Lesson 14
Maxwell's Equations

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Overview-1

The four fundamental postulates of electrostatics & magnetostatics:

$\nabla \times \vec{E} = 0$ $\nabla \cdot \vec{D} = \rho$	Indep. ⟷	$\nabla \times \vec{H} = \vec{J}$ $\nabla \cdot \vec{B} = 0$

Overview-2

Time-varying, $\Rightarrow$ the two curl equations have to be modified to meet:

1. Faraday's law: $\mathcal{V}' \equiv \oint_C \vec{E}' \cdot d\vec{l} = -\frac{d}{dt}\Phi_{dynm}$
2. Equation of continuity: $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

$\nabla \times \vec{E} = 0$	coupled	$\nabla \times \vec{H} = \vec{J}$
$\nabla \cdot \vec{D} = \rho$	⟷	$\nabla \cdot \vec{B} = 0$

Overview-3

EM fields $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ have to be solved simultaneously

The solutions behave like waves, $\Rightarrow$ EM waves

$\nabla \times \vec{E} = 0$ $\nabla \cdot \vec{D} = \rho$	coupled ⟷	$\nabla \times \vec{H} = \vec{J}$ $\nabla \cdot \vec{B} = 0$

Outline

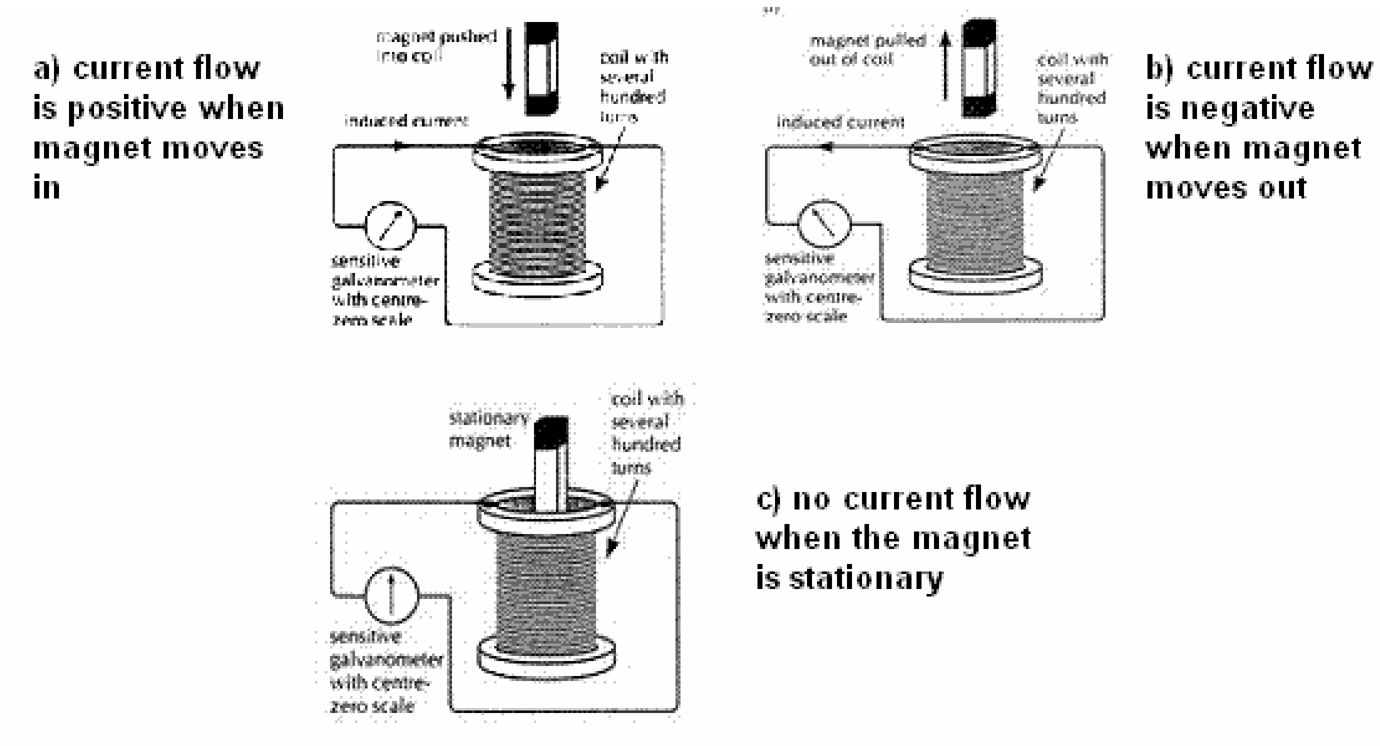
- Faraday’s law
- Maxwell’s equations

Outline

- Faraday’s law
  1. Modified fundamental postulates
  2. Stationary circuit in time-varying field
  3. Moving circuit in static field
  4. Moving circuit in time-varying field
- Maxwell’s equations

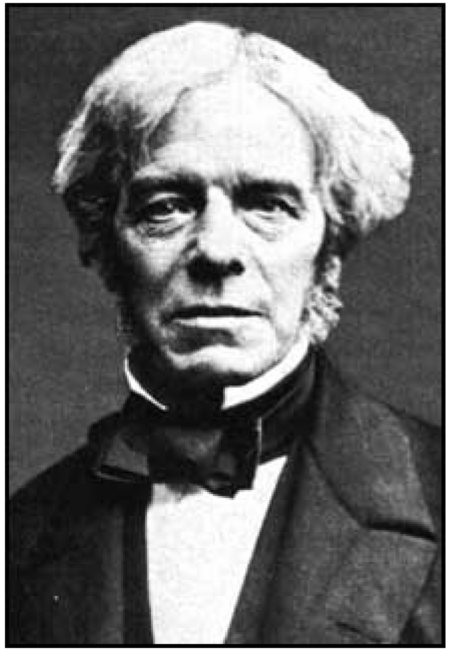
Modified fundamental postulate-1

A current will be induced in a conducting loop if the magnetic flux over that loop is changed



Michel Faraday

(1791-1867)



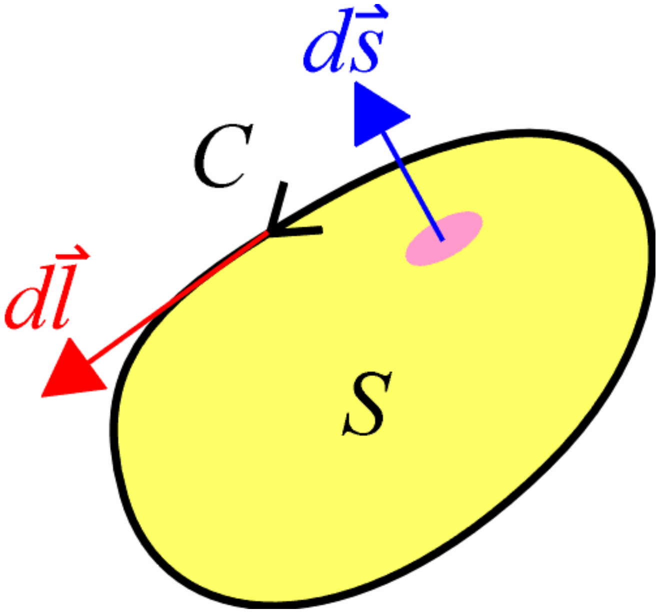
Modified fundamental postulate-2

Mathematical description of Faraday's law:

$$\nabla \times \vec{E} = 0 \longrightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Integral form:

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$



Modified fundamental postulate-3

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

…Time-varying B as vortex source of E

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

…Work done by induced E is nonzero (non-conservative)

Modified fundamental postulate-4

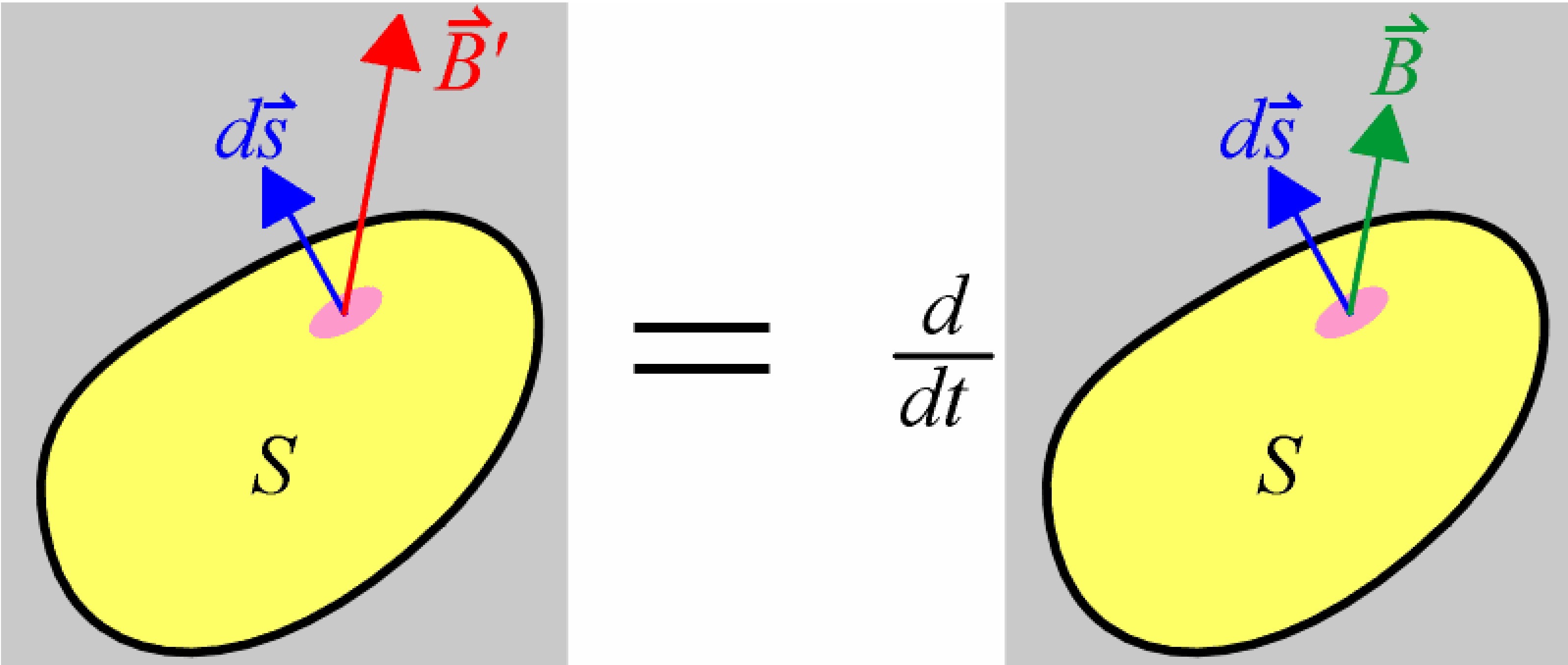
No well-defined electric potential value for any point in space

$$\mathcal{V} \equiv \oint_C \vec{E} \cdot d\vec{l}$$ …induced emf

describe the “tendency” of driving current in a conducting loop

Stationary circuit in a time-varying magnetic field-1

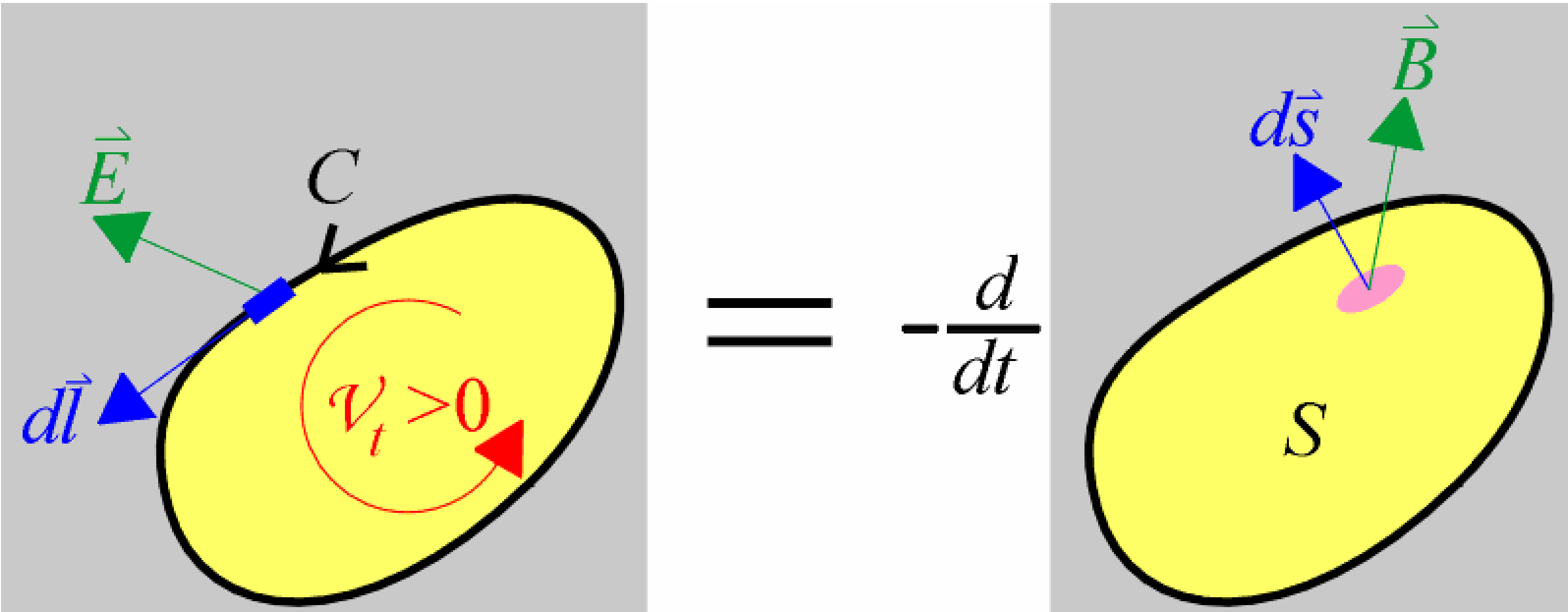
If the observation surface S is stationary, the order of time-derivative and surface-integral can be exchanged: $\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = \frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} = \frac{d}{dt} \Phi_{stat}$



Stationary circuit in a time-varying magnetic field-2

Transformer emf:

$$\mathcal{V}_t \equiv \oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = - \frac{d}{dt} \Phi_{stat}$$



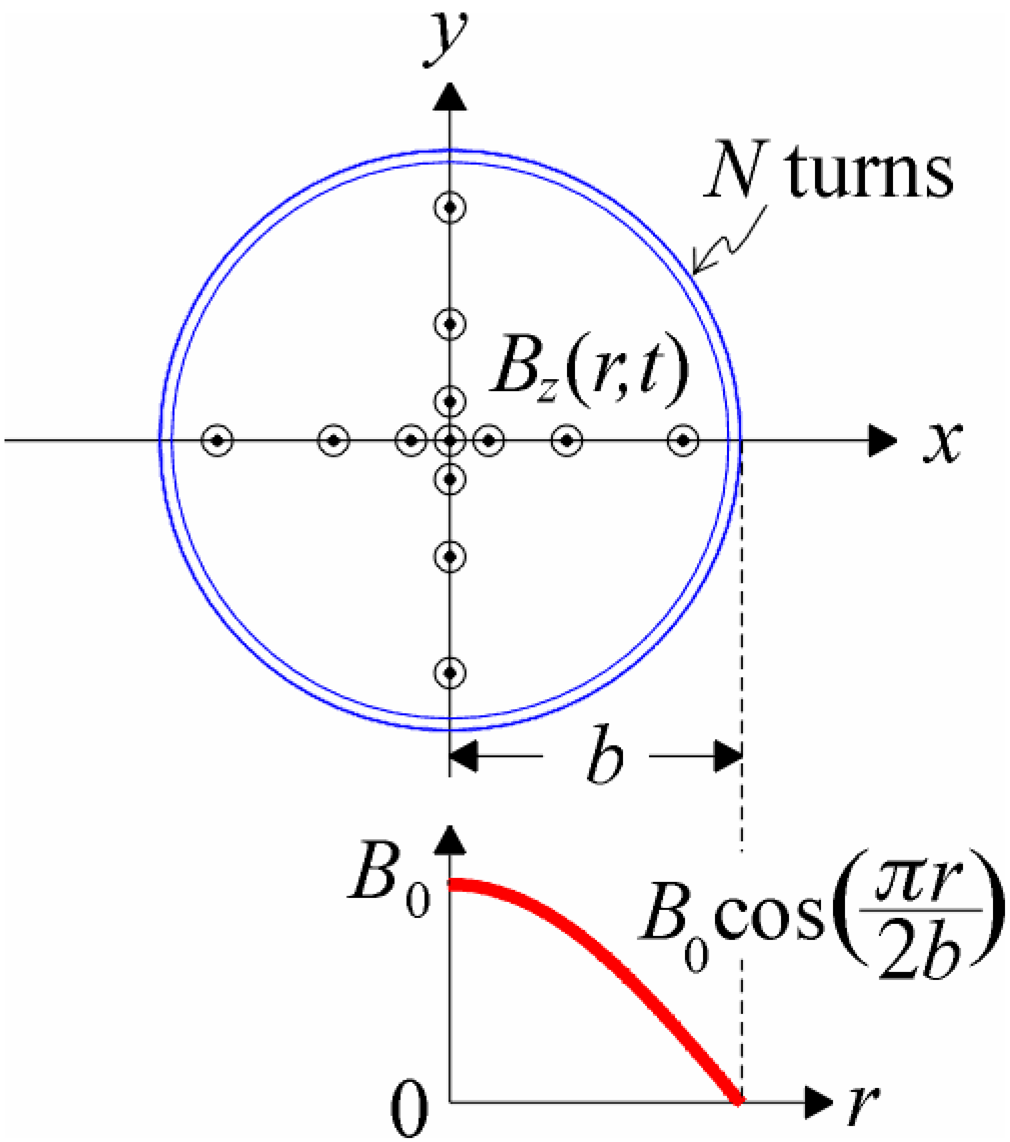
Example 14-1: Conducting loop in a time-varying M-field (1)

Consider N circular conducting loops placed in the xy-plane where a time-varying M-field:

$$\vec{B} = \vec{a}_z B_z(r,t)$$

$$= \vec{a}_z B_0 \cos\left(\frac{\pi r}{2b}\right) \sin \omega t$$

Find the emf?



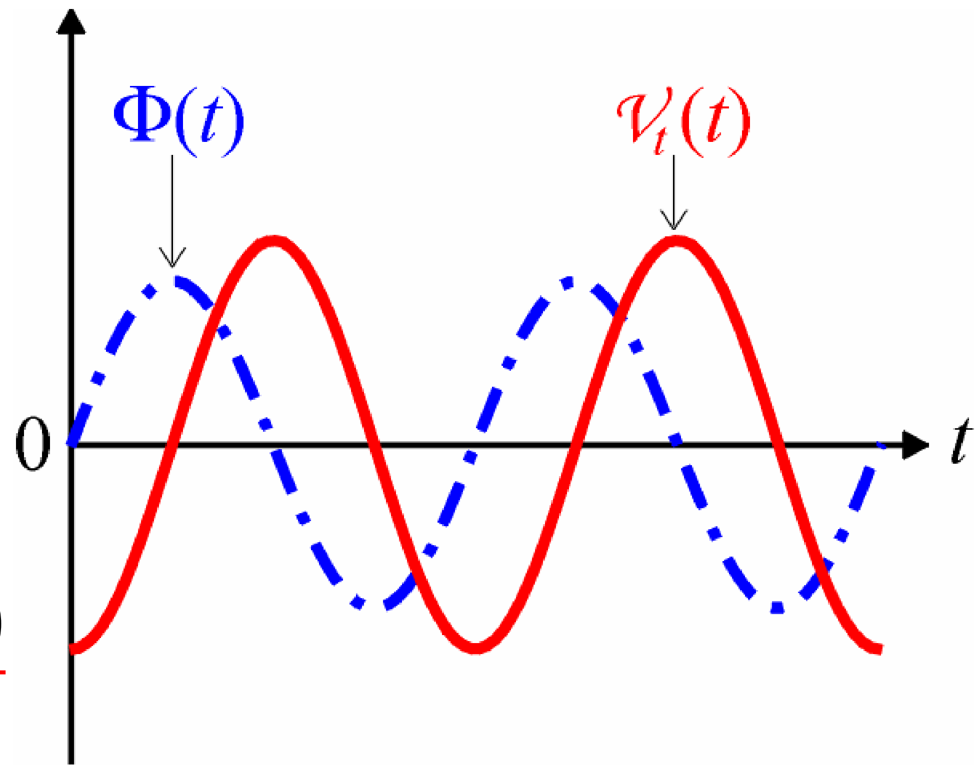
Example 14-1: Conducting loop in a time-varying M-field (2)

$$\Phi(t) = \int_S \vec{B}\cdot d\vec{s} = \int_0^b \left[\vec{a}_z B_0 \left(\cos\frac{\pi r}{2b} \right) \sin\omega t \right] \cdot \left(\vec{a}_z 2\pi r dr \right)$$

$$= \frac{8b^2}{\pi}\left(\frac{\pi}{2}-1\right) B_0 \underline{\sin\omega t}$$

$$\mathcal{V}_t = -N\frac{d\Phi}{dt}$$

$$= \frac{8Nb^2}{\pi}\left(\frac{\pi}{2}-1\right) B_0 \omega \underline{(-\cos\omega t)}$$

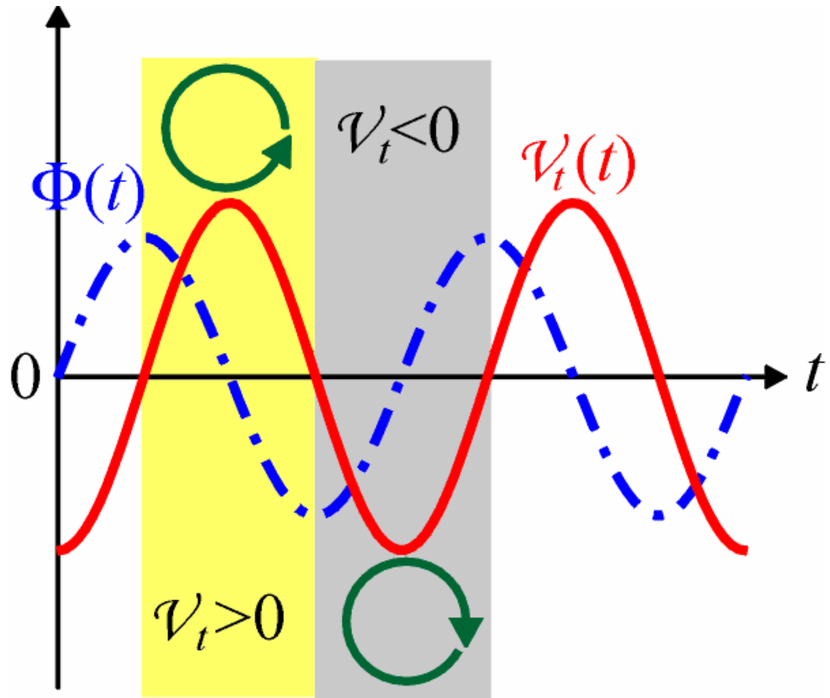


Comments-1

$$\Phi(t)=\int_S \vec{B}\cdot d\vec{s}=\int_0^b\left[\vec{a}_z B_0\left(\cos\frac{\pi r}{2b}\right)\sin\omega t\right]\cdot\left(\vec{a}_z 2\pi r dr\right)$$

Choosing $d\vec{s}\,//+\vec{a}_z$ implies the sense of contour C is counterclockwise (CCLK),

$\Rightarrow \mathcal{V}_t>0(<0)$ means tends to drive a current in CCLK(CLK) sense

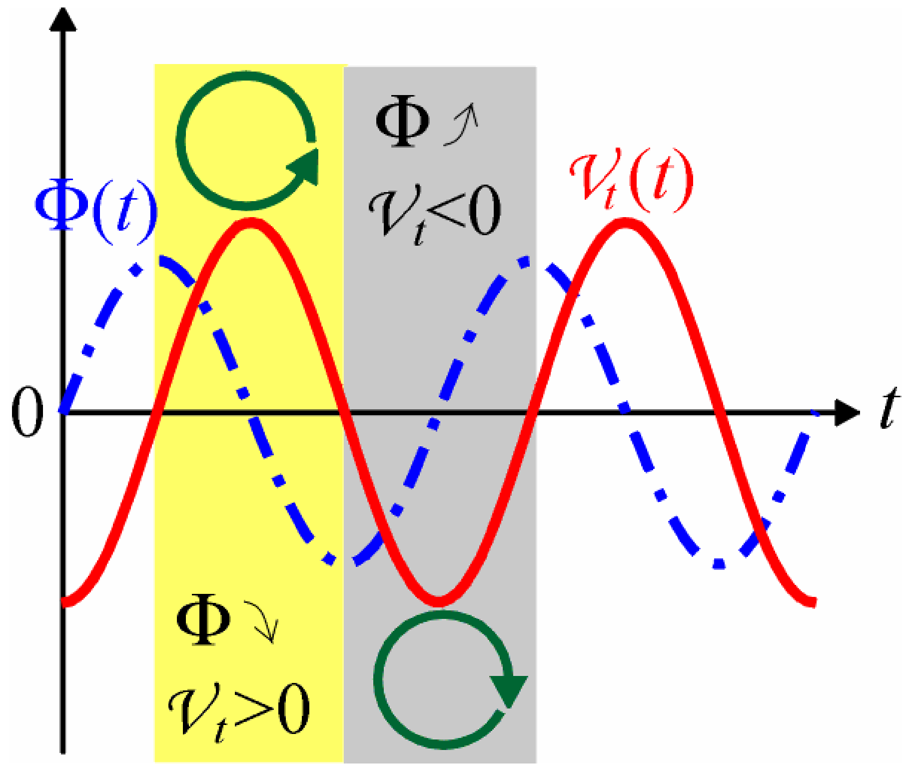


Comments-2

When Φ is decreasing(increasing), $\Rightarrow \mathcal{V}_t > 0(< 0)$ tends to drive a current in CCLK(CLK) sense,

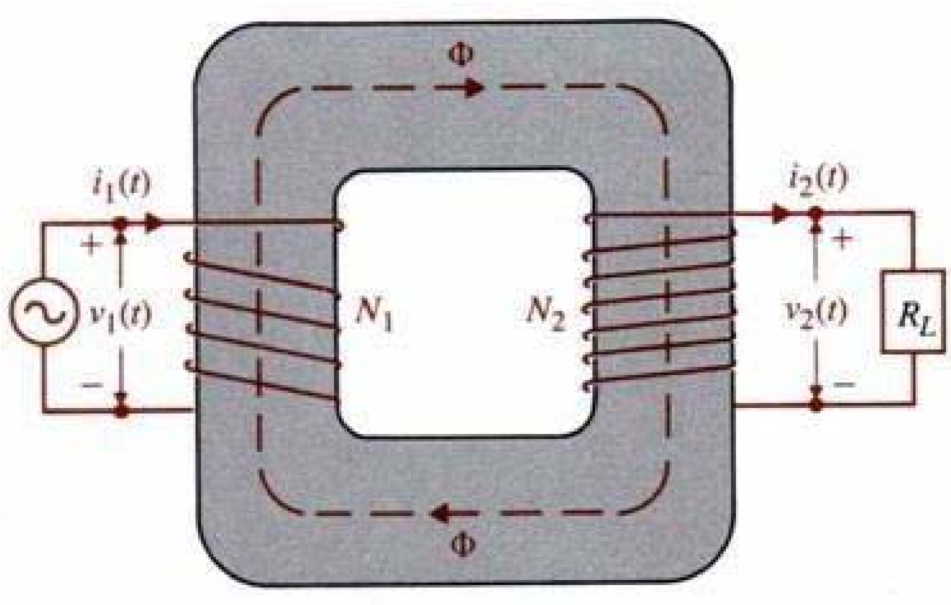
$\Rightarrow$ B in +z(-z) direction,

$\Rightarrow$ increasing(decreasing) Φ (Lentz law)



Applications: Transformer

Primary & secondary coils with N_1, N_2 turns, ferromagnetic core ($\mu >> \mu_0$) to guide the flux (reduce leakage)



Current $i_1(t)$, $\Rightarrow$ B, mutual flux linkage

$$\sum_j N_j I_j = \sum_k R_k \Phi_k$$

$$\Rightarrow N_1 i_1 - N_2 i_2 = R\Phi$$

$$R = l/\mu S$$

Applications: Ideal transformer

$$\mu \to \infty, \quad R = \frac{l}{\mu S} \to 0, \quad \Rightarrow N_1 i_1 - N_2 i_2 = R\Phi \to 0$$

$$\Rightarrow N_1 i_1 = N_2 i_2, \quad \boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}}$$

$$v_i = -\frac{d}{dt}\Lambda_i = -N_i \frac{d\Phi}{dt}, \quad \Rightarrow \quad \boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$

$$v_1 i_1 = v_2 i_2$$

$$p_1 = p_2$$

$$\begin{cases} (R_1)_{eff} \equiv v_1 / i_1 \\ R_L = v_2 / i_2 \end{cases} \Rightarrow \quad \boxed{(R_1)_{eff} = \left(\frac{N_1}{N_2}\right)^2 R_L}$$

Applications: Real transformer

$$N_1 i_1 - N_2 i_2 = R\Phi, \quad \Rightarrow \Phi = \frac{N_1 i_1 - N_2 i_2}{R}$$

$$\Rightarrow \Lambda_1 = N_1 \Phi = \frac{1}{R}\left(N_1^2 i_1 - N_1 N_2 i_2\right)$$

$$\Lambda_2 = N_2 \Phi = \frac{1}{R}\left(N_1 N_2 i_1 - N_2^2 i_2\right)$$

$$v_i = \frac{d}{dt}\Lambda_i, \quad \Rightarrow \begin{cases} v_1 = L_1 \dfrac{di_1}{dt} - L_{12}\dfrac{di_2}{dt} \\ v_2 = L_{12}\dfrac{di_1}{dt} - L_2 \dfrac{di_2}{dt} \end{cases}$$

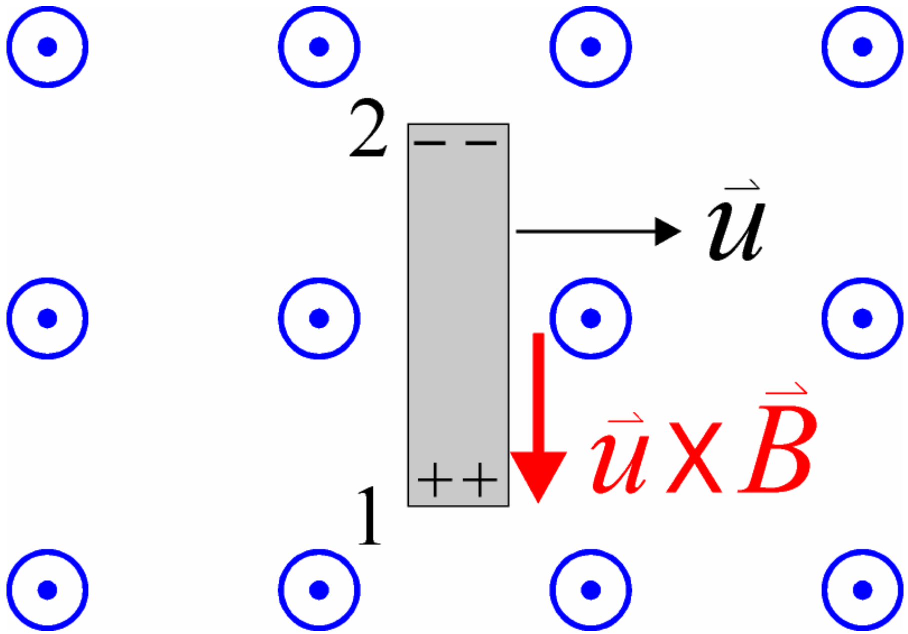
$$L_1 = \frac{\mu S}{l} N_1^2$$

$$L_{12} = \frac{\mu S}{l} N_1 N_2$$

$$L_2 = \frac{\mu S}{l} N_2^2$$

Moving circuit in a static magnetic field-1

Free charges are driven by magnetic force $\vec{F}_m = q\vec{u} \times \vec{B}$, $\Rightarrow$ accumulation of + & - charges at two ends until balanced by Columbian force



Moving circuit in a static magnetic field-2

For an observer moving with the conductor,

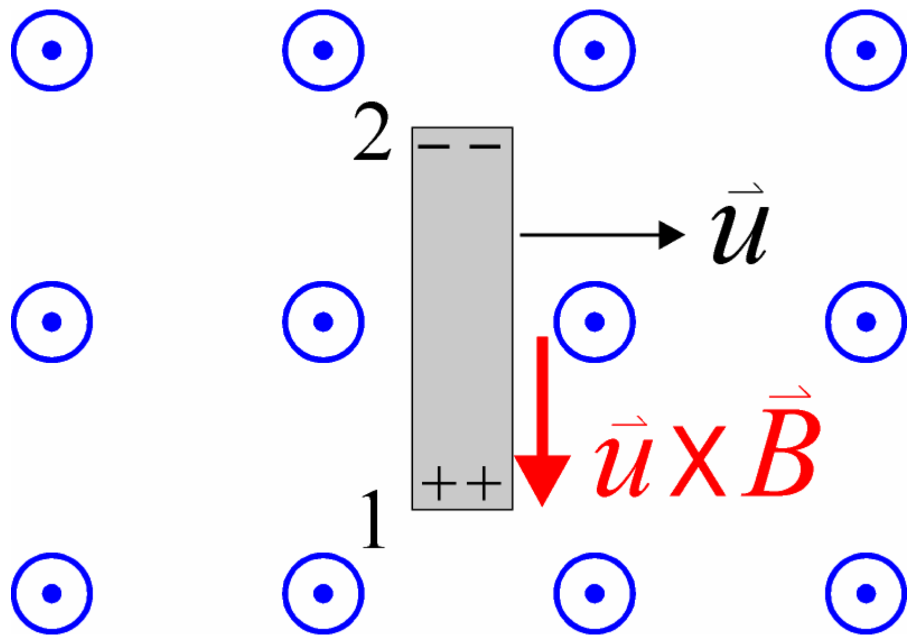
effect of $\vec{F}_m \leftrightarrow$ (non-conservative) impressed

E-field $\vec{E}_m \equiv \vec{F}_m / q = \vec{u} \times \vec{B}$

$\Rightarrow$ Voltage:

$$V_{12} = V_1 - V_2$$

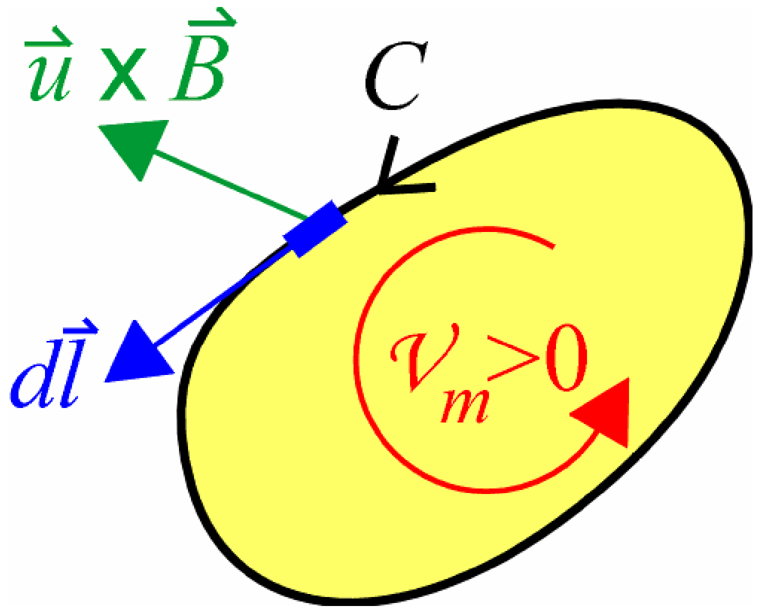
$$= \int_2^1 \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l}$$



Moving circuit in a static magnetic field-3

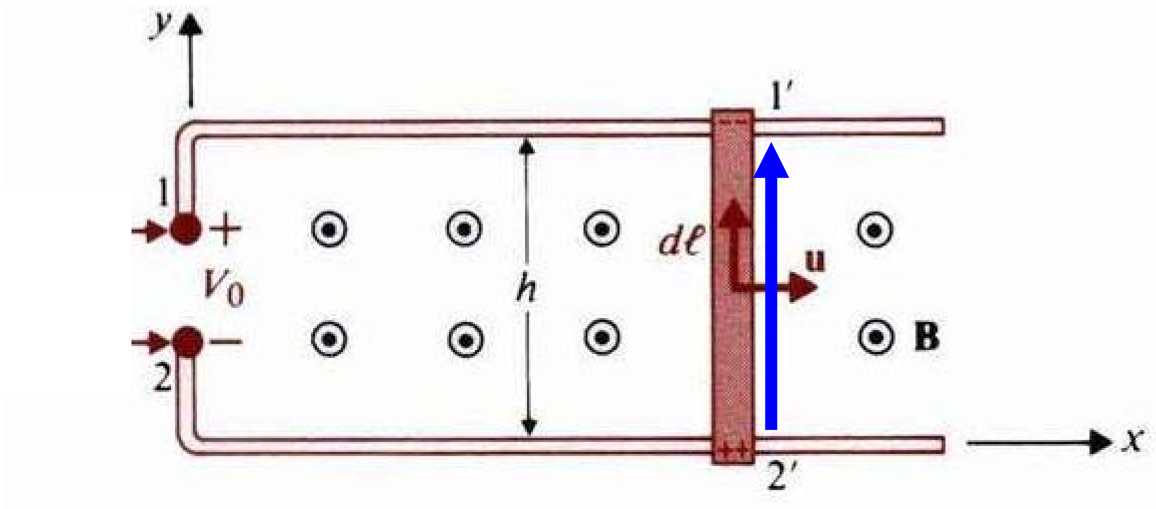
For a closed moving circuit C :

Motional emf: $\mathcal{V}_m \equiv \oint_C \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l}$



Example 14-2: Metal bar sliding over conducting rails (1)

Find: open ckt voltage V_0

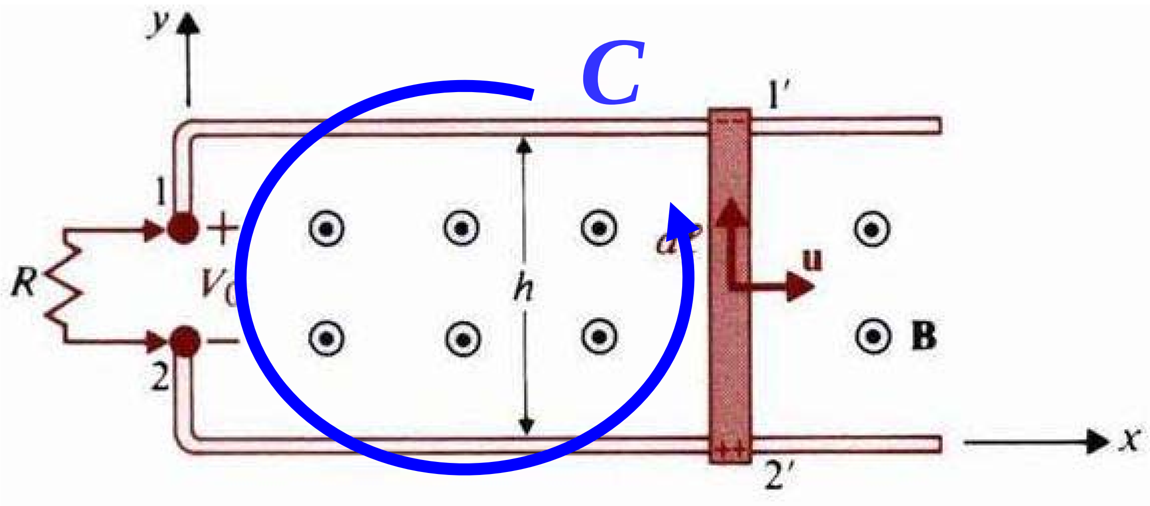


$$V_1 = V_{1'},\ V_2 = V_{2'},\ \Rightarrow$$

$$V_0 = V_{1'} - V_{2'} = \int_{2'}^{1'} \left(\vec{a}_x u \times \vec{a}_z B_0\right) \cdot \left(\vec{a}_y dy\right) = -uB_0 h$$

Example 14-2: Metal bar sliding over conducting rails (2)

When loaded with resistance R



$$\mathcal{V}_m \equiv \oint_C \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l}$$

$$= -uB_0 h < 0$$

(only 2'→1' counts)

$\Rightarrow \mathcal{V}_m$ drives a CLK current $I = \dfrac{\mathcal{V}_m}{R} = \dfrac{uB_0 h}{R}, \Rightarrow$

$P_e = I^2 R = \left(uB_0 h\right)^2 / R$ …equal to mechanical pwr

Moving circuit in a time-varying magnetic field-1

A charge q on a ckt C moving with velocity $\vec{u}$ in a region where $\vec{E}$, $\vec{B}$ coexist experiences a force: $\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$

For an observer moving with q, the force can be regarded as a result of effective E-field:

$$\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$$

Moving circuit in a time-varying magnetic field-2

$$\oint_C \vec{E}' \cdot d\vec{l} = \oint_C \vec{E} \cdot d\vec{l} + \oint_C \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l}$$

$$= -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l}$$

transformer emf: $\mathcal{V}_t$ motional emf: $\mathcal{V}_m$

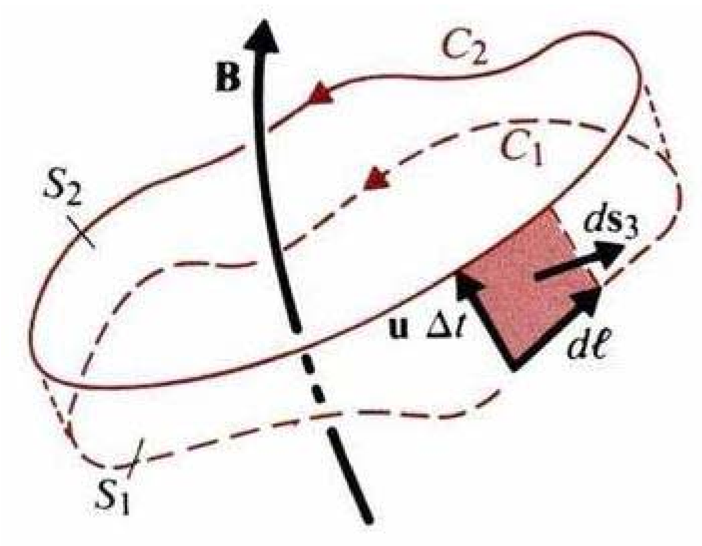
Total emf, work done by $\vec{E}'$ over C, is:

$$\mathcal{V}' \equiv \oint_C \vec{E}' \cdot d\vec{l} = \mathcal{V}_t + \mathcal{V}_m$$

Total emf = the time derivative of dynamic flux-1

$$\mathcal{V}' = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} = -\frac{d}{dt}\underline{\Phi_{dynm}}$$

Proof: Consider a ckt moving from C_1 at t to C_2 at $t+\Delta t$ in a time varying M-field $\vec{B}(\vec{r},t)$:

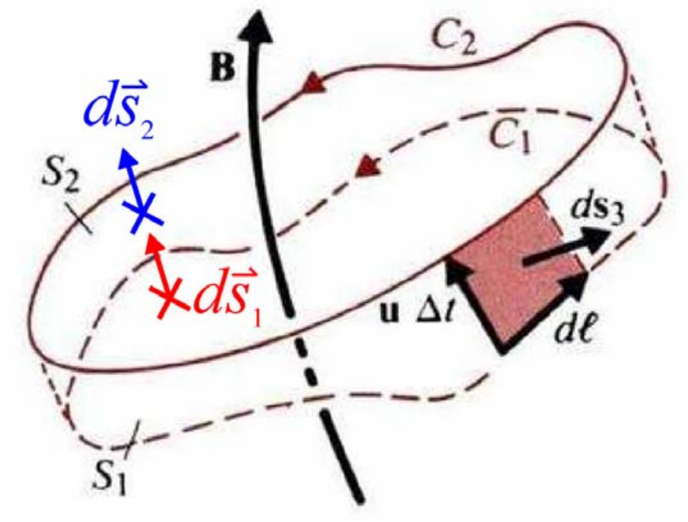


Total emf equals the time derivative of dynamic flux-2

$$\frac{d}{dt}\Phi_{dynm} = \frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \left(\neq \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t+\Delta t)\cdot d\vec{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\vec{s}_1 \right]$$

Different from Φ_{stat}



Total emf equals the time derivative of dynamic flux-3

By $\vec{B}(t+\Delta t) \approx \vec{B}(t) + \dfrac{\partial \vec{B}(t)}{\partial t}\Delta t$:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t+\Delta t)\cdot d\vec{s}_2\right] \approx \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t)\cdot d\vec{s}_2 + \int_{S_2}\left(\frac{\partial \vec{B}(t)}{\partial t}\Delta t\right)\cdot d\vec{s}_2\right]$$

$$= \lim_{\Delta t \to 0}\left[\frac{1}{\Delta t}\int_{S_2} \vec{B}(t)\cdot d\vec{s}_2 + \int_{S_2} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{s}_2\right] \approx \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{S_2} \vec{B}(t)\cdot d\vec{s}_2 + \int_{S} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{s}$$

$$\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t+\Delta t)\cdot d\vec{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\vec{s}_1\right]$$

$$\approx \int_S \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{s} + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \vec{B}(t)\cdot d\vec{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\vec{s}_1\right]$$

Total emf equals the time derivative of dynamic flux-4

$$\int_V \left(\nabla\cdot\vec{B}\right)dv = \oint_S \vec{B}\cdot d\vec{s}$$

$$d\vec{s}_3 = d\vec{l}\times\vec{u}\Delta t$$

$$ds_3 = \left|d\vec{l}\times\vec{u}\Delta t\right| = dl\cdot u\Delta t\cdot\sin\theta$$

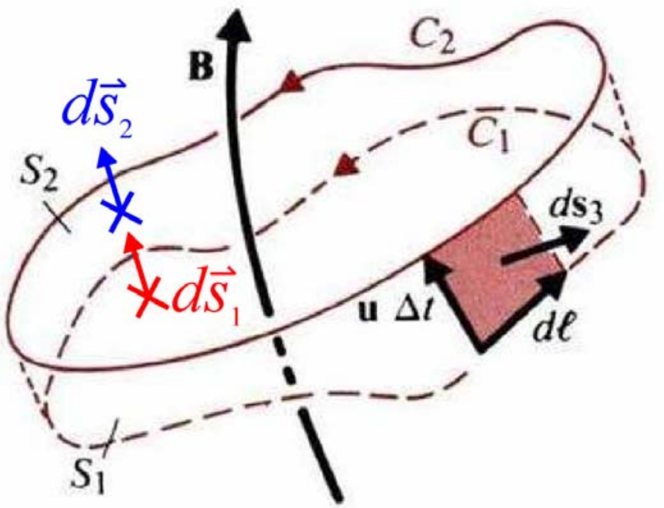
$$= \int_{S_2} \vec{B}(t)\cdot d\vec{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\vec{s}_1 + \int_{S_3} \vec{B}(t)\cdot d\vec{s}_3 = 0$$

$$\vec{B}\cdot d\vec{s}_3 = \Delta t\vec{B}\cdot\left(d\vec{l}\times\vec{u}\right) = \Delta t d\vec{l}\cdot\left(\vec{u}\times\vec{B}\right)$$

$$\int_{S_3} \vec{B}(t)\cdot d\vec{s}_3 = \Delta t\int_{S_3} d\vec{l}\cdot\left(\vec{u}\times\vec{B}\right)$$

$$\rightarrow \Delta t\oint_C d\vec{l}\cdot\left(\vec{u}\times\vec{B}\right)$$

$\Delta t \rightarrow 0$



Total emf equals the time derivative of dynamic flux-5

$$\int_{S_2} \vec{B}(t)\cdot d\vec{s}_2 - \int_{S_1} \vec{B}(t)\cdot d\vec{s}_1 =$$

$$-\int_{S_3} \vec{B}(t)\cdot d\vec{s}_3 \rightarrow \underline{-\Delta t \oint_C \left(\vec{u}\times\vec{B}\right)\cdot d\vec{l}}$$

$$\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \approx \int_S \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\underline{\int_{S_2}\vec{B}(t)\cdot d\vec{s}_2 - \int_{S_1}\vec{B}(t)\cdot d\vec{s}_1}\right]$$

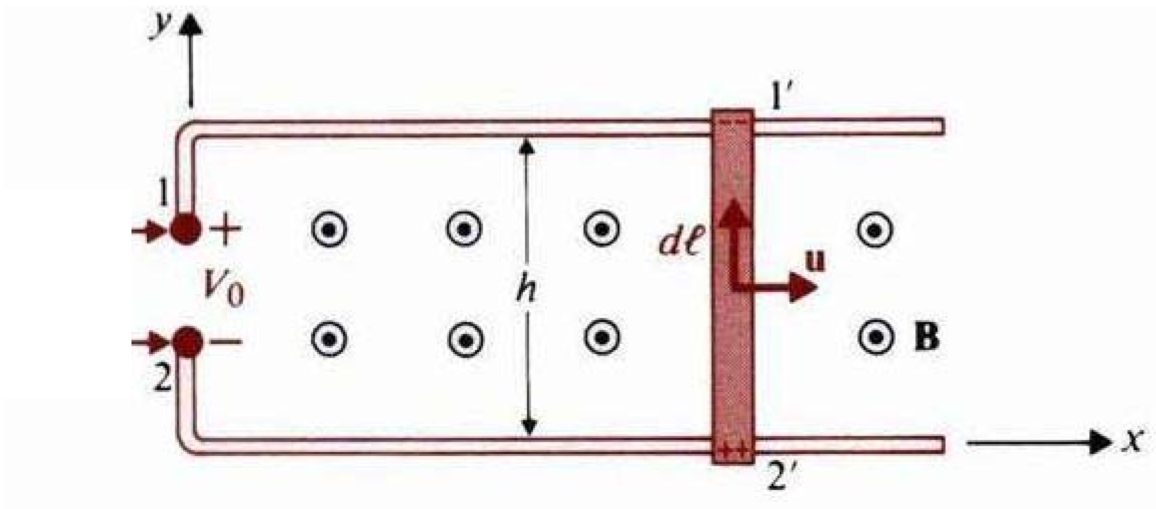
$$\approx \int_S \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[-\Delta t\oint_C \left(\vec{u}\times\vec{B}\right)\cdot d\vec{l}\right] = -\mathcal{V}'$$

$-\mathcal{V}_t$ $\quad$ $-\mathcal{V}_m$

$$\Rightarrow \mathcal{V}' = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} = -\frac{d}{dt}\Phi_{dynm}$$

Example 14-3: Metal bar sliding over conducting rails

Find: open ckt voltage V_0



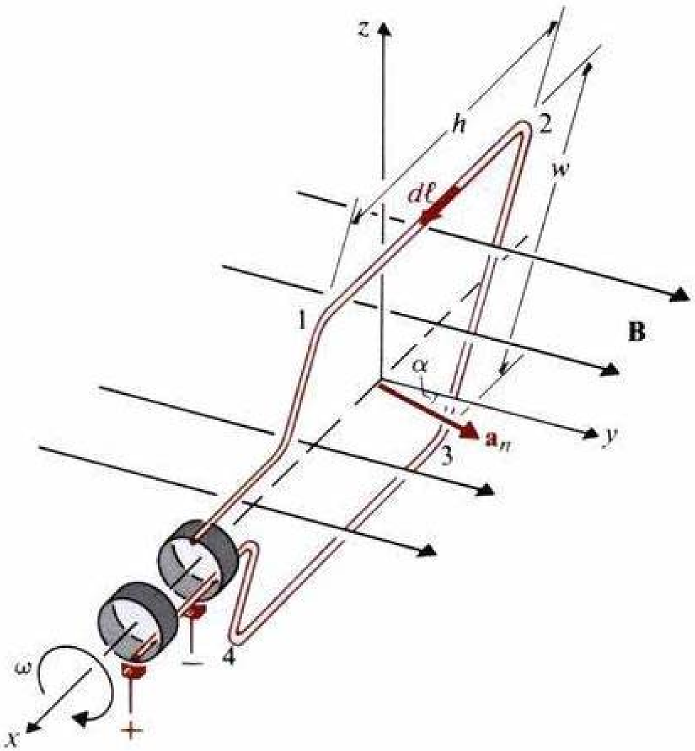
$$\Phi_{dynm} = \int_S \vec{B} \cdot \underline{d\vec{s}} = B_0 hut, \quad \Rightarrow V_0 = -\frac{d}{dt}\Phi_{dynm} = -B_0 hu$$

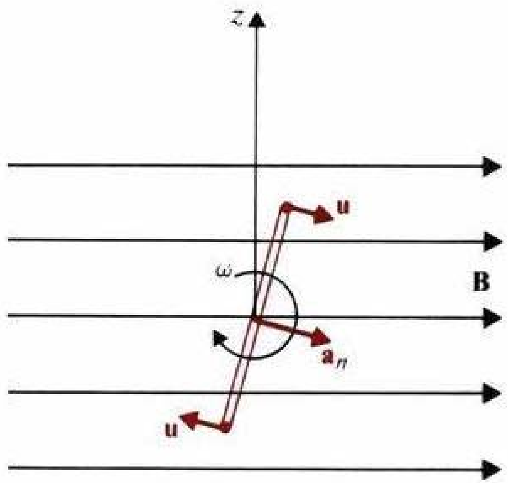
$\sim +d\vec{z}$

$V_0 < 0$, $\Rightarrow$ CLK current

Example 14-4: AC generator (1)

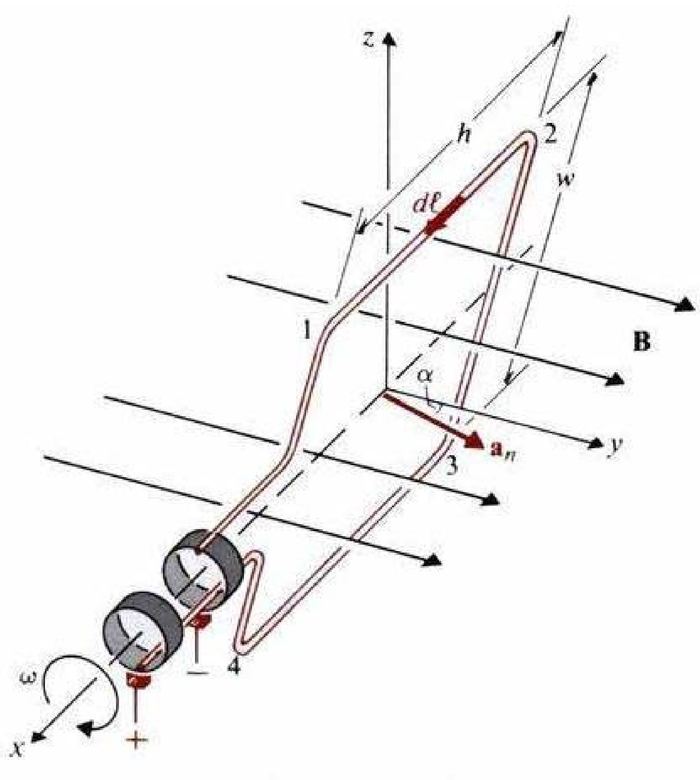
Rectangular ($h \times w$) loop in a time-varying M-field $\vec{B} = \vec{a}_y B_0 \sin \omega t$, rotate about x-axis with angular velocity ω. Find emf?





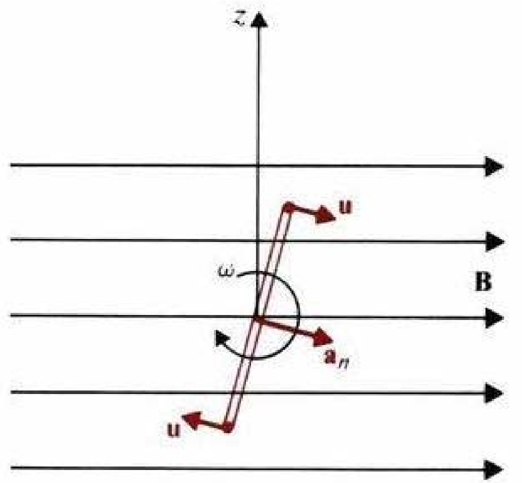
Example 14-4: AC generator (2)

(Method 1) At t, unit normal vector $\vec{a}_n$ makes an angle $\alpha = \omega t$ w.r.t $\vec{a}_y$



$$\Phi_{dynm} = \int_S \vec{B} \cdot d\vec{s} = \left(\vec{a}_y B_0 \sin \omega t\right) \cdot \left(\vec{a}_n hw\right)$$

$$= B_0 hw \cdot \underline{\sin \omega t} \cdot \underline{\cos \alpha(t)}$$



Example 14-4: AC generator (3)

The static flux Φ_{stat} is the flux assuming the loop is stationary (α is constant)

$$\Phi_{stat} = B_0 hw \cdot \underline{\sin \omega t} \cdot \cos \alpha$$

$$\alpha = \omega t$$

$$\mathcal{V}_t = -\frac{d}{dt}\Phi_{stat} = -B_0 hw\omega \cdot \cos \omega t \cdot \cos \alpha$$

or $$\mathcal{V}_t = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = -\left(\vec{a}_y B_0 \omega \cos \omega t\right) \cdot \left(\vec{a}_n hw\right)$$

$$= -B_0 hw\omega \cdot \cos \omega t \cdot \cos \alpha$$

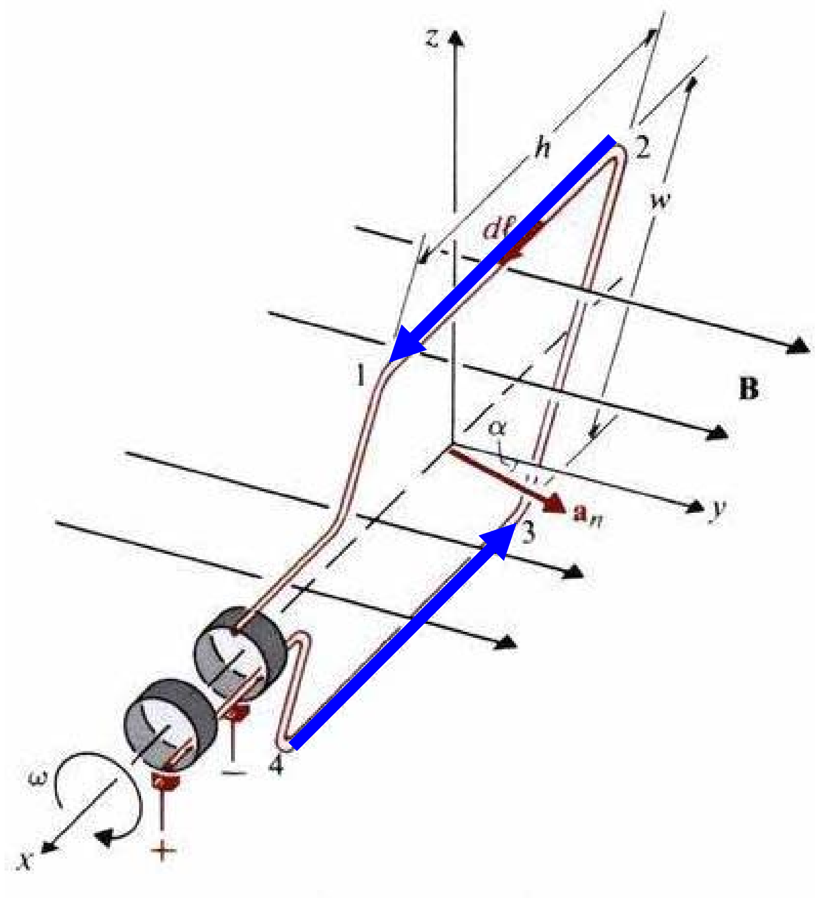
Example 14-4: AC generator (4)

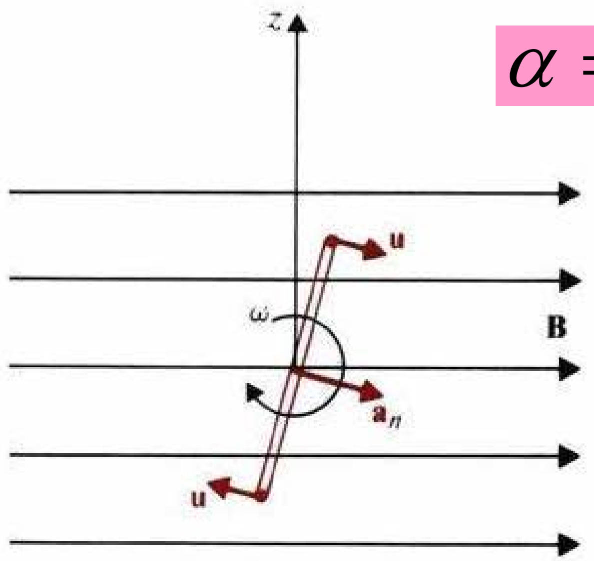
$$\mathcal{V}_m = \oint_{14321} \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l} = \int_2^1 \left[\left(\vec{a}_n \frac{w}{2}\omega\right) \times \left(\vec{a}_y B_0 \sin \omega t\right)\right] \cdot (\vec{a}_x dx)$$

$$+ \int_4^3 \left[\left(-\vec{a}_n \frac{w}{2}\omega\right) \times \left(\vec{a}_y B_0 \sin \omega t\right)\right] \cdot (-\vec{a}_x dx)$$

$$= B_0 h w \omega \cdot \sin \omega t \cdot \sin \alpha$$

$$\alpha = \omega t$$





Example 14-4: AC generator (5)

$$\begin{cases} \mathcal{V}_t = -B_0 hw\omega \cdot \cos\omega t \cdot \cos\alpha \\ \mathcal{V}_m = B_0 hw\omega \cdot \sin\omega t \cdot \sin\alpha \end{cases}$$

$$\Rightarrow \quad \mathcal{V}' = \mathcal{V}_t + \mathcal{V}_m = -B_0 hw\omega \cdot \cos 2\omega t$$

Example 14-4: AC generator (6)

(Method 2) Directly differentiate Φ_{dymn}:

$$\Phi_{dynm} = B_0 hw \cdot \sin \omega t \cdot \cos \omega t = \frac{B_0 hw \cdot \sin 2\omega t}{2}$$

$$\Rightarrow \mathcal{V}' = -\frac{d}{dt}\Phi_{dynm} = -B_0 hw\omega \cdot \cos 2\omega t$$

Comments-1

Faraday's law $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ couples:

1) Electric field with magnetic field

2) Space with time

which enables electromagnetic waves

Comments-2

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \ \oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$ always valid

$$\begin{cases} \mathcal{V}_t \equiv \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\Phi_{stat} \\ \mathcal{V}_m \equiv \oint_C \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l} \\ \oint_C \vec{E}' \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l} \end{cases}$$

useful only in the presence of conducting loop

Outline

- Faraday's law
- Maxwell's equations

  1. Equations

  2. Boundary conditions

Contradiction between Ampère's circuital law & equation of continuity

$$\begin{cases} \nabla \times \vec{H} = \vec{J} \\ \nabla \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t} \end{cases} \Rightarrow \nabla \cdot \left(\nabla \times \vec{H} \right) = -\frac{\partial \rho}{\partial t}$$

in violation of the vector identity: $\nabla \cdot \left(\nabla \times \vec{A} \right) = 0$,

if charge density ρ is time-varying: $\dfrac{\partial \rho}{\partial t} \neq 0$

Modified Ampère's circuital law

To maintain the consistency, we demand:

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \left(= 0\right)$$

$$\nabla \cdot \vec{D} = \rho$$

$$= \nabla \cdot \vec{J} + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}\right) = \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

$$\Rightarrow \boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

$\vec{J}_D$ …displacement current density

Maxwell's equations-differential form

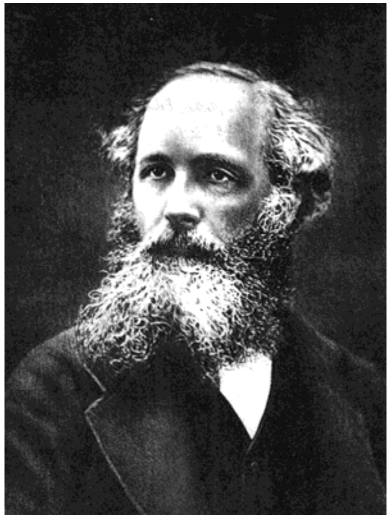
$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \text{...Faraday's law of EM induction} \\ \nabla \cdot \vec{D} = \rho & \text{...Gauss's law} \\ \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} & \text{...Modified Ampère's circuital law} \\ \nabla \cdot \vec{B} = 0 & \text{...Inexistence of magnetic charge} \end{cases}$$

and $\nabla \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t}$, $\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$, that's it!

Maxwell's equations-integral form

James Maxwell

(1864)



$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow \oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\nabla \cdot \vec{D} = \rho \longrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q$$

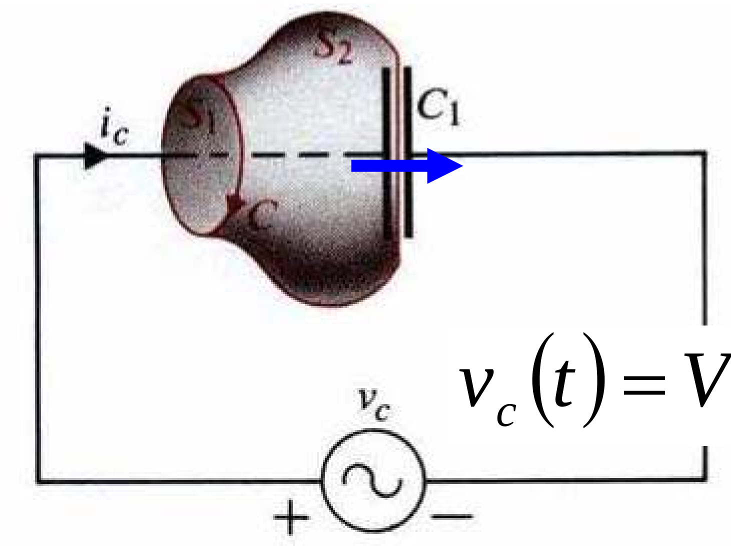
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow \oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$$

$$\nabla \cdot \vec{B} = 0 \longrightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$$

I_D…displacement current

Example 14-5: AC driven parallel-plate capacitor (1)

Find: displacement current $i_d(t)$, magnetic field intensity $\vec{H}$ everywhere



$$v_c(t) = V_0 \sin \omega t$$

Electrostatics gives: $E(t) = v_c(t)/d$

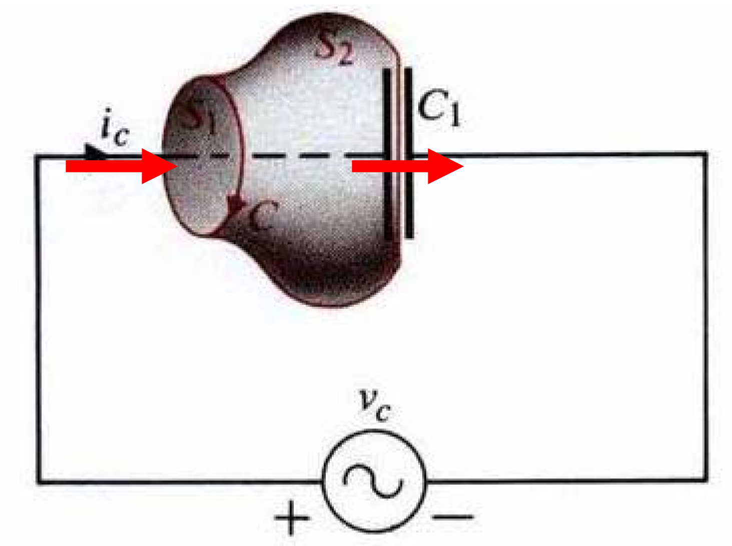
$$\Rightarrow D(t) = \varepsilon E(t) = \varepsilon v_c(t)/d$$

Example 14-5: AC driven parallel-plate capacitor (2)

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t}, \quad D(t) = \varepsilon \frac{v_c(t)}{d}$$

$$i_d(t) = \int_S \frac{\partial \vec{D}(t)}{\partial t} \cdot d\vec{s} = \left(\varepsilon \frac{S}{d} \right) \frac{d}{dt} v_c(t) = C \frac{d}{dt} v_c(t)$$

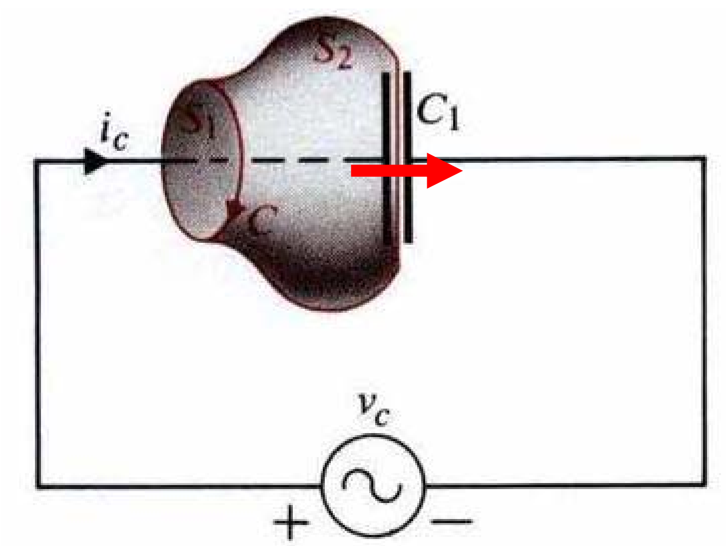
$$= CV_0 \omega \cos \omega t = i_c(t)$$ …conduction current



Example 14-5: AC driven parallel-plate capacitor (3)

$$\oint_C \vec{H} \cdot d\vec{l} = 2\pi r H_\phi(r,t) = i_c(t) = i_d(t)$$

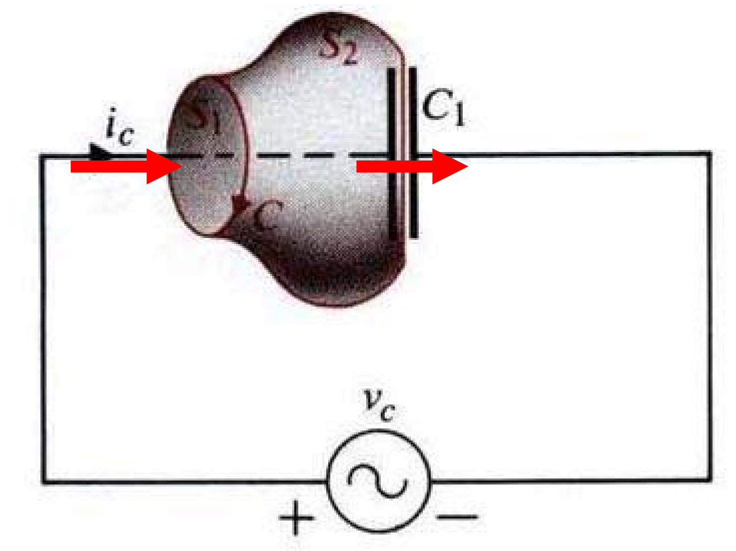
$$\Rightarrow \vec{H} = \vec{a}_\phi \frac{CV_0}{2\pi r} \omega \cos \omega t$$ …everywhere



Comments-1

$$i_d(t) = i_c(t), \Rightarrow$$

Total current is continuous across the circuit

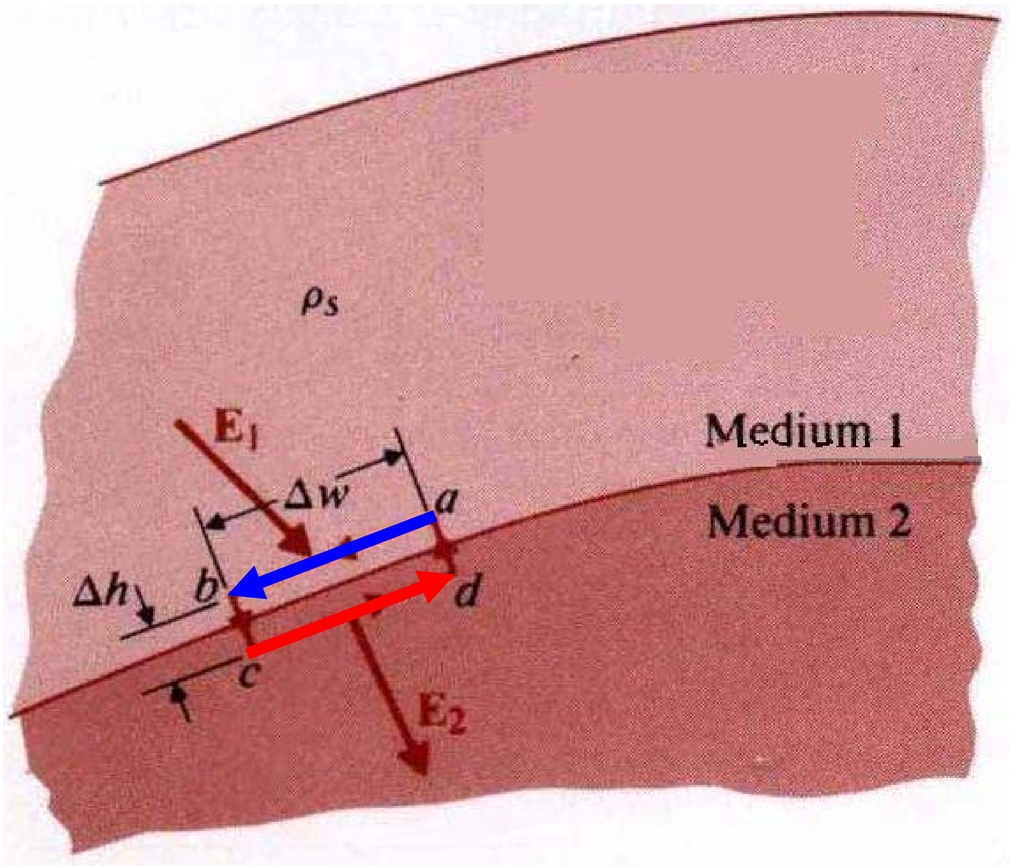


Comments-2

$$\vec{H} = \vec{a}_\phi \frac{CV_0}{2\pi r} \omega \cos \omega t, \Rightarrow$$

the magnitude of magnetic field is proportional to the frequency of the driving source

Tangential BC of dynamic E-field



$$\oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

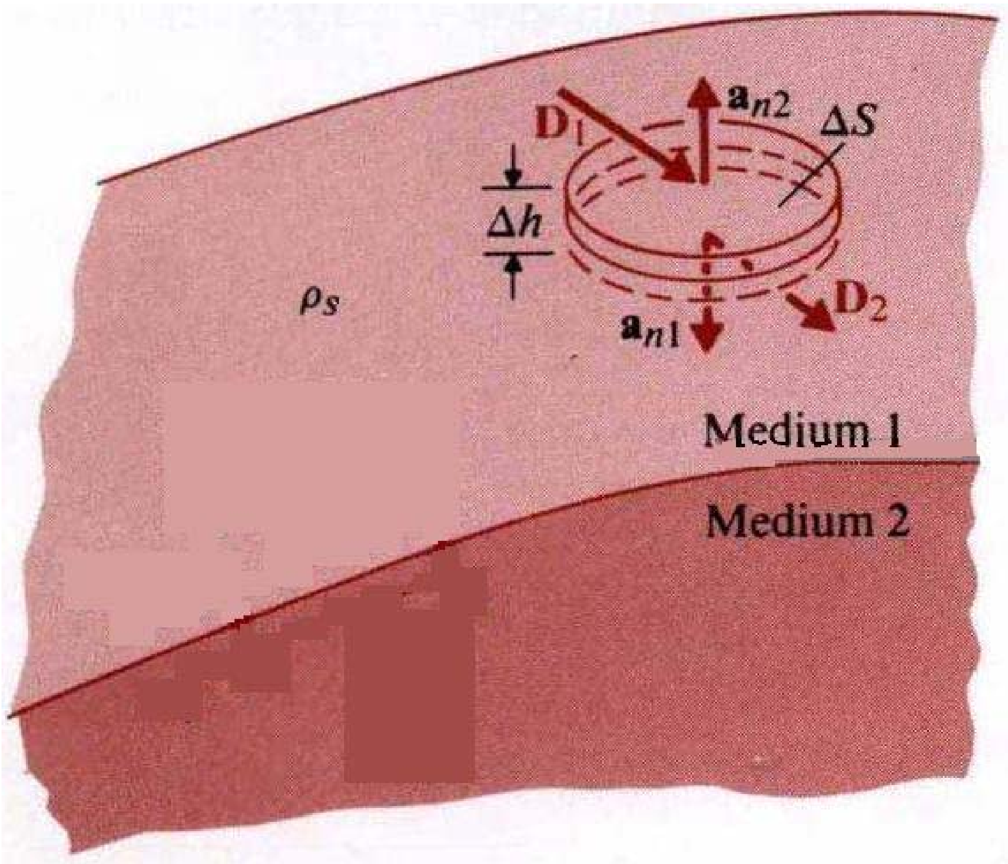
$$\oint_{abcda} \vec{E}\cdot d\vec{l}\,\Big|_{\Delta h\to 0}$$

$$= E_{1t}\cdot(-\Delta w) + E_{2t}\cdot(\Delta w)$$

$$= 0,$$

$$\Rightarrow \boxed{E_{1t} = E_{2t}}$$

Normal BC of dynamic E-field



$$\oint_S \vec{D}\cdot d\vec{s} = Q$$

$$\oint_S \vec{D}\cdot d\vec{s}\Big|_{\Delta h\to 0}$$

$$= \left(\vec{D}_1\cdot\vec{a}_{n2} + \vec{D}_2\cdot\vec{a}_{n1}\right)(\Delta S)$$

$$= \rho_s\cdot\Delta S$$

$$\Rightarrow \boxed{\vec{a}_{n2}\cdot\left(\vec{D}_1 - \vec{D}_2\right) = \rho_s}$$

$$\Rightarrow \quad D_{1n} - D_{2n} = \rho_s$$

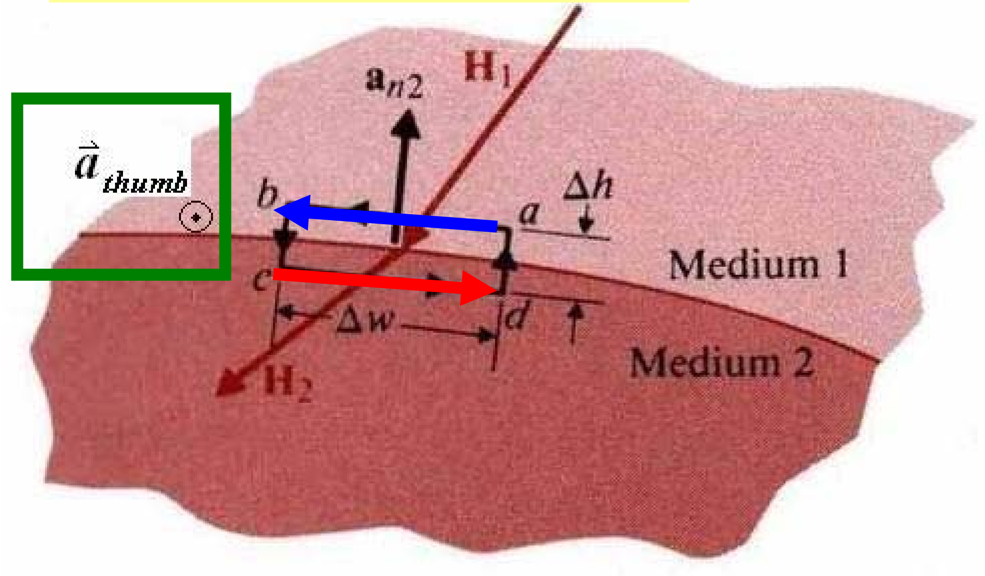
Tangential BC of dynamic M-field

$$\oint_C \vec{H}\cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{s}, \Rightarrow \oint_{abcda} \vec{H}\cdot d\vec{l} = \vec{H}_1\cdot\Delta\vec{w} + \vec{H}_2\cdot(-\Delta\vec{w})$$

$$= H_{1t}\cdot\Delta w - H_{2t}\cdot\Delta w = J_{sn}\Delta w + 0$$

Component of $\vec{H}_i$ in the *ab*-direction

Component of $\vec{J}_s$ in $\vec{a}_{thumb}$



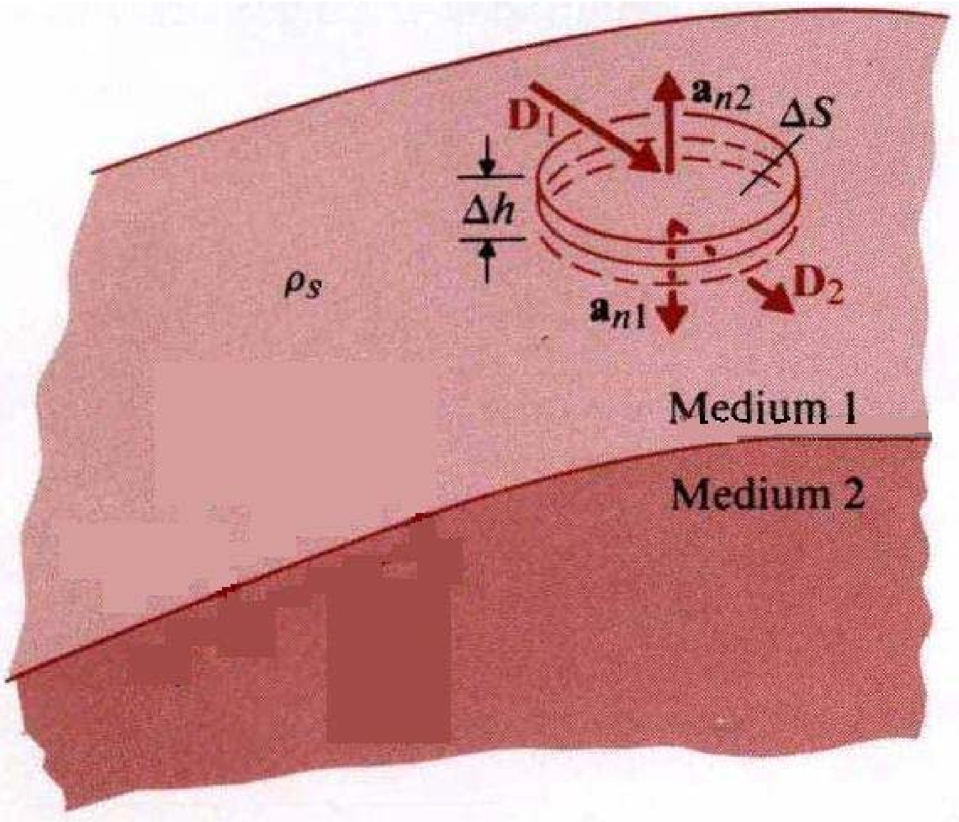
$$\Rightarrow H_{1t} - H_{2t} = J_{sn}$$

In general,

$$\boxed{\vec{a}_{n2}\times\left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s}$$

Normal BC of dynamic M-field

$$\oint_S \vec{B}\cdot d\vec{s} = 0, \quad \Rightarrow \oint_S \vec{B}\cdot d\vec{s} = \left(\vec{B}_1\cdot\vec{a}_{n2} - \vec{B}_2\cdot\vec{a}_{n2}\right)(\Delta S) = 0$$



$$\Rightarrow \boxed{B_{1n} = B_{2n}}$$