# **Lesson 15** Wave Equations

## 15.1 Wave Equations of Fields

■ Homogeneous (source-free) wave equations in time domain

In a simple (linear, isotropic, homogeneous), charge-free ($\rho = 0$), nonconducting ($\sigma = 0$, $\vec{J} = 0$) medium, the four Maxwell's equations reduce to:

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{15.1}$$

$$\nabla \cdot \vec{E} = 0 \tag{15.2}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{15.3}$$

$$\nabla \cdot \vec{H} = 0 \tag{15.4}$$

Take curl for eq. (15.1) and employ eq. (15.3),

$$\nabla \times \nabla \times \vec{E} = \nabla \times \left( -\mu \frac{\partial \vec{H}}{\partial t} \right) = -\mu \frac{\partial}{\partial t} \left( \nabla \times \vec{H} \right) = -\mu \frac{\partial}{\partial t} \left( \varepsilon \frac{\partial \vec{E}}{\partial t} \right) = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}.$$

By eq's (5.31) and (15.2),

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}.$$

Therefore, we arrive at a second-order partial differential equation for vector field $\vec{E}$:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{15.5}$$

Similarly, combining eq's (15.1), (15.3), (15.4) leads to:

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{15.6}$$

Since eq's (15.5-6) are the same as the equations of lossless transmission lines [eq's (2.3-5)], the solutions $\vec{E}(\vec{r},t)$ $\vec{H}(\vec{r},t)$ must behave properties of waves propagating with phase velocity:

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} \tag{15.7}$$

Instead of investigating how electromagnetic waves are generated from time-varying sources $\left(\rho, \vec{J}\right)$, we are often concerned with how they propagate, where eq's (15.5-7) are useful.

■ Why time-harmonic (sinusoidal)?

1) Any periodic (aperiodic) function can be expressed by superposition of discrete (continuous) sinusoidal functions using Fourier series (integral).

2) Maxwell's equations are linear differential equations. $\Rightarrow$ (1) Sinusoidal sources $(\rho, \vec{J})$ will produce sinusoidal fields ($\vec{E}, \vec{H}$) of the same frequency in steady state. (2) Total field can be derived by superposition of individual sinusoidal responses.

3) Easy to operate if "phasors" are used: $\frac{\partial}{\partial t} \rightarrow j\omega$, $\int dt \rightarrow \frac{1}{j\omega}$.

Scalar phasors $V(z)$ and $I(z)$ of voltages and currents are sufficient to describe steady-state response of transmission lines [eq. (4.1)]. However, we need to use vector phasors $\vec{E}(x,y,z)$ and $\vec{H}(x,y,z)$ to represent time-harmonic electromagnetic fields:

$$\vec{E}(x,y,z,t) = \mathrm{Re}\{\vec{E}(x,y,z)e^{j\omega t}\}, \quad \vec{H}(x,y,z,t) = \mathrm{Re}\{\vec{H}(x,y,z)e^{j\omega t}\} \tag{15.8}$$

■ Homogeneous (source-free) wave equations in frequency domain

Replacing $\vec{E}$, $\vec{H}$ and $\frac{\partial}{\partial t}$ in eq's (15.1-4) by corresponding vector phasors and $j\omega$, we can get the time-harmonic Maxwell's equations in simple, charge-free, nonconducting media:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{15.9}$$

$$\nabla \cdot \vec{E} = 0 \tag{15.10}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \tag{15.11}$$

$$\nabla \cdot \vec{H} = 0 \tag{15.12}$$

Eq's (15.9-12) can be combined to get the time-harmonic version of eq's (15.5) and (15.6):

$$\nabla^2\vec{E} + k^2\vec{E} = 0 \tag{15.13}$$

$$\nabla^2\vec{H} + k^2\vec{H} = 0 \tag{15.14}$$

where

$$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u_p} = \frac{2\pi}{\lambda} \tag{15.15}$$

denotes the wavenumber (counterpart of "angular frequency" in space). The solutions to eq's (15.13), (15.14) represent propagating waves [e.g. $E_x(z) = \exp(-jkz)$], which will be the subject of subsequent lessons.

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Analogy can be found between the following pairs of equations: (15.13) ↔ (4.4), (15.14) ↔ (4.5), (15.15) ↔ (4.6). ⇒ Electromagnetic fields exhibit all "wave" properties of voltages and currents in transmission lines.

■ Electromagnetic fields in lossy medium

If the simple medium is conducting $(\sigma \neq 0)$, the presence of electric field $\vec{E}$ results in free conduction currents ($\vec{J} = \sigma\vec{E}$). Eq. (15.11) is generalized to:

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega\varepsilon_c\vec{E},$$

where

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} \tag{15.16}$$

represents the complex permittivity. As a result, the wavenumber [eq. (15.15)] also becomes complex:

$$k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu\varepsilon(1 - j\tan\delta_c)} \tag{15.17}$$

where

$$\tan\delta_c = \frac{\sigma}{\omega\varepsilon} \tag{15.18}$$

is called the loss tangent. If $\tan\delta_c << 1$, $k_c \approx \omega\sqrt{\mu\varepsilon} \in R$, the medium behaves like a "dielectric". If $\tan\delta_c >> 1$, the imaginary part of $k_c$ is not negligible and the medium behaves like a "good conductor". As will be elucidated in Lesson 16, $\text{Im}\{k_c\}$ represents the power loss when wave propagates through the medium. Physically, the loss results from: (1) The inertia of bound charges prevents the polarization $\vec{P}$ from keeping in phase with the external time-varying field $\vec{E}$, causing frictional damping; (2) Ohmic losses due to collision among free charges and atoms [eq. (10.12)]. The power loss, according to eq. (15.18), depends on both frequency $\omega$ and material properties $(\sigma, \varepsilon)$.

Example 15-1: A moist ground has $\varepsilon = 10\varepsilon_0$, and $\sigma = 10^{-2}$ (S/m). At a frequency of 1 KHz, $\tan\delta_c \sim 10^4$, it is like a good conductor. At a frequency of 10 GHz, $\tan\delta_c \sim 10^{-3}$, it is like a dielectric.

Example 15-2: A microwave oven produces a sinusoidal electric field with amplitude 250 (V/m) at 2.45 GHz. Find the averaged power density dissipated in a beef steak with $\varepsilon = 40\varepsilon_0$, and $\tan\delta_c = 0.35$.

Ans: By eq. (15.18), $\sigma = \omega\varepsilon\tan\delta_c = (2\pi \cdot 2.45\times10^9)\cdot\left(40\dfrac{10^{-9}}{36\pi}\right)\cdot(0.35) = 1.9$ (S/m). By eq. (10.12), the instantaneous power density is: $p(t) = \vec{E}(t)\cdot\vec{J}(t) = \sigma\left|\vec{E}(t)\right|^2$. The time-averaged value is: $\dfrac{1}{2}\sigma E_0^2 \approx 60$ (kW/m$^3$).

## 15.2 (*)Wave Equations of Potentials

- Fields & potentials

In the presence of time-varying fields:

1) $\nabla \cdot \vec{B} = 0$ [eq. (11.2)] is the same as that in magnetostatics, $\Rightarrow$ the relation between magnetic field and magnetic potential remains unchanged: $\vec{B} = \nabla \times \vec{A}$ [eq. (11.8)].

2) $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \neq 0$ [eq. (14.1)] is different from that in electrostatics, $\Rightarrow$ the relation between electric field and electric potential is no longer $\vec{E} = -\nabla V$ [eq. (6.11)]. Substituting eq. (11.8) into eq. (14.1):

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right), \Rightarrow \ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0\,,$$

from which we can define an artificial scalar potential $V$ such that $\vec{E} + \dfrac{\partial \vec{A}}{\partial t} = -\nabla V$. $\Rightarrow$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \tag{15.19}$$

This means that $\vec{E}$ can be decomposed into conservative and nonconservative components, contributed by charge distribution $\left(-\nabla V\right)$ and time-varying current $-\dfrac{\partial \vec{A}}{\partial t}$, respectively.

■ Nonhomogeneous (driven) wave equations of potentials $\vec{A}$, $V$ in time domain

For simple media, $\nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$ [eq. (14.12)] becomes:

$$\nabla \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}\,.$$

By eq's (11.8) and (15.19): $\nabla \times \vec{B} = \nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$, $\dfrac{\partial \vec{E}}{\partial t} = -\nabla\left(\dfrac{\partial V}{\partial t}\right) - \dfrac{\partial^2 \vec{A}}{\partial t^2}$, $\Rightarrow$

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla\left(\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right).$$

Since there is no physical restriction about $\nabla \cdot \vec{A}$, we choose Lorentz gauge:

$$\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0 \tag{15.20}$$

such that $\vec{A}$ is decoupled with $V$:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \tag{15.21}$$

For simple media, $\nabla \cdot \vec{D} = \rho$ [eq. (7.8)] becomes $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon}$. Substitute into eq. (15.19):

$$\nabla \cdot \vec{E} = -\nabla \cdot (\nabla V) - \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\nabla^2 V - \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right).$$

By eq. (15.20), $\nabla \cdot \vec{E} = -\nabla^2 V + \mu\varepsilon \dfrac{\partial^2 V}{\partial t^2} = \dfrac{\rho}{\varepsilon}, \Rightarrow$

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{15.22}$$

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1) Given charge and current distributions $\rho$ and $\vec{J}$, we can solve for the potential distributions $V$ and $\vec{A}$ by eq's (15.21), (15.22), then derive the fields $\vec{E}$ and $\vec{B}$ by eq's (15.19), (11.8).

2) In static cases: (1) Lorentz gauge $\nabla \cdot \vec{A} + \mu\varepsilon \dfrac{\partial V}{\partial t} = 0$ [eq. (15.20)] reduces to Coulomb's gauge $\nabla \cdot \vec{A} = 0$ [eq. (11.9)]. (2) Nonhomogeneous wave equations of potentials [eq's (15.21), (15.22)] reduce to Poisson's equations [eq's (11.9), (8.1)].

■ Solutions to nonhomogeneous wave equations of potentials in time domain

Consider a single point source at the origin (with spherical symmetry), the wave equation of electric potential $\nabla^2 V - \mu\varepsilon \dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\varepsilon}$ [eq. (15.22)] becomes homogeneous:

$$\frac{1}{R^2}\frac{\partial}{\partial R} R^2\left(\frac{\partial V}{\partial R}\right) - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0$$

except for the origin. Define $U(R,t) = R \cdot V(R,t), \Rightarrow$

$$\frac{\partial^2 U}{\partial R^2} - \mu\varepsilon \frac{\partial^2 U}{\partial t^2} = 0,$$

which is a standard wave equation as eq. (2.3). The general solution must be of the form:

$$U(R,t) = f(\tau),$$

where $f(\cdot)$ is any function of variable $\tau = t - R/u_p$, representing a wave traveling in the $+\vec{a}_R$ direction with velocity $u_p = \dfrac{1}{\sqrt{\mu\varepsilon}}$.

Since eq. (15.22) is linear, we can find the solution $\Delta V$ (impulse response) to a point charge at origin $\rho(t)dv'$ (impulse), then derive the solution $V$ to arbitrary charge distribution $\rho(\vec{r}',t)$ by superposition (integral). By eq. (6.13), the potential due to a "static" point charge $\rho dv'$ at the origin is: $\Delta V(R) = \dfrac{\rho dv'}{4\pi\varepsilon R}$ ($R$ is the radius in spherical coordinates). As a result, the particular solution to eq. (15.22) due to a "time-varying" point charge $\rho(t)dv'$ at origin should be:

$$\Delta V(R,t) = \frac{\rho(t - R/u_p)dv'}{4\pi\varepsilon R}$$

such that the form of general solution $U(R,t) = R \cdot \Delta V(R,t) = f(\tau)$ can be met. By superposition, the potential due to a charge distribution $\rho(\vec{r}',t)$ over a volume $V'$ is:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(\vec{r}', t - R'/u_p)}{R'} dv' \tag{15.23}$$

where $\vec{r}$ and $\vec{r}'$ denote the position vectors of observation and source points, while $R' = |\vec{r} - \vec{r}'|$.

Similarly, the vector potential due to a time-varying current density distribution $\vec{J}(\vec{r}',t)$ is:

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}', t - R'/u_p)}{R'} dv' \tag{15.24}$$

Eq's (15.23), (15.24) indicate that the potential of an observation point at time $t$ depends on the source value at an earlier time $t - R'/u_p$. In other words, the time-varying electromagnetic potentials (thus fields) can propagate with finite speed $u_p$.

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In circuit theory where the sources are assumed to be of low-frequencies, wavelength $\lambda$ is much longer than the linear dimension $L$ of the circuit, $\Rightarrow$ the time retardation $t_d = R'/u_p < L/u_p << \lambda/u_p = T$ $(T = 1/f)$ is negligible, $\Rightarrow$ eq's (15.23) and (15.24) reduce to eq's (6.15) and (11.11), respectively.

■ Nonhomogeneous (driven) wave equations of potentials in frequency domain

For time-harmonic (sinusoidal) waves, eq's (15.22) and (15.21) can be replaced by:

$$\nabla^2 V + k^2 V = -\frac{\rho}{\varepsilon} \tag{15.25}$$

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} \tag{15.26}$$

where $V$, $\rho$, $\vec{A}$, $\vec{J}$ are the corresponding scalar and vector phasors, and wavenumber $k$ is given in eq. (15.15). Solutions to eq's (15.25), (15.26) can be derived by taking Fourier transform for eq's (15.23), (15.24), where the time retardation $t_d = R'/u_p$ in a sinusoidal wave is substituted by a phase lead of $\omega \cdot t_d = kR'$:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(\vec{r}\,')e^{-jkR'}}{R'} dv' \tag{15.27}$$

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}\,')e^{-jkR'}}{R'} dv' \tag{15.28}$$

The Lorentz gauge eq. (15.20) becomes:

$$\nabla \cdot \vec{A} + j\omega\mu\varepsilon V = 0 \tag{15.29}$$

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The time retardation or phase lead becomes negligible if the low-frequency condition is satisfied: $t_d << T$, $R' << \lambda$, $kR' << 1$.

To determine electromagnetic fields due to time-harmonic charge and current distributions:

1) Find the phasors of potentials $V(\vec{r})$, $\vec{A}(\vec{r})$ by eq's (15.25), (15.26).

2) Find the phasors of fields $\vec{E}(\vec{r})$, $\vec{B}(\vec{r})$ by eq's (15.19), (11.8).

3) Find the instantaneous fields by $\vec{E}(\vec{r},t) = \mathrm{Re}\{\vec{E}(\vec{r})e^{j\omega t}\}$, $\vec{B}(\vec{r},t) = \mathrm{Re}\{\vec{B}(\vec{r})e^{j\omega t}\}$.

## 15.3 (*) Electromagnetic Spectrum

The frequencies (left axis), wavelengths (right axis), and common names (middle column) of different electromagnetic bands are illustrated in Fig. 11.1, followed by brief descriptions about the corresponding applications.

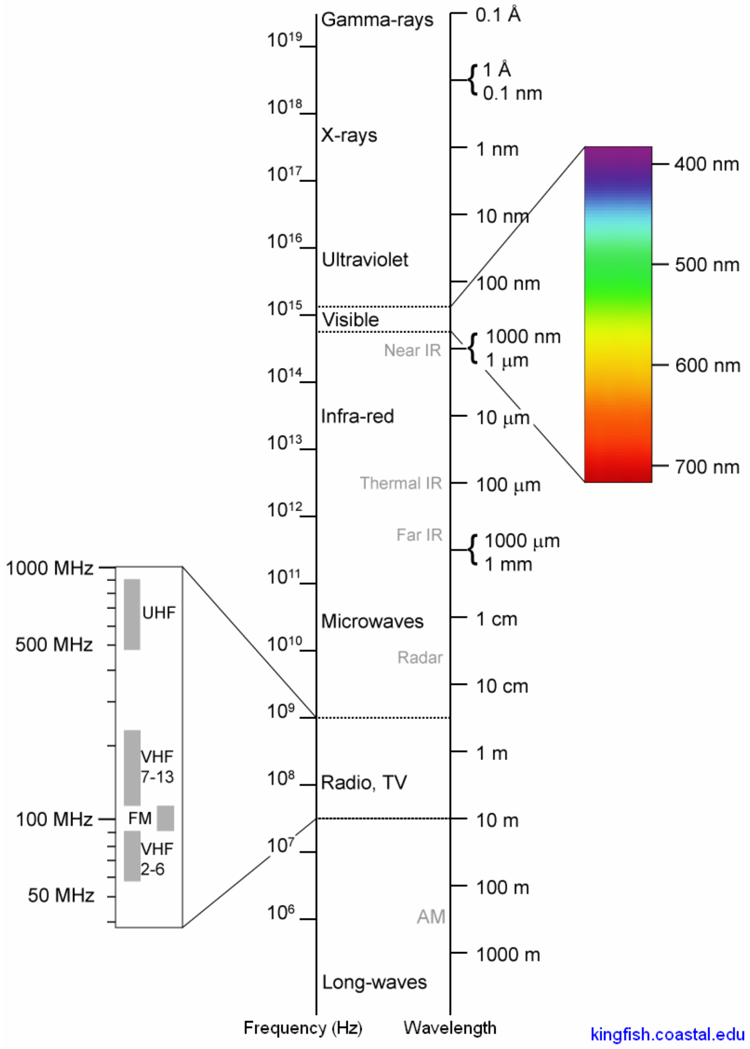


Fig. 15-1. Electromagnetic spectrum (after kingfish.coastal.edu).

Electromagnetics
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1) Extremely low frequency (ELF): $f = 3{-}30$ Hz. Global communications with submerged submarines (e.g. US Seafarer system: 76 Hz, wavelength $\lambda \approx 3900$ km).
2) Very low frequency (VLF): $f = 3{-}30$ kHz. submarine communications at shallow ($\approx 20$ m) depth.
3) Medium frequency (MF): $f = 0.3{-}3$ MHz. Amplitude modulation (AM) broadcast (0.53−1.61 MHz). Global maritime distress safety system ($\approx 0.5$ MHz).
4) High frequency (HF, or shortwave): $f = 3{-}30$ MHz. Amateur radio (by ionosphere reflection).
5) Very high frequency (VHF): $f = 30{-}300$ MHz. Frequency modulation (FM) broadcast (88−108 MHz). TV broadcast.
6) Ultra high frequency (UHF): $f = 300$ MHz−3 GHz. TV broadcast. Cell phone (0.9, 1.8 GHz). Cordless telephone (0.9, 2.4 GHz). GPS (1.6 GHz). Microwave oven (2.45 GHz).
7) Super high frequency (SHF): $f = 3{-}30$ GHz. Radar. Wireless local area network.
8) Terahertz: $f \approx 10^{12}$ Hz. Safe medical imaging. High-altitude (e.g. aircraft-to-satellite) communications. Security screening (e.g. plastic explosive).
9) Infrared (IR): $f = 10^{13}{-}10^{14}$ Hz, $\lambda = 0.7{-}100$ μm. Fiber communications ($\lambda \approx 1.55$ μm). Spectroscopy. Night vision. Thermal imaging (black body, live human $\lambda = 9.5$ μm). Missile guidance ($\lambda = 3{-}5$ μm, 8−13 μm). Astronomy (cool planet).
10) Visible (VIS): $f \approx 10^{15}$ Hz, $\lambda = 0.4{-}0.7$ μm. Lighting. Photography.
11) Ultraviolet (UV): $\lambda = 10{-}400$ nm. Photolithography ($\lambda = 248$ nm, 193 nm). Fluorescent lamp (mercury vapor discharges emit $\lambda = 254$ nm).
12) X-rays: $\lambda \approx 0.1$ Å−10 nm. Photon energy ~ keV. Crystallography. Medical diagnostics.
13) $\gamma-$rays: $\lambda < 0.1$ Å. Photon energy > MeV. Radiation therapy and diagnostics.