Lesson 13 Inductance, Magnetic Energy, Force, & Torque

13.1 Inductance

■ Physical meaning

As shown in Fig. 13-1, a closed loop C_1 carrying a current I_1 will create a magnetic field $\vec{B}_1$, causing a magnetic flux of $\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot d\vec{s}$ over the loop itself. It will cause a flux linkage of $\Lambda_{11} = N_1\Phi_{11}$ if C_1 has N_1 turns. If the loop current is changed to $I_1' = rI_1$ (r is a constant), eq. (11.13) predicts that the resulting magnetic field will be $\vec{B}_1' = r\vec{B}_1$ (i.e., the magnitude change proportionally while the spatial distribution remains intact) as long as the field exists in some linear medium (i.e., μ is independent of current). As a result, the "self"-flux $\Phi_{11}' = \int_{S_1} \vec{B}_1' \cdot d\vec{s} = r\Phi_{11}$ and the flux linkage $\Lambda_{11}' = r\Lambda_{11}$ are proportional to the loop current. The constant ratio of self flux linkage to loop current is defined as the "self-inductance" of the loop:

$$L_{11} = \frac{\Lambda_{11}}{I_1} \quad \text{(H)} \tag{13.1}$$

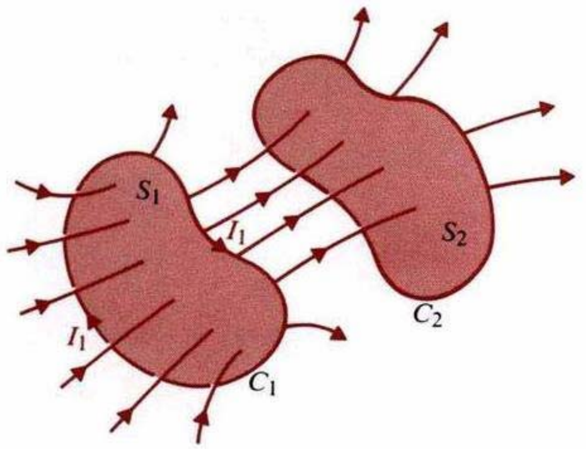

Fig. 13-1. Definition of inductance (after DKC).

In the presence of another loop C_2 of N_2 turns, the magnetic field $\vec{B}_1$ created by C_1 will also pass through C_2, causing a mutual flux linkage of $\Lambda_{12} = N_2\Phi_{12}$, where the mutual flux $\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{s}$ is proportional to current I_1. The "mutual inductance" between the two loops is defined as:

$$L_{12} = \frac{\Lambda_{12}}{I_1} \quad \text{(H)} \tag{13.2}$$

In general, inductance depends on the geometry of the loops and the material where they lie within.

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1) By eq's (11.8), (11.12), $\vec{B}_1 = \nabla \times \vec{A}_1$, $\vec{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1'} \frac{d\vec{l}_1}{R}$, $\Rightarrow$

$$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \left(\nabla \times \vec{A}_1\right) \cdot d\vec{s} = \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2 = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{R}.$$

Similarly, $L_{21} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{R}$, $\Rightarrow$ $L_{12} = L_{21}$.

2) In Lesson 14, we will see that a time-varying current I_1 will induce a voltage (electromotive force) in C_2 through L_{12} (Faraday's law of EM induction), justifying the name of "inductance".

■ Evaluation procedures

1) (1) Assume a current I for the loop. (2) Find $\vec{B}$ by Ampère's circuital law [eq. (12.6)] or Biot-Savart law [eq. (11.13)]. (3) Find flux linkage Λ by integration: $\Lambda = N\int_S \vec{B} \cdot d\vec{s}$.

(4) Find L by eq's (13.1), (13.2).

2) (1) Assume a current I for the loop. (2) Find $\vec{H}$ and $\vec{B}$ by previous methods. (3) Find the stored magnetic energy W_m. (4) Find L by $W_m = \frac{1}{2}LI^2$ [see eq. (13.3)].

<u>Example 13-1</u>: Consider a hollow solenoid inductor with cross-sectional area S, and n turns per unit length (Fig. 13-2). Find the inductance per unit length.

Ans: (1) Assume current I. (2) By eq. (11.7), there is a uniform magnetic flux density of magnitude $B = \mu_0 nI$ along axial direction. (3) For a unit length ($l = 1$), the flux linkage is:

$$\Lambda = n \cdot \Phi = n \cdot (\mu_0 nI) \cdot S .$$

(4) By eq. (13.1), $L = n^2 \mu_0 S$.

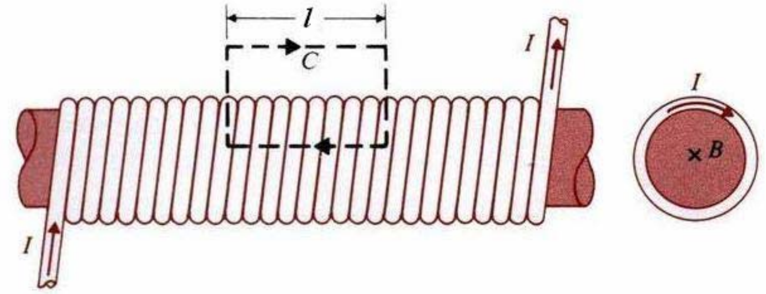

Fig. 13-2. Schematic of a solenoid (after DKC).

Example 13-2: Consider two coils C_1, C_2 with N_1, N_2 turns and lengths l_1, l_2, respectively (Fig. 13-3). They are wound concentrically on a thin cylindrical core of radius a ($<< l_1$, l_2) with permeability μ. Find the mutual inductance.

Ans: (1) Assume C_1, C_2 have currents I_1, I_2, respectively. (2) By eq. (11.7), current I_1 will produce a uniform magnetic flux density inside C_1 of magnitude $B_1 = \mu \dfrac{N_1}{l_1} I_1$. (3) The mutual flux linkage of C_2 due to C_1 is: $\Lambda_{12} = N_2 \cdot \Phi_{12} = N_2 \cdot B_1 \cdot S$. (4) By eq. (13.2), $\Rightarrow$

$$L_{12} = \mu \frac{N_1 N_2}{l_1} \pi a^2 .$$

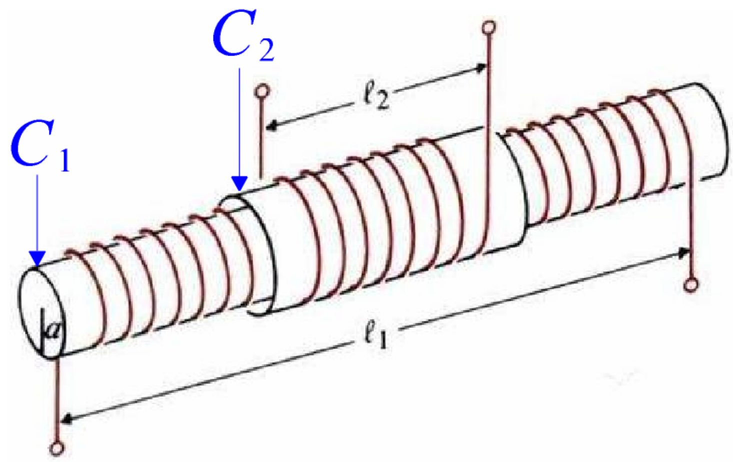

Fig. 13-3. Two coils wound on a common core (after DKC).

13.2 Magnetic Energy

■ Magnetic energy of assembling current loops

Consider a single closed loop C_1 with self-inductance L_1. If the loop current i_1 increases from zero to I_1 slowly (such that the linear dimension of C_1 is much smaller than the wavelength, i.e., "quasi-static" condition), a voltage (emf) of:

$$v_1 = -\frac{d\Phi_{11}}{dt} = L_1 \frac{di_1}{dt}$$

will be induced on the loop to oppose the change of current i_1 (Faraday's law and Lenz's law). The work done to overcome the induced v_1 and enforce the change of i_1 is:

$$W_1 = \int v_1 i_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2,$$

which is stored as magnetic energy:

$$W_1 = \frac{1}{2} L_1 I_1^2 \qquad (13.3)$$

Insert a second closed loop C_2 with self-inductance L_2 and mutual inductance L_{21}. If we maintain $i_1 = I_1$ while increase i_2 from zero to I_2 slowly, a voltage (emf) of:

$$v_{21} = -\frac{d\Phi_{21}}{dt} = L_{21} \frac{di_2}{dt}$$

will be induced on C_1 in an attempt to change i_1 (away from I_1). The work done to maintain $i_1 = I_1$ is:

$$W_{21} = \int v_{21} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2 .$$

Meanwhile, a voltage (emf) of:

$$v_2 = -\frac{d\Phi_{22}}{dt} = L_2 \frac{di_2}{dt}$$

will be induced on C_2 to oppose the change of current i_2. The work done to overcome the induced v_2 and enforce the change of i_2 is:

$$W_{22} = \frac{1}{2} L_2 I_2^2 .$$

The total magnetic energy stored in the system of two current loops is:

$$W_2 = \frac{1}{2}L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2}L_2 I_2^2 .$$

The magnetic energy stored in a system of N loops carrying currents I_1, I_2, …, I_N can be generalized to:

$$W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k , \tag{13.4}$$

By eq's (13.1), (13.2), the flux of loop C_k due to all the N current loops is:

$$\Phi_k = \sum_{j=1}^{N} L_{jk} I_j , \Rightarrow$$

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k \tag{13.5}$$

Assume the magnetic flux density and vector potential created by a system of continuous current density distribution $\vec{J}(\vec{r})$ in a volume V' are $\vec{B}$ and $\vec{A}$, respectively. The system can be decomposed into N elementary current loops C_k ($k = 1, 2, …, N$), each has a current ΔI_k and a "filamentary" cross-sectional area Δa_k. The magnetic flux of the kth loop is:

$$\Phi_k = \int_{S_k} \vec{B}\cdot d\vec{s} = \oint_{C_k} \vec{A}\cdot d\vec{l}_k ,$$

where S_k is the surface bounded by C_k. By eq. (13.5),

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2}\sum_{k=1}^{N} \Delta I_k \oint_{C_k} \vec{A}\cdot d\vec{l}_k .$$

Since $\Delta I_k \cdot d\vec{l}_k = \left(\left|\vec{J}\right|\cdot \Delta a_k\right)\cdot d\vec{l}_k = \vec{J}\cdot\left(\Delta a_k \cdot \left|d\vec{l}_k\right|\right) = \vec{J}\cdot \Delta v_k$, where $\Delta v_k = \Delta a_k \cdot \left|d\vec{l}_k\right|$ is the differential volume, $\Rightarrow$

$$W_m = \frac{1}{2}\int_{V'} \left(\vec{A}\cdot\vec{J}\right) dv \tag{13.6}$$

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1) Compare eq's (9.5) and (13.5), $W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \leftrightarrow W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k$, $\Rightarrow$ (1) charge $Q \leftrightarrow$ current I, (2) voltage $V \leftrightarrow$ flux Φ.

2) Compare eq's (9.6) and (13.6), $W_e = \frac{1}{2}\int_{V'} \rho V dv \leftrightarrow W_m = \frac{1}{2}\int_{V'} (\vec{A}\cdot\vec{J}) dv$, $\Rightarrow$ (1) charge density $\rho \leftrightarrow$ current density $\vec{J}$, (2) electric potential $V \leftrightarrow$ magnetic potential $\vec{A}$.

■ Magnetic energy of magnetic fields

In terms of real applications of electromagnetism (especially electromagnetic waves), sources are usually far away from the region of interest and only the resulting fields are given. It becomes more convenient to express the magnetic energy W_m by the magnetic field quantities $\vec{B}$ and $\vec{H}$ in the absence of the current distribution $\vec{J}$.

(1) Substituting $\vec{J} = \nabla \times \vec{H}$ [eq. (12.4)] into eq. (13.6), $\Rightarrow$

$$W_m = \frac{1}{2}\int_{V'} \vec{A}\cdot(\nabla\times\vec{H}) dv,$$

where V' is a volume containing all the source currents.

(2) By eq. (11.8) and the vector identity:

$$\nabla\cdot(\vec{A}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{H}) = \vec{H}\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{H}), \Rightarrow$$

$$W_m = \frac{1}{2}\int_{V'} (\vec{H}\cdot\vec{B}) dv - \frac{1}{2}\int_{V'} \nabla\cdot(\vec{A}\times\vec{H}) dv.$$

(3) By divergence theorem [eq. (5.24)], $\int_{V'} \nabla\cdot(\vec{A}\times\vec{H}) dv = \oint_{S'} (\vec{A}\times\vec{H})\cdot d\vec{s}$, where S' is the closed surface of V'. $\Rightarrow$

$$W_m = I_1 - I_2, \text{ where } I_1 = \frac{1}{2}\int_{V'} (\vec{H}\cdot\vec{B}) dv, \quad I_2 = \frac{1}{2}\oint_{S'} (\vec{A}\times\vec{H})\cdot d\vec{s}.$$

(4) One can choose S' as a spherical surface centered at the origin with an infinite radius $R \to \infty$, such that all the source currents are definitely enclosed. For an observation point (on

S') far away from the source (at the origin), the magnitude of the vector potential $|\vec{A}| \propto R^{-1}$, and the field magnitude $|\vec{H}| \propto R^{-2}$. $\Rightarrow$

$$I_2 = \frac{1}{2}\oint_{S'} (\vec{A} \times \vec{H}) \cdot d\vec{s} \approx \frac{1}{2}|\vec{A}(R)||\vec{H}(R)| \cdot 4\pi R^2 \propto \frac{1}{R} \cdot \frac{1}{R^2} \cdot R^2 \propto \frac{1}{R} \to 0, \quad W_m = I_1. \Rightarrow$$

$$W_m = \int_{V'} w_m dv, \quad w_m = \frac{1}{2}\vec{H} \cdot \vec{B} \quad (\text{J/m}^3) \tag{13.7}$$

where V' has to cover everywhere with nonzero magnetic field, and w_m represents the magnetostatic energy density.

Example 13-3: A coaxial transmission line with solid inner conductor ($\mu = \mu_0$) of radius a, thin outer conductor of inner radius b, and filled with air in between. A uniform current I flows in the inner conductor and returns in the outer conductor. Find the inductance per unit length.

Ans: By the result of Example 11-1: $\vec{B} = \begin{cases} \vec{a}_\phi B_{\phi 1}(r), \text{if } r < a \\ \vec{a}_\phi B_{\phi 2}(r), \text{if } a < r < b \end{cases}$, where

$$B_{\phi 1}(r) = \frac{\mu_0 I}{2\pi a^2} r, \quad B_{\phi 2}(r) = \frac{\mu_0 I}{2\pi r}. \Rightarrow \quad H_{\phi 1}(r) = \frac{I}{2\pi a^2} r, \quad H_{\phi 2}(r) = \frac{I}{2\pi r}.$$

By eq. (13.7), $\Rightarrow$ the magnetic energy density w_m and stored energy W_m inside the inner conductor and in the region between the two conductors are:

$$w_{m1} = \frac{\mu_0 I^2}{8\pi^2 a^4} r^2, \quad w_{m2} = \frac{\mu_0 I^2}{8\pi^2 r^2}, \quad dv = 2\pi r \cdot dr, \Rightarrow$$

$$W_{m1} = \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi}, \quad W_{m2} = \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi}\ln\left(\frac{b}{a}\right).$$

By eq. (13.3), the inductance is:

$$L = \frac{2(W_{m1} + W_{m2})}{I^2} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right).$$

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1) The same result can be obtained by a more tedious way: (1) The flux linkage due to a

"loop" bounded by the outer conductor ($r' = b$) and a thin cylindrical shell of radius $r < r' < r + dr$ inside the inner conductor ($0 < r < a$) is derived by:

$$d\Phi(r) = \int_r^a B_{\phi 1}(r') \cdot 1dr' + \int_a^b B_{\phi 2}(r') \cdot 1dr', \Rightarrow \ d\Lambda(r) = \frac{2\pi r dr}{\pi a^2} \cdot d\Phi(r).$$

(2) The total flux linkage is: $\Lambda = \int_{r=0}^{r=a} d\Lambda(r)$. (3) By eq. (13.1), $L = \Lambda / I$.

2) $\frac{\mu_0}{8\pi}$ and $\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$ represent the internal and external inductance per unit length. In high-frequency cases, current only flows near the conducting surface (skin effect), and internal inductance approaches zero.

13.3 Magnetic Forces and Torques

■ Forces on current-carrying conductors

Consider an elemental current-carrying wire of cross-sectional area S, which is represented by a differential displacement vector $d\vec{l}$ (Fig. 13-4). In the presence of an external magnetic field $\vec{B}$, free charges within the wire of volume charge density ρ (C/m^3) move with velocity $\vec{u}$ (in the same direction of $d\vec{l}$). By eq. (11.1), the magnetic force exerted on the element is:

$$d\vec{F}_m = \rho S \left|d\vec{l}\right| \left(\vec{u} \times \vec{B}\right).$$

Since $\left|d\vec{l}\right|\vec{u} = \left|\vec{u}\right| d\vec{l}$, $\vec{J} = \rho \vec{u}$ [eq. (10.1)], $\Rightarrow$ $d\vec{F}_m = \rho S \left|\vec{u}\right| d\vec{l} \times \vec{B} = JS\left(d\vec{l} \times \vec{B}\right)$,

$$d\vec{F}_m = I\left(d\vec{l} \times \vec{B}\right), \tag{13.8}$$

where $I = JS$ denotes the current in the element.

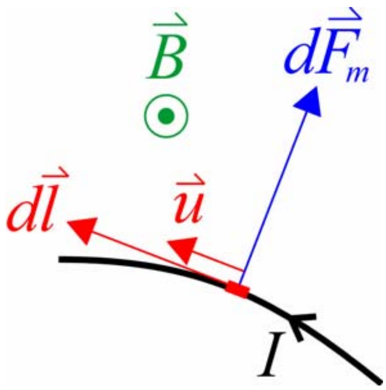

Fig. 13-4. Magnetic force experienced by an elemental current-carrying wire.

For a closed loop C, eq. (13.8) becomes:

$$\vec{F}_m = I\oint_C d\vec{l} \times \vec{B} \tag{13.9}$$

If $\vec{B}$ is created by another closed loop C_2 carrying a current I_2, the force exerted on the loop C_1 carrying a current I_1 is:

$$\vec{F}_{21} = I_1\oint_{C_1} d\vec{l}_1 \times \vec{B}_{21}, \text{ where } \vec{B}_{21} = \frac{\mu_0 I_2}{4\pi}\oint_{C_2} \frac{d\vec{l}_2 \times \vec{a}_{R_{21}}}{R_{21}^2} \text{ [eq. (11.13)].} \Rightarrow$$

$$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\vec{l}_1 \times \left(d\vec{l}_2 \times \vec{a}_{R21}\right)}{R_{21}^2} = -\vec{F}_{12} \tag{13.10}$$

Eq. (13.10) is the Ampère's law of force between two current-carrying loops, which is the counterpart of Coulomb's law of force between two electric charges [eq. (6.8)].

Example 13-4: Consider two infinitely long, parallel wires separated by a distance d, and carrying currents I_1, I_2 in the same direction (Fig. 13-5). Find the force per unit length between them.

Ans: By the result of Example 11-1, the magnetic field on wire 2 due to wire 1 is:

$$\vec{B}_{12} = -\vec{a}_x \frac{\mu_0 I_1}{2\pi d}.$$

By eq. (13.9), the force exerted on wire 2 is:

$$\vec{F}_{12} = I_2\int_0^1 d\vec{l}_2 \times \vec{B}_{12} = I_2\int_0^1 (\vec{a}_z dz) \times \left(-\vec{a}_x \frac{\mu_0 I_1}{2\pi d}\right) = -\vec{a}_y \frac{\mu_0 I_1 I_2}{2\pi d},$$

which is a force of attraction.

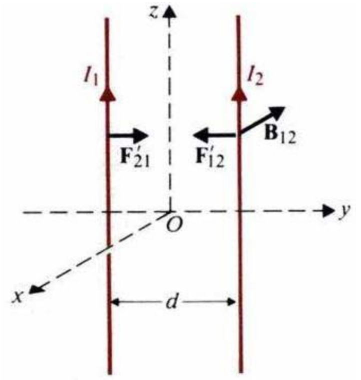

Fig. 13-5. Two parallel current-carrying wires (after DKC).

■ Torques on current-carrying conductors

Example 13-5: Consider a circular loop of radius b and carrying a clockwise current I in a "uniform" magnetic filed $\vec{B} = \vec{B}_\perp + \vec{B}_\parallel$, where $\vec{B}_\perp = -\vec{a}_z B_\perp$ and $\vec{B}_\parallel = \vec{a}_y B_\parallel$ are perpendicular and parallel to the plane of the loop (*xy*-plane). Find the force and torque exerted on the loop.

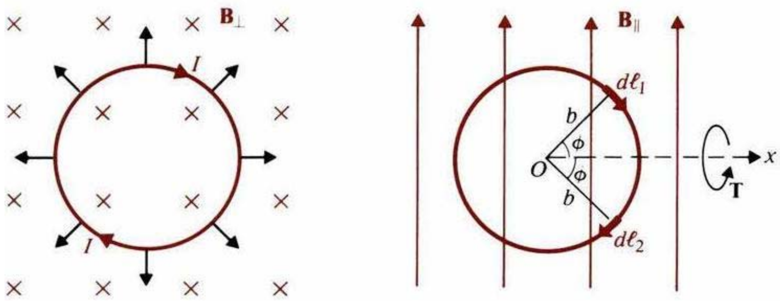

Fig. 13-6. Circular current-carrying loop in a uniform magnetic field (after DKC).

By eq. (13.8), the forces exerted on a differential current element $d\vec{l} = -\vec{a}_\phi b d\phi$ on the loop due to $\vec{B}_\perp$ and $\vec{B}_\parallel$ are:

$$d\vec{F}_\perp = I\left(-\vec{a}_\phi b d\phi\right)\times\left(-\vec{a}_z B_\perp\right) = \vec{a}_r IbB_\perp d\phi\,;$$

$$d\vec{F}_\parallel = I\left(-\vec{a}_\phi b d\phi\right)\times\left(\vec{a}_y B_\parallel\right) = IbB_\parallel d\phi\left(\vec{a}_x \sin\phi - \vec{a}_y \cos\phi\right)\times\left(\vec{a}_y\right) = \vec{a}_z IbB_\parallel \sin\phi d\phi\,.$$

In either case, the net force is zero (i.e. the loop will not move), because:

$$\vec{F}_\perp = \int_0^{2\pi} d\vec{F}_\perp = IbB_\perp\left(\int_0^{2\pi} \vec{a}_r d\phi\right) = 0\,,$$

$$\vec{F}_{\parallel} = \int_0^{2\pi} d\vec{F}_{\parallel} = \vec{a}_z IbB_{\parallel}\left(\int_0^{2\pi} \sin\phi \cdot d\phi\right) = 0 .$$

The corresponding torques are:

$$\vec{T}_{\perp} = \int_0^{2\pi} d\vec{F}_{\perp} \times \left(-\vec{a}_r b\right) = -Ib^2 B_{\perp}\left(\int_0^{2\pi} \vec{a}_r \times \vec{a}_r\right) = 0 ,$$

$$\vec{T}_{\parallel} = \int_0^{2\pi} d\vec{F}_{\parallel} \times \left(-\vec{a}_r b\right) = Ib^2 B_{\parallel}\int_0^{2\pi} \left[\vec{a}_z \times \left(-\vec{a}_r\right)\right]\sin\phi d\phi = Ib^2 B_{\parallel}\int_0^{2\pi} \left(-\vec{a}_{\phi}\right)\sin\phi d\phi$$

$$Ib^2 B_{\parallel} \cdot \left[\vec{a}_x \int_0^{2\pi} \sin^2\phi d\phi - \vec{a}_y \int_0^{2\pi} \sin\phi \cdot \cos\phi \cdot d\phi\right] = \vec{a}_x \left(I\pi b^2\right) B_{\parallel} = \vec{a}_x m B_{\parallel} ,$$

where $m = I\pi b^2$ is the magnitude of magnetic dipole moment $\vec{m}$. $\Rightarrow$

$$\vec{T} = \vec{T}_{\perp} + \vec{T}_{\parallel} = \vec{T}_{\parallel} = \vec{a}_x (-m_z) B_y .$$

In general, the torque is determined by:

$$\vec{T} = \vec{m} \times \vec{B} \tag{13.11}$$