\mu\varepsilon}} = \dfrac{1}{\sqrt{\mu\varepsilon}}$, which is the same as eq. (15.7).

2) As in transmission lines, reflected wave $E_x^-(z) = E_0^- e^{+jkz}$ can emerge if the medium is nonhomogeneous along the z-direction (i.e. ε or μ changes with z).

■ Characteristics of plane waves

The electrical field of a plane wave propagating in an arbitrary direction $\vec{a}_k$ is of the form (Fig. 16-2):

$$\vec{E} = \vec{e}E_0 e^{-j\vec{k}\cdot\vec{r}} = (\vec{a}_x\cos\alpha + \vec{a}_y\cos\beta + \vec{a}_z\cos\gamma)E_0 e^{-j\vec{k}\cdot\vec{r}} \tag{16.4}$$

(1) The phase evolution term kz in eq. (16.3) is generalized by $\vec{k}\cdot\vec{r}$, where

$$\vec{k} = \vec{a}_x k_x + \vec{a}_y k_y + \vec{a}_z k_z = \vec{a}_k k$$

is the wavevector, and $\vec{r} = \vec{a}_x x + \vec{a}_y y + \vec{a}_z z$ is the position vector of the observation point. (2) The direction of electric field (polarization direction) $\vec{a}_x$ in eq. (16.1) is generalized by a unit vector:

$$\vec{e} = \vec{a}_x \cos\alpha + \vec{a}_y \cos\beta + \vec{a}_z \cos\gamma$$

where $\{\alpha, \beta, \gamma\}$ are the included angles between $\vec{e}$ and $\{\vec{a}_x, \vec{a}_y, \vec{a}_z\}$, respectively. For any point $\vec{r}$ on an infinite plane perpendicular to the direction of propagation $\vec{a}_k$ (intensity & phase front, Fig. 16-2), $\vec{k}\cdot\vec{r} = kr\cos\theta = \text{constant}$, $\Rightarrow$ $\vec{E} = \vec{e}E_0 e^{-j\vec{k}\cdot\vec{r}}$ is a constant vector. Therefore, eq. (16.4) does represent a plane wave.

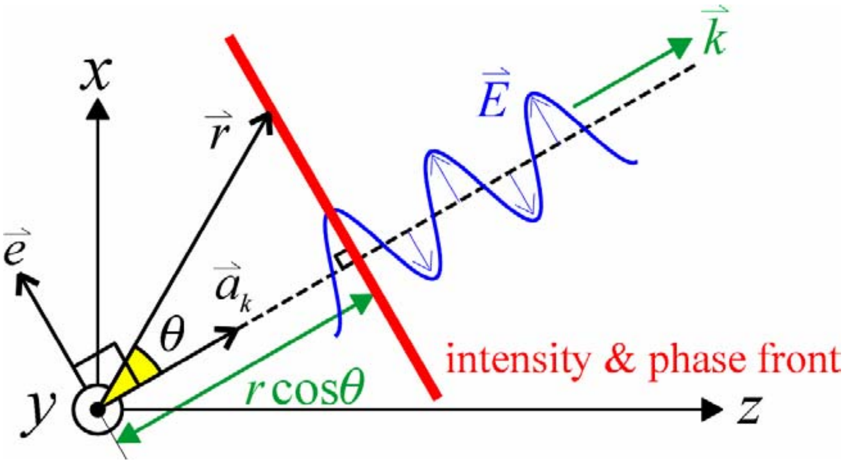

Fig. 16-2. A plane wave propagates in an arbitrary direction on the *xz*-plane.

The three vectors $\vec{E}$, $\vec{H}$, $\vec{k}$ of a time-harmonic plane wave are mutually orthogonal in a charge-free simple medium.

<u>Proof</u>:

(1) In a charge-free ($\rho = 0$) simple medium, eq. (7.8) reduces to:

$$\nabla\cdot\vec{E} = 0\,.$$

(2) By the vector identity $\nabla\cdot(\psi\vec{A}) = \psi(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla\psi)$ (where ψ, $\vec{A}$ are arbitrary scalar and vector fields, respectively), eq. (16.4) reduces to:

$$\nabla\cdot\vec{E} = \nabla\cdot\left(\vec{e}E_0 e^{-j\vec{k}\cdot\vec{r}}\right) = E_0 e^{-j\vec{k}\cdot\vec{r}}(\nabla\cdot\vec{e}) + \vec{e}\cdot\nabla\left(E_0 e^{-j\vec{k}\cdot\vec{r}}\right) = \vec{e}\cdot\left[E_0\nabla\left(e^{-j\vec{k}\cdot\vec{r}}\right)\right]\,.$$

(3) By the formula of gradient in Cartesian coordinate system,

$$\nabla\left(e^{-j\vec{k}\cdot\vec{r}}\right) = \left(\vec{a}_x\frac{\partial}{\partial x} + \vec{a}_y\frac{\partial}{\partial y} + \vec{a}_z\frac{\partial}{\partial z}\right)e^{-j(k_x x + k_y y + k_z z)} = -j\left(\vec{a}_x k_x + \vec{a}_y k_y + \vec{a}_z k_z\right)e^{-j\vec{k}\cdot\vec{r}} = -j\vec{k}e^{-j\vec{k}\cdot\vec{r}}\,.$$

(4) $\nabla \cdot \vec{E} = \vec{e} \cdot \left[E_0\left(-j\vec{k}e^{-j\vec{k}\cdot\vec{r}}\right)\right] = -jkE_0 e^{-j\vec{k}\cdot\vec{r}}\left(\vec{e}\cdot\vec{a}_k\right) = 0$, $\Rightarrow$ $\vec{e} \perp \vec{a}_k$, i.e.,

$$\vec{E} \perp \vec{k} \tag{16.5}$$

If $\vec{E}$ has been known, one can derive $\vec{H}$ by $\nabla \times \vec{E} = -j\omega\mu\vec{H}$ [eq. (15.9)]:

(5) By the formula of curl in Cartesian coordinate system,

$$\vec{H} = \frac{-1}{j\omega\mu}\left(\nabla\times\vec{E}\right) = \frac{-1}{j\omega\mu}\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\alpha & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\beta & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\gamma \end{vmatrix}$$

$$= \frac{-1}{j\omega\mu}\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ -jk_x & -jk_y & -jk_z \\ E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\alpha & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\beta & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\gamma \end{vmatrix}$$

$$= \frac{-1}{j\omega\mu}\left(-j\vec{k}\right)\times\vec{E} = \frac{\vec{k}\times\vec{E}}{\omega\mu} = \frac{\vec{a}_k\times\vec{E}}{\left(\omega\mu/k\right)}, \Rightarrow$$

$$\vec{H} = \frac{\vec{a}_k\times\vec{E}}{\eta}, \tag{16.6}$$

where $\eta = \dfrac{\omega\mu}{k} = \dfrac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\dfrac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega) \tag{16.7}$$

For vacuum, $\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 377\,(\Omega)$. Eq. (16.6) implies that $\vec{H} \perp \vec{k}$ and $\vec{H} \perp \vec{E}$. Combined with eq. (16.5), $\Rightarrow$ $\vec{E}$, $\vec{H}$, $\vec{k}$ of a time-harmonic plane wave are mutually orthogonal in a charge-free simple medium. The fact that both $\vec{E}$ and $\vec{H}$ are perpendicular to the direction of propagation ($\vec{a}_k$) makes the plane wave a transverse electromagnetic (TEM) wave.

<u>Example 16-1</u>: (1) If $\vec{E} = \vec{a}_x E_x^+(z) = \vec{a}_x E_0^+ e^{-jkz}$, $\Rightarrow$ $\vec{k} = \vec{a}_z k$. By eq. (16.6), $\Rightarrow$ $\vec{H} = \dfrac{\vec{a}_z \times \vec{a}_x E_x^+(z)}{\eta} = \vec{a}_y \dfrac{E_x^+(z)}{\eta}$, where $H_y^+(z) = \dfrac{E_x^+(z)}{\eta}$. (2) If $\vec{E} = \vec{a}_x E_x^-(z) = \vec{a}_x E_0^- e^{+jkz}$, $\Rightarrow$

$\vec{k} = -\vec{a}_z k$, $\vec{H} = \dfrac{-\vec{a}_z \times \vec{a}_x E_x^-(z)}{\eta} = \vec{a}_y \dfrac{E_x^-(z)}{-\eta}$, where $H_y^-(z) = \dfrac{E_x^-(z)}{-\eta}$. (3) If $\vec{E} = \vec{a}_y E_y^+(z) = \vec{a}_y E_0^+ e^{-jkz}$, $\Rightarrow$ $\vec{k} = \vec{a}_z k$, $\vec{H} = \dfrac{\vec{a}_z \times \vec{a}_y E_y^+(z)}{\eta} = \vec{a}_x \dfrac{E_y^+(z)}{-\eta}$, where $H_x^+(z) = \dfrac{E_y^+(z)}{-\eta}$.

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1) Intrinsic impedance η [eq. (16.7)] of plane wave is analogous to the characteristic impedance Z_0 [eq. (2.8)] in transmission lines:

$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta} \leftrightarrow I^+ = \frac{V^+}{Z_0} \text{ implies } \{\vec{E} \leftrightarrow v,\ \vec{H} \leftrightarrow i\}.$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \leftrightarrow Z_0 = \sqrt{\frac{L}{C}} \text{ implies } \{\mu \leftrightarrow L,\ \varepsilon \leftrightarrow C\}.$$

2) Not all TEM waves are plane waves. **E.g.** Coaxial transmission lines support TEM waves ($\vec{E} // \vec{a}_r$, $\vec{H} // \vec{a}_\phi$, $\vec{k} // \vec{a}_z$) with $r-$dependent (non-uniform) fields.

■ What is the state of polarization of time-harmonic plane waves?

By eq. (16.4), the electric field phasor of a time-harmonic plane wave propagating in the +*z*-direction is:

$$\vec{E} = \vec{e} E_0 e^{-j\vec{k}\cdot\vec{r}} = (\vec{a}_x \cos\alpha + \vec{a}_y \cos\beta) E_0 e^{-jkz}.$$

At $z = z_0$, the time-dependent electric field vector:

$$\vec{E}(z_0, t) = \mathrm{Re}\{\vec{E}(z_0) e^{j\omega t}\} = \vec{e} E_0 \cos(\omega t - kz_0)$$

will traverse a line of length $2E_0$ along $\vec{e}$ in one period $T = 2\pi/\omega$ (Fig. 16-3a). In general, the electric field of a time-harmonic plane wave propagating in the +*z*-direction can have *x*- and *y*-components ($\vec{E} \perp \vec{k}$ is still satisfied), and their superposition will traverse a tilted ellipse (Fig. 16-3b).

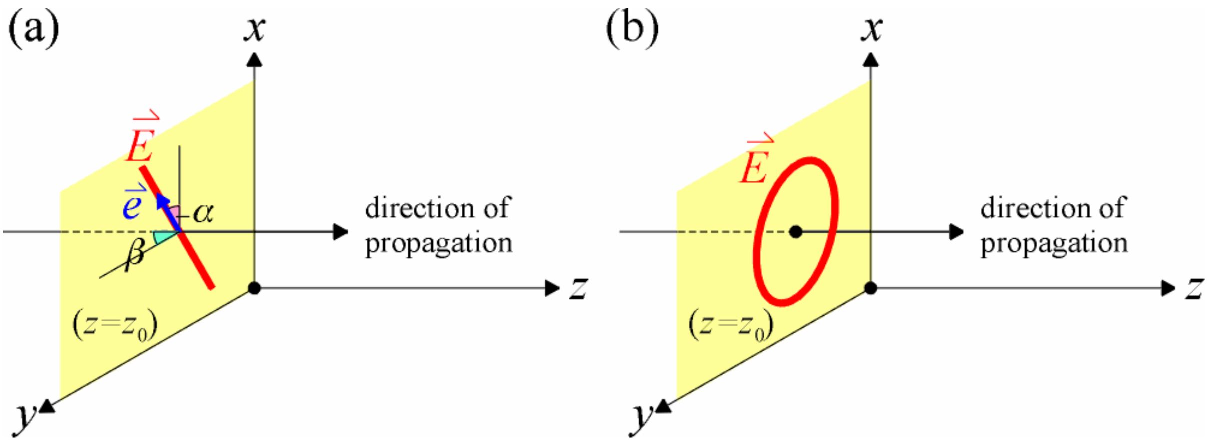

Fig. 16-3. Trajectories of the electric fields of (a) linearly polarized, and (b) elliptically polarized time-harmonic plane waves propagating in the +z-direction.

The state of polarization, represented by geometric parameters (orientation angle θ and ellipticity b/a, see Fig. 16-4) and sense of rotation (clockwise or counterclockwise) of the trajectory, is determined by the relative magnitude and phase of the two constituent components (see below). Polarization state is critical for interference, wave propagation through oblique boundaries and in anisotropic medium (crystal) or waveguide.

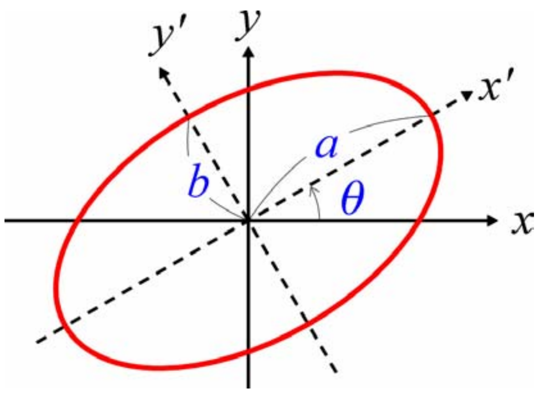

Fig. 16-4. Trajectory of the electric field of a time-harmonic wave propagating in the +z-direction.

■ Analysis of the state of polarization

The vector phasor of a general time-harmonic plane wave propagating in the +z-direction is:

$$\vec{E}(z) = \vec{a}_x E_x e^{-jkz} + \vec{a}_y E_y e^{-jkz},$$

where $E_x = |E_x| e^{j\phi_x}$, $E_y = |E_y| e^{j\phi_y}$ are arbitrary complex numbers. At $z = 0$, the time-dependent electric field is:

$$\vec{E}(t) = \vec{a}_x E_x(t) + \vec{a}_y E_y(t),$$

where $E_x(t) = |E_x|\cos(\omega t + \phi_x)$, $E_y(t) = |E_y|\cos(\omega t + \phi_y)$ are time-harmonic functions.

(1) The parametric representation of the trajectory of $\vec{E}(t)$ is:

$$x = |E_x|\cos(\omega t + \phi_x), \quad y = |E_y|\cos(\omega t + \phi_y).$$

However, the "absolute" phases ϕ_x, ϕ_y only influence the initial point of the trajectory $\vec{E}(t=0)$. The geometry and sense of rotation of the trajectory can be determined by:

$$x = |E_x|\cos(\omega t), \quad y = |E_y|\cos(\omega t + \phi),$$

where $\phi = \phi_y - \phi_x$ denotes the relative phase between $E_x(t)$ and $E_y(t)$.

(2) To eliminate the variable t, we use $\dfrac{x}{|E_x|} = \cos\omega t$, $\dfrac{y}{|E_y|} = \cos\omega t \cdot \cos\phi - \sin\omega t \cdot \sin\phi$, $\Rightarrow$

$$\left(\frac{y}{|E_y|} - \cos\omega t \cdot \cos\phi\right)^2 = \sin^2\omega t \cdot \sin^2\phi$$

$$\left(\frac{y}{|E_y|} - \frac{x}{|E_x|}\cos\phi\right)^2 = \left(1 - \frac{x^2}{|E_x|^2}\right)\sin^2\phi, \Rightarrow$$

$$\left(\frac{x}{|E_x|\sin\phi}\right)^2 + \left(\frac{y}{|E_y|\sin\phi}\right)^2 - \frac{2\cos\phi}{|E_x E_y|\sin^2\phi}xy = 1 \tag{16.8}$$

which represents a tilted ellipse.

The geometric parameters a, b, θ (Fig. 16-4) can be determined by rotating the coordinate axes for a proper angle θ, such that the trajectory becomes a "right" ellipse in the new coordinates (x', y'):

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1, \tag{16.9}$$

By substituting $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} x' \\ y' \end{bmatrix}$ into eq. (16.8) and making the coefficient of $x'y'$ term vanishing, we derive:

$$\tan 2\theta = \frac{2|E_x E_y|\cos\phi}{|E_x|^2 - |E_y|^2} \tag{16.10}$$

$$a = \frac{1}{2}\left(\sqrt{|E_x|^2 + 2|E_x E_y|\sin\phi + |E_y|^2} + \sqrt{|E_x|^2 - 2|E_x E_y|\sin\phi + |E_y|^2} \right) \tag{16.11}$$

$$b = \frac{1}{2}\left| \sqrt{|E_x|^2 + 2|E_x E_y|\sin\phi + |E_y|^2} - \sqrt{|E_x|^2 - 2|E_x E_y|\sin\phi + |E_y|^2} \right| \tag{16.12}$$

The sense of rotation is clockwise (counterclockwise) if $0 < \phi < \pi \, (-\pi < \phi < 0)$. If the wave propagates along +*z*-direction, clockwise (counterclockwise) rotation follows the left-hand (right-hand) rule and is also called left-hand (right-hand) polarized.

<u>Example 16-2</u>: Let the vector phasor of the electric field at $z = 0$ be: $\vec{E} = E_0(\vec{a}_x - j\vec{a}_y)$, $\Rightarrow$ $\{|E_x| = |E_y| = E_0,\ \phi = -\pi/2\}$, $\{E_x(t) = E_0 \cos\omega t,\ E_y(t) = E_0 \sin\omega t\}$. By eq's (16.10-12), $\theta = ?$, $a = b = E_0$. Since $\phi = -\pi/2 \in (-\pi, 0)$, the sense of rotation is counterclockwise. $\Rightarrow$ The state of polarization is right-hand circularly polarized, in counterclockwise sense.

16.2 Time-harmonic Plane Waves in Lossy Media

■ Complex propagation constant

If the simple medium is conducting $(\sigma \neq 0)$, permittivity ε_c and wavenumber k_c become complex [eq's (15.16), (15.17)]. It is customary to define the propagation constant as:

$$\gamma \equiv jk_c = \alpha + j\beta \tag{16.13}$$

In this way, the electric field of an *x*-polarized time-harmonic plane wave propagating in the $+z$ direction is described by a phasor:

$$\vec{E}(z) = \vec{a}_x E_x(z), \quad E_x(z) = E_0^+ e^{-\gamma z} = E_0^+ e^{-\alpha z} e^{-j\beta z},$$

where $\alpha\left(N_p/\mathrm{m}\right)$ and $\beta\left(\mathrm{rad/m}\right)$ represent the attenuation and phase shift per unit length, respectively. The phase velocity $u_p = \frac{\omega}{k}$ in eq. (15.15)] and intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ in eq. (16.7) are modified as:

$$u_p = \frac{\omega}{\beta} \in R \tag{16.14}$$

$$\eta_c = \frac{\omega\mu}{k_c} = \sqrt{\frac{\mu}{\varepsilon_c}} = |\eta| e^{j\theta_\eta} \in C \tag{16.15}$$

By eq. (16.6), a complex intrinsic impedance means that $\vec{E}$ and $\vec{H}$ are not in phase when the wave propagates through a lossy medium.

- Low-loss dielectrics

When the loss tangent [eq. (15.18)] is small, $\frac{\sigma}{\omega\varepsilon} << 1$:

(1) $\gamma = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \approx j\omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{2\omega\varepsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right], \Rightarrow$

$$\alpha \approx \frac{\sigma\eta}{2}, \quad \beta \approx \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right] \approx \omega\sqrt{\mu\varepsilon}\,.$$

(2) By eq. (16.14), $u_p \approx \frac{1}{\sqrt{\mu\varepsilon}}\left[1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right] < \frac{1}{\sqrt{\mu\varepsilon}}$.

(3) By eq. (16.15), $\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon}}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{-1/2} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right)$.

■ Good conductors

When the loss tangent is large, $\dfrac{\sigma}{\omega\varepsilon} >> 1$,

(1) $\gamma = j\omega\sqrt{\mu\varepsilon\left(1 - j\dfrac{\sigma}{\omega\varepsilon}\right)} \approx j\omega\sqrt{\mu\varepsilon\left(-j\dfrac{\sigma}{\omega\varepsilon}\right)} = e^{j\frac{\pi}{2}}\omega\sqrt{e^{-j\frac{\pi}{2}}\dfrac{\mu\sigma}{\omega}} = e^{j\frac{\pi}{4}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\mu\sigma}$,

where $\omega = 2\pi f$, $\Rightarrow$ $\alpha = \beta \approx \sqrt{\pi f\mu\sigma}$.

(2) By eq. (16.14), $u_p = \dfrac{\omega}{\beta} \approx \sqrt{\dfrac{2\omega}{\mu\sigma}} << \dfrac{1}{\sqrt{\mu\varepsilon}}$.

(3) By eq. (16.15), $\eta_c = \sqrt{\dfrac{\mu}{\varepsilon_c}} = \sqrt{\dfrac{\mu}{\varepsilon[1 - j(\sigma/\omega\varepsilon)]}} \approx \sqrt{\dfrac{\mu}{\varepsilon[-j(\sigma/\omega\varepsilon)]}} = \sqrt{\dfrac{\omega\mu}{e^{-j\pi/2}\sigma}} = (1+j)\sqrt{\dfrac{\pi f\mu}{\sigma}}$,

$\Rightarrow$ $\vec{E}$ and $\vec{H}$ have 45° phase difference.

Since $\alpha \propto \sqrt{f}$, high-frequency EM wave is attenuated very rapidly as it travels through a good conductor. The distance δ through which the amplitude of a plane wave decays by a factor of $e^{-1} = 0.368$ is called the skin depth:

$$\delta \equiv \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}} \tag{16.16}$$

Since $\alpha \approx \beta$ in good conductors, $\Rightarrow$ $\delta \approx \dfrac{1}{\beta} = \dfrac{\lambda}{2\pi}$, EM wave only penetrates a depth shorter than one wavelength.

<u>Example 16-3</u>: Consider sea water: $\sigma = 4$ (S/m), $\varepsilon = 72\varepsilon_0$, $\mu_0 = 4\pi \times 10^{-7}$ (H/m). At $f = 3$ MHz, $u_p \approx 2.7 \times 10^{6}$ (m/s), $\delta = 14$ cm. $\Rightarrow$ Impractical for submarine detection or communications.

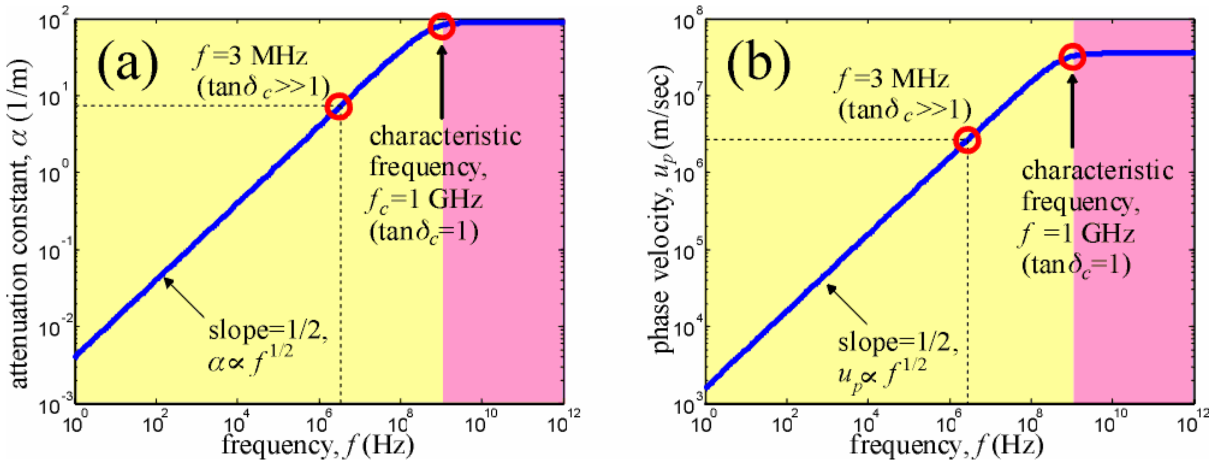

Fig. 16-5. Frequency response (log-log plot) of (a) attenuation constant, (b) phase velocity of sea water.

16.3 Power Flow of Electromagnetic Waves

- Poynting vector

By the vector identity $\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$ and eq's (14.1), (14.13):

$$\nabla\cdot(\vec{E}\times\vec{H})=\vec{H}\cdot(\nabla\times\vec{E})-\vec{E}\cdot(\nabla\times\vec{H})=-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{E}\cdot\vec{J}-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}.$$

For electromagnetic waves in simple media,

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}=\vec{H}\cdot\left(\mu\frac{\partial\vec{H}}{\partial t}\right)=\mu H\cdot\frac{\partial H}{\partial t}=\frac{\mu}{2}\frac{\partial H^2}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}\right)=\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right),\quad \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right),$$

$$\Rightarrow\ \nabla\cdot(\vec{E}\times\vec{H})=-\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right)-\vec{E}\cdot\vec{J}-\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right).$$

By integrating both sides of equality over a volume V bounded by a closed surface S:

$$\oint_S(\vec{E}\times\vec{H})\cdot d\vec{s}+\int_V\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right)dv+\int_V\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right)dv+\int_V(\vec{E}\cdot\vec{J})dv=0 \qquad (16.17)$$

(1) By eq's (9.7) and (13.7), $w_e=\frac{1}{2}\vec{D}\cdot\vec{E}$, $w_m=\frac{1}{2}\vec{H}\cdot\vec{B}$ (J/m^3) represent the electric and magnetic energy densities, respectively. $\Rightarrow$ $\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right)$, $\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right)$ (W/m^3), and $\int_V\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right)dv$, $\int_V\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right)dv$ (W) represent the electromagnetic power densities and total electromagnetic powers stored within the volume V, respectively.

(2) By eq. (10.12), $\int_V (\vec{E}\cdot\vec{J})dv$ represents the dissipated power (energy transferred from the electromagnetic field into the medium through collision among free electrons and immobile ions) within the volume V.

(3) To meet the requirement of energy conservation, eq. (16.17) implies that $\oint_S (\vec{E}\times\vec{H})\cdot d\vec{s}$ must mean the net power flow "out of" the volume V. The integrand, defined as the Poynting vector:

$$\vec{P}=\vec{E}\times\vec{H} \quad (\text{W/m}^2), \tag{16.18}$$

must represent the instantaneous directed power density of the electromagnetic wave. Note that eq. (16.18) is valid for arbitrary electromagnetic waves and static electromagnetic fields, not just limited to plane waves.

<u>Example 16-4</u>: Find the Poynting vector on the surface of a conducting wire of radius b and conductivity σ, whose cross-section flows a uniform dc current I (Fig. 16-5).

Ans: $\vec{J}=\vec{a}_z\dfrac{I}{\pi b^2}$. By eq. (10.3), $\vec{E}=\dfrac{\vec{J}}{\sigma}=\vec{a}_z\dfrac{I}{\sigma\pi b^2}$, everywhere inside and on the surface of the wire. By eq. (12.6) and cylindrical symmetry, $\vec{H}=\vec{a}_\phi\dfrac{I}{2\pi b}$ on the surface of the wire. By eq. (16.18), the surface power density is:

$$\vec{P}=\left(\vec{a}_z\frac{I}{\sigma\pi b^2}\right)\times\left(\vec{a}_\phi\frac{I}{2\pi b}\right)=-\vec{a}_r\frac{I^2}{2\sigma\pi^2 b^3} \quad (\text{W/m}^2).$$

The total power flowing into the surface is:

$$-\oint_S \vec{P}\cdot d\vec{s}=\left(\vec{a}_r\frac{I^2}{2\sigma\pi^2 b^3}\right)\cdot(\vec{a}_r 2\pi bL)=I^2\frac{L}{\sigma\pi b^2}=I^2R \quad (\text{W}),$$

which is consistent with the ohmic power loss inside the conducting wire.

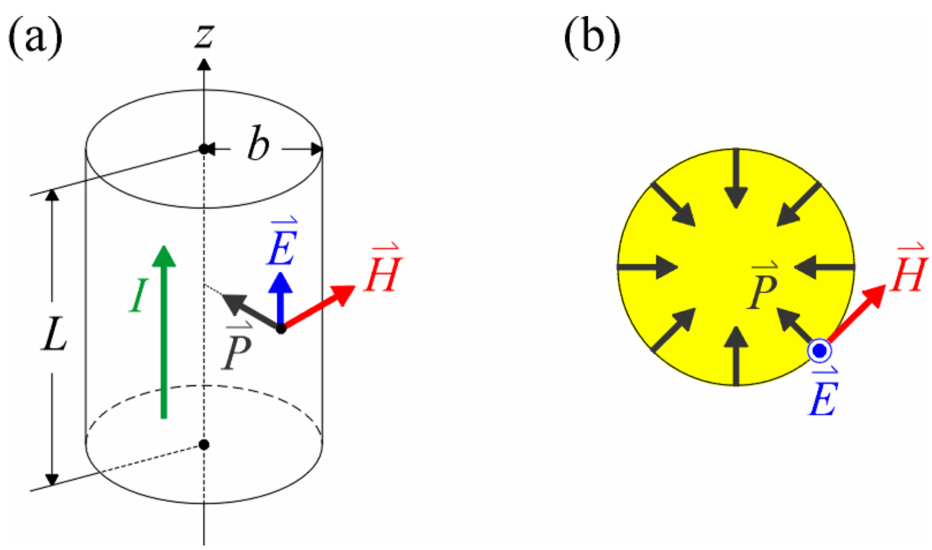

Fig. 16-6. (a) A current-carrying wire and corresponding EM fields and Poynting vector. (b) Top view of (a).

■ Time-averaged Poynting vector

For a general time-harmonic wave, the instantaneous Poynting vector is formulated as:

$$\vec{P}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right] \times \mathrm{Re}\left[\vec{H}(\vec{r})e^{j\omega t}\right],$$

where $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ are vector phasors (will be denoted as $\vec{E}$ and $\vec{H}$ for simplicity).

By the relation $\mathrm{Re}(\vec{A}) \times \mathrm{Re}(\vec{B}) = \frac{1}{2}\mathrm{Re}(\vec{A} \times \vec{B}^* + \vec{A} \times \vec{B}), \Rightarrow$

$$\vec{P}(\vec{r},t) = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^* + \vec{E} \times \vec{H}e^{j2\omega t}\right),$$

consisting of a time-independent term $\frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*)$ and an oscillating term $\frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}e^{j2\omega t})$ of angular frequency 2ω. The time-averaged Poynting vector $\vec{P}_{av}(\vec{r})$ can be derived by integrating $\vec{P}(\vec{r},t)$ over one oscillating period $T = \pi/\omega$,

$$\vec{P}_{av}(\vec{r}) = \frac{1}{T}\int_0^T \vec{P}(\vec{r},t)dt, \Rightarrow$$

$$\vec{P}_{av}(\vec{r}) = \frac{1}{2}\mathrm{Re}\left\{\vec{E} \times \vec{H}^*\right\} \ (\mathrm{W/m^2}) \quad (16.19)$$

Eq. (16.19) is valid for all time-harmonic waves. For time-harmonic plane waves propagating in charge-free $(\rho = 0)$, conducting $(\sigma \neq 0)$, simple media, the formula can be simplified.

Substituting eq. (16.6) into eq. (16.19), and apply the vector identity

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{C}\cdot\vec{A}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right),$$

we have:

$$\vec{P}_{av}(\vec{r})=\frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\left(\frac{\vec{a}_k\times\vec{E}^*}{\eta_c^*}\right)\right\}=\frac{1}{2}\mathrm{Re}\left\{\vec{a}_k\frac{\left|\vec{E}\right|^2}{\eta_c^*}-\vec{E}^*\left(\frac{\vec{E}\cdot\vec{a}_k}{\eta_c^*}\right)\right\}=\frac{1}{2}\mathrm{Re}\left\{\vec{a}_k\frac{\left|\vec{E}\right|^2}{\eta_c^*}\right\}=\vec{a}_k\frac{\left|\vec{E}(\vec{r})\right|^2}{2|\eta|}\cos\theta_\eta,$$

where $\vec{E}\cdot\vec{a}_k=0$ [eq. (16.5)], and $\eta_c=|\eta|e^{j\theta_\eta}$ represents the complex intrinsic impedance of the medium [eq. (16.15)]. Without loss of generality, assume the time-harmonic plane wave is x-polarized and propagates in the $+z$ direction, $\Rightarrow$ $\vec{a}_k=\vec{a}_z$, $\vec{E}(\vec{r})=\vec{a}_x E_0^+ e^{-\alpha z}e^{-j\beta z}$ [eq. (16.13)], $\Rightarrow$

$$\vec{P}_{av}(z)=\vec{a}_z\frac{\left|E_0^+\right|^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta \quad (\mathrm{W/m^2}) \tag{16.20}$$

Eq. (16.20) shows that: (1) The power is transmitted in the direction of wavevector. (2) If the medium is lossless $(\sigma=0)$, $\Rightarrow$ $\{\gamma,\eta_c\}\in R$, $\{\alpha=0,\theta_\eta=0\}$, $\Rightarrow$

$$\vec{P}_{av}(z)=\vec{a}_z\frac{\left|E_0^+\right|^2}{2|\eta|}.$$

The plane wave has a constant power density (no propagation loss) proportional to the modulus square of the electric field. (3) If the medium is lossy $(\sigma\neq 0)$, the power density decays with propagation due to the field attenuation ($\alpha>0$) and the phase mismatch between $\vec{E}$ and $\vec{H}$ ($\theta_\eta\neq 0$).

16.4 Plane Wave Packets in Dispersive Media

■ Why to discuss wave packets?

Though a perfect sinusoidal wave is useful in power delivery [eq. (16.20)], it does not carry any "information". We normally modulate the amplitude, frequency, and/or phase of a sinusoidal wave (the carrier) to create electromagnetic wave packets to transmit information. For example, digital information can be carried by a train of wave packets with amplitude (Fig. 16-7a) or phase (Fig. 16-7b) modulation.

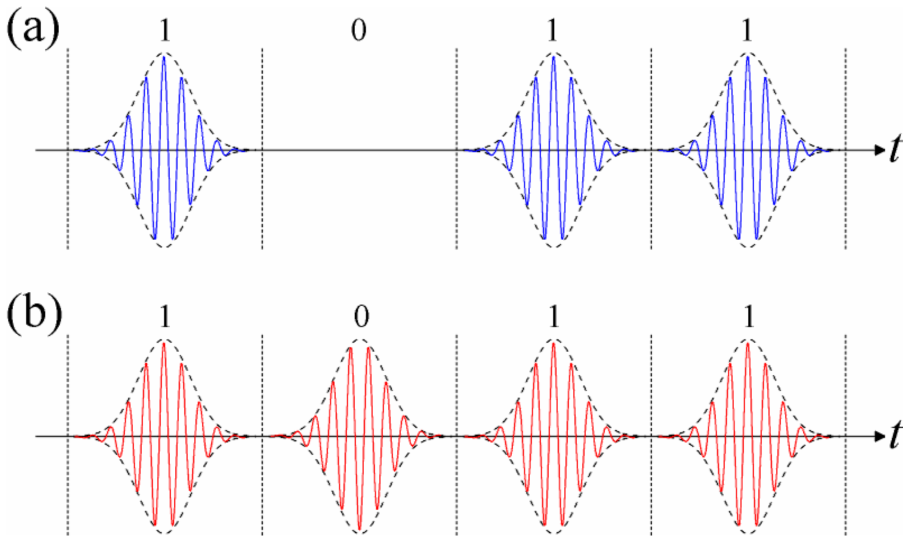

Fig. 16-7. Trains of wave packets with (a) amplitude, and (b) phase modulation used to carry four-bit (1011) data.

■ Fundamentals of dispersion

In terms of Fourier analysis, a wave packet must consist of multiple frequency components, When propagating through a dispersive medium (with frequency-dependent ε and/or μ), different frequency components can have different velocities $u_p(\omega) = 1/\sqrt{\mu(\omega)\varepsilon(\omega)}$ [eq. (15.7)], resulting in wave packet distortion. In this section, we will quantitatively analyze the distortion of linearly polarized plane waves propagating in the +z-direction through a source-free ($\rho = 0$, $\vec{J} = 0$), simple, non-magnetic ($\mu = \mu_0$), dispersive medium. In this case, the electric field in the frequency domain can be formulated by:

$$\vec{E} = \vec{e}E(z,\omega), \tag{16.21}$$

which is a solution to the vector wave equation [eq. (15.13)]. Substituting eq. (16.21) into eq.

(15.13) shows that the scalar field $E(z,\omega)$ satisfies with the scalar wave equation:

$$\frac{d^2}{dz^2}E(z,\omega)+k^2(\omega)E(z,\omega)=0 \tag{16.22}$$

where the wavenumber [eq. (15.15)] is generalized to a nonlinear function of ω:

$$k(\omega)=\omega\sqrt{\mu_0\varepsilon(\omega)} \tag{16.23}$$

The general solution to eq. (16.22) representing a wave propagating in the +z-direction is:

$$E(z,\omega)=E_0(\omega)e^{-jk(\omega)z} \tag{16.24}$$

Eq. (16.24) means that (1) A time-harmonic wave of angular frequency ω_1 will experience a phase shift $-k(\omega_1)L$ after propagating a distance L. (2) A wave packet with spectrum $E_0(\omega)$ at $z=0$ will experience a spectral phase modulation of:

$$\Delta\psi(\omega)=-k(\omega)L \tag{16.25}$$

at $z=L$. This will result in distortion of wave packet in time domain:

$$e(z=0,t)=F^{-1}\{E_0(\omega)\}\neq F^{-1}\{E_0(\omega)e^{j\Delta\psi(\omega)}\}=e(z=L,t)$$

as long as $\Delta\psi(\omega)$ is a nonlinear function of ω.

■ Beat wave of two frequency components

For simplicity, consider the superposition of two time-harmonic waves with the same amplitude E_0 and direction of propagation ($+\vec{a}_z$) but different angular frequencies ω_1, ω_2:

$$e(z,t)=E_0\cos(\omega_1 t-k_1 z)+E_0\cos(\omega_2 t-k_2 z),$$

where $k_i=k(\omega_i)=\omega\sqrt{\mu_0\varepsilon(\omega_i)}$, $i=1,2$ [eq. (16.23)]. By the relation

$$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right),\Rightarrow$$

$$e(z,t)=2E_0\cos(\Delta\omega\cdot t-\Delta k\cdot z)\cos(\omega_m t-k_m z), \tag{16.26}$$

where $\Delta\omega=\dfrac{\omega_1-\omega_2}{2}$, $\Delta k=k(\Delta\omega)$, $\omega_m=\dfrac{\omega_1+\omega_2}{2}$, $k_m=k(\omega_m)$, respectively. Eq. (16.26) represents a beat wave, which is the product of an envelope wave $e_n(z,t)=\cos(\Delta\omega\cdot t-\Delta k\cdot z)$ and a carrier wave $e_c(z,t)=\cos(\omega_m t-k_m z)$. For example, Fig. 16-8a shows the two constituent time-harmonic waves with frequencies $f_1=120$ Hz (solid), and $f_2=100$ Hz (dashed) observed at $z=0$. Fig. 16-8b shows the corresponding envelope (dashed, $\Delta f=10$ Hz), carrier (dotted, $f_m=110$ Hz), and beat (solid) waves, respectively.

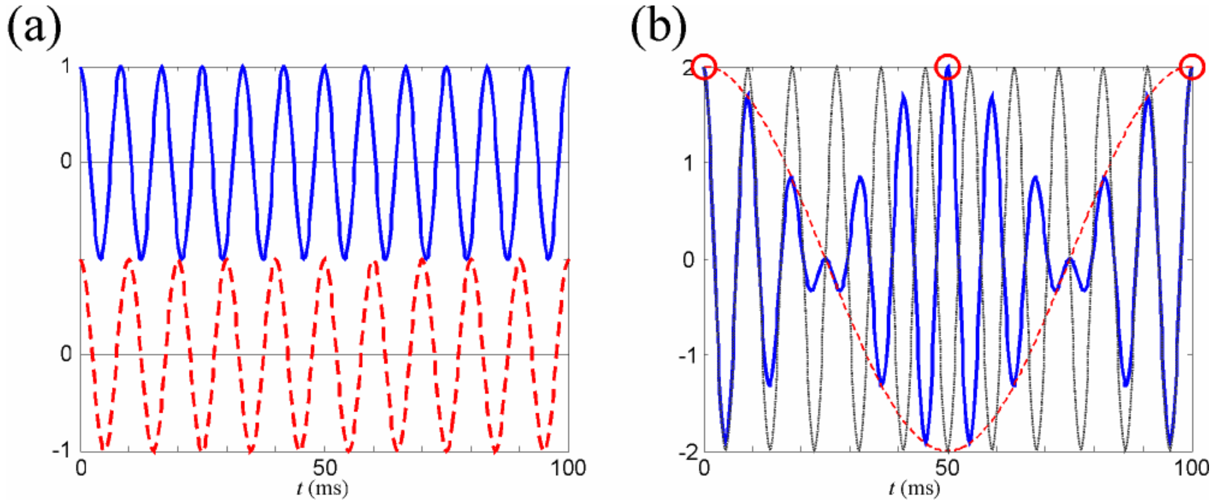

Fig. 16-8. (a) Two time-harmonic waves, and the corresponding (b) envelope (dashed), carrier (dotted), and beat (solid) waves. Note that the peaks (open circles) of beat wave coincide with the max or min of the envelope.

■ Phase velocity & group velocity

Since $e_c(z,t)=\cos(\omega_m[t-z/(\omega_m/k_m)])$, the carrier wave propagates with phase velocity [eq. (15.7)] evaluated at the carrier (average) frequency $\omega_m=\dfrac{\omega_1+\omega_2}{2}$:

$$u_p=\frac{\omega_m}{k_m}=\frac{1}{\sqrt{\mu_0\varepsilon(\omega_m)}} \tag{16.27}$$

On the other hand, $e_n(z,t)=\cos(\Delta\omega[t-z/(\Delta\omega/\Delta k)])$, the envelope wave propagated with group velocity:

$$u_g=\frac{\Delta\omega}{\Delta k} \tag{16.28}$$

For wave packets in real applications (such as those in Fig. 16-7), the bandwidth $\Delta\omega$ is typically much smaller than the carrier frequency ω_m, and eq. (16.28) becomes:

$$u_g = \lim_{\Delta\omega\to 0}\frac{\Delta\omega}{\Delta k} = \frac{1}{k'(\omega_m)} \tag{16.29}$$

■ Propagation of beat waves

1) If the medium is dispersion-free [$\varepsilon \neq \varepsilon(\omega)$], $\Rightarrow$ $k(\omega) = \omega\sqrt{\mu_0\varepsilon} \propto \omega$, $k'(\omega) = \sqrt{\mu_0\varepsilon}$, $u_g = \frac{1}{k'(\omega_m)} = u_p$, the envelope and carrier waves propagate with the same velocity. As a result, the beat waveform $e(z,t)$ will be independent of z. For example, the solid curve in Fig. 16-9a represents the beat wave propagating through a dispersion-free medium over some distance. It is the same as the solid curve in Fig. 16-8b except for a time shift.

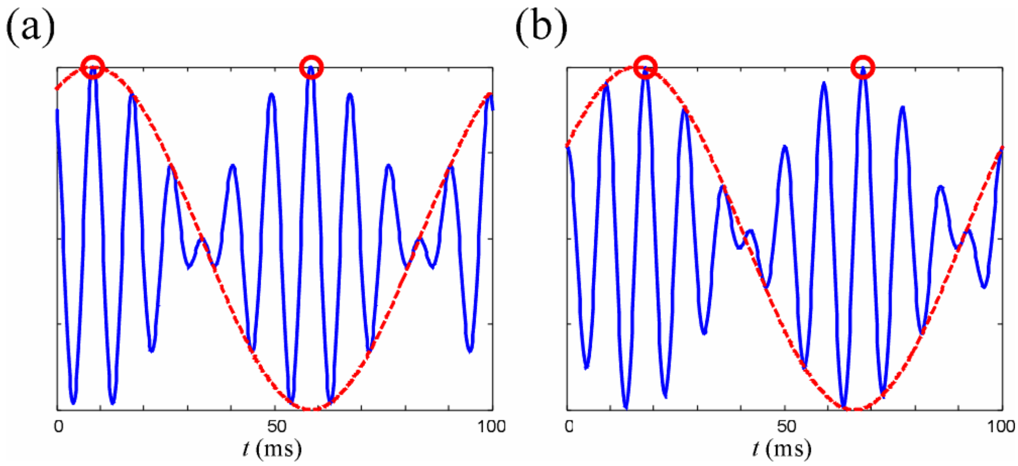

Fig. 16-9. Envelope (dashed), and beat (solid) waves after propagating through a (a) dispersion-free, and (b) dispersive medium over some distance. Note that the peaks (open circles) of beat wave still coincide with the max or min of the envelope in (a), but are mismatched in (b).

2) If the medium is dispersive [$\varepsilon = \varepsilon(\omega)$], $\Rightarrow$ $k(\omega) = \omega\sqrt{\mu_0\varepsilon(\omega)}$ [eq. (16.23)] is a nonlinear function of ω, $k'(\omega) \neq \sqrt{\mu_0\varepsilon(\omega)}$, $u_g = \frac{1}{k'(\omega_m)} \neq u_p$ (Fig. 16-10), the envelope and carrier waves propagate with different velocities. As a result, the beat waveform $e(z,t)$ will vary with z. For example, the solid curve in Fig. 16-9b represents the beat wave propagating through a dispersive medium [$\varepsilon(\omega_1) \neq \varepsilon(\omega_2)$] over

some distance. It is different from that in Fig. 16-8b for the peaks (open circles) are deviated from the maximum or minimum of the envelope.

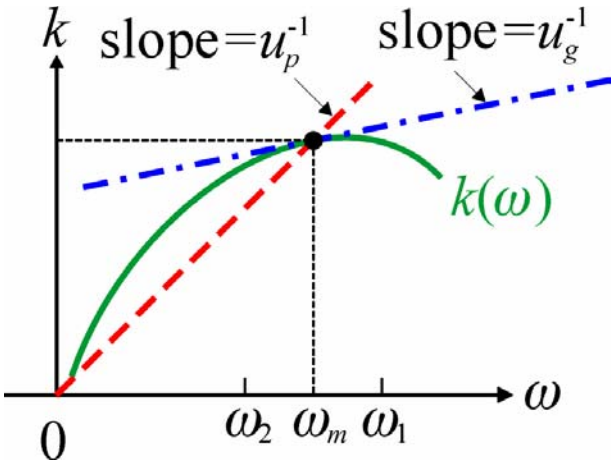

Fig. 16-10. Frequency-dependent wavenumber $k(\omega)$ (solid) and the secant (dashed), tangential (dashed-dotted) lines at carrier angular frequency ω_m. The slopes represent u_p^{-1} and u_g^{-1}, respectively.

■ Propagation of general wave packets

In the model of beat wave, dispersion only changes the relative position between the envelope and the carrier while the envelope's shape is intact (Fig. 16-9). However, wave packets in real applications consist of many frequency components or even a continuous band. In this case, analysis via spectral phase modulation [eq. (16.25)] will show that dispersion will cause changes of: (1) the relative position between the envelope and the carrier, (2) the envelope's shape, and (3) the fringe density distributions (Fig. 16-11b).

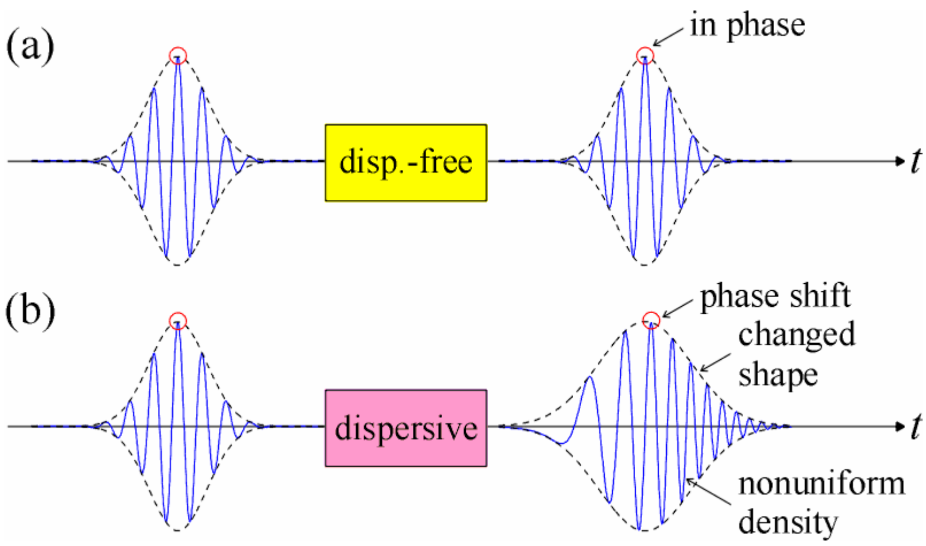

Fig. 16-11. Wave packets before (left) and after (right) propagating through a (a) dispersion-free, and (b) dispersive medium, respectively.