Lesson 14 Maxwell's Equations

■ Overview

The fundamental postulates of electrostatics and magnetostatics are summarized as:

$$\nabla \times \vec{E} = 0 \qquad (6.2)$$

$$\nabla \cdot \vec{D} = \rho \qquad (7.8)$$

$$\nabla \times \vec{H} = \vec{J} \qquad (12.4)$$

$$\nabla \cdot \vec{B} = 0 \qquad (11.2)$$

where $\vec{E}$ and $\vec{D}$ are independent of $\vec{H}$ and $\vec{B}$.

In the presence of time-varying fields, eq's (6.2) and (12.4) have to be modified to meet Faraday's law of electromagnetic induction and the equation of continuity [eq. (10.8)], respectively. As a result, $\vec{E}$ and $\vec{D}$ are coupled with $\vec{H}$ and $\vec{B}$, i.e. they must be solved simultaneously. The solutions of $\vec{E}$, $\vec{D}$, $\vec{H}$, $\vec{B}$ behave like waves, justifying the existence of electromagnetic waves.

14.1 Faraday's Law

■ Modified fundamental postulate

M. Faraday experimentally observed that a current will be induced in a conducting loop if the magnetic flux of the loop is changed. This fact can be mathematically described by modifying eq. (6.2) as:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad (14.1)$$

which is applicable in both free space and materials. The integral form of eq. (14.1) is:

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}, \qquad (14.2)$$

where S is the open surface bounded by contour C (the normal direction of S and the sense of rotation of C satisfy the right hand rule). Eq's (14.1), (14.2) imply that: (1) The time-varying $\vec{B}$ serves as the "vortex source" of $\vec{E}$. (2) The work done by the induced $\vec{E}$ along a closed loop is nonzero, i.e., the induced $\vec{E}$ is non-conservative. As a result, there is no well-defined potential value for any point in space (as opposed to the case of conservative $\vec{E}$ due to charges). Instead, the magnitude and sign of the induced electromotive force (emf)

$$\mathcal{V} \equiv \oint_C \vec{E} \cdot d\vec{l}$$

are used to describe the "tendency" of driving current in a conducting loop.

■ Stationary circuit in a time-varying magnetic field

If the observation surface S (and C) is stationary (time-independent), the order of time-derivative and surface-integral can be exchanged:

$$\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = \frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} = \frac{d}{dt} \Phi_{stat},$$

where Φ_{stat} is the magnetic flux over the stationary surface S. By eq. (14.2), the induced "transformer emf" is:

$$\mathcal{V}_t \equiv \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \Phi_{stat} \tag{14.3}$$

If $\mathcal{V}_t > 0$, the induced emf tends to drive a current flowing in the same sense of rotation with contour C. The minus sign of eq. (14.3) means the induced emf tends to drive a current flowing in such a sense as to oppose the change of flux (H. Lenz, 1833).

<u>Example 14-1</u>: Consider N circular conducting loops of radius b placed in the *xy*-plane where a time-varying magnetic field $\vec{B} = \vec{a}_z B_z(r,t) = \vec{a}_z B_0 \cos\left(\frac{\pi r}{2b}\right) \sin \omega t$ exists (Fig. 14-1ab). Find the induced emf.

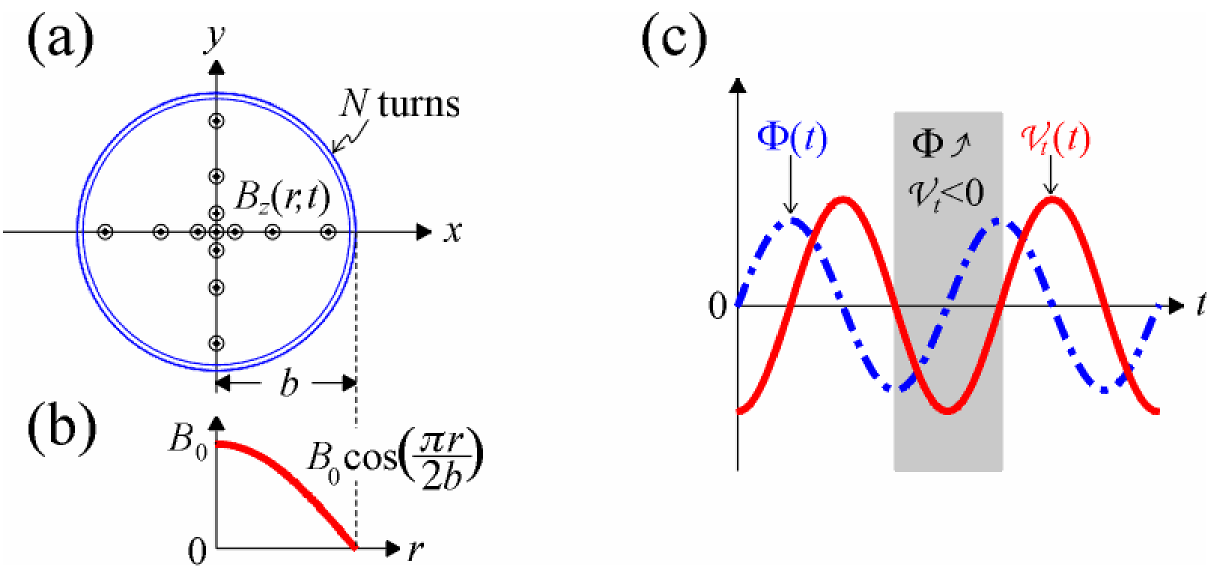

Fig. 14-1. (a) The geometry of a stationary conducting loop, and (b) the spatial amplitude of magnetic flux density passing through it. (c) The resulting magnetic flux (dash-dot) and transformer emf (solid).

Ans: $\Phi(t)=\int_S \vec{B}\cdot d\vec{s}=\int_0^b\left[\vec{a}_z B_0\left(\cos\frac{\pi r}{2b}\right)\sin\omega t\right]\cdot\left(\vec{a}_z 2\pi r dr\right)=\frac{8b^2}{\pi}\left(\frac{\pi}{2}-1\right)B_0\sin\omega t$ (dash-dot, Fig. 14-1c). By eq. (14.3), $\Rightarrow$ $\mathcal{V}_t=-N\frac{d\Phi}{dt}=\frac{8Nb^2}{\pi}\left(\frac{\pi}{2}-1\right)B_0\omega(-\cos\omega t)$ (solid, Fig. 14-1c).

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1) Choosing $d\vec{s}=+\vec{a}_z 2\pi r dr$ implies that the sense of contour C in eq. (14.3) is counterclockwise, $\Rightarrow$ $\mathcal{V}_t>0$ tends to drive a current in counterclockwise sense.

2) When the flux Φ is increasing, $\Rightarrow$ $\mathcal{V}_t<0$ tends to drive a current in clockwise sense, causing $\vec{B}$ in the $-z$ direction and reducing the total magnetic field (shaded region, Fig. 14-1c).

■ Transformers

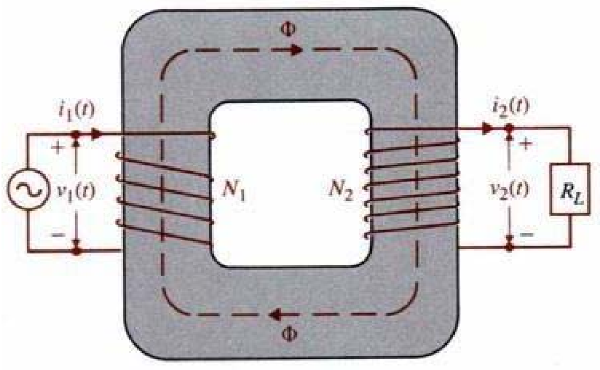

Fig. 14-2. A transformer (after DKC).

Consider a typical transformer shown in Fig. 14-2. The primary and secondary coils with N_1 and N_2 turns are wound around a ferromagnetic core (grey region) of high permeability $\mu >> \mu_0$ such that a common magnetic flux Φ passes through both coils with negligible leakage. Current $i_1(t)$ flowing through the primary coil will create a magnetic field, which will pass through the secondary coil and contribute to the mutual flux linkage. By eq. (12.10),

$$N_1 i_1 - N_2 i_2 = R\Phi ,$$

where $R = \dfrac{l}{\mu S}$ is the reluctance of the core with length l, and cross-sectional area S.

1) Ideal transformer ($\mu \to \infty$): $R \to 0$, $N_1 i_1 = N_2 i_2$, $\Rightarrow$

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{14.4}$$

By eq. (14.3), the induced emf on the *i*th coil is: $v_i = -\dfrac{d}{dt}\Lambda_i = -N_i \dfrac{d\Phi}{dt}$, $\Rightarrow$

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \tag{14.5}$$

The effective load seen by the source connected to the primary coil is: $(R_1)_{eff} \equiv v_1 / i_1$. By eq's (14.4), (14.5) and $R_L = v_2 / i_2$, $\Rightarrow$

$$(R_1)_{eff} = \left(\frac{N_1}{N_2}\right)^2 R_L \tag{14.6}$$

Eq's (14.4), (14.5), (14.6) indicate that a transformer can transform voltages, currents, and impedances.

2) Real transformer: The flux linkages of the primary and secondary coils are:

$$\Lambda_1 = N_1\Phi = \frac{1}{R}\left(N_1^2 i_1 - N_1 N_2 i_2\right),\ \ \Lambda_2 = N_2\Phi = \frac{1}{R}\left(N_1 N_2 i_1 - N_2^2 i_2\right).$$

By $v_i = \frac{d}{dt}\Lambda_i$ (not $-\frac{d}{dt}\Lambda_i$, for the polarities of v_i, i_i are defined in Fig. 14-2), $\Rightarrow$

$$v_1 = L_1\frac{di_1}{dt} - L_{12}\frac{di_2}{dt},\quad v_2 = L_{12}\frac{di_1}{dt} - L_2\frac{di_2}{dt} \tag{14.7}$$

where $L_1 = \frac{\mu S}{l}N_1^2$, $L_2 = \frac{\mu S}{l}N_2^2$, $L_{12} = \frac{\mu S}{l}N_1 N_2$ are the self and mutual inductances. $\Rightarrow$ $\{v_i,\ i_i\}$ are coupled and have to be solved simultaneously.

■ Moving conductor in a static magnetic field

Consider a conducting bar moving with velocity $\vec{u}$ in a magnetic field $\vec{B}$ (Fig. 14-3). The free charges inside the bar are driven by magnetic force: $\vec{F}_m = q\vec{u}\times\vec{B}$, resulting in accumulation of positive and negative charges at the two ends until balanced by Coulombian force at equilibrium.

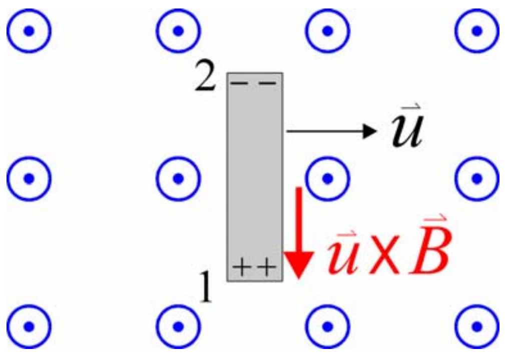

Fig. 14-3. A conductor moving in a magnetic field will induce motional emf.

For an observer moving with the conductor, the effect of magnetic force can be interpreted as an (non-conservative) "impressed" electric field:

$$\vec{E}_m \equiv \vec{F}_m / q = \vec{u}\times\vec{B},$$

causing a voltage $V_{12} = V_1 - V_2 = \int_2^1 + (\vec{u} \times \vec{B}) \cdot d\vec{l}$ (the plus sign of the integrand is because $\vec{u} \times \vec{B}$ is an impressed electric field). If the moving conductor is part of a closed circuit C, there is an induced motional emf:

$$\mathcal{V}_m \equiv \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l} \tag{14.8}$$

$\mathcal{V}_m > 0$ tends to drive a current flowing in the same sense of rotation of the contour C.

<u>Example 14-2</u>: A metal bar slides over a pair of conducting rails in a uniform magnetic field $\vec{B} = \vec{a}_z B_0$ with a constant velocity $\vec{u}$ (Fig. 14-4). Find (1) the open-circuit voltage V_0, (2) the dissipated power P_e when loaded with resistance R.

Ans: (1) $V_0 = V_{1'} - V_{2'} = \int_{2'}^{1'} (\vec{a}_x u \times \vec{a}_z B_0) \cdot (\vec{a}_y dy) = -uB_0 h$.

(2) Choose $1'122'1'$ as contour C in eq. (14.8), $\mathcal{V}_m = \int_{2'}^{1'} (\vec{a}_x u \times \vec{a}_z B_0) \cdot (\vec{a}_y dy) = -uB_0 h (< 0)$,

$\Rightarrow$ the emf drives a current $I = \frac{\mathcal{V}_m}{R} = \frac{uB_0 h}{R}$ in clockwise sense. $\Rightarrow$ $P_e = I^2 R = (uB_0 h)^2 / R$, which is equal to the required mechanical power.

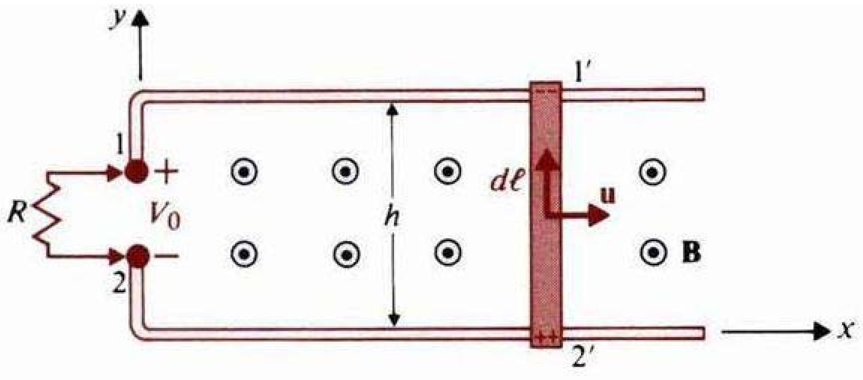

Fig. 14-4. DC generator made by a moving metal bar (after DKC).

■ Moving circuit in a time-varying magnetic field

Consider a circuit (conducting loop) C moving with velocity $\vec{u}$ in a region where $\vec{E}$ and

$\vec{B}$ coexist. A charge q on the circuit experiences a Lorentz's force $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ [eq. (11.1)]. For an observer moving with q, the force can be regarded as a result of an effective electric field:

$$\vec{E}' = \vec{E} + \vec{u} \times \vec{B} \tag{14.9}$$

Perform contour integral for eq. (14.9) over C and substitute eq. (14.2) into it, we have:

$$\oint_C \vec{E}' \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l} \ , \tag{14.10}$$

where S is the moving surface bounded by C. By eq's (14.3), (14.8), we know the right hand side of eq. (14.10) is the summation of the transformer emf $\mathcal{V}_t$ and motional emf $\mathcal{V}_m$. This means that the total emf induced on the moving circuit C, which is defined as the work done by the effective electric field $\vec{E}'$ over C, becomes:

$$\mathcal{V}' \equiv \oint_C \vec{E}' \cdot d\vec{l} = \mathcal{V}_t + \mathcal{V}_m \ , \tag{14.11}$$

In the following, we will prove that the total emf is equal to the time derivative of the "dynamic" magnetic flux Φ_{dynm} over the moving surface S:

$$\mathcal{V}' = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} = -\frac{d}{dt} \Phi_{dynm} \tag{14.12}$$

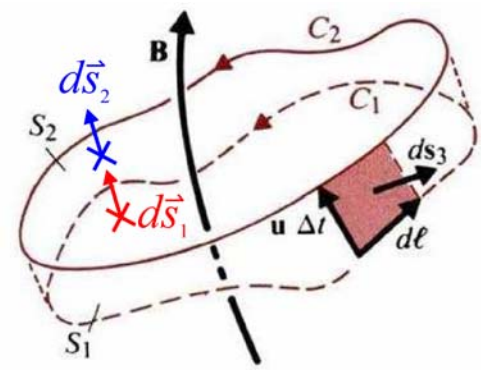

Fig. 14-5. A moving circuit in a magnetic field (after DKC).

Proof: Consider a circuit C moving from C_1 (dashed, Fig. 14-5) at time t to C_2 (solid) at time $t + \Delta t$ in a time-varying magnetic field $\vec{B}(\vec{r}, t)$ (denoted by $\vec{B}(t)$ for simplicity).

(1) $\frac{d}{dt} \Phi_{dynm} = \frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{S_2} \vec{B}(t + \Delta t) \cdot d\vec{s}_2 - \int_{S_1} \vec{B}(t) \cdot d\vec{s}_1 \right]$, where the direction of

$d\bar{s}_2$ is essentially the same as that of $d\bar{s}_1$ at the corresponding position (Fig. 14-5). By first-order Taylor series approximation: $\vec{B}(t+\Delta t) \approx \vec{B}(t) + \frac{\partial \vec{B}(t)}{\partial t}\Delta t$, $\Rightarrow$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t+\Delta t)\cdot d\bar{s}_2\right] \approx \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t)\cdot d\bar{s}_2 + \int_{S_2}\left(\frac{\partial \vec{B}(t)}{\partial t}\Delta t\right)\cdot d\bar{s}_2\right]$$

$$= \lim_{\Delta t \to 0}\left[\frac{1}{\Delta t}\int_{S_2} \vec{B}(t)\cdot d\bar{s}_2 + \int_{S_2}\frac{\partial \vec{B}(t)}{\partial t}\cdot d\bar{s}_2\right] = \lim_{\Delta t \to 0}\frac{1}{\Delta t}\int_{S_2} \vec{B}(t)\cdot d\bar{s}_2 + \int_{S}\frac{\partial \vec{B}(t)}{\partial t}\cdot d\bar{s}, \Rightarrow$$

$$\frac{d}{dt}\int_S \vec{B}\cdot d\bar{s} \approx \int_S \frac{\partial \vec{B}(t)}{\partial t}\cdot d\bar{s} + \lim_{\Delta t \to 0}\frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t)\cdot d\bar{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\bar{s}_1\right].$$

(2) By eq. (11.4), $\int_V (\nabla\cdot\vec{B})dv = \oint_S \vec{B}\cdot d\bar{s} = \int_{S_2} \vec{B}(t)\cdot d\bar{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\bar{s}_1 + \int_{S_3} \vec{B}(t)\cdot d\bar{s}_3 = 0$. A differential element on the side surface S_3 is represented by $d\bar{s}_3 = d\vec{l}\times\bar{u}\Delta t$ (Fig. 14-5), $\Rightarrow$

$$\vec{B}\cdot d\bar{s}_3 = \Delta t\vec{B}\cdot\left(d\vec{l}\times\bar{u}\right) = \Delta t d\vec{l}\cdot\left(\bar{u}\times\vec{B}\right). \Rightarrow$$

$$\int_{S_3} \vec{B}(t)\cdot d\bar{s}_3 = \Delta t\int_{S_3} d\vec{l}\cdot\left(\bar{u}\times\vec{B}\right) \to \Delta t\oint_C d\vec{l}\cdot\left(\bar{u}\times\vec{B}\right) \text{ in the limit of } \Delta t \to 0. \Rightarrow$$

$$\int_{S_2} \vec{B}(t)\cdot d\bar{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\bar{s}_1 = -\int_{S_3} \vec{B}(t)\cdot d\bar{s}_3 \to -\Delta t\oint_C \left(\bar{u}\times\vec{B}\right)\cdot d\vec{l}\ .$$

Combine (1) and (2), $\Rightarrow$ eq. (14.12) is proved.

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Note the surface S is moving (time-dependent), $\Rightarrow$ $\frac{d}{dt}\int_S \vec{B}\cdot d\bar{s} \neq \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\bar{s}$.

Example 14-3: The motional emf in Example 14-2 can also be derived by eq. (14.12): $\Phi_{dynm} = \int_S \vec{B}\cdot d\bar{s} = B_0 hut$, $\Rightarrow$ $V_0 = -\frac{d}{dt}\Phi_{dynm} = -B_0 hu$. Since $d\bar{s}$ is chosen in the +*z*-direction, $V_0 < 0$ implies the emf tends to drive a current in clockwise sense.

Example 14-4: A rectangular ($h\times w$) conducting loop is situated in a time-varying magnetic field $\vec{B} = \bar{a}_y B_0 \sin\omega t$ and rotates with angular velocity ω about the *x*-axis (Fig. 14-6).

Find the induced emf.

Ans: (Method 1) At time t, the unit normal vector $\vec{a}_n$ of the loop makes an angle $\alpha = \omega t$ with respect to $\vec{a}_y$. $\Rightarrow$ The dynamic flux due to time-varying magnetic field and moving circuit is:

$$\Phi_{dynm} = \int_S \vec{B} \cdot d\vec{s} = \left(\vec{a}_y B_0 \sin \omega t\right) \cdot \left(\vec{a}_n hw\right) = B_0 hw \cdot \sin \omega t \cdot \cos \alpha(t) .$$

By eq. (14.3), the transformer emf is: $\mathcal{V}_t = -\dfrac{d}{dt}\Phi_{stat}$, where the static flux Φ_{stat} is the flux assuming the loop is stationary (α is constant):

$$\Phi_{stat} = B_0 hw \cdot \sin \omega t \cdot \cos \alpha .$$

As a result,

$$\mathcal{V}_t = -\frac{d}{dt}\Phi_{stat} = -B_0 hw\omega \cdot \cos \omega t \cdot \cos \alpha ,$$

where the angle α is not differentiated. One can also arrive at the same result by:

$$\mathcal{V}_t = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = -\left(\vec{a}_y B_0 \omega \cos \omega t\right) \cdot \left(\vec{a}_n hw\right) = -B_0 hw\omega \cdot \cos \omega t \cdot \cos \alpha .$$

By eq. (14.8), the motional emf is:

$$\mathcal{V}_m = \oint_{14321} \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l}$$

$$= \int_2^1 \left[\left(\vec{a}_n \frac{w}{2}\omega\right) \times \left(\vec{a}_y B_0 \sin \omega t\right)\right] \cdot \left(\vec{a}_x dx\right) + \int_4^3 \left[\left(-\vec{a}_n \frac{w}{2}\omega\right) \times \left(\vec{a}_y B_0 \sin \omega t\right)\right] \cdot \left(\vec{a}_x dx\right)$$

$$= B_0 hw\omega \cdot \sin \omega t \cdot \sin \alpha .$$

By eq. (14.11) and $\alpha = \omega t$, $\Rightarrow$

$$\mathcal{V}' = \mathcal{V}_t + \mathcal{V}_m = -B_0 hw\omega \cdot \cos 2\omega t .$$

(Method 2) By eq. (14.12),

$$\mathcal{V}' = -\frac{d}{dt}\Phi_{dynm} = -B_0 hw\omega \cdot \cos 2\omega t ,$$

where $\Phi_{dynm} = B_0 hw \cdot \sin \omega t \cdot \cos \omega t = \dfrac{B_0 hw \cdot \sin 2\omega t}{2}$ (the angle $\alpha = \omega t$ is differentiated).

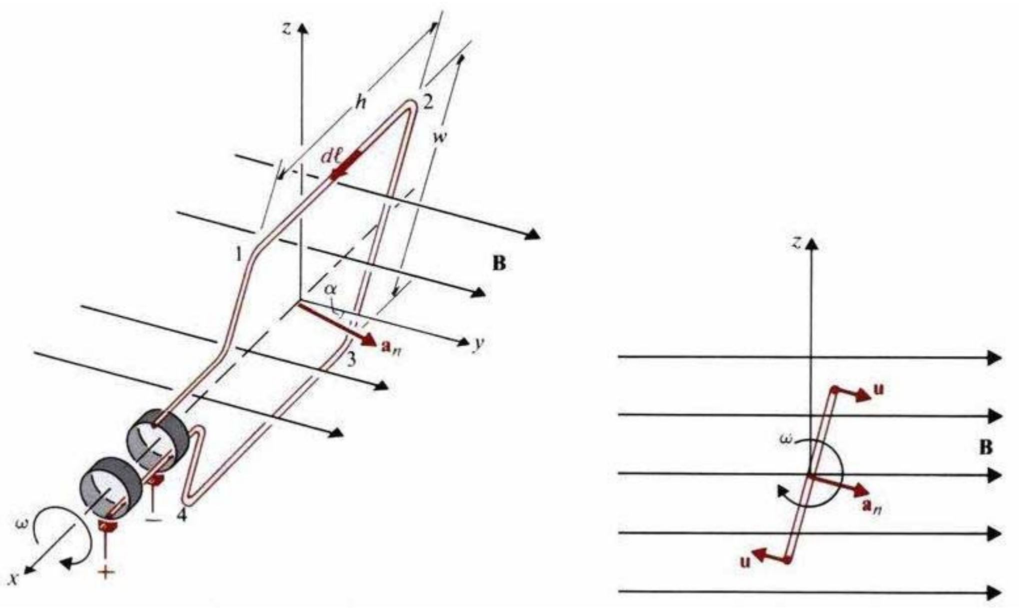

Fig. 14-6. AC generator made by a rotating a metal loop in time-varying M-field (after DKC).

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1) Faraday's law [eq. (14.1)] couples (1) electric field with magnetic field, (2) space with time, which enables electromagnetic waves.

2) Eq's (14.1), (14.2) are always valid, while eq's (14.3), (14.8), and (14.10) are useful only in the presence of conducting loop.

14.2 Maxwell's Equations

■ Modified Ampère's circuital law

If we take divergence for $\nabla\times\vec{H}=\vec{J}$ [eq. (12.4)] and employ the equation of continuity $\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$ [eq. (10.8)], ⇒

$$\nabla\cdot\left(\nabla\times\vec{H}\right)=\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}.$$

The above equation is in violation of the vector identity $\nabla\cdot\left(\nabla\times\vec{A}\right)=0$ [eq. (5.35)] if the charge density ρ is time-varying ($\frac{\partial\rho}{\partial t}\neq 0$). Mathematically, one can maintain the consistency by demanding:

$$\nabla\cdot\left(\nabla\times\vec{H}\right)=\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}.$$

By $\nabla\cdot\vec{D}=\rho$ [eq. (7.8)], $\Rightarrow$ $\nabla\cdot\left(\nabla\times\vec{H}\right)=\nabla\cdot\vec{J}+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{D}\right)=\nabla\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)$,

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad (14.13)$$

The term $\frac{\partial\vec{D}}{\partial t}$ is called the displacement current density:

$$\vec{J}_D=\frac{\partial\vec{D}}{\partial t} \quad (\text{A/m}^2) \qquad (14.14)$$

In other words, a time-varying electric field is equivalent to a current source, which can produce a magnetic field in the absence of a free current $\vec{J}$.

■ Maxwell's equations

Eq's (6.2), (7.8), (12.4), (11.2) are now modified as coupled differential equations when the fields are time-varying (J. Maxwell, 1864):

$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$, Faraday's law of EM induction (14.1)

$\nabla\cdot\vec{D}=\rho$, Gauss's law (7.8)

$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$, Modified Ampère's circuital law (14.13)

$\nabla\cdot\vec{B}=0$, Inexistence of magnetic charge (11.2)

One can use these four equations along with equation of continuity $\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$ [eq. (10.8)] and Lorentz's force equation $\vec{F}=q\left(\vec{E}+\vec{u}\times\vec{B}\right)$ [eq. (11.1)] to describe all macroscopic electromagnetic phenomena.

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The four Maxwell's equations are consistent but not independent. E.g. Eq's (7.8), (11.2) can be derived by eq's (14.1), (14.13), and (10.8).

By taking the surface integral of eq's (14.1) and (14.13) over an open surface S bounded by a contour C, and volume integral of eq's (7.8) and (11.2) over a volume V bounded by a closed surface S, we get integral forms of the four Maxwell's equations:

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \tag{14.2}$$

$$\oint_S \vec{D} \cdot d\vec{s} = Q \tag{7.10}$$

$$\oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s} \tag{14.15}$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \tag{11.4}$$

<u>Example 14-5</u>: Consider a parallel-plate capacitor driven by a sinusoidal voltage $v_c(t) = V_0 \sin \omega t$ (Fig. 14-7). Find the displacement current $i_d(t)$ flowing through the capacitor, and the magnetic field intensity $\vec{H}$ everywhere.

Ans: (1) Borrow the result derived in electrostatics, ⇒

$$E(t) = \frac{v_c(t)}{d}, \quad D(t) = \varepsilon E(t) = \varepsilon \frac{v_c(t)}{d}.$$

By eq. (14.14), ⇒

$$i_d(t) = \int_S \frac{\partial \vec{D}(t)}{\partial t} \cdot d\vec{s} = \left(\varepsilon \frac{S}{d}\right) \frac{d}{dt} v_c(t) = C \frac{d}{dt} v_c(t) = C V_0 \omega \cos \omega t,$$

which is equal to the conduction current $i_c(t)$.

(2) By eq. (14.15) and cylindrical symmetry, ⇒

$$\oint_C \vec{H} \cdot d\vec{l} = 2\pi r H_\phi(r,t) = i_c(t) = i_d(t).$$

In either case (surrounding the conducting wire or the capacitor),

$$\vec{H} = \vec{a}_\phi \frac{C V_0}{2\pi r} \omega \cos \omega t.$$

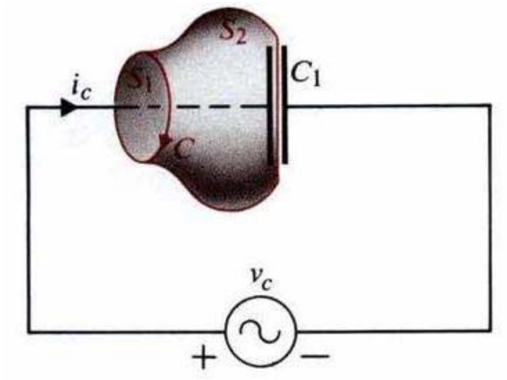

Fig. 14-7. Displacement current in a parallel-plate capacitor (after DKC).

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1) Since $i_c(t) = i_d(t)$, $\Rightarrow$ the "total" current across the circuit is continuous.

2) $\vec{H} = \vec{a}_\phi \dfrac{CV_0}{2\pi r}\omega\cos\omega t$, $\Rightarrow$ the magnitude of magnetic field is proportional to the frequency of the driving source.

■ Boundary conditions

As in electromagneto-statics, boundary conditions of electromagneto-dynamics are derived by applying the integral form of Maxwell's equations to: (1) a closed path for curl equations, (2) a pillbox for divergence equations across the boundary (Fig. 7-8), then letting the thickness Δh approach zero. Since the contributions from the modified terms, $-\int_S \dfrac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$ in eq. (14.2) and $\int_S \dfrac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$ in eq. (14.15), are zero as $\Delta h \to 0$, boundary conditions are the same as those in electromagneto-statics:

$$E_{1t} = E_{2t} \tag{7.14}$$

$$\vec{a}_{n2} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s \tag{7.15}$$

$$\vec{a}_{n2} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s \tag{12.11}$$

$$B_{1n} = B_{2n} \tag{12.12}$$

where $\vec{a}_{21}$ is the unit normal vector pointing from medium-2 to medium-1.

<Comment>

In time-varying case, eq. (7.14) is equivalent to eq. (12.12), and eq. (7.15) is equivalent to eq. (12.11). (DKC p330)