Lesson 16
Plane Waves in Homogeneous Media

楊尚達 Shang-Da Yang

Institute of Photonics Technologies
Department of Electrical Engineering
National Tsing Hua University, Taiwan

What is plane wave? (1)

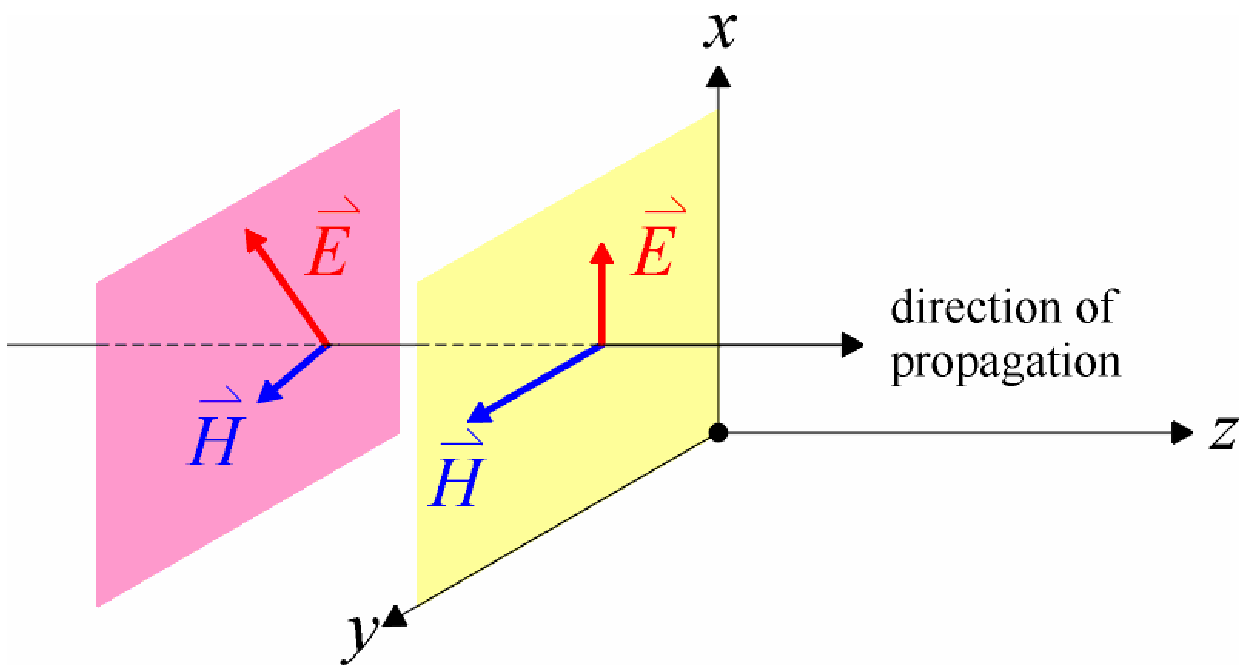
A particular solution to the homogeneous vector wave equations:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \qquad \nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

where every point on an infinite plane normal to the direction of propagation has the same E-field and H-field (same magnitude and same direction).

What is plane wave? (2)

At $t = t_0$, EM-fields are vector functions of space. For plane waves, they can be simpler.



E(H) is constant throughout a transversal plane, but may differ at different planes.

Sec. 16-1
Plane Waves in Vacuum

1. Time-harmonic plane waves
2. State of polarization (SOP)

Most simplified time-harmonic plane waves-1

The E-field vector phasor of a time-harmonic wave satisfies with:

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0, \quad \text{where} \quad k = \omega\sqrt{\mu\varepsilon}$$

For simplicity, consider a time-harmonic plane wave propagating in the z-direction and polarized in the x-direction, $\Rightarrow$ $\vec{E} = \vec{a}_x E_x(z)$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \longrightarrow \boxed{\frac{d^2 E_x}{dz^2} + k^2 E_x = 0}$$

(Vector PDE) (Scalar ODE)

Most simplified time-harmonic plane waves-2

If there is no boundary, the general solution to

$$\frac{d^2E_x}{dz^2}+k^2E_x=0 \text{ is: } E_x(z)=\underbrace{E_0^+e^{-jkz}}_{+z\text{ wave}}+\underbrace{E_0^-e^{+jkz}}_{-z\text{ wave}},$$

where $E_0^+=\left|E_0^+\right|e^{j\phi^+}$, $E_0^-=\left|E_0^-\right|e^{j\phi^-}$ are complex.

$$\underbrace{E_x^+(z,t)}_{+z\text{ wave}}=\mathrm{Re}\left\{E_x^+(z)e^{j\omega t}\right\}=\left|E_0^+\right|\cos\left(\omega t-kz+\phi^+\right)$$

function of $\tau=t-\dfrac{z}{\omega/k}$

phase velocity (ω/k)

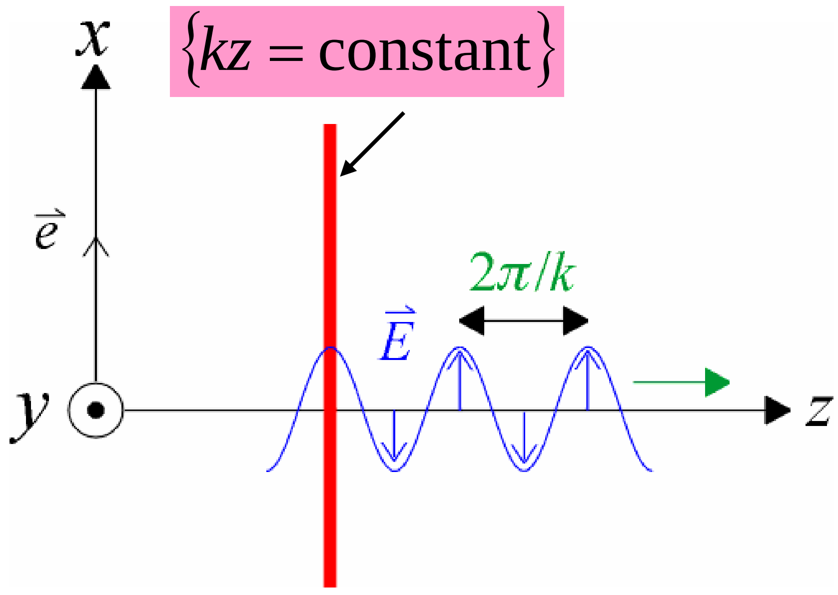
Time-harmonic plane waves propagating along $+z$

The E-field vector phasor of a time-harmonic plane wave linearly polarized along $\vec{a}_x$ and propagating along $+\vec{a}_z$ is:

$$\vec{E} = \vec{a}_x E_0^+ e^{-jkz}$$

Every point on an infinite transversal plane $z = z_0$ has the same field (for $kz = kz_0 =$ constant), $\Rightarrow z = z_0$ is called the intensity and phase front.

Peak-to-peak spacing is $2\pi/k$, $\Rightarrow k$ is called the wavenumber ("angular frequency" in space).



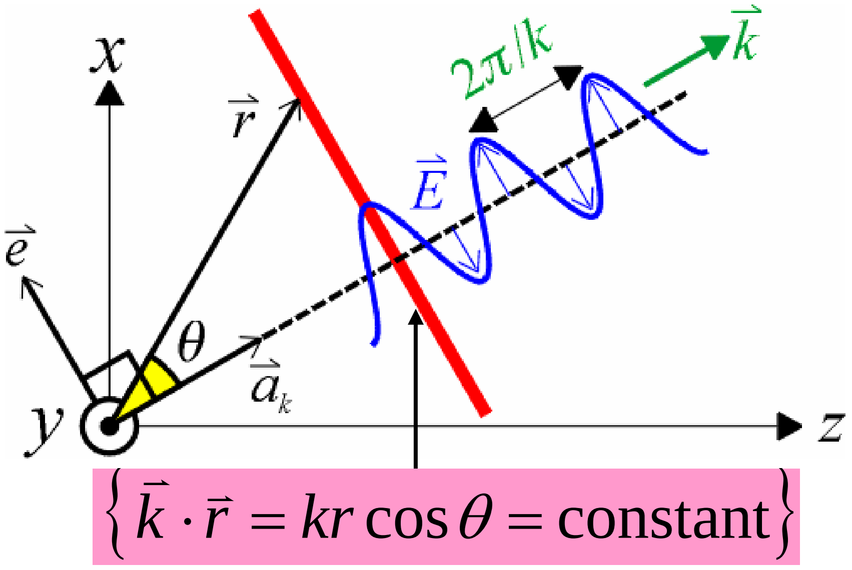
The E-field vector phasor of a time-harmonic plane wave linearly polarized along $\vec{e}$ and propagating along $\vec{a}_k$ is:

$$\vec{E} = \vec{e}E_0 e^{-j\vec{k}\cdot\vec{r}}, \quad \vec{k} = \vec{a}_x k_x + \vec{a}_y k_y + \vec{a}_z k_z = \vec{a}_k k$$

Set of points $\{\vec{k}\cdot\vec{r} = \text{constant}\}$ share the same field, forming the intensity and phase front.

It is still a transversal plane for $\{\vec{k}\cdot\vec{r} = kr\cos\theta = \text{constant}\}$

Peak-to-peak spacing is still $2\pi/k$, $\Rightarrow$ $\vec{k}$ is called the wavevector.



Time-harmonic plane waves are TEM waves (1)

In a charge-free, simple medium:

$$\nabla \cdot \vec{D} = \rho \xrightarrow[\vec{D} = \varepsilon \vec{E}]{\rho = 0} \nabla \cdot \vec{E} = 0$$

By $\nabla \cdot (\psi \vec{A}) = \psi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \psi), \Rightarrow$

$$\nabla \cdot \vec{E} = \nabla \cdot \left(\vec{e} E_0 e^{-j\vec{k}\cdot\vec{r}} \right)$$

$$= E_0 e^{-j\vec{k}\cdot\vec{r}} (\nabla \cdot \vec{e}) + \vec{e} \cdot \nabla \left(E_0 e^{-j\vec{k}\cdot\vec{r}} \right) = \vec{e} \cdot \left[E_0 \nabla \left(e^{-j\vec{k}\cdot\vec{r}} \right) \right]$$

Time-harmonic plane waves are TEM waves (2)

By the formula of gradient in Cartesian coordinate system:

$$\nabla\left(e^{-j\vec{k}\cdot\vec{r}}\right)=\left(\vec{a}_x\frac{\partial}{\partial x}+\vec{a}_y\frac{\partial}{\partial y}+\vec{a}_z\frac{\partial}{\partial z}\right)e^{-j(k_x x+k_y y+k_z z)}$$

$$-jk_x e^{-j\vec{k}\cdot\vec{r}} \quad -jk_y e^{-j\vec{k}\cdot\vec{r}} \quad -jk_z e^{-j\vec{k}\cdot\vec{r}}$$

$$=-j\left(\vec{a}_x k_x+\vec{a}_y k_y+\vec{a}_z k_z\right)e^{-j\vec{k}\cdot\vec{r}}=-j\vec{k}e^{-j\vec{k}\cdot\vec{r}}$$

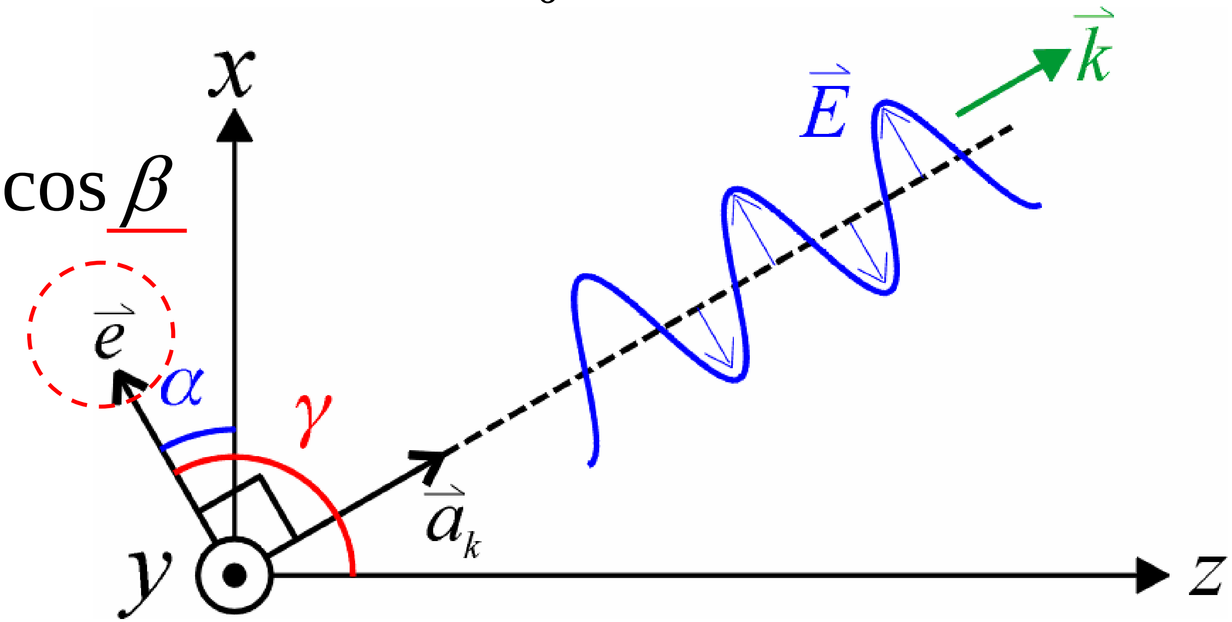
$$\nabla\cdot\vec{E}=\vec{e}\cdot\left[E_0\left(-j\underbrace{\vec{k}}_{\vec{a}_k k}e^{-j\vec{k}\cdot\vec{r}}\right)\right]=-jkE_0e^{-j\vec{k}\cdot\vec{r}}\left(\vec{e}\cdot\vec{a}_k\right)=0$$

$\Rightarrow \vec{e}\perp\vec{a}_k,\ \boxed{\vec{E}\perp\vec{k}}$ … E-field must be transversal.

Direction of the E-field vector

If a time-harmonic plane wave is linearly polarized along $\vec{e}$: $\vec{E} = \vec{e} E_0 e^{-j\vec{k}\cdot\vec{r}}$,

$$\vec{a}_x \cos\underline{\alpha} + \vec{a}_y \cos\underline{\beta} + \vec{a}_z \cos\underline{\gamma}$$



By $\vec{E} \perp \vec{k}$, E-field is only required to lie on a plane normal to the direction of propagation.

Time-harmonic plane waves are TEM waves (3)

By $\nabla \times \vec{E} = -j\omega\mu\vec{H}$, $\Rightarrow \vec{H} = \dfrac{-1}{j\omega\mu}\left(\nabla \times \vec{E}\right)$

$$\vec{E} = \vec{e}E_0 e^{-j\vec{k}\cdot\vec{r}}, \quad \vec{e} = \vec{a}_x \cos\alpha + \vec{a}_y \cos\beta + \vec{a}_z \cos\gamma$$

$$= \frac{-1}{j\omega\mu}\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\alpha & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\beta & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\gamma \end{vmatrix}$$

$$= \frac{-1}{j\omega\mu}\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ -jk_x & -jk_y & -jk_z \\ E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\alpha & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\beta & E_0 e^{-j\vec{k}\cdot\vec{r}}\cos\gamma \end{vmatrix}$$

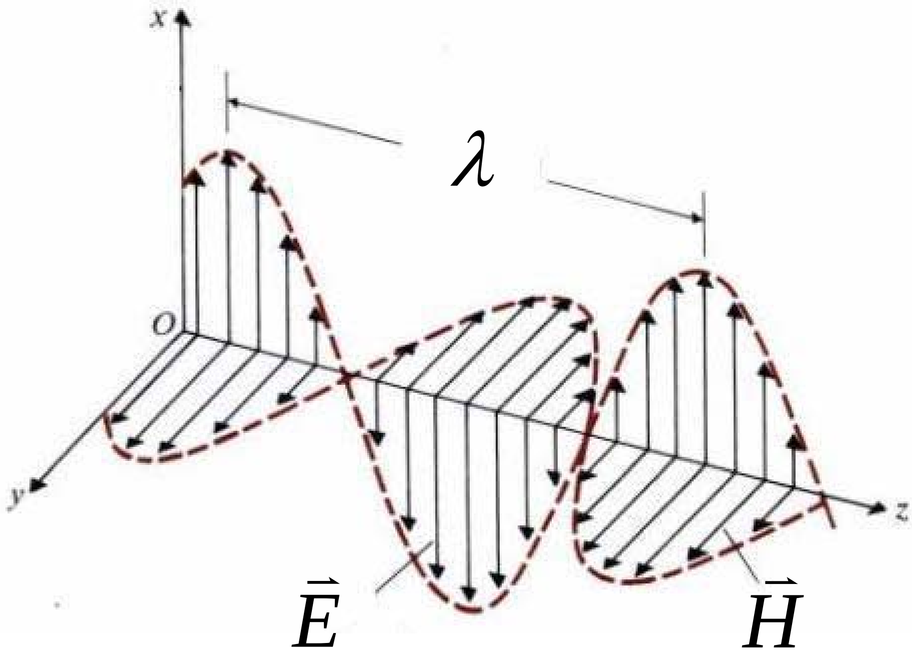
$$= \frac{-1}{j\omega\mu}\left(-j\vec{k}\right)\times\vec{E} = \frac{\vec{k}\times\vec{E}}{\omega\mu} = \frac{\vec{a}_k \times \vec{E}}{\boxed{\omega\mu/k}}$$

Time-harmonic plane waves are TEM waves (4)

$$\Rightarrow \boxed{\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta}} \quad \ldots \quad \vec{H} \perp \vec{k}, \; \vec{H} \perp \vec{E}$$

$\Rightarrow \{\vec{E}, \vec{H}, \vec{k}\}$ are mutually orthogonal (given $\vec{E} \perp \vec{k}$);

$\Rightarrow \{\vec{E}, \vec{H}\} \perp \vec{k}$, H-field must be also transversal it's a transverse electromagnetic (TEM) wave.



Time-harmonic plane waves are TEM waves (5)

$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta},$$

$$\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$ …intrinsic impedance (Ω)

In vacuum, $\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 377\,(\Omega)$. It means the ratio of E-field to H-field, not the impedance of a resistor!

Example 16-1: Time-harmonic plane wave (1)

1. For an x-polarized (E-field is in the x-direction) time-harmonic plane wave propagating in the $+z$-direction: $\vec{E} = \vec{a}_x E_x^+(z) = \vec{a}_x E_0^+ e^{-jkz}$, $\vec{k} = \vec{a}_z k$

$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta} = \frac{\vec{a}_z \times \vec{a}_x E_x^+(z)}{\eta} = \vec{a}_y \frac{E_x^+(z)}{\eta}$$

$\Rightarrow$ H-field is y-polarized, and η is the ratio of the scalar phasors E/H.

2. For an x-polarized time-harmonic plane wave propagating in the $-z$-direction:

$$\vec{E} = \vec{a}_x E_x^-(z) = \vec{a}_x E_0^- e^{+jkz}, \quad \vec{k} = -\vec{a}_z k$$

$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta} = \frac{-\vec{a}_z \times \vec{a}_x E_x^-(z)}{\eta} = \vec{a}_y \frac{E_x^-(z)}{-\eta}$$

$\Rightarrow$ H-field is y-polarized, and the ratio of the scalar phasors E/H is $-\eta$.

Example 16-1: Time-harmonic plane wave (3)

3. For an y-polarized time-harmonic plane wave propagating in the $+z$-direction:

$$\vec{E} = \vec{a}_y E_y^+(z) = \vec{a}_y E_0^+ e^{-jkz}, \quad \vec{k} = \vec{a}_z k$$

$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta} = \frac{\vec{a}_z \times \vec{a}_y E_y^+(z)}{\eta} = \vec{a}_x \frac{E_y^+(z)}{-\eta}$$

$\Rightarrow$ H-field is x-polarized, and the ratio of the scalar phasors E/H is $-\eta$.

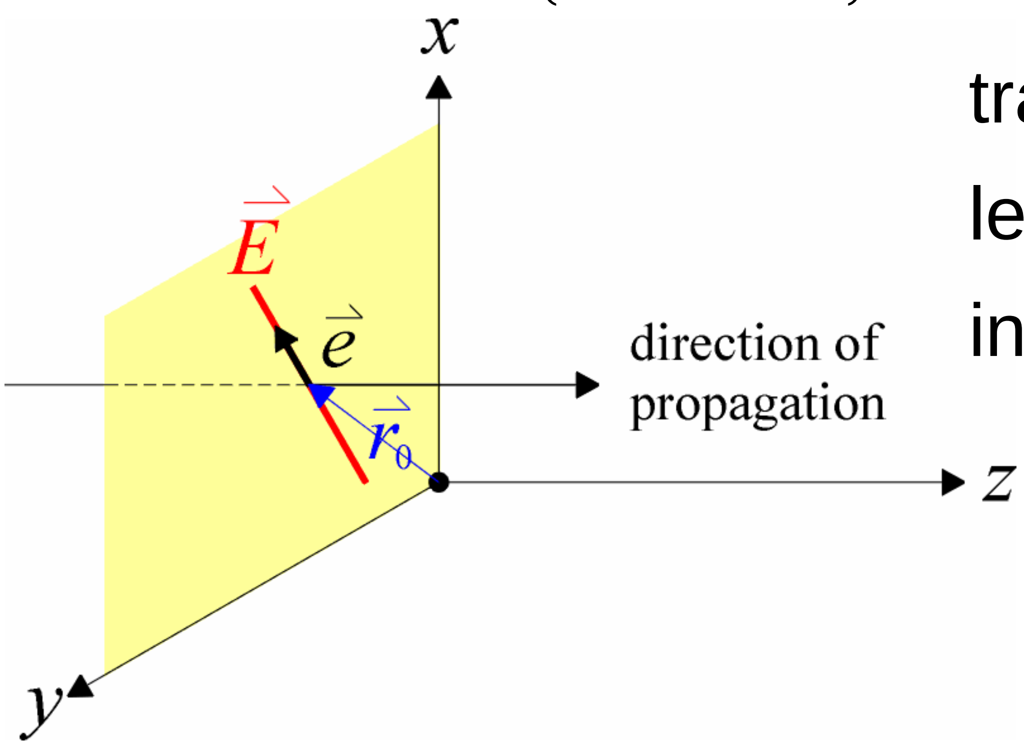
Comment

Plane waves		TX lines
$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$	$\longleftrightarrow$	$$Z_0 = \sqrt{\frac{L}{C}}$$
$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta}$$		$$I^+ = \frac{V^+}{Z_0}$$

For a time-harmonic plane wave with phasor

$\vec{E} = \vec{e} E_0 e^{-j\vec{k}\cdot\vec{r}}$, the tip of the time-varying field

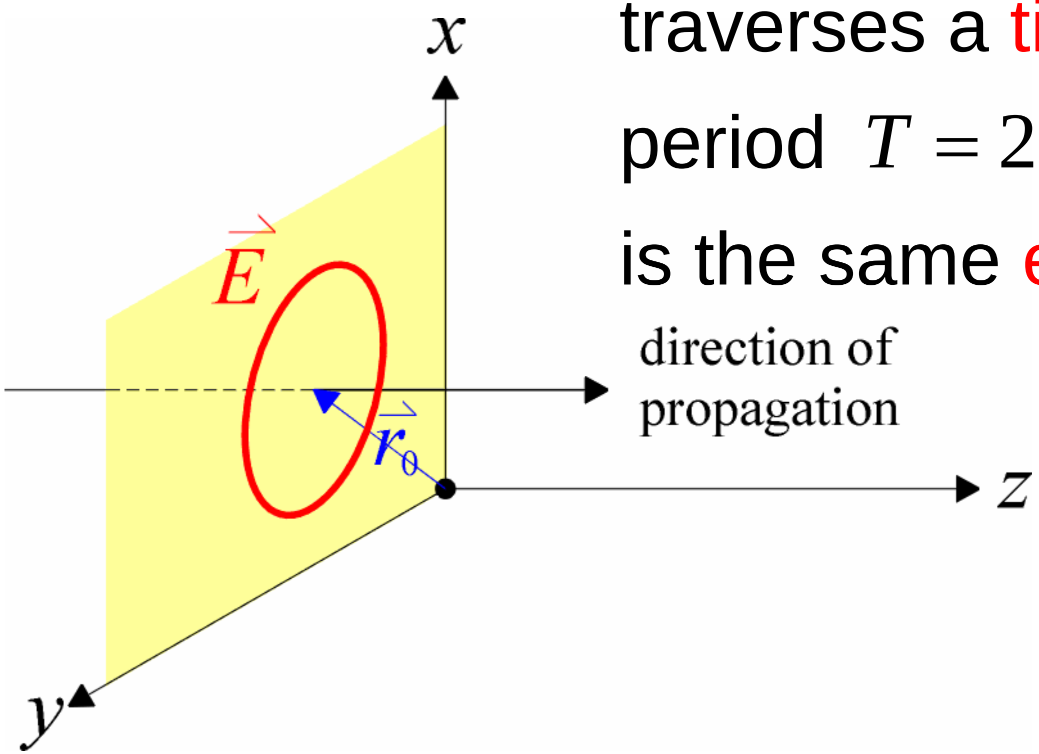
$$\vec{E}(\vec{r}_0, t) = \mathrm{Re}\left\{\vec{E}(\vec{r}_0) e^{j\omega t}\right\} = \vec{e} E_0 \cos\left(\omega t - \vec{k}\cdot\vec{r}_0\right)$$



traverses a line of length $2E_0$ along $\vec{e}\left(\perp \vec{k}\right)$ in one period $T = 2\pi/\omega$.

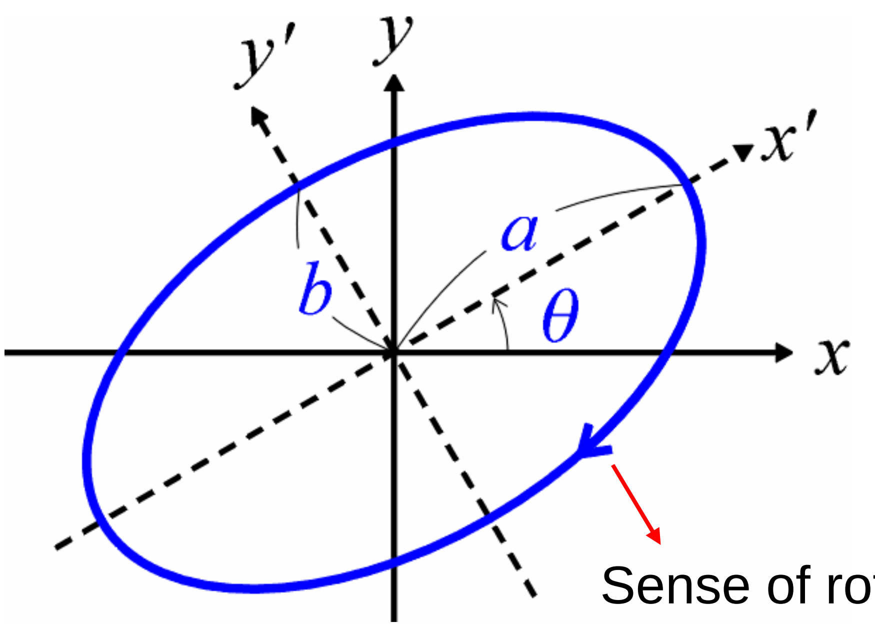
What is the state of polarization? (2)

For a general time-harmonic plane wave, the E-field $\left(\perp \vec{k}\right)$ can be decomposed into two mutually orthogonal time-harmonic components, and its tip traverses a tilted ellipse in one period $T = 2\pi/\omega$. The trajectory is the same everywhere.



What is the state of polarization? (3)

The state of polarization (SOP) is represented by (1) orientation angle θ, (2) ellipticity b/a, (3) sense of rotation (CLK or CCLK), which is determined by the relative magnitude and phase of the two constituent components.



Sense of rotation is defined when the observer confronts with the direction of propagation.

Analysis of the state of polarization-1

The general vector phasor of a time-harmonic plane wave propagating in the $+z$-direction is:

$$\vec{E}(z) = \vec{a}_x E_x e^{-jkz} + \vec{a}_y E_y e^{-jkz}$$

$$E_x \rightarrow |E_x| e^{j\phi_x} \qquad E_y \rightarrow |E_y| e^{j\phi_y}$$

At $z = 0$, $\vec{E} = \vec{a}_x E_x + \vec{a}_y E_y$,

$$\Rightarrow \vec{E}(t) = \text{Re}\left\{\vec{E} e^{j\omega t}\right\} = \vec{a}_x E_x(t) + \vec{a}_y E_y(t)$$

$$E_x(t) \rightarrow |E_x| \cos(\omega t + \phi_x) \qquad E_y(t) \rightarrow |E_y| \cos(\omega t + \phi_y)$$

Analysis of the state of polarization-2

The absolute phases ϕ_x, ϕ_y only influence the initial point of the trajectory $\vec{E}(t=0)$, the geometry and sense of rotation of the trajectory can be determined by the parametric representation:

$$x = |E_x| \cos(\omega t), \quad y = |E_y| \cos(\omega t + \phi)$$

relative phase: $\phi_y - \phi_x$

$$\Rightarrow \quad \frac{x}{|E_x|} = \cos \omega t, \quad \frac{y}{|E_y|} = \cos \omega t \cdot \cos \phi - \sin \omega t \cdot \sin \phi$$

To get the trajectory, eliminate the variable t by:

$$\frac{x}{|E_x|} = \cos\omega t, \qquad \frac{y}{|E_y|} = \cos\omega t \cdot \cos\phi - \sin\omega t \cdot \sin\phi$$

$$\Rightarrow \left(\frac{y}{|E_y|} - \cos\omega t \cdot \cos\phi \right)^2 = \sin^2\omega t \cdot \sin^2\phi$$

$$\Rightarrow \left(\frac{y}{|E_y|} - \frac{x}{|E_x|}\cos\phi \right)^2 = \left(1 - \frac{x^2}{|E_x|^2} \right)\sin^2\phi$$

Analysis of the state of polarization-4

$$\left(\frac{y}{|E_y|}-\frac{x}{|E_x|}\cos\phi\right)^2=\left(1-\frac{x^2}{|E_x|^2}\right)\sin^2\phi$$

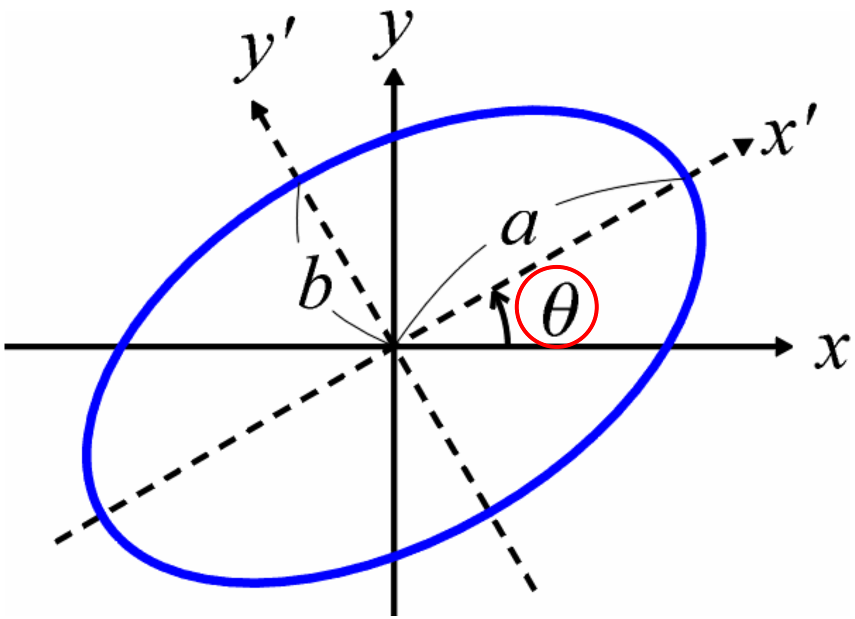
$$\Rightarrow \frac{y^2}{|E_y|^2}-2\frac{xy}{|E_xE_y|}\cos\phi+\frac{x^2}{|E_x|^2}\cos^2\phi=\sin^2\phi-\frac{x^2}{|E_x|^2}\sin^2\phi$$

$$\Rightarrow \frac{y^2}{|E_y|^2\sin^2\phi}-2\frac{xy\cos\phi}{|E_xE_y|\sin^2\phi}+\frac{x^2}{|E_x|^2\sin^2\phi}=\frac{\sin^2\phi}{\sin^2\phi}$$

Tilted ellipse:

$$\left(\frac{x}{|E_x|\sin\phi}\right)^2+\left(\frac{y}{|E_y|\sin\phi}\right)^2-\frac{2\cos\phi}{|E_xE_y|\sin^2\phi}xy=1$$

Rotate the coordinates by θ to get a right ellipse:



Substitute

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} x' \\ y' \end{bmatrix}$$

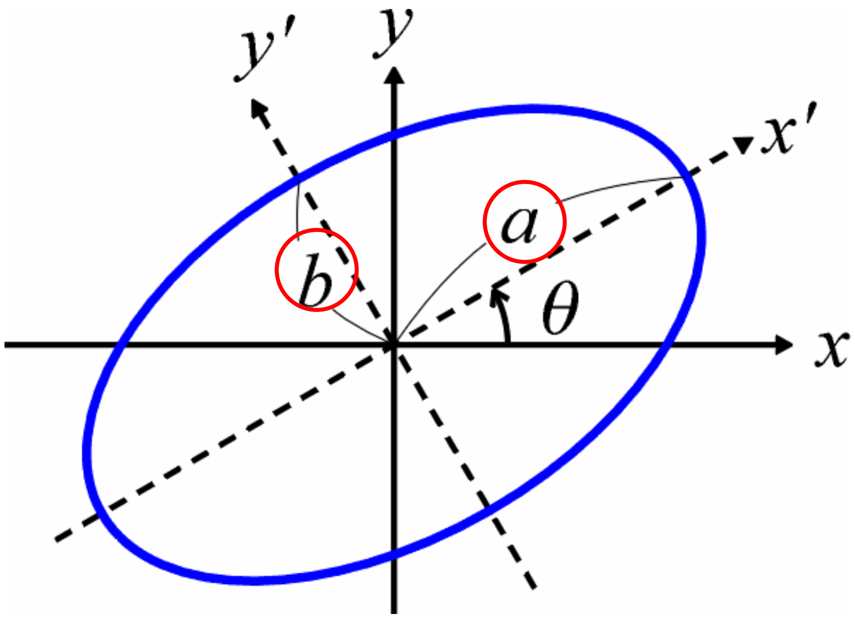
into the tilted ellipse eq.,

have coefficient of $x'y'$ zero,

$$\Rightarrow \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1,$$

$$\boxed{\tan 2\theta = \frac{2\left|E_x E_y\right|\cos\phi}{\left|E_x\right|^2 - \left|E_y\right|^2}}$$

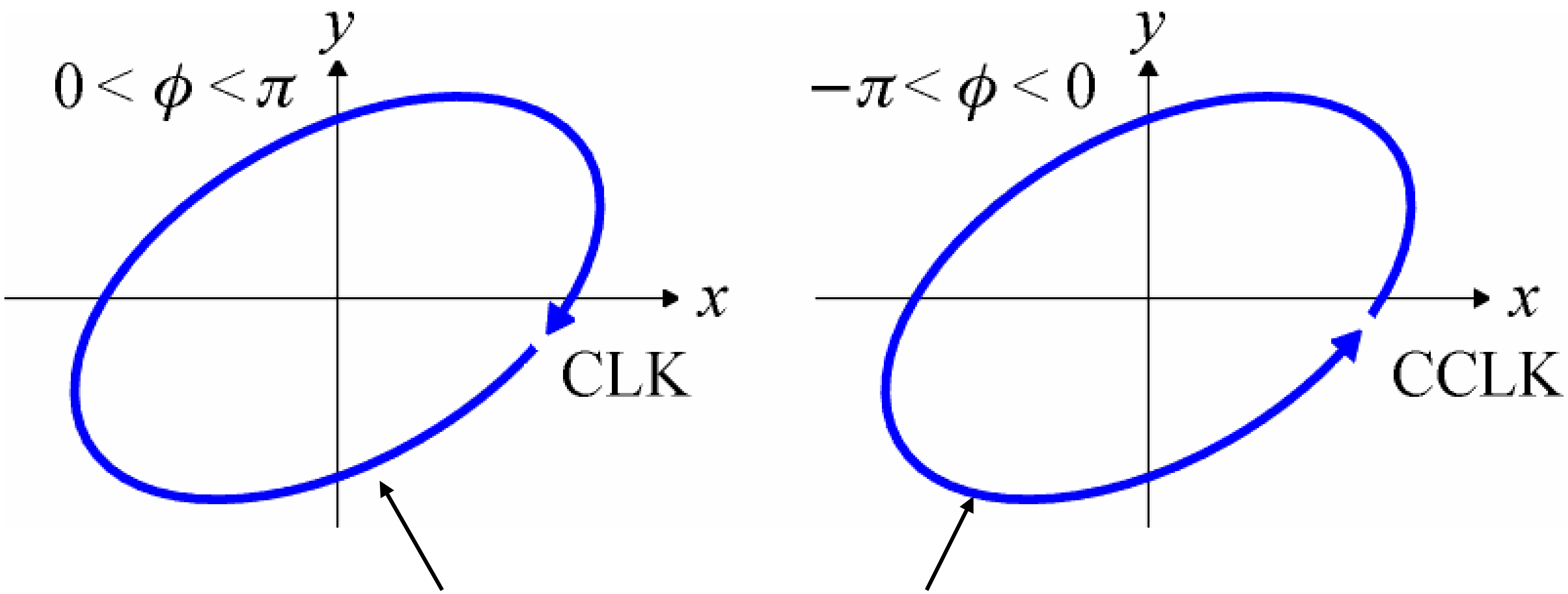
The values of a, b can be evaluated by:



$$a = \frac{1}{2}\left(\sqrt{|E_x|^2 + 2|E_x E_y|\sin\phi + |E_y|^2} + \sqrt{|E_x|^2 - 2|E_x E_y|\sin\phi + |E_y|^2}\right)$$

$$b = \frac{1}{2}\left|\sqrt{|E_x|^2 + 2|E_x E_y|\sin\phi + |E_y|^2} - \sqrt{|E_x|^2 - 2|E_x E_y|\sin\phi + |E_y|^2}\right|$$

The sense of rotation, defined when the observer confronts with the wave, is determined by ϕ:



(Assuming that wave is propagating along $+z$).

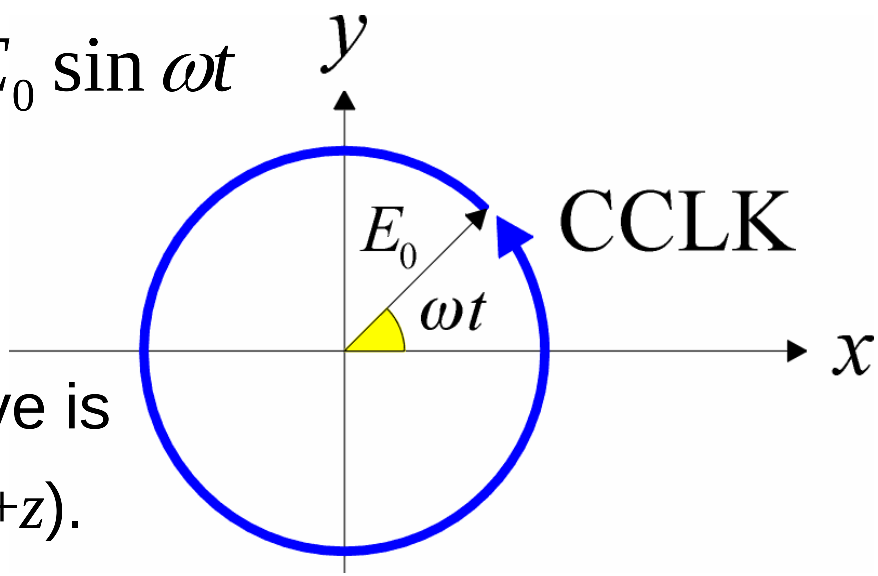
Example 16-2: Right-hand circular polarization-1

Let the vector phasor of the E-field at $z = 0$ be:

$$\vec{E} = E_0\left(\vec{a}_x - j\vec{a}_y\right), \Rightarrow$$

$$\phi_x = 0,\ \phi_y = -\pi/2 = \phi$$

$$\begin{cases} E_x(t) = E_0 \cos\left(\omega t + 0\right) = E_0 \cos \omega t \\ E_y(t) = E_0 \cos\left(\omega t - \pi/2\right) = E_0 \sin \omega t \end{cases}$$



(Assuming that wave is propagating along $+z$).

Example 16-2: Right-hand circular polarization-2

By the formulas:

$$\vec{E} = E_0\left(\vec{a}_x - j\vec{a}_y\right), \Rightarrow \left\{\left|E_x\right| = \left|E_y\right| = E_0,\ \phi = -\pi/2\right\}$$

$$\phi_y - \phi_x$$

$$\tan 2\theta = \frac{2\left|E_x E_y\right|\cos\phi}{\left|E_x\right|^2 - \left|E_y\right|^2} = \frac{2\left|E_0 E_0\right|\cos\left(-\pi/2\right)}{\left|E_0\right|^2 - \left|E_0\right|^2},\quad \theta = ?$$

$$\phi = -\pi/2 \in \left(-\pi, 0\right), \Rightarrow \text{CCLK}$$

Example 16-2: Right-hand circular polarization-3

$$\left\{ |E_x| = |E_y| = E_0,\ \phi = -\pi/2 \right\},\ \Rightarrow$$

$$a = \frac{1}{2}\left(\sqrt{|E_x|^2 + 2|E_x E_y|\sin\phi + |E_y|^2} + \sqrt{|E_x|^2 - 2|E_x E_y|\sin\phi + |E_y|^2} \right)$$

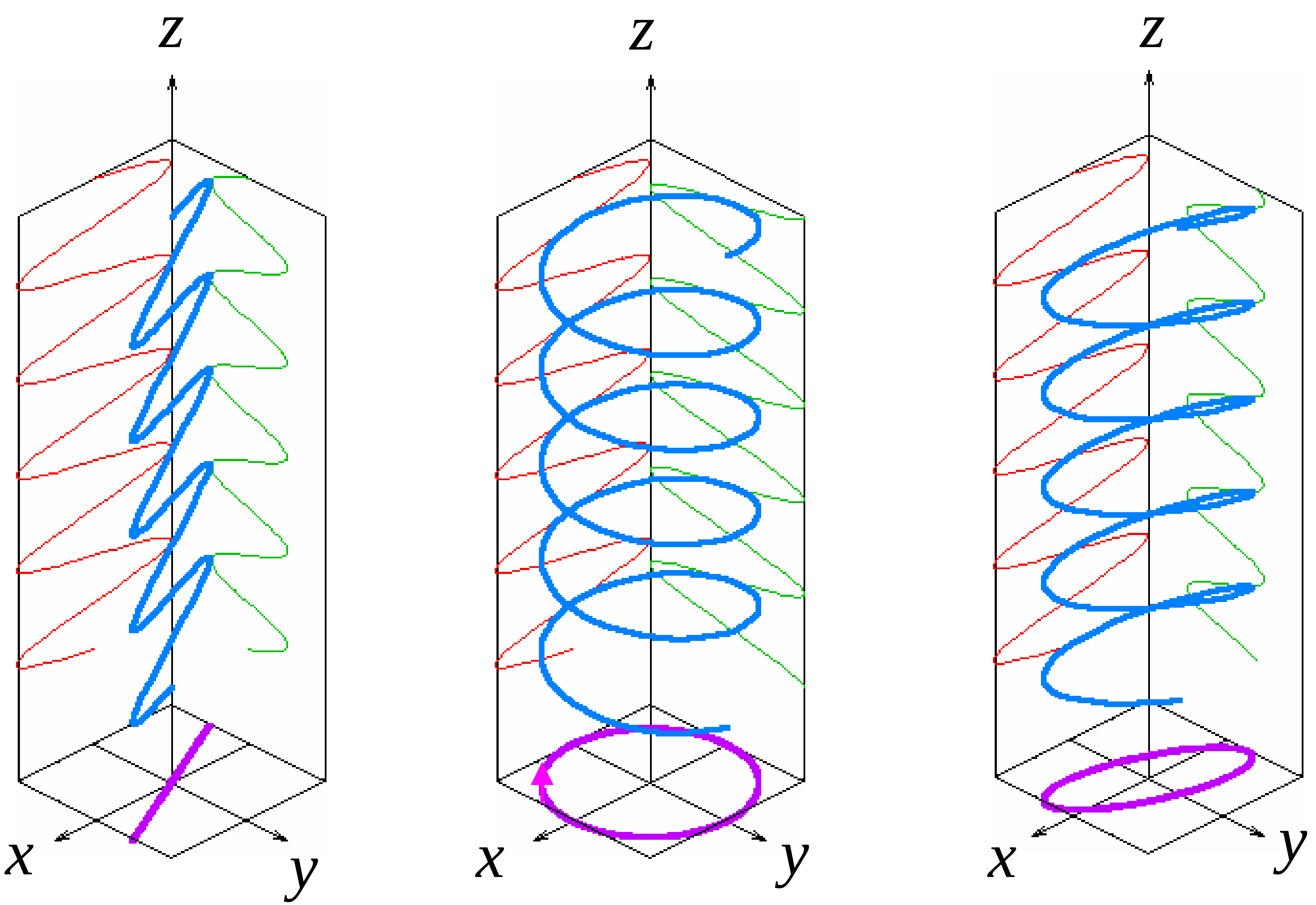
$$= \frac{1}{2}\left(\sqrt{|E_0|^2 + 2|E_0 E_0|\underline{\sin(-\pi/2)} + |E_0|^2} + \sqrt{|E_0|^2 - 2|E_0 E_0|\underline{\sin(-\pi/2)} + |E_0|^2} \right)$$

$$= \frac{1}{2}\left(\sqrt{|E_0|^2 - 2|E_0|^2 + |E_0|^2} + \sqrt{|E_0|^2 + 2|E_0|^2 + |E_0|^2} \right) = |E_0|$$

$$b = \frac{1}{2}\left| \sqrt{|E_x|^2 + 2|E_x E_y|\sin\phi + |E_y|^2} - \sqrt{|E_x|^2 - 2|E_x E_y|\sin\phi + |E_y|^2} \right|$$

$$= \frac{1}{2}\left| \sqrt{|E_0|^2 - 2|E_0|^2 + |E_0|^2} - \sqrt{|E_0|^2 + 2|E_0|^2 + |E_0|^2} \right| = \frac{1}{2}\left| -2|E_0| \right| = |E_0|$$

Illustration of different states of polarization



$\phi_x = \phi_y$

$\phi_y - \phi_x = \pi/2$

Sec. 16-2
Plane Waves in Lossy Media

1. Complex permittivity & wavenumber
2. Modified phase velocity & impedance
3. Wave behavior in dielectrics
4. Wave behavior in good conductors

EM fields in lossy media - Complex permittivity

If the medium is conducting ($\sigma \neq 0$), the presence of $\vec{E}$ results in conduction currents $\vec{J} = \sigma\vec{E}$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t};$$ if time-harmonic, simple medium

$$\Rightarrow \nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega\left(\frac{\sigma}{j\omega} + \varepsilon\right)\vec{E} = j\omega\varepsilon_c\vec{E},$$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

EM fields in lossy media - Complex wavenumber

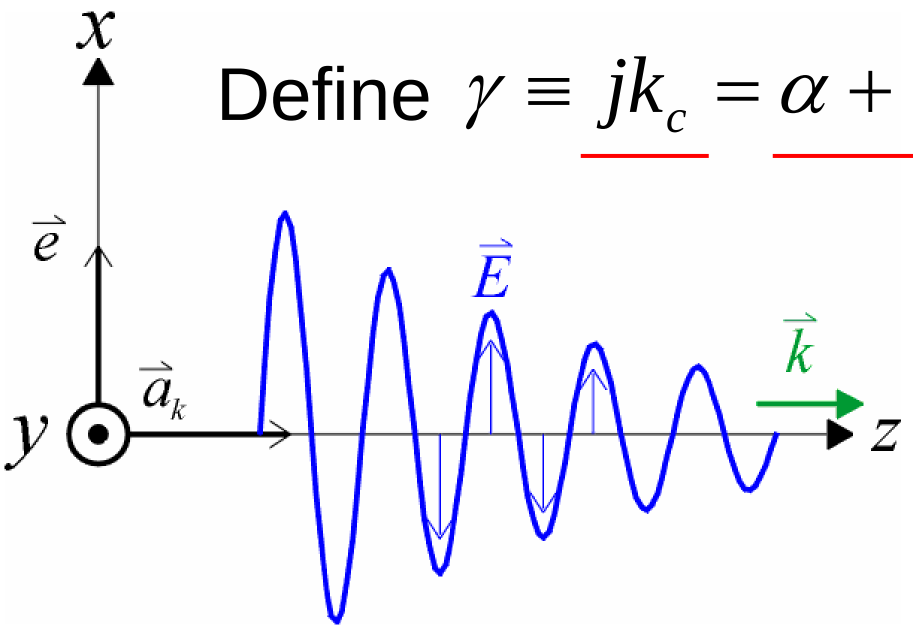
$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon(1 - j\tan\delta_c)$$

$\tan\delta_c$ → Loss tangent

$k = \omega\sqrt{\mu\varepsilon}$ ↓

$$k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu\varepsilon(1 - j\tan\delta_c)} \in C$$

Define $\gamma \equiv jk_c = \alpha + j\beta$ as propagation constant.



$$\vec{E}(z) = \vec{a}_x E_x(z),$$

$$E_x(z) = E_0^+ e^{-jk_c z}$$

$$= E_0^+ e^{-\gamma z} = E_0^+ e^{-\alpha z} e^{-j\beta z}$$

EM fields in lossy media - Modified phase velocity

For x-polarized, z-propagating plane waves in lossless media, $\Rightarrow$

$$E_x^+(z,t) = \mathrm{Re}\left\{E_x^+(z)e^{j\omega t}\right\} = \left|E_0^+\right|\cos\left(\underline{\omega t - kz} + \phi^+\right)$$

function of $\tau = t - \dfrac{z}{\omega/k}$

In lossy media, $\Rightarrow$

$$E_x^+(z,t) = \left|E_0^+\right|\underline{e^{-\alpha z}}\cos\left(\underline{\omega t - \beta z} + \phi^+\right), \quad \boxed{u_p = \frac{\omega}{\beta}} \in R$$

EM fields in lossy media - Modified wave impedance

For plane waves, $\Rightarrow \vec{H} = \dfrac{\vec{a}_k \times \vec{E}}{\eta},$

In lossless media, $\Rightarrow \eta = \dfrac{\omega\mu}{k} = \sqrt{\dfrac{\mu}{\varepsilon}} \in R$

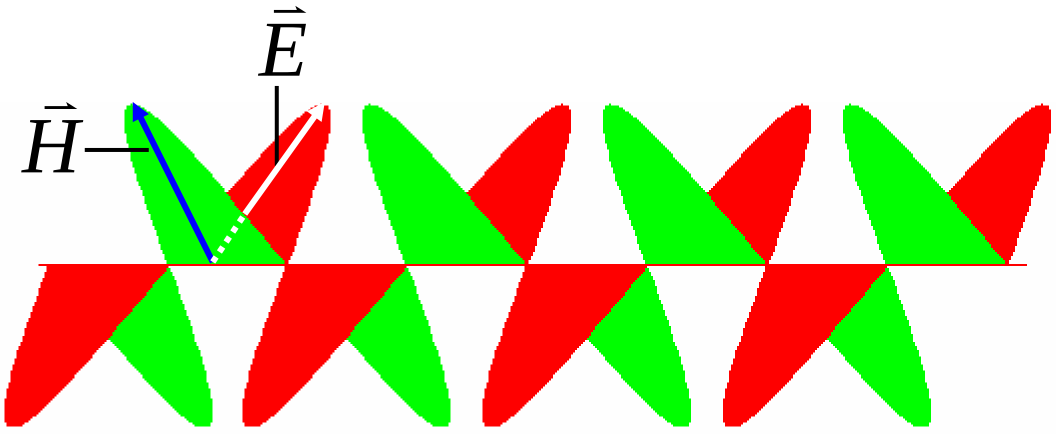
$\Rightarrow \vec{E}, \vec{H}$ are in-phase.

In lossy media, $\Rightarrow \boxed{\eta_c = \dfrac{\omega\mu}{k_c} = \sqrt{\dfrac{\mu}{\varepsilon_c}}} = |\eta| e^{j\theta_\eta} \in C$

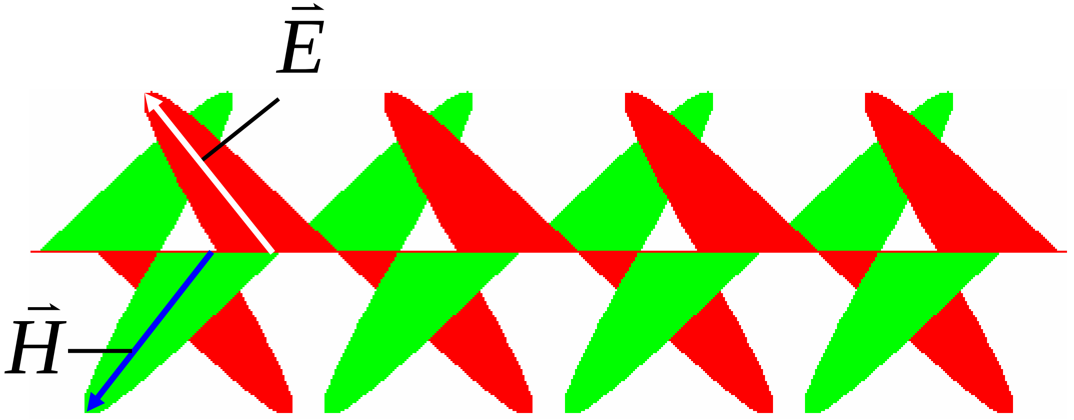
$\Rightarrow \theta_\eta \neq 0$ means $\vec{E}, \vec{H}$ are out of phase.

EM waves with/without phase shift between E & H

$\theta_\eta = 0,\ \vec{E}\ ,\ \vec{H}$ are in-phase.



$\theta_\eta \neq 0,\ \vec{E}\ ,\ \vec{H}$ are out of phase.



(www.blackholeformulas.com)

"Low-loss(?)" dielectrics (1)

If $\tan\delta_c << 1$, $\Rightarrow k_c \approx \omega\sqrt{\mu\varepsilon} \in R$, the medium behaves like a dielectric.

$\gamma \equiv jk_c$

$\varepsilon_c = \varepsilon[1 - j(\sigma/\varepsilon\omega)]$

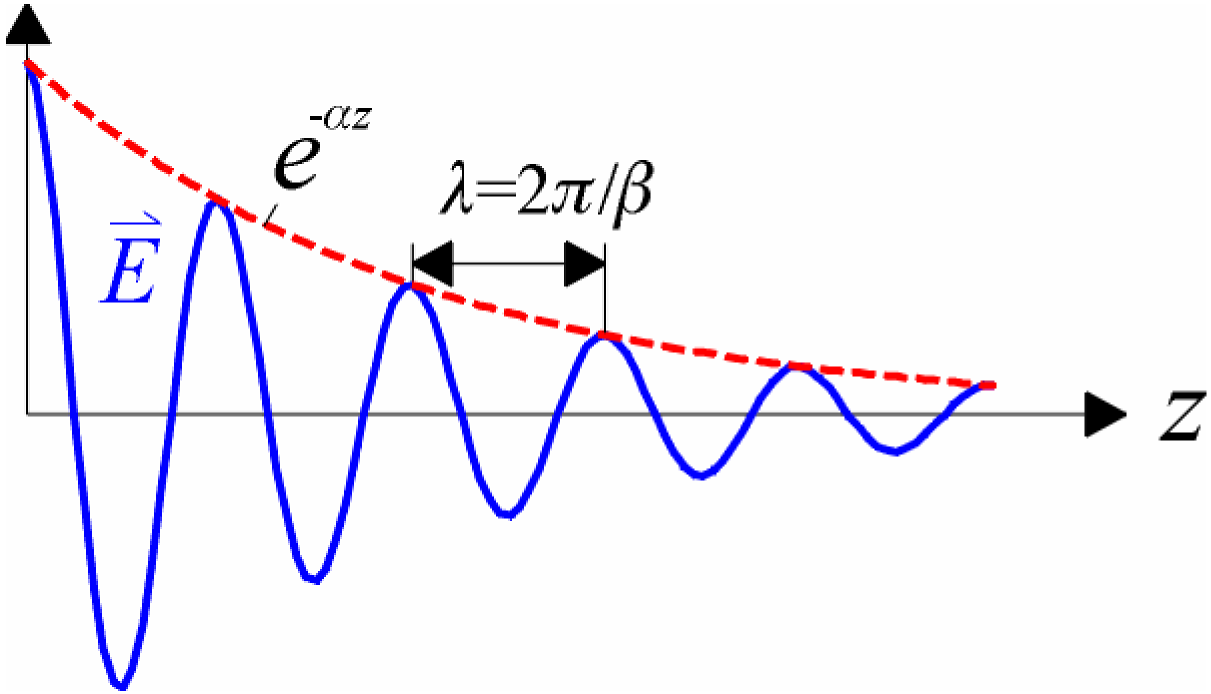
$$\gamma = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$(1-x)^{1/2} \approx 1 - \frac{1}{2}x + \frac{1}{8}x^2$

$$\approx j\omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{2\omega\varepsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right] = \alpha + j\beta$$

$$\Rightarrow \boxed{\alpha \approx \frac{\sigma\eta}{2}} \quad \boxed{\beta \approx \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right] \approx \omega\sqrt{\mu\varepsilon}}$$

“Low-loss(?)” dielectrics (2)



$\alpha \approx \dfrac{\sigma\eta}{2}$ …amplitude decay is indep. of freq., but can be quite large!

$\beta \approx \omega\sqrt{\mu\varepsilon}$ …$\lambda \propto f^{-1}$, similar to that in lossless media.

Dielectrics (3)

$$u_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon}}\left[1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right] < \frac{1}{\sqrt{\mu\varepsilon}}$$

…velocity is slightly slower.

$$\beta \approx \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

$$(1-x)^{-1/2} \approx 1 + \frac{1}{2}x$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon}}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{-1/2} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$\varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

… $\vec{E}$, $\vec{H}$ are slightly not in-phase.

Good conductors (1)

If $\tan\delta_c >> 1$, $\Rightarrow \mathrm{Im}\{k_c\}$ is non-negligible, the medium behaves like a good conductor.

$$\varepsilon_c = \varepsilon[1 - j(\sigma/\varepsilon\omega)]$$

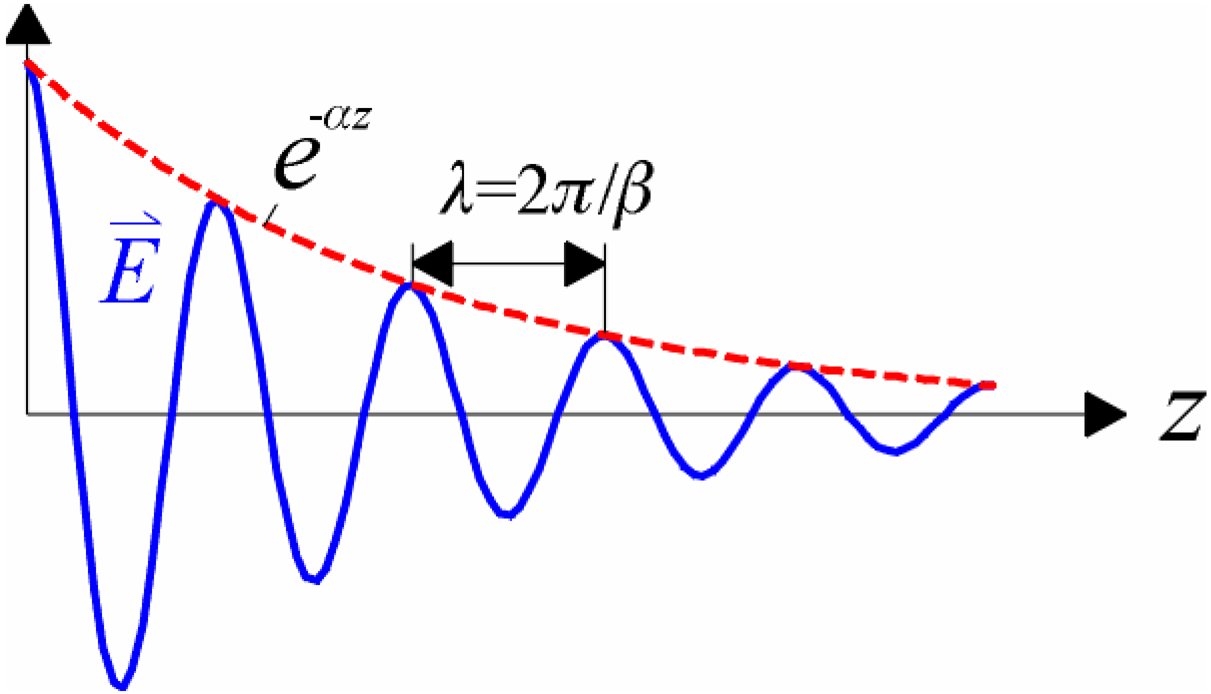
$$\gamma = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \approx j\omega\sqrt{\mu\varepsilon\left(-j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$= e^{j\frac{\pi}{2}}\omega\sqrt{e^{-j\frac{\pi}{2}}\frac{\mu\sigma}{\omega}} = e^{j\frac{\pi}{4}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta$$

$$\omega = 2\pi f$$

$$\Rightarrow \boxed{\alpha = \beta \approx \sqrt{\pi f\mu\sigma}}$$

Good conductors (2)



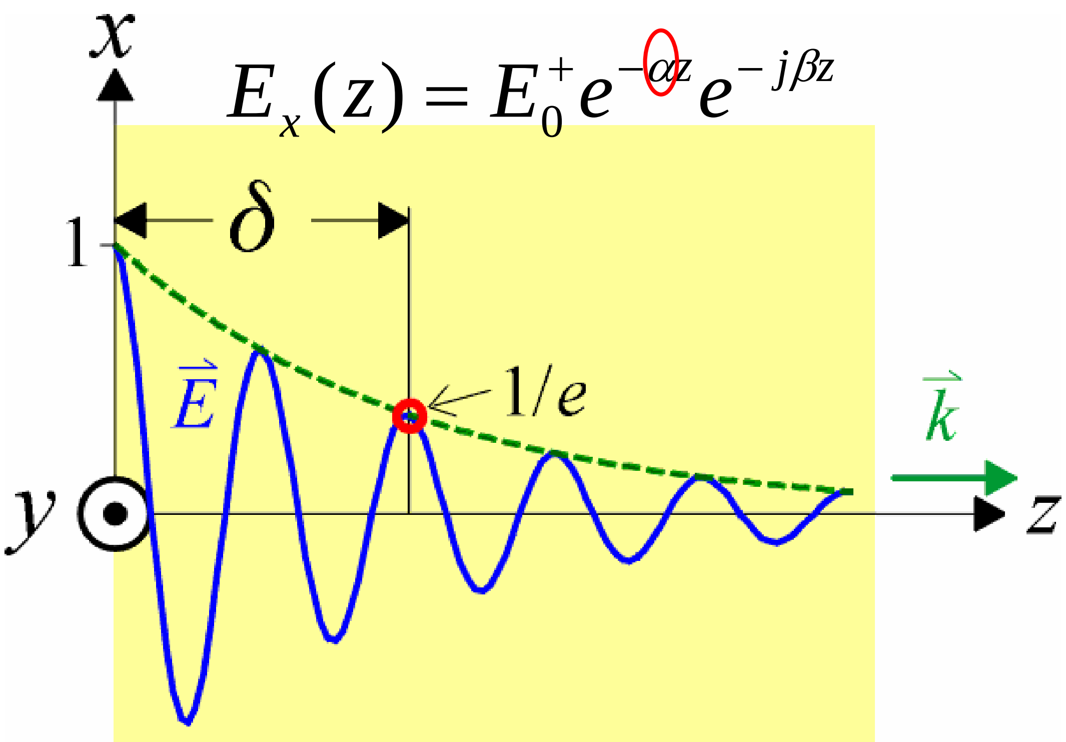
$\alpha \approx \sqrt{\pi f \mu \sigma}$ …amplitude decay increases with freq.

$\beta \approx \sqrt{\pi f \mu \sigma}$ …λ is not $\propto f^{-1}$, unlike that in lossless media.

Good conductors (3)

Since $\alpha \approx \sqrt{\pi f \mu \sigma} \propto \sqrt{f}$, high-frequency EM waves are attenuated rapidly as it propagates through a good conductor.

$$E_x(z) = E_0^+ e^{-\alpha z} e^{-j\beta z}$$



Skin depth:

$$\boxed{\delta \equiv \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}}$$

$$\because \alpha \approx \beta, \Rightarrow$$

$$\delta \approx \frac{1}{\beta} = \frac{\lambda}{2\pi}$$

Good conductors (4)

$$u_p = \frac{\omega}{\beta} \approx \sqrt{\frac{2\omega}{\mu\sigma}} << \frac{1}{\sqrt{\mu\varepsilon}}$$ …velocity is much slower.

$$\beta \approx \sqrt{\pi f \mu \sigma}$$

$$\varepsilon_c = \varepsilon\left[1 - j(\sigma/\omega\varepsilon)\right]$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon\left[1 - j(\sigma/\omega\varepsilon)\right]}} \approx \sqrt{\frac{\mu}{\varepsilon\left[-j(\sigma/\omega\varepsilon)\right]}}$$

$$= \sqrt{\frac{\omega\mu}{e^{-j\pi/2}\sigma}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

… $\vec{E}$, $\vec{H}$ have 45° phase difference.

$$e^{j\pi/4} \approx (1+j)/\sqrt{2}$$

Example 16-3: Attenuation of EM waves in sea water (1)

Sea water: $\sigma = 4\,(\mathrm{S/m})$, $\varepsilon = 72\varepsilon_0$:

At f =3 MHz, $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{4}{2\pi\left(3\times10^{6}\right)\left(72\dfrac{10^{-9}}{36\pi}\right)} \approx 333 \underline{>>1}$

good conductor

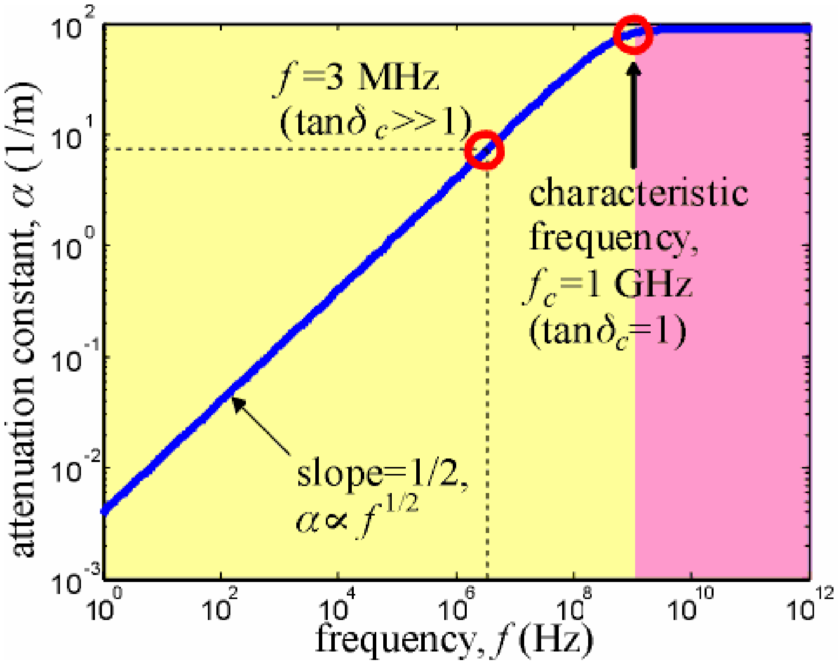
$$u_p \approx \sqrt{\frac{2\omega}{\mu\sigma}} = 2.7\times10^{6}\,(\mathrm{m/sec}) = \frac{c}{111}$$

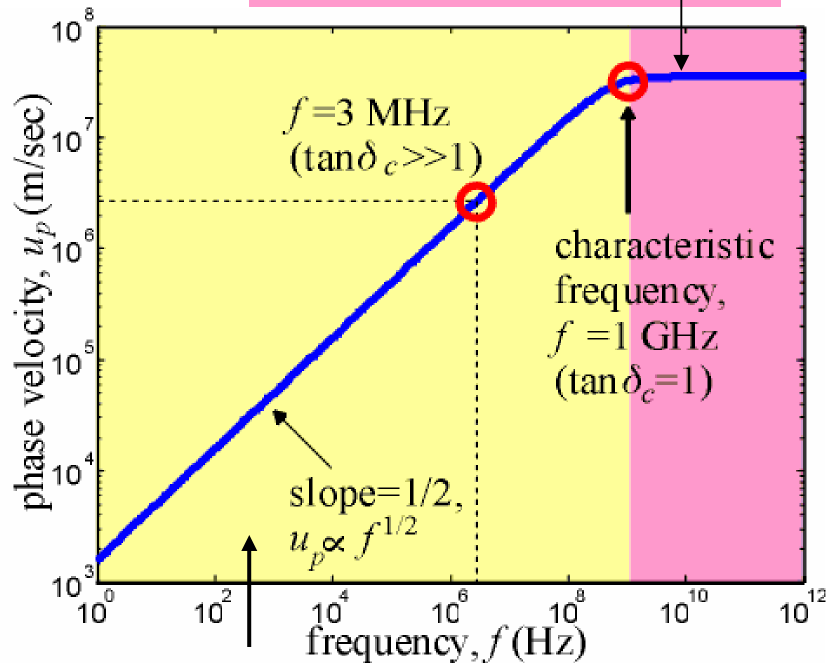
$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} \approx 14\,(\mathrm{cm})$$

Example 16-3: Attenuation of EM waves in sea water (2)

The loss of EM waves in “dielectric” is larger than in “good conductor”!

$$\alpha \approx \frac{\sigma\eta}{2}, \beta \approx \omega\sqrt{\mu\varepsilon}$$





$$\alpha = \beta \approx \sqrt{\pi f \mu \sigma}$$

Sec. 16-3
Power Flow of EM Waves

1. Instantaneous power
2. Time-averaged power

Physical meaning of Poynting vector-1

By vector identity: $\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)$

$$\Rightarrow \nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \left(\nabla \times \vec{E}\right) - \vec{E} \cdot \left(\nabla \times \vec{H}\right)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$= -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t},$$ in simple media:

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \vec{H} \cdot \left(\mu \frac{\partial \vec{H}}{\partial t}\right) = \mu H \cdot \frac{\partial H}{\partial t} = \frac{\mu}{2} \frac{\partial H^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}\right) = \frac{\partial}{\partial t}\left(\frac{\vec{H} \cdot \vec{B}}{2}\right)$$

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\vec{D} \cdot \vec{E}}{2}\right)$$

Physical meaning of Poynting vector-2

$$\Rightarrow \nabla\cdot\left(\vec{E}\times\vec{H}\right)=-\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right)-\vec{E}\cdot\vec{J}-\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right)$$

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s}+\int_V \frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right)dv+\int_V \frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right)dv+\int_V \left(\vec{E}\cdot\vec{J}\right)dv=0$$

$w_e\ (\mathrm{J/m^3})$ $w_m\ (\mathrm{J/m^3})$ Ohmic power dissipation

total power stored and dissipated in volume V

Physical meaning of Poynting vector-3

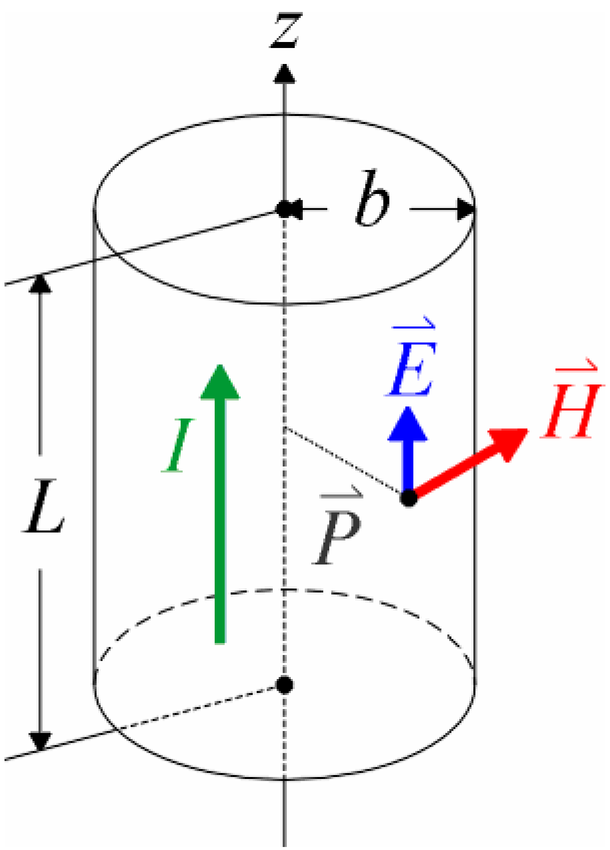
$$\Rightarrow \oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s}$$ …power flow out of volume V

$$\Rightarrow \boxed{\vec{P}=\vec{E}\times\vec{H}\ \ \left(\mathrm{W/m^2}\right)}$$ …Poynting vector, instantaneous directed power density.

(Valid for all EM waves, not just plane waves)

Example 16-4: Insight of a conducting wire (1)

Consider a conducting wire of radius b, conductivity σ, flowing uniform dc current I, $\Rightarrow \vec{P} = ?$



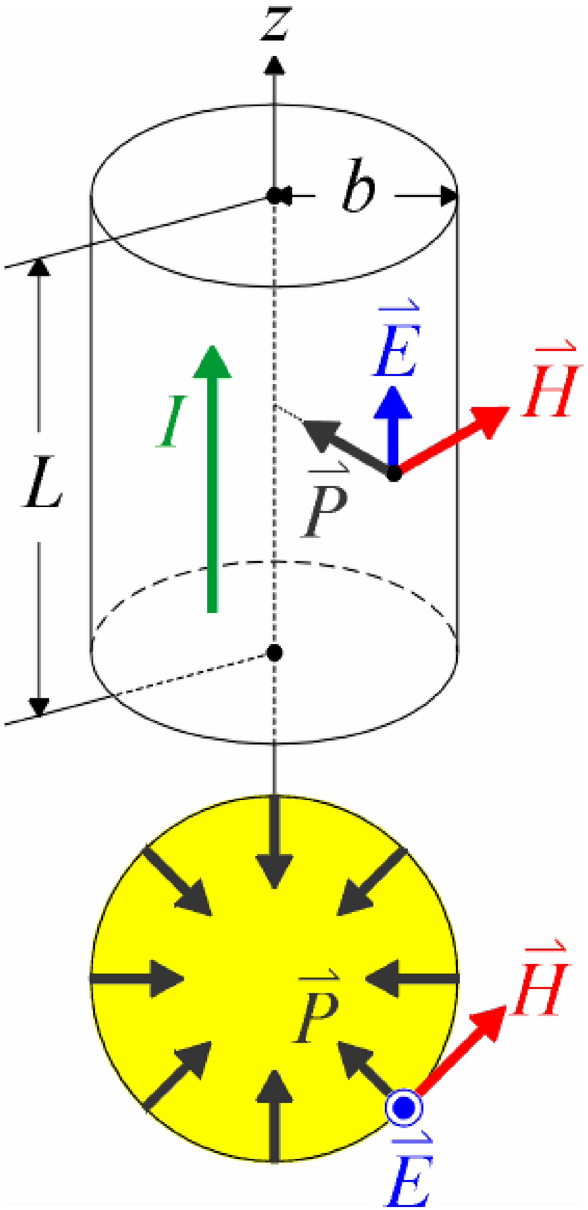
$$\vec{J} = \vec{a}_z \frac{I}{\pi b^2}, \quad \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = \vec{a}_z \frac{I}{\sigma \pi b^2}$$

everywhere

$$\oint_C \vec{H} \cdot d\vec{l} = I, \quad \Rightarrow \vec{H} = \vec{a}_\phi \frac{I}{2\pi b}$$

surface

Example 16-4: Insight of a conducting wire (2)



$$\vec{P} = \left(\vec{a}_z \frac{I}{\sigma\pi b^2}\right) \times \left(\vec{a}_\phi \frac{I}{2\pi b}\right)$$

$$= -\vec{a}_r \frac{I^2}{2\sigma\pi^2 b^3}\left(\frac{\text{W}}{\text{m}^2}\right)$$

not $+\vec{a}_z$

$$P_{tot} = -\oint_S \vec{P}\cdot d\vec{s} = \left(\vec{a}_r \frac{I^2}{2\sigma\pi^2 b^3}\right)\cdot\left(\vec{a}_r 2\pi bL\right)$$

$$= I^2 \frac{L}{\sigma\pi b^2} = I^2 R\ (\text{W})$$

Time-averaged Poynting vector of time-harmonic waves

For a general time-harmonic (non-plane) wave:

$$\vec{P}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = \mathrm{Re}\left[\underline{\vec{E}(\vec{r})}e^{j\omega t}\right] \times \mathrm{Re}\left[\underline{\vec{H}(\vec{r})}e^{j\omega t}\right]$$

vector phasors

By $\mathrm{Re}(\vec{A}) \times \mathrm{Re}(\vec{B}) = \frac{1}{2}\mathrm{Re}\left(\vec{A} \times \vec{B}^* + \vec{A} \times \vec{B}\right)$

$$\Rightarrow \vec{P}(\vec{r},t) = \frac{1}{2}\mathrm{Re}\left(\underline{\vec{E} \times \vec{H}^*} + \underline{\vec{E} \times \vec{H}e^{j2\omega t}}\right)$$

t-indep. oscillating with 2ω

$$\vec{P}_{av}(\vec{r}) = \frac{1}{T}\int_0^T \vec{P}(\vec{r},t)dt, \Rightarrow \boxed{\vec{P}_{av}(\vec{r}) = \frac{1}{2}\mathrm{Re}\left\{\vec{E} \times \vec{H}^*\right\}}$$

Time-averaged Poynting vector of time-harmonic plane waves-1

For a general time-harmonic plane wave:

$$\vec{P}_{av}(\vec{r}) = \frac{1}{2}\text{Re}\left\{\vec{E}\times\vec{H}^*\right\}$$

$$\vec{H} = \frac{\vec{a}_k \times \vec{E}}{\eta}$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{C}\cdot\vec{A}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

$$= \frac{1}{2}\text{Re}\left\{\vec{E}\times\left(\frac{\vec{a}_k\times\vec{E}^*}{\eta_c^*}\right)\right\} = \frac{1}{2}\text{Re}\left\{\vec{a}_k\frac{\left|\vec{E}\right|^2}{\eta_c^*} - \vec{E}^*\left(\frac{\vec{E}\cdot\vec{a}_k}{\eta_c^*}\right)\right\}$$

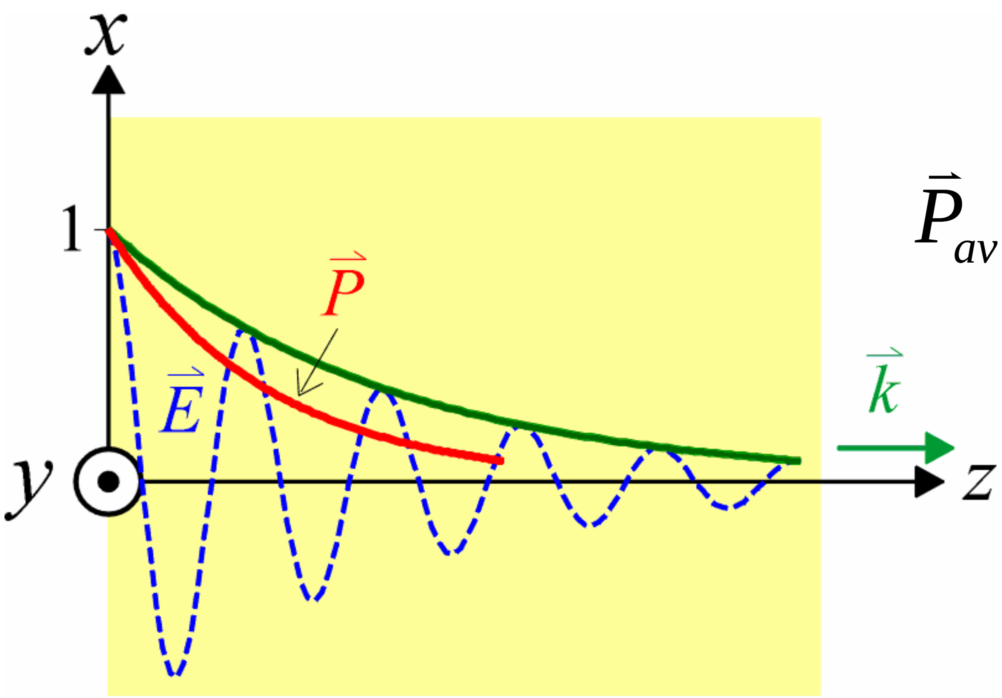
$$= \frac{1}{2}\text{Re}\left\{\vec{a}_k\frac{\left|\vec{E}\right|^2}{\eta_c^*}\right\} = \vec{a}_k\frac{\left|\vec{E}\right|^2}{2|\eta|}\cos\theta_\eta$$

$$\eta_c = |\eta|e^{j\theta_\eta}$$

Time-averaged Poynting vector of time-harmonic plane waves-2

For a time-harmonic plane wave x-polarized and propagates in the $+z$-direction:

$$\vec{a}_k = \vec{a}_z, \; \vec{E}(\vec{r}) = \vec{a}_x E_0^+ \underline{e^{-\alpha z}} e^{-j\beta z}, \quad \vec{P}_{av}(\vec{r}) = \vec{a}_k \frac{\left|\vec{E}(\vec{r})\right|^2}{2|\eta|} \cos\theta_\eta$$



$$\vec{P}_{av}(z) = \vec{a}_z \frac{\left|E_0^+\right|^2}{2|\eta|} \underline{e^{-2\alpha z}} \cos\theta_\eta$$

Time-averaged Poynting vector of time-harmonic plane waves-3

$$\vec{P}_{av}(z) = \vec{a}_z \frac{\left|E_0^+\right|^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta$$ means:

1. Power is transmitted along wavevector direction.

2. If $\sigma=0$, $\{\alpha = 0, \theta_\eta = 0\}$, $\vec{P}_{av}(z) = \vec{a}_z \frac{\left|E_0^+\right|^2}{2|\eta|}$,

   plane wave has constant power density (no attenuation).

3. If $\sigma \neq 0$, power density decays due to field attenuation ($\alpha > 0$), E-H phase mismatch $(\theta_\eta \neq 0)$

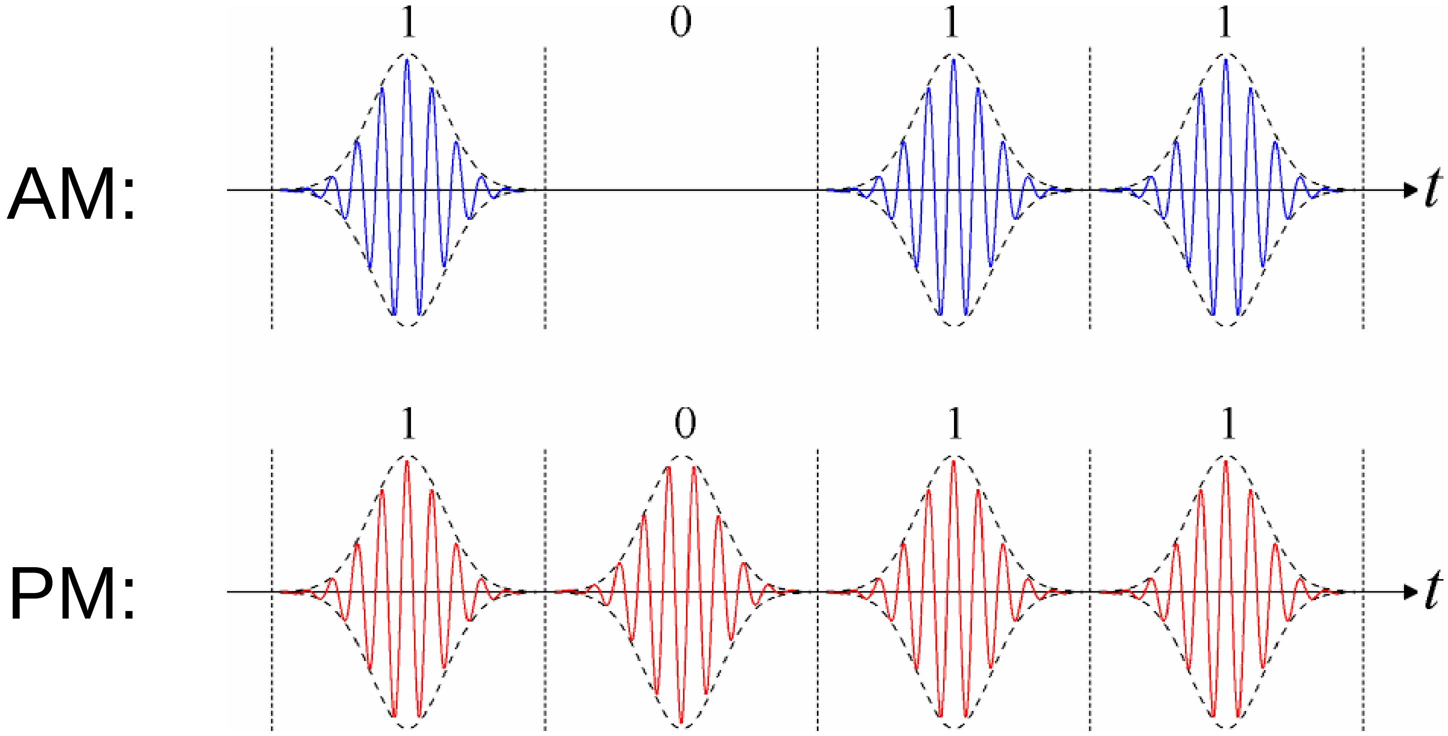
Sec. 16-4
Plane Waves in Dispersive Media

1. Fundamentals
2. Propagation of beat waves
3. Propagation of general wave packets

Why to discuss wave packets?

Sinusoidal waves are useful in power delivery, but carry no information.

To carry information, some type of modulation (causing wave packets) is required.



What is dispersion?

A dispersive medium means $\{\varepsilon, \mu\}$ depend on ω

$\Rightarrow$ phase velocity depends on ω: $u_p(\omega) = \dfrac{1}{\sqrt{\mu(\omega)\varepsilon(\omega)}}$

A wave packet consists of multiple frequency components, and will distort during propagation.

Analysis of dispersion-1

Consider linearly polarized plane waves propagating in the $+z$-direction through a source-free $\left(\rho = 0, \vec{J} = 0\right)$, simple, non-magnetic $\left(\mu = \mu_0\right)$, dispersive medium:

$$\vec{E} = \vec{e}E(z,\omega) \longrightarrow \nabla^2 \vec{E} + k^2 \vec{E} = 0$$

…Freq. domain vector wave eq.

$$\Rightarrow \frac{d^2}{dz^2} E(z,\omega) + \underline{k^2(\omega)} E(z,\omega) = 0$$

…Freq. domain scalar wave eq.

$$k(\omega) = \omega\sqrt{\mu_0 \varepsilon(\omega)}$$

Analysis of dispersion-2

The general solution (propagation in $+z$-direction)

$$E(z,\omega) = E_0(\omega) e^{-jk(\omega)z}$$

1. A time-harmonic wave of ω_1 will experience a a phase shift $-k(\omega_1)L$ after a distance L.

2. A wave packet with spectrum $E_0(\omega)$ at z=0 will experience a spectral phase modulation:

$$\boxed{\Delta\psi(\omega) = -k(\omega)L}$$

after a distance L, $\Rightarrow$ distortion in time domain.

Beat wave of two frequency components-1

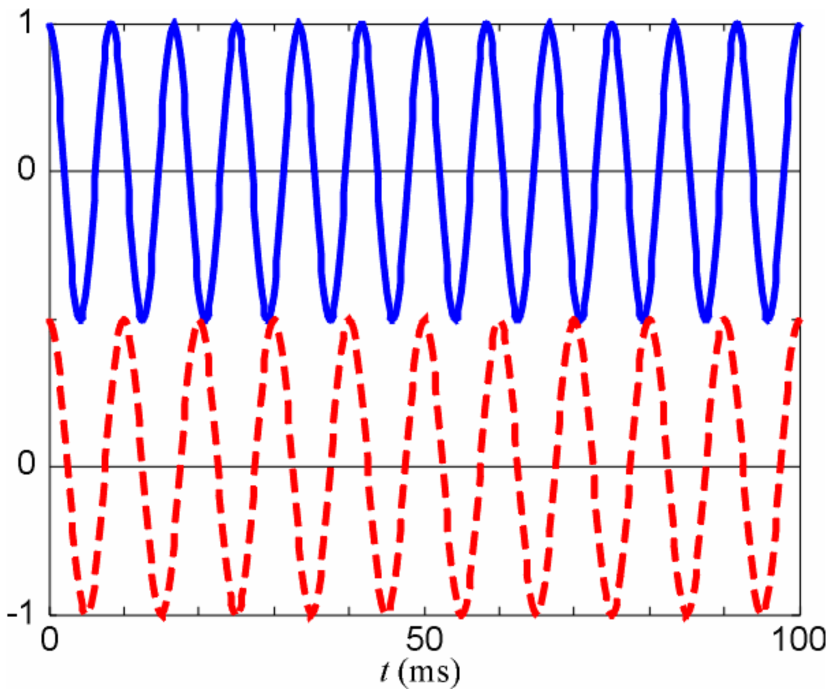
Consider the superposition of 2 time-harmonic waves: $e(z,t) = E_0 \cos(\omega_1 t - k_1 z) + E_0 \cos(\omega_2 t - k_2 z)$

$$k_i = k(\omega_i) = \omega\sqrt{\mu_0 \varepsilon(\omega_i)}$$

$z = 0:$

$f_1 = 120\,(\text{Hz})$

$f_2 = 100\,(\text{Hz})$



Beat wave of two frequency components-2

$$e(z,t) = E_0 \cos(\omega_1 t - k_1 z) + E_0 \cos(\omega_2 t - k_2 z)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Delta k = k(\Delta\omega) \qquad k_m = k(\omega_m)$$

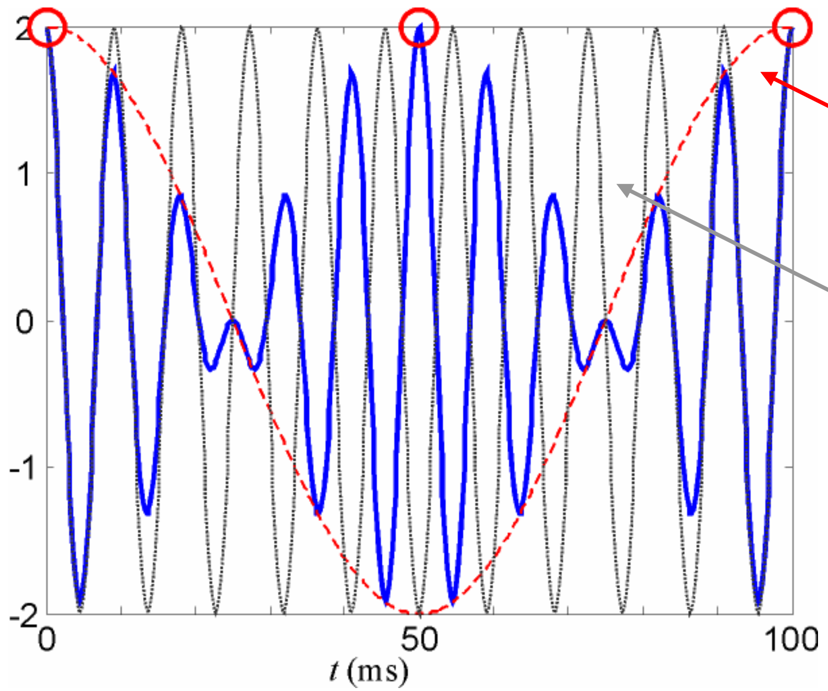
$$\boxed{e(z,t) = 2E_0 \cos(\Delta\omega \cdot t - \Delta k \cdot z)\cdot\cos(\omega_m t - k_m z)}$$

$$\Delta\omega = \frac{\omega_1 - \omega_2}{2} \qquad \omega_m = \frac{\omega_1 + \omega_2}{2}$$

Beat wave of two frequency components-3

$$e(z,t) = 2E_0 \cos(\Delta\omega \cdot t - \Delta k \cdot z) \cdot \cos(\omega_m t - k_m z)$$

envelope: $e_n(z,t)$ carrier: $e_c(z,t)$



$\Delta f = 10\ (\text{Hz})$

$f_m = 110\ (\text{Hz})$

Phase velocity & group velocity

$$e_c(z,t) = \cos\left(\omega_m\left[t - z/\left(\omega_m/k_m\right)\right]\right) = f(\tau)$$

Carrier wave propagates with phase velocity:

$$u_p = \frac{\omega_m}{k_m} = \frac{1}{\sqrt{\mu_0 \varepsilon(\omega_m)}}$$

$$e_n(z,t) = \cos\left(\Delta\omega\left[t - z/\left(\Delta\omega/\Delta k\right)\right]\right) = f(\tau)$$

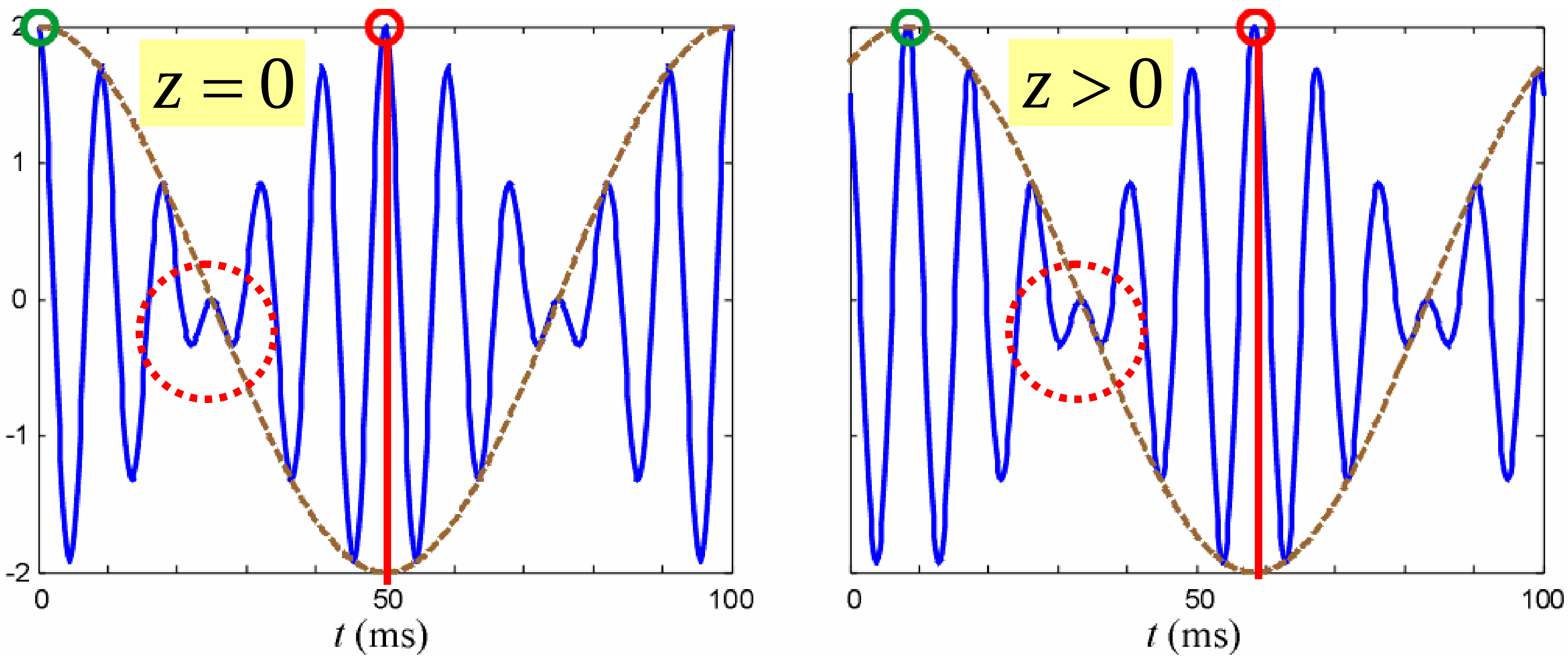
Envelope wave propagates with group velocity:

$$u_g = \frac{\Delta\omega}{\Delta k} \rightarrow \lim_{\Delta\omega \to 0} \frac{\Delta\omega}{\Delta k} = \frac{1}{k'(\omega_m)}$$

Propagation of beat wave in dispersion-free media

$$\varepsilon \neq \varepsilon(\omega), \quad \Rightarrow k(\omega) = \omega\sqrt{\mu_0\varepsilon} \propto \omega, \quad k'(\omega) = \sqrt{\mu_0\varepsilon},$$

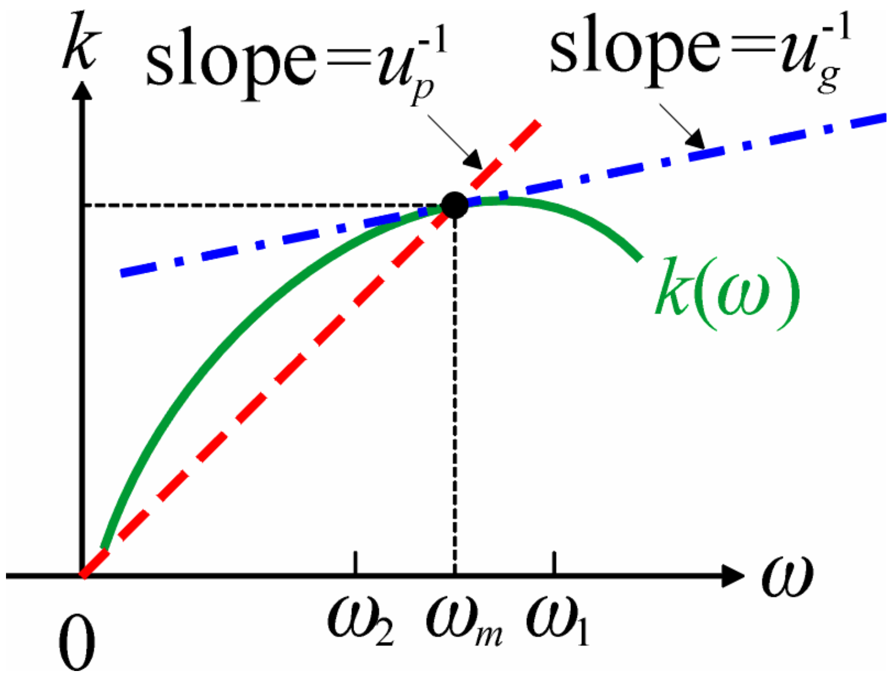
$$\Rightarrow u_g = \frac{1}{k'(\omega_m)} = u_p$$, beat waveform indep. of z



Propagation of beat wave in dispersive media-1

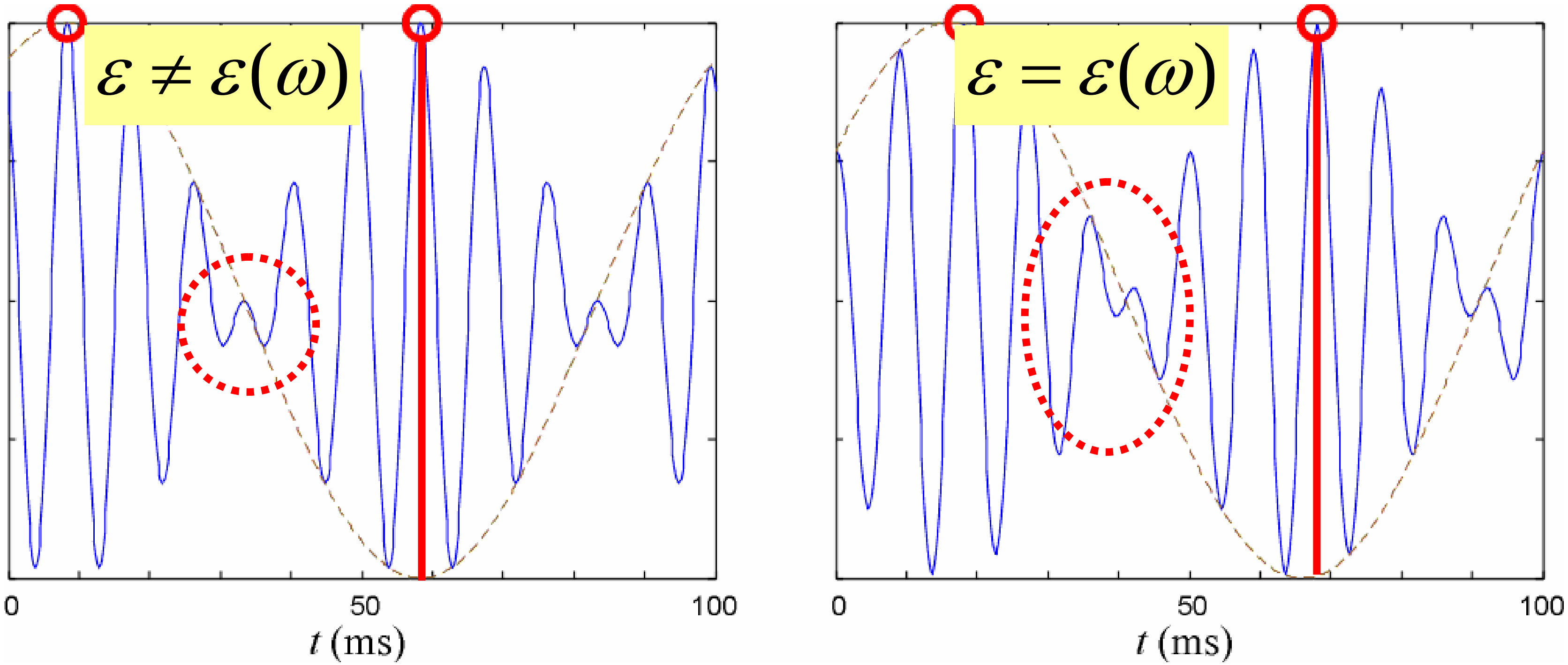
$$\varepsilon = \varepsilon(\omega), \Rightarrow k(\omega) = \omega\sqrt{\mu_0 \varepsilon(\omega)}, \quad k'(\omega) \neq \sqrt{\mu_0 \varepsilon(\omega)},$$

$$\Rightarrow u_g = \frac{1}{k'(\omega_m)} \neq u_p$$

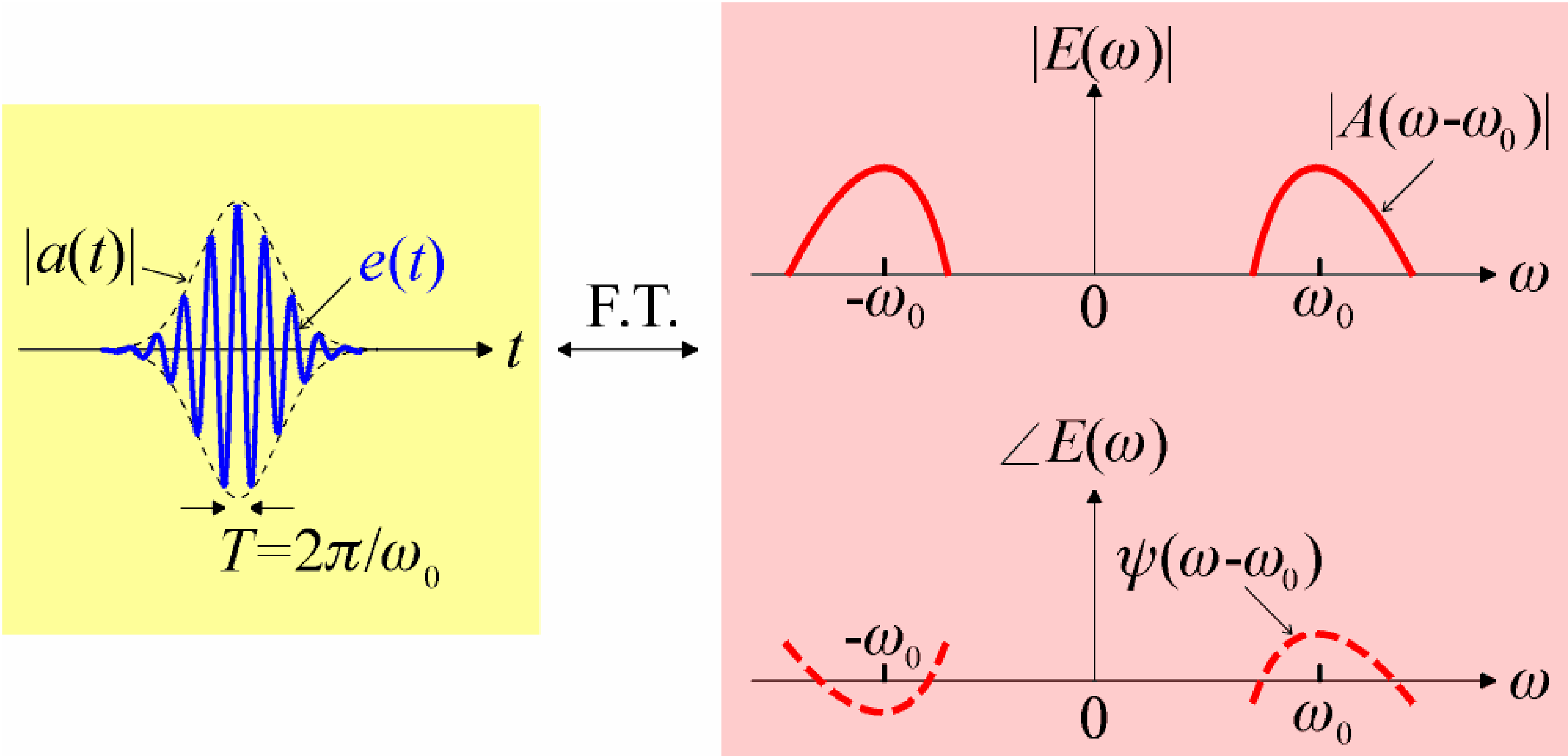


Propagation of beat wave in dispersive media-2

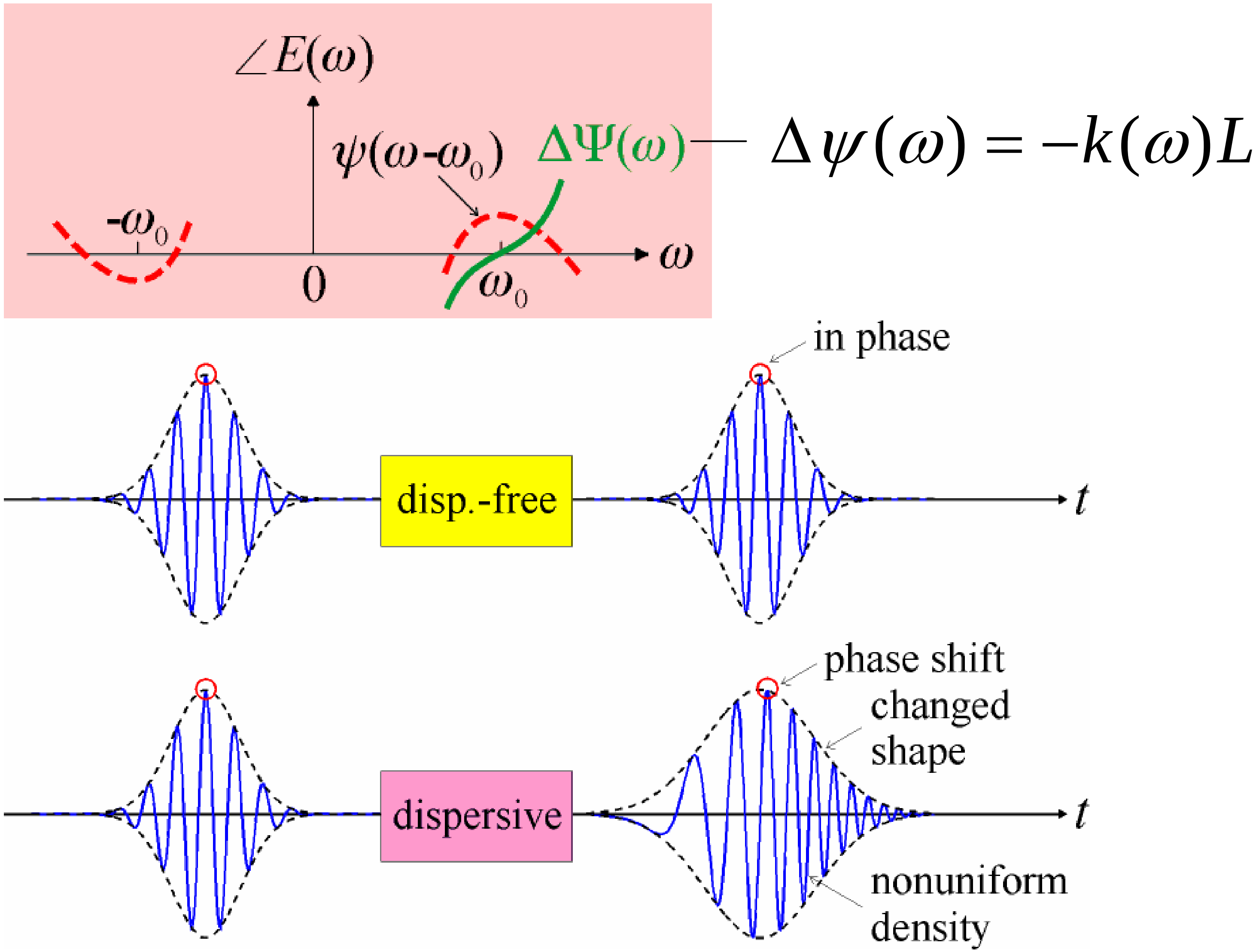
$$u_g = \frac{1}{k'(\omega_m)} \neq u_p, \Rightarrow \text{beat waveform varies with } z.$$



Carrier-envelope representation of general wave packets



Propagation of general wave packets thru dispersive media



Thanks for your attention,

& contributions!