Lesson 13
Inductance, Magnetic energy /force /torque

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Outline

- Inductance
- Magnetic energy
- Magnetic force
- Magnetic torque

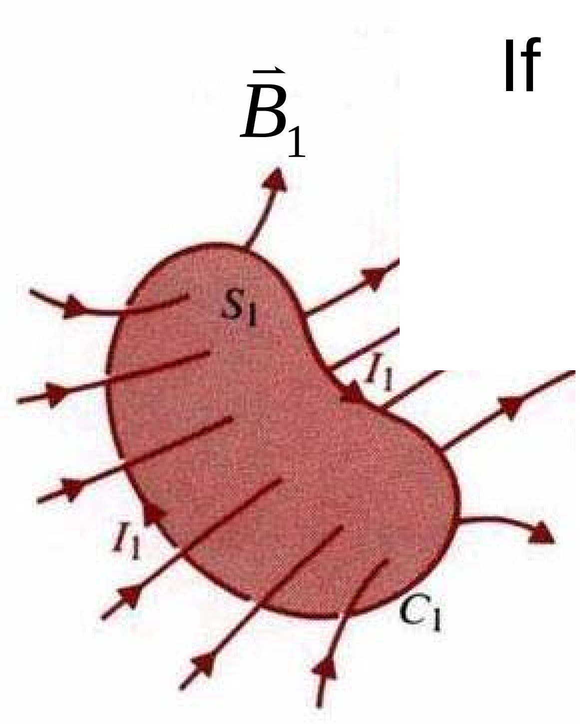
Sec. 13-1
Inductance

1. Self & mutual inductances
2. Evaluation procedures

Definition-1

Closed loop C_1 carrying current I_1 will create $\vec{B}_1$

$\Rightarrow$ flux: $\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot d\vec{s}$, flux linkage: $\Lambda_{11} = N_1 \Phi_{11}$



If $I_1' = rI_1$, by $\vec{B} = \oint_{C'} \frac{\mu_0 I}{4\pi} \boxed{\frac{d\vec{l}\,' \times \vec{a}_R}{R^2}}$ Only depend on geometry

$\Rightarrow \vec{B}_1' = r\vec{B}_1$, $\Phi_{11}' = \int_{S_1} \vec{B}_1' \cdot d\vec{s} = r\Phi_{11}$,

$\Lambda_{11}' = r\Lambda_{11}$

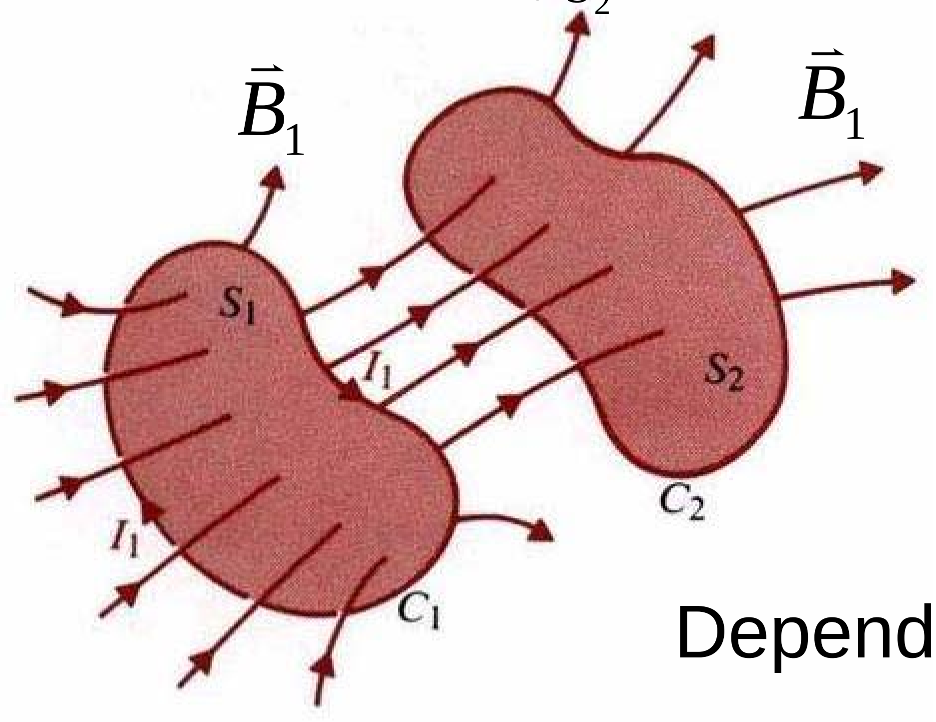
$\Rightarrow$ "Self-inductance" of the loop C_1:

$$\boxed{L_{11} = \frac{\Lambda_{11}}{I_1}}$$

Definition-2

In the presence of another loop C_2, $\vec{B}_1$ will pass through C_2, $\Rightarrow$ mutual flux linkage: $\Lambda_{12} = N_2 \Phi_{12}$

where $\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{s} \propto I_1$



$\Rightarrow$ "Mutual-inductance" between the 2 loops:

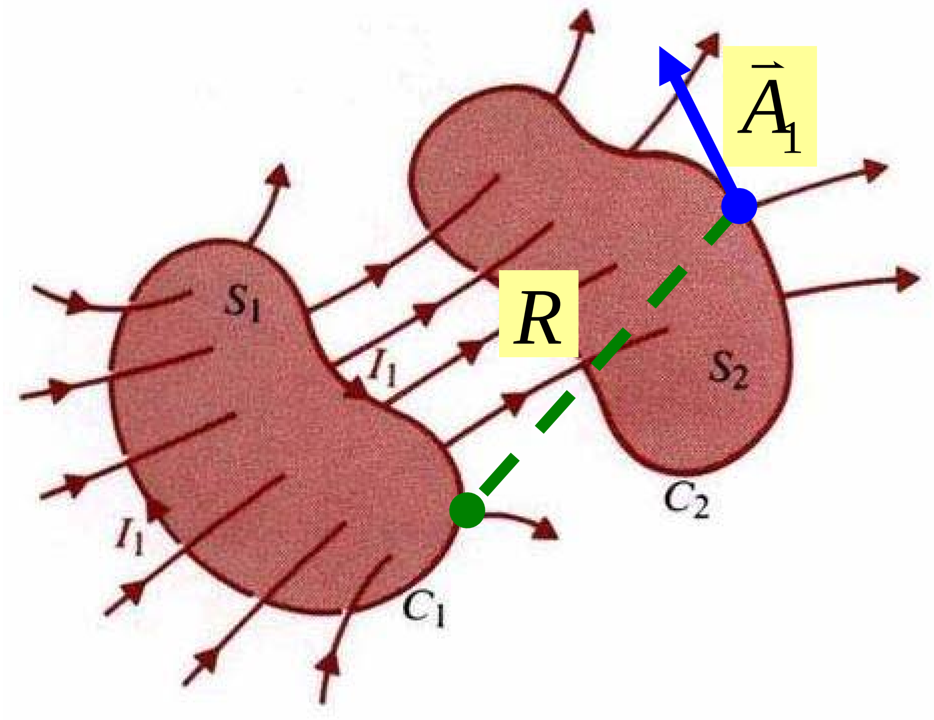
$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1}}$$

Depend on geometry & material.

Comment

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')}, \qquad \Rightarrow \vec{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi}\oint_{C_1}\frac{d\vec{l}_1}{R}$$

$$L_{12} = \frac{N_2\Phi_{12}}{I_1} = \frac{N_2}{I_1}\int_{S_2}\left(\overset{\substack{\vec{B}_1 \\ \uparrow}}{\nabla\times\vec{A}_1}\right)\cdot d\vec{s} = \frac{N_2}{I_1}\oint_{C_2}\vec{A}_1\cdot d\vec{l}_2$$



$$= \frac{\mu_0 N_1 N_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{l}_1\cdot d\vec{l}_2}{R}$$

$$L_{21} = \frac{\mu_0 N_1 N_2}{4\pi}\oint_{C_2}\oint_{C_1}\frac{d\vec{l}_1\cdot d\vec{l}_2}{R}$$

$$\Rightarrow \boxed{L_{12} = L_{21}}$$

Evaluation of inductance (Method 1)

1. Assume current $\boxed{I}$ flowing on the loop.

2. Find $\vec{B}$ by Ampere's law or Biot-Savart law:

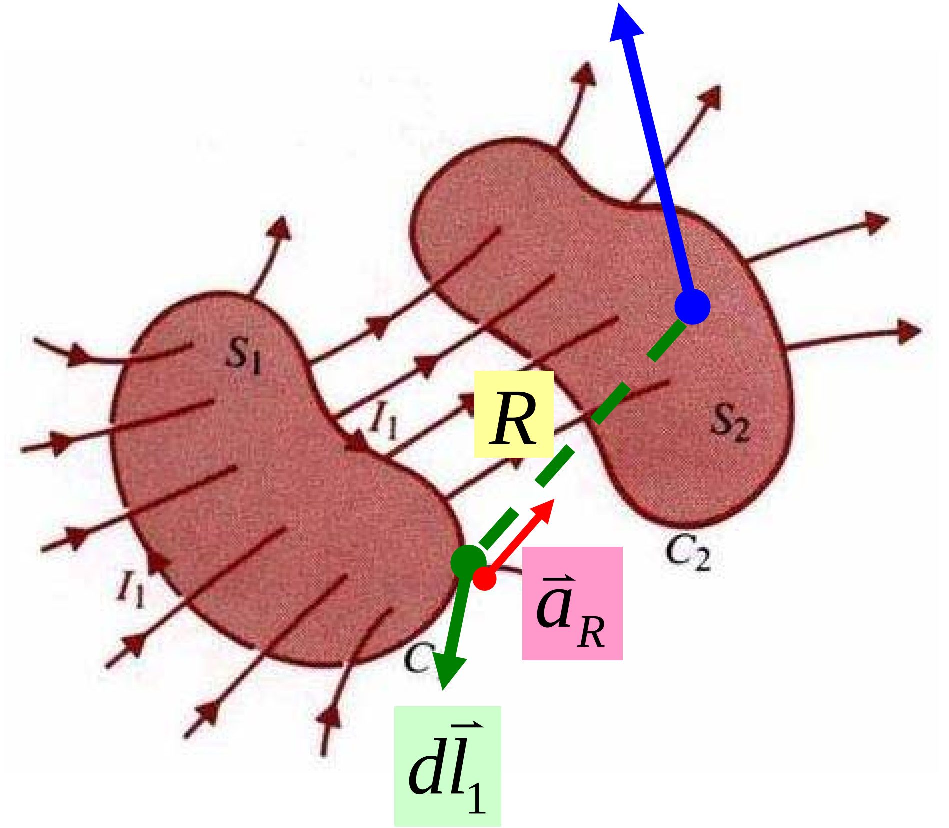
$$\oint_C \vec{H}\cdot d\vec{l} = I, \quad \vec{B} = \oint_{C'} \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\,'\times\vec{a}_R}{R^2}$$

3. Find $\Lambda\underline{\left(\propto I\right)}$ by $\Lambda = N\int_S \vec{B}\cdot d\vec{s}$

4. Find L by $L = \dfrac{\Lambda}{I}$, independent of I

Evaluation of inductance-reference figure

$$\vec{B}_1 = \oint_{C_1} \frac{\mu_0 I}{4\pi} \frac{d\vec{l}_1 \times \vec{a}_R}{R^2}$$



Evaluation of inductance (Method 2)

1. Assume current I flowing on the loop.

2. Find $\vec{H}$ and $\vec{B}$ by Method 1

3. Find the stored energy

$$W_m = \frac{1}{2}\int_{V'} \left(\vec{H}\cdot\vec{B}\right) dv \propto I^2$$

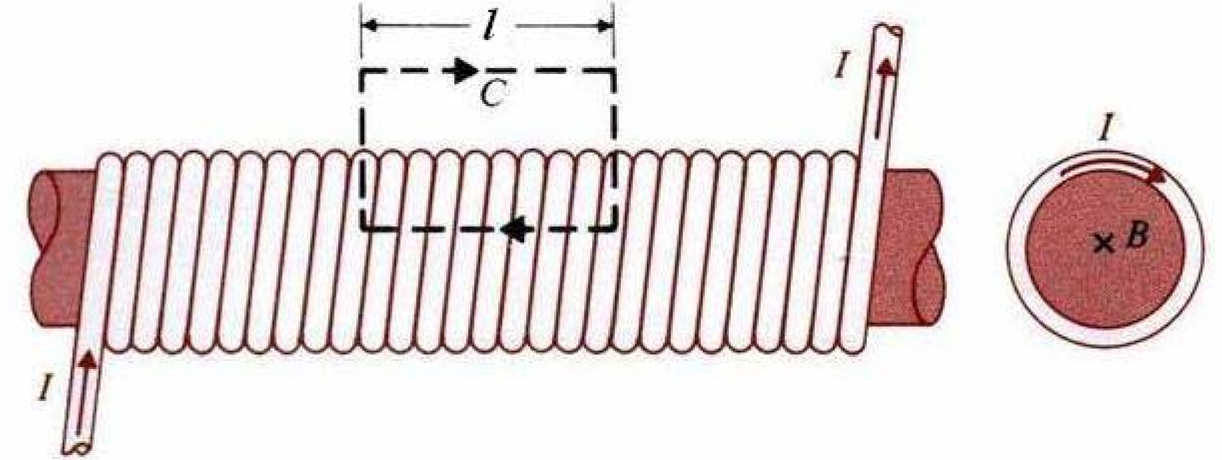
4. Find L by $W_m = \frac{1}{2} L I^2$

Example 13-1: Solenoid inductor (1)

Consider a hollow solenoid with cross-sectional area ***S***, ***n*** turns per unit length. Find the inductance per unit length ***L***.

1. Assume current I flowing on the loop.

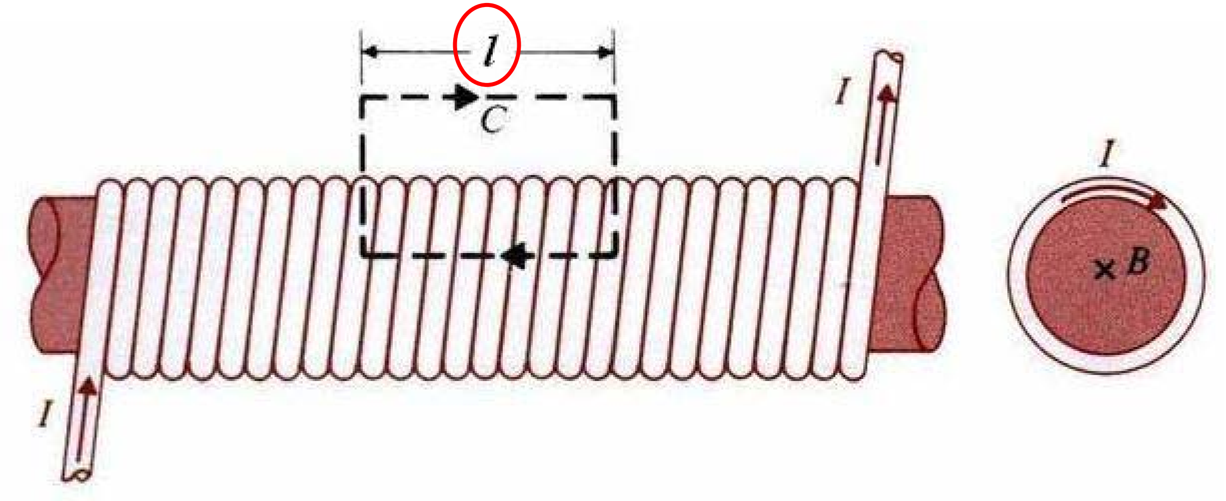
2. By Ampere's law: $B = \mu_0 nI$



Example 13-1: Solenoid inductor (2)

3. For unit length ($l=1$), $\Lambda = n \cdot \Phi = n \cdot \left(\mu_0 n I\right) \cdot S$

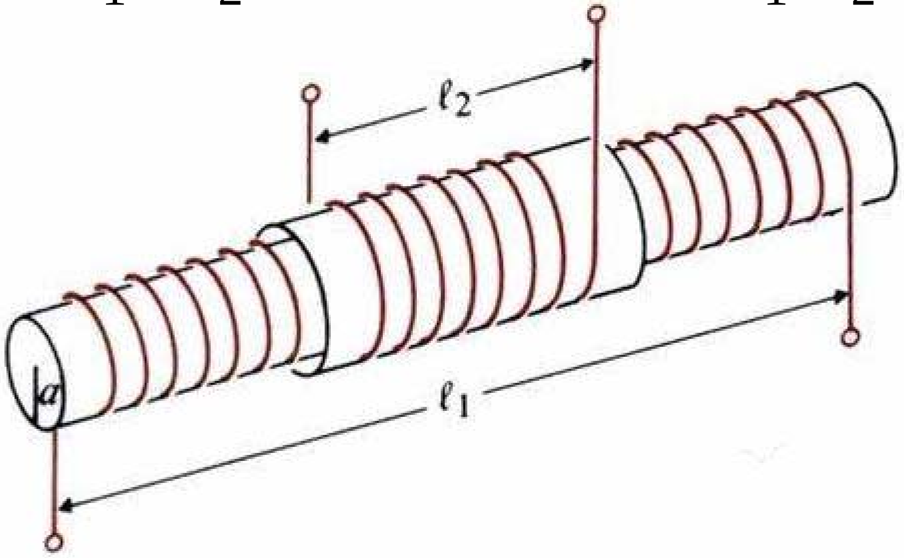
4. By definition: $L = \dfrac{\Lambda}{I} = \dfrac{n\left(\mu_0 n I\right)S}{I} = \boxed{n^2 \mu_0 S}$



Example 13-2: Two concentric coils (1)

Consider two coils C_1, C_2 with N_1, N_2 turns and lengths l_1, l_2. They are wound concentrically on a thin cylindrical core of radius a with permeability μ. Find the mutual inductance L_{12}.

1. Assume C_1, C_2 have currents I_1, I_2

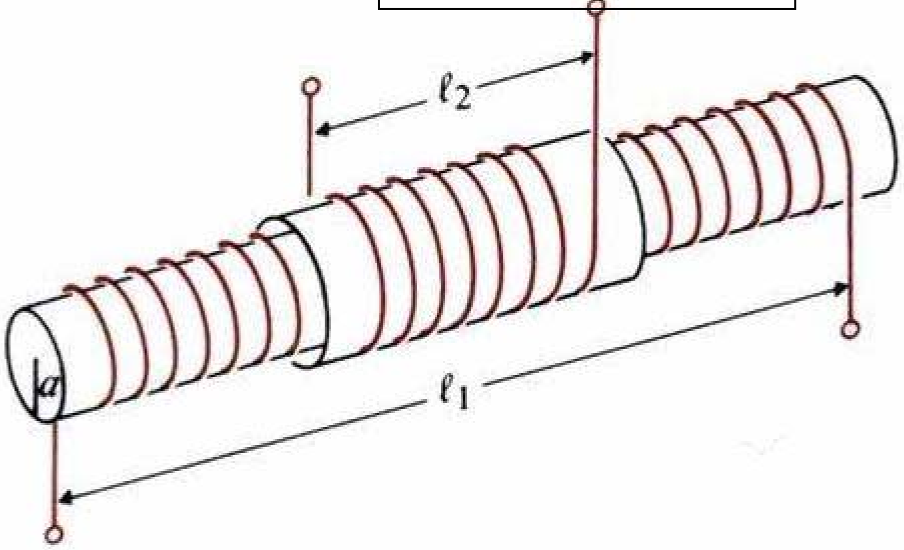


Example 13-2: Two concentric coils (2)

2. By Ampere's law, uniform field $B_1 = \mu \frac{N_1}{l_1} I_1$

3. Flux linkage of C_2 due to C_1: $\Lambda_{12} = N_2 \cdot \Phi_{12} = N_2 \cdot B_1 \cdot S$

4. By definition: $L_{12} = \boxed{\mu \frac{N_1 N_2}{l_1} \pi a^2}$



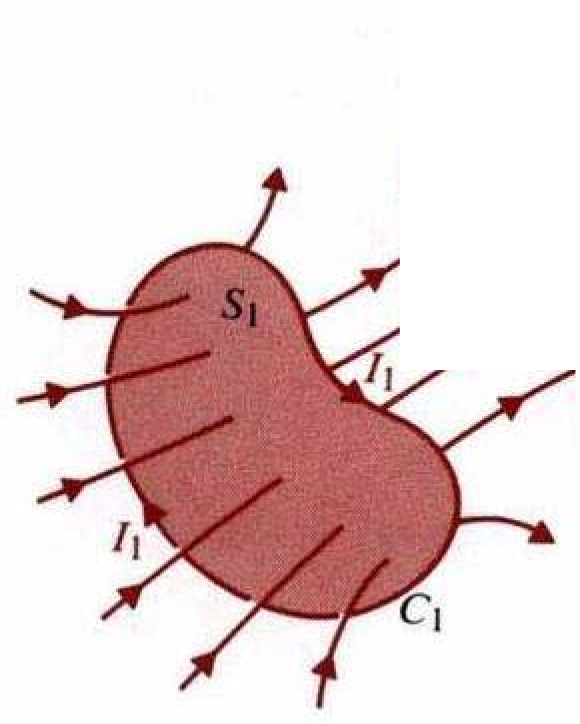
Sec. 13-2
Magnetic Energy

1. Energy of assembling current loops
2. Energy of magnetic fields

Energy of assembling current loops-One loop (1)

Closed loop C_1 with self-inductance L_1. If the loop current i_1 increases from **0** to I_1 slowly (quasi-static), an emf of:

$$v_1 = -\frac{d\Phi_{11}}{dt} = L_1 \frac{di_1}{dt}$$

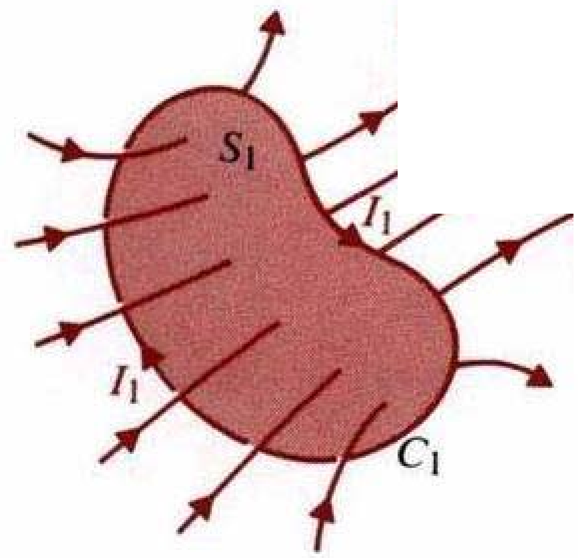


will be induced on C_1 to oppose the change of i_1 (Faraday’s law, Lenz’s law).

Energy of assembling current loops-One loop (2)

The work done to overcome the induced v_1 and enforce the change of i_1 is:

$$W_1 = \int_0^{\infty} v_1(t) i_1(t) dt = \int_0^{\infty} L_1 \frac{di_1}{\cancel{dt}} i_1 \cancel{dt} = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$



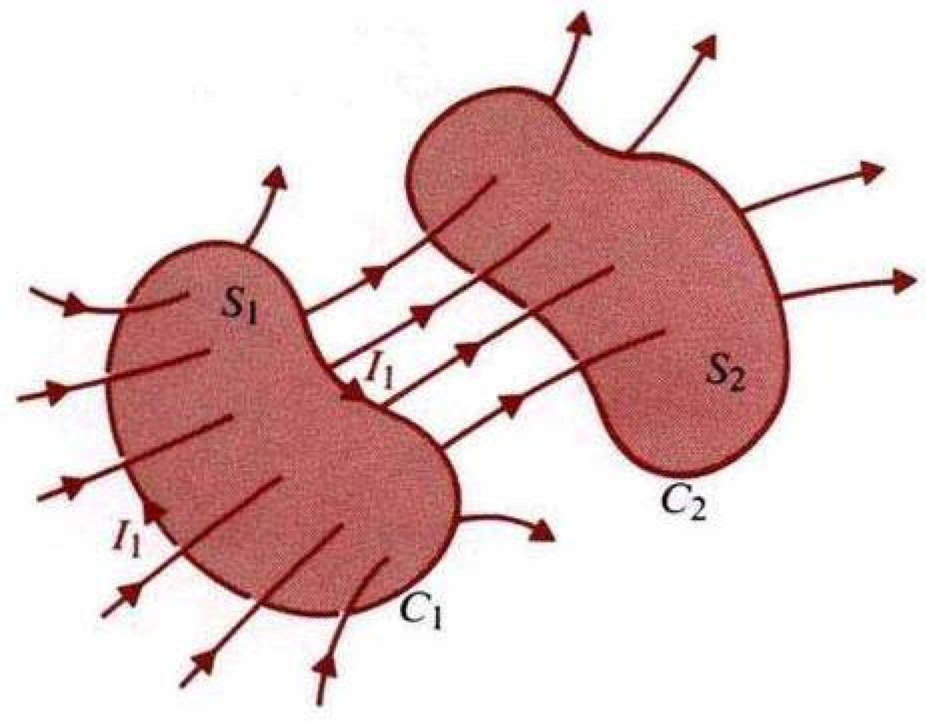
which is stored as magnetic energy:

$$W_1 = \frac{1}{2} L_1 I_1^2$$

one loop

Energy of assembling current loops-Two loops (1)

Insert loop C_2 with self-inductance $\boldsymbol{L_2}$, mutual inductance $\boldsymbol{L_{21}}$. If we maintain $\boldsymbol{i_1=I_1}$, while $\boldsymbol{i_2}$ increases from **0** to $\boldsymbol{I_2}$ slowly, an emf of:



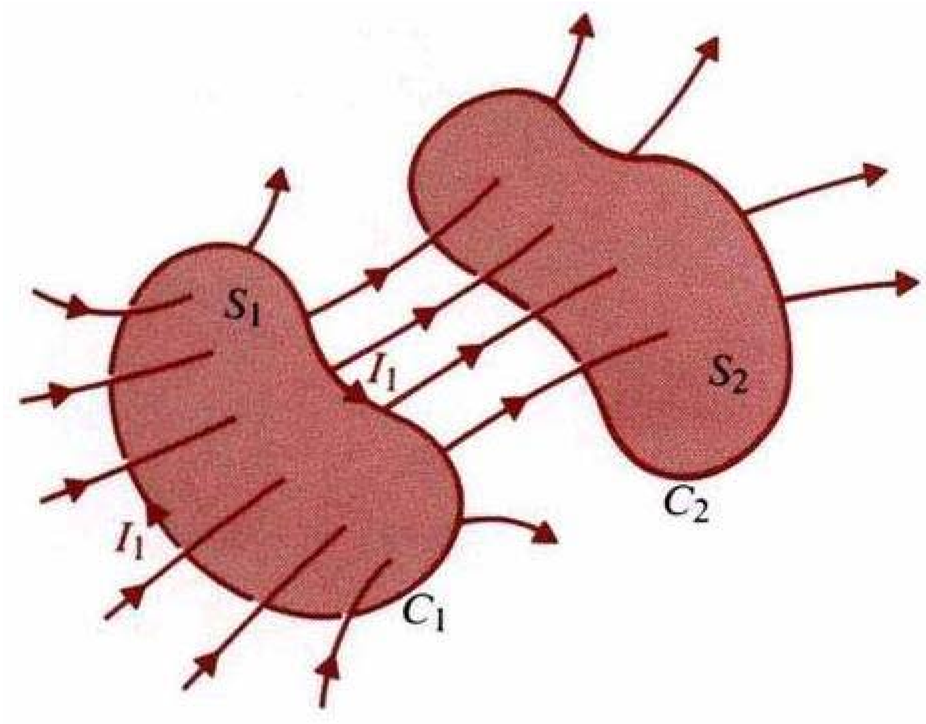
$$v_{21} = -\frac{d\Phi_{21}}{dt} = L_{21}\frac{di_2}{dt}$$

will be induced on $\boldsymbol{C_1}$ in an attempt to change i_1 away from I_1

Energy of assembling current loops-Two loops (2)

The work done to maintain $i_1 = I_1$ is:

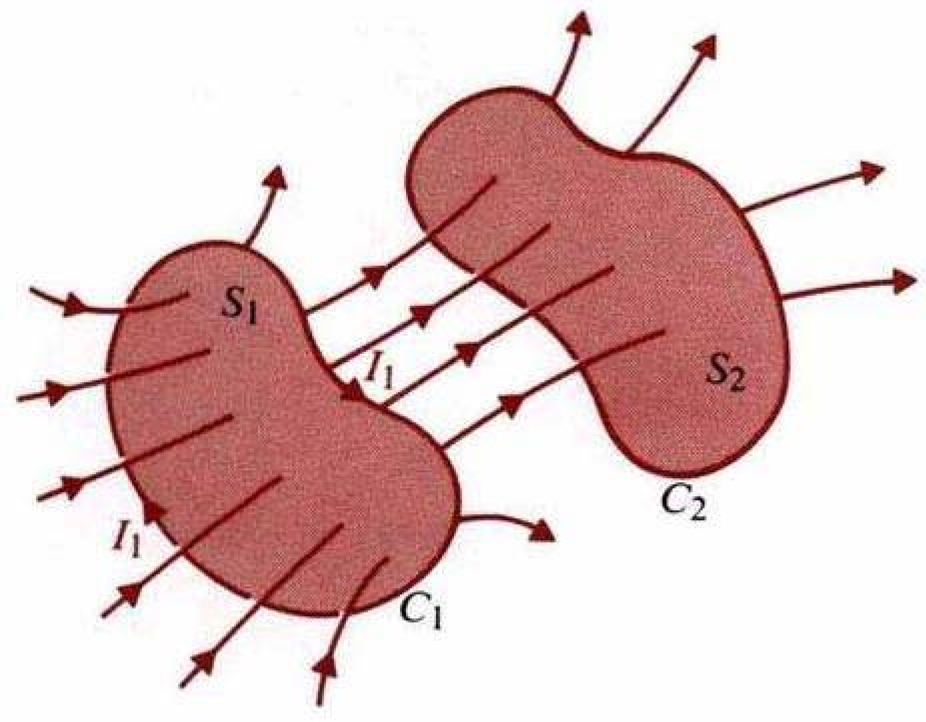
$$W_{21} = \int_0^\infty v_{21}(t) I_1 dt = \int_0^\infty L_{21} \frac{di_2}{\cancel{dt}} I_1 \cancel{dt} = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$$



Energy of assembling current loops-Two loops (3)

Meanwhile, an emf of: $v_2 = -\frac{d\Phi_{22}}{dt} = L_2 \frac{di_2}{dt}$

will be induced on C_2 to oppose the change of i_2 (from 0 to I_2).



The work done to overcome v_2 and enforce the change of i_2 is:

$$W_{22} = \frac{1}{2} L_2 I_2^2$$

Energy of assembling current loops-Two loops (4)

The total magnetic energy stored in the system of two current loops is:

$$W_{\underset{\substack{\downarrow \\ \text{two loops}}}{\textcircled{2}}} = \underline{\frac{1}{2}L_1 I_1^2} + \underline{L_{21} I_1 I_2 + \frac{1}{2}L_2 I_2^2}$$

Energy of assembling current loops-N loops

The total magnetic energy stored in the system of N current loops carrying currents I_1, I_2, ….., I_N , is:

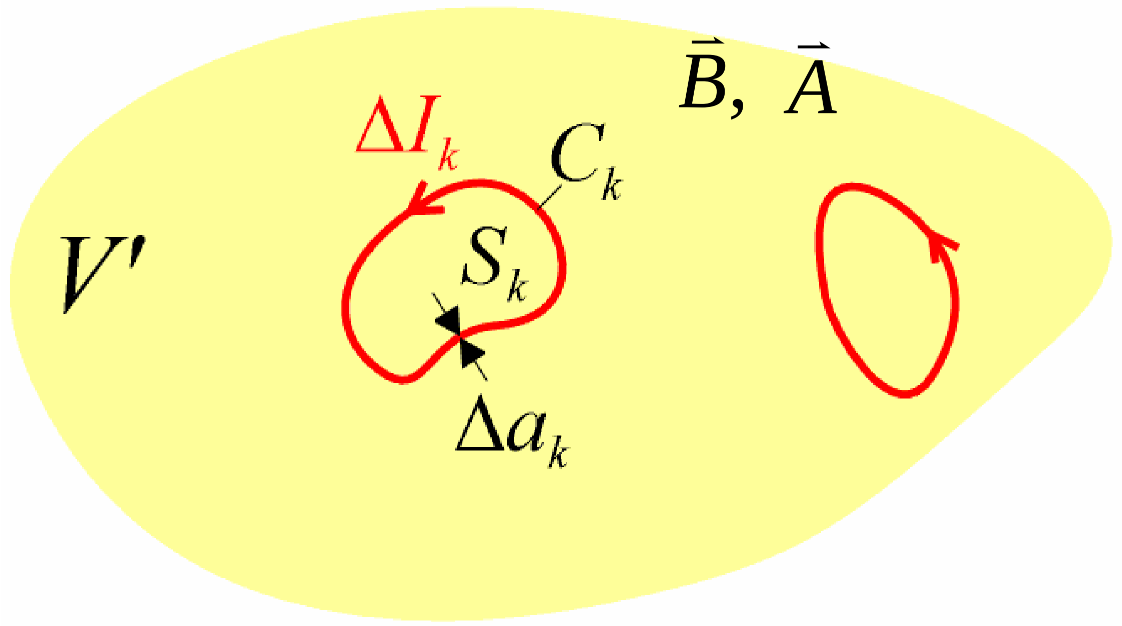
$$W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k$$

By $L_{12} = \Lambda_{12}/I_1$, the flux (linkage) of loop C_k due to all the N current loops:

$$\Phi_k = \sum_{j=1}^{N} L_{jk} I_j , \quad \Rightarrow \quad \boxed{W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k}$$

Energy of continuous current distributions-1

Decompose a system of continuous current distribution $\vec{J}(\vec{r})$ in a volume V' into N elementary current loops C_k, each has current ΔI_k and filamentary cross-sectional area Δa_k



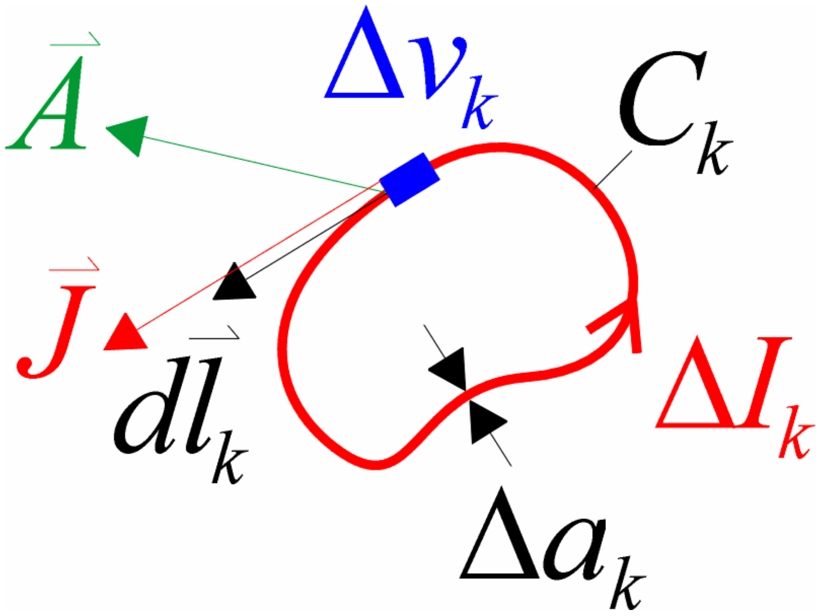
$$\Phi_k = \int_{S_k} \vec{B} \cdot d\vec{s}$$

$$= \oint_{C_k} \vec{A} \cdot d\vec{l}_k$$

Energy of continuous current distributions-2

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2}\sum_{k=1}^{N} \underline{\Delta I_k} \oint_{C_k} \vec{A}\cdot \underline{d\vec{l}_k}$$

$$\Delta I_k \cdot d\vec{l}_k = \left(\left|\vec{J}\right|\Delta a_k\right) d\vec{l}_k = \vec{J}\left(\Delta a_k \left|d\vec{l}_k\right|\right) = \underline{\vec{J}\Delta v_k}$$



$$\Rightarrow W_m = \frac{1}{2}\sum_{k=1}^{N} \oint_{C_k} \vec{A}\cdot \vec{J}\Delta v_k ,$$

$$\boxed{W_m = \frac{1}{2}\int_{V'} \left(\vec{A}\cdot \vec{J}\right) dv}$$

Comments

$$W_e = \frac{1}{2}\sum_{k=1}^{N}\underline{Q_k}\,\underline{V_k} \quad\longleftrightarrow\quad W_m = \frac{1}{2}\sum_{k=1}^{N}\underline{I_k}\,\underline{\Phi_k}$$

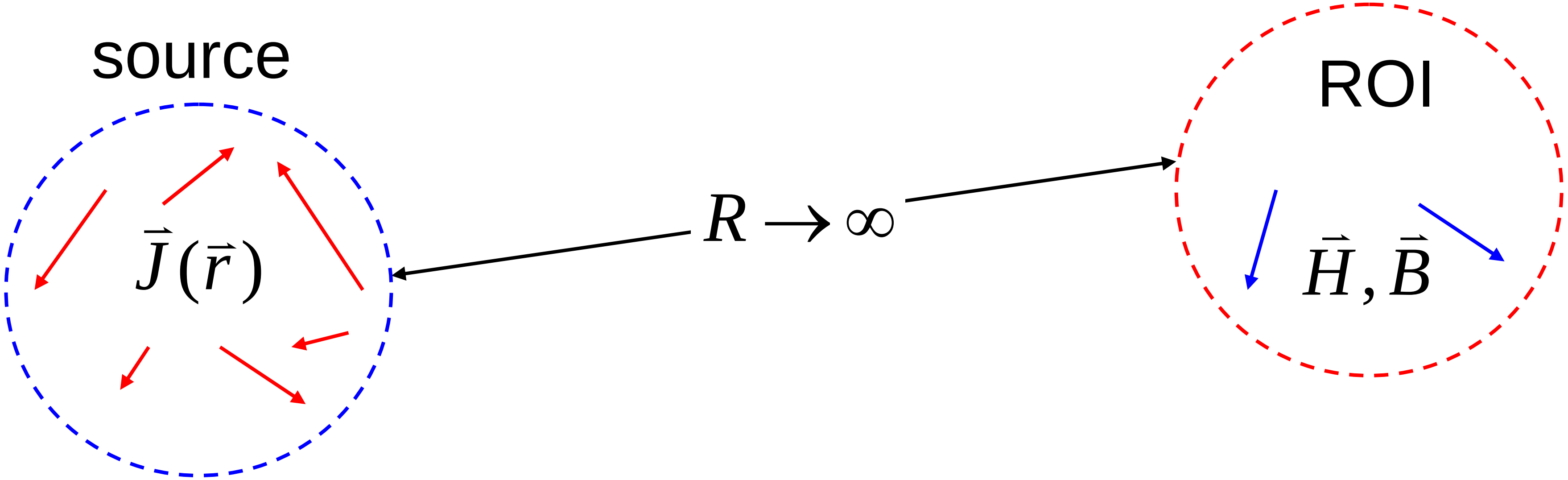
$$W_e = \frac{1}{2}\int_{V'}\left(\underline{\rho}\,\underline{V}\right)dv \quad\longleftrightarrow\quad W_m = \frac{1}{2}\int_{V'}\left(\underline{\vec{A}}\cdot\underline{\vec{J}}\right)dv$$

Electrostatics

Magnetostatics

Energy of magnetic fields-1

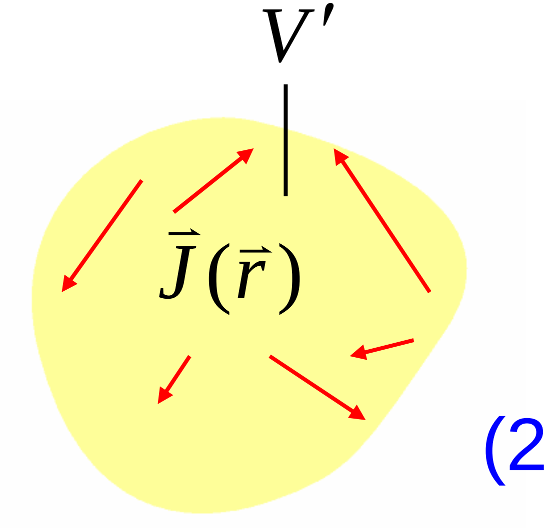
In real applications (especially electromagnetic waves), sources are usually far away from the region of interest, only the fields are given



Energy of magnetic fields-2

(1) $$W_m = \frac{1}{2}\int_{V'}\left(\vec{A}\cdot\underbrace{\vec{J}}_{\vec{J} = \nabla\times\vec{H}}\right)dv = \frac{1}{2}\int_{V'}\vec{A}\cdot\left(\nabla\times\vec{H}\right)dv$$

contain all the source currents



By vector identity:

$$\nabla\cdot\left(\vec{A}\times\vec{H}\right) = \vec{H}\cdot\underline{\left(\nabla\times\vec{A}\right)} - \vec{A}\cdot\left(\nabla\times\vec{H}\right)$$

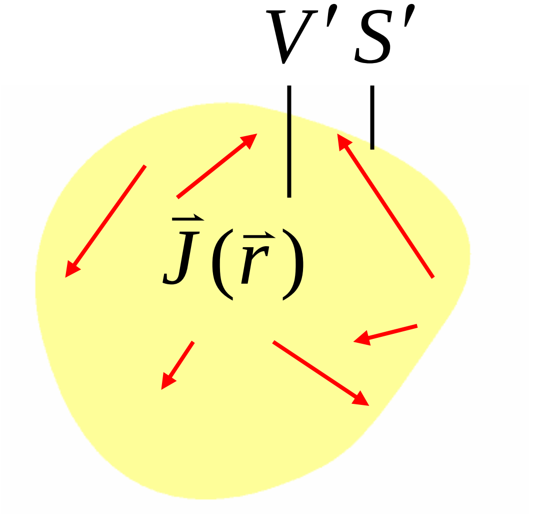
$$= \vec{H}\cdot\vec{B} - \vec{A}\cdot\left(\nabla\times\vec{H}\right)$$

(2) $$W_m = \frac{1}{2}\int_{V'}\left(\vec{H}\cdot\vec{B}\right)dv - \frac{1}{2}\int_{V'}\nabla\cdot\left(\vec{A}\times\vec{H}\right)dv$$

Energy of magnetic fields-3

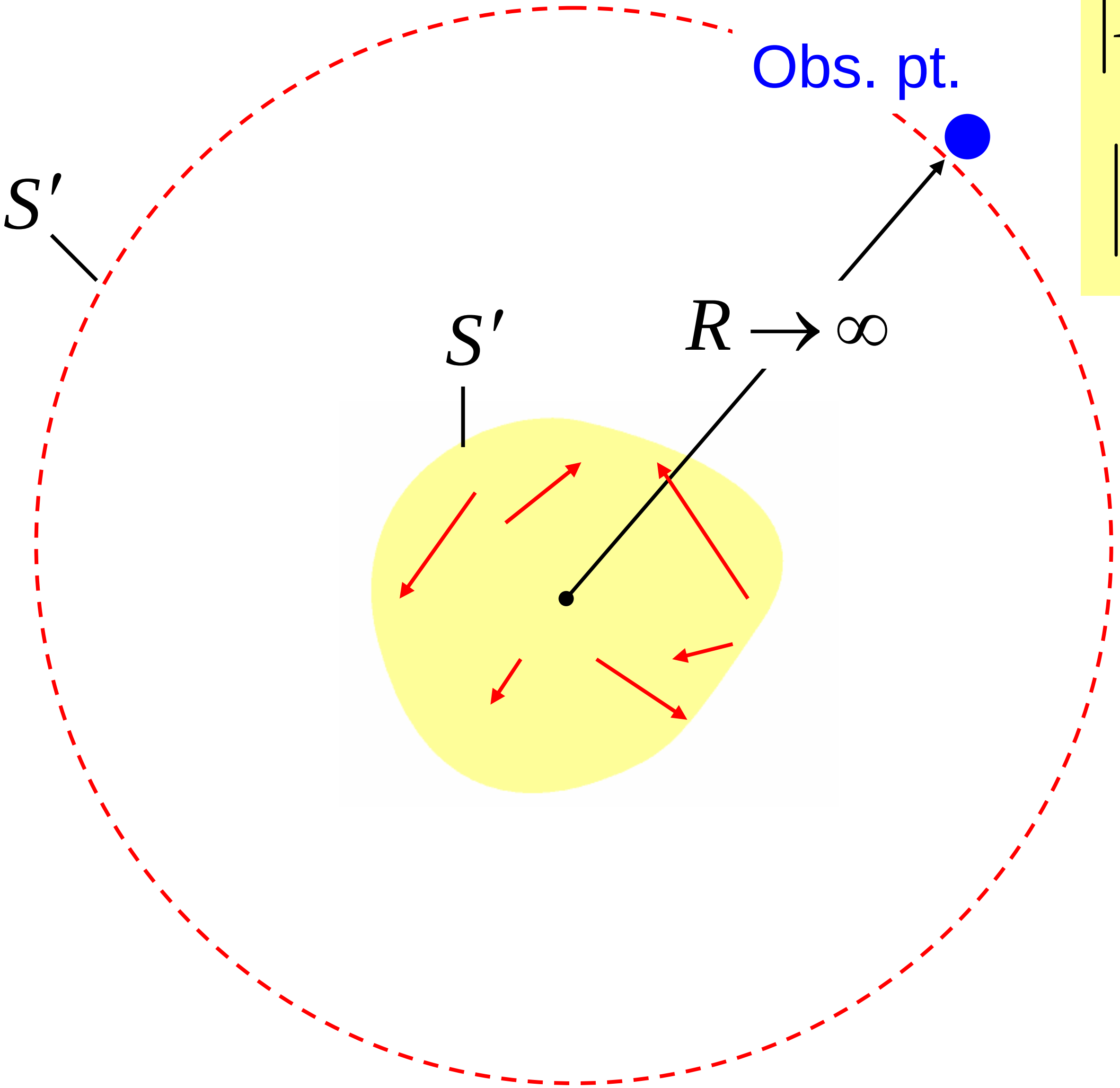
$$\because \oint_S \vec{A}\cdot d\vec{s} = \int_V \left(\nabla\cdot\vec{A}\right)dv,$$

$$\Rightarrow \int_{V'} \nabla\cdot\left(\vec{A}\times\vec{H}\right)dv = \oint_{S'}\left(\vec{A}\times\vec{H}\right)\cdot d\vec{s}$$



(3) $$W_m = \underbrace{\frac{1}{2}\int_{V'}\left(\vec{H}\cdot\vec{B}\right)dv}_{I_1} - \underbrace{\frac{1}{2}\oint_{S'}\left(\vec{A}\times\vec{H}\right)\cdot d\vec{s}}_{I_2}$$

Energy of magnetic fields-4



$$\left|\vec{H}\right| \propto \frac{1}{R^2}$$

$$\left|\vec{A}\right| \propto \frac{1}{R}$$

Energy of magnetic fields-5

$$I_2 = \frac{1}{2}\oint_{S'}\left(\vec{A}\times\vec{H}\right)\cdot d\vec{s} \approx \frac{1}{2}\left|\vec{A}(R)\right|\left|\vec{H}(R)\right|\cdot 4\pi R^2$$

$$\propto \frac{1}{R}\cdot\frac{1}{R^2}\cdot R^2 \propto \frac{1}{R} \to 0$$

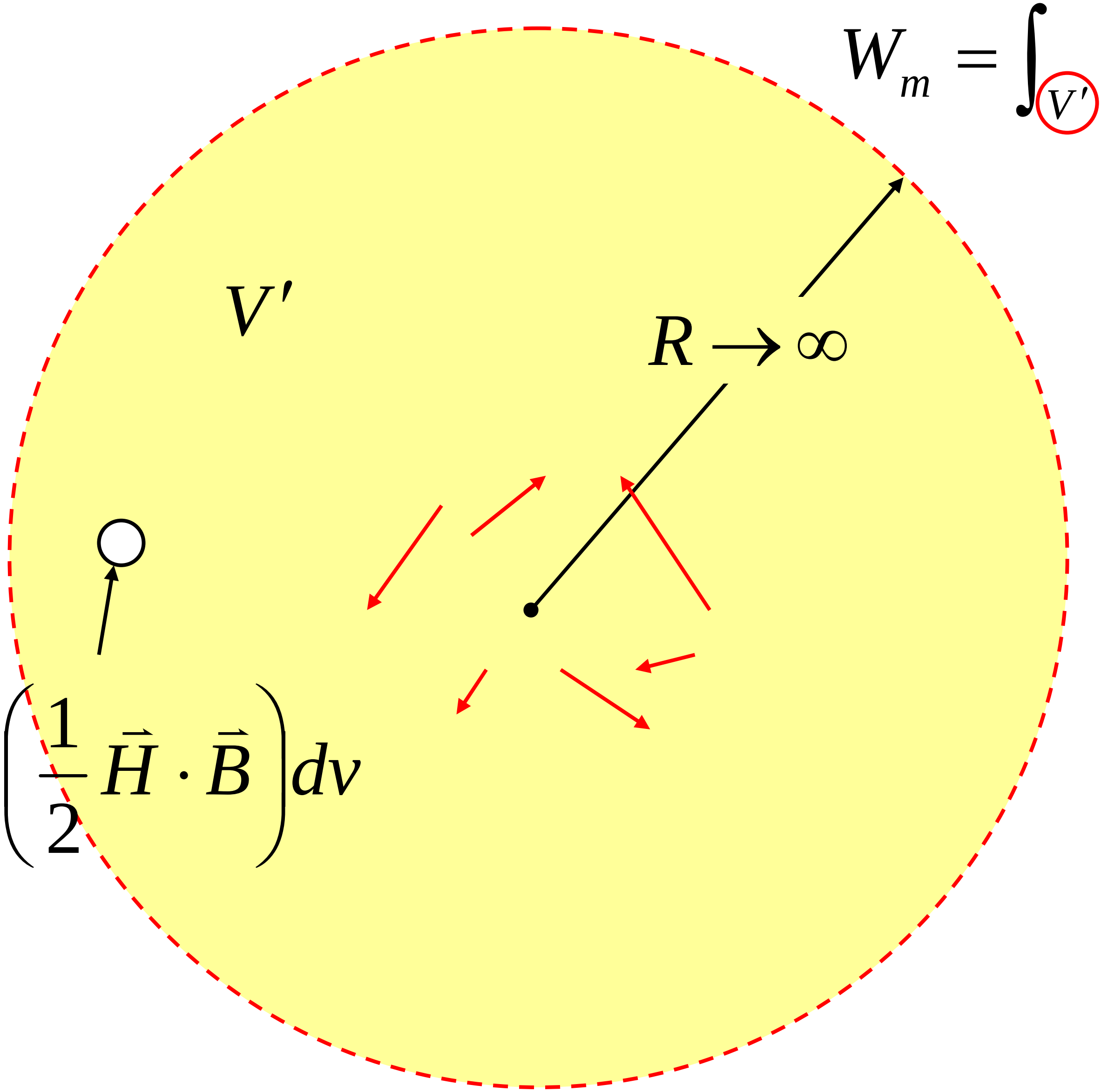
$$\Rightarrow W_m = I_1 = \int_{V'} \underline{w_m(\vec{r})}\,dv$$

$$\downarrow$$

$$\boxed{\frac{1}{2}\vec{H}\cdot\vec{B}\,(\mathrm{J/m^3})}$$ …energy density

Energy of magnetic fields-6

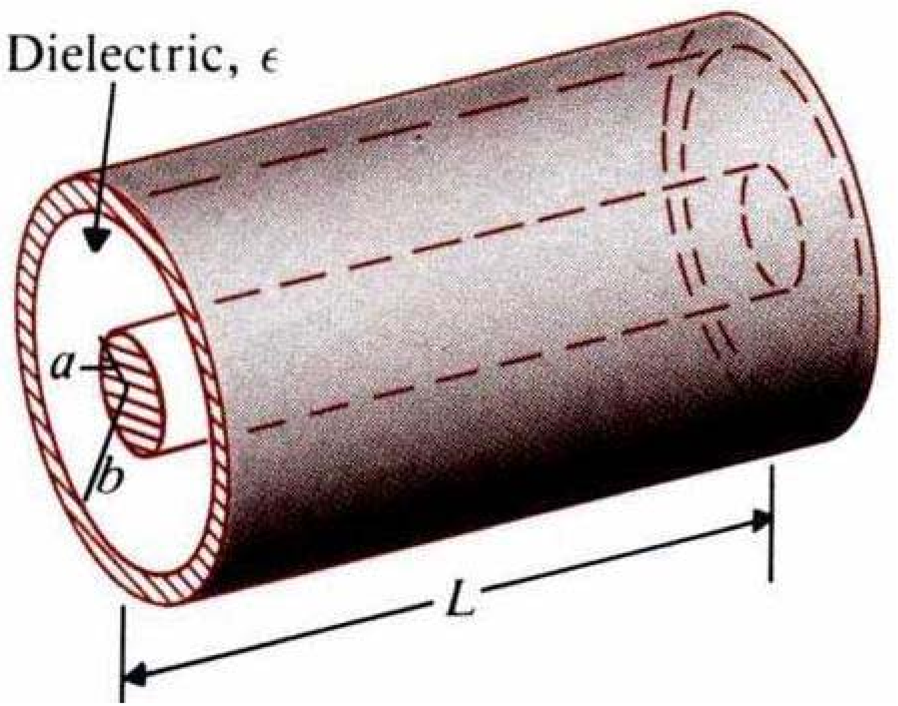
$$W_m = \int_{V'} \left(\frac{1}{2} \vec{H} \cdot \vec{B} \right) dv$$



$$dW_m = \left(\frac{1}{2} \vec{H} \cdot \vec{B} \right) dv$$

Example 13-3: Coaxial cable inductor (1)

Find the stored magnetostatic energy and inductance per unit length of:



Cylindrical symmetry,

Ampere's law, $\Rightarrow$

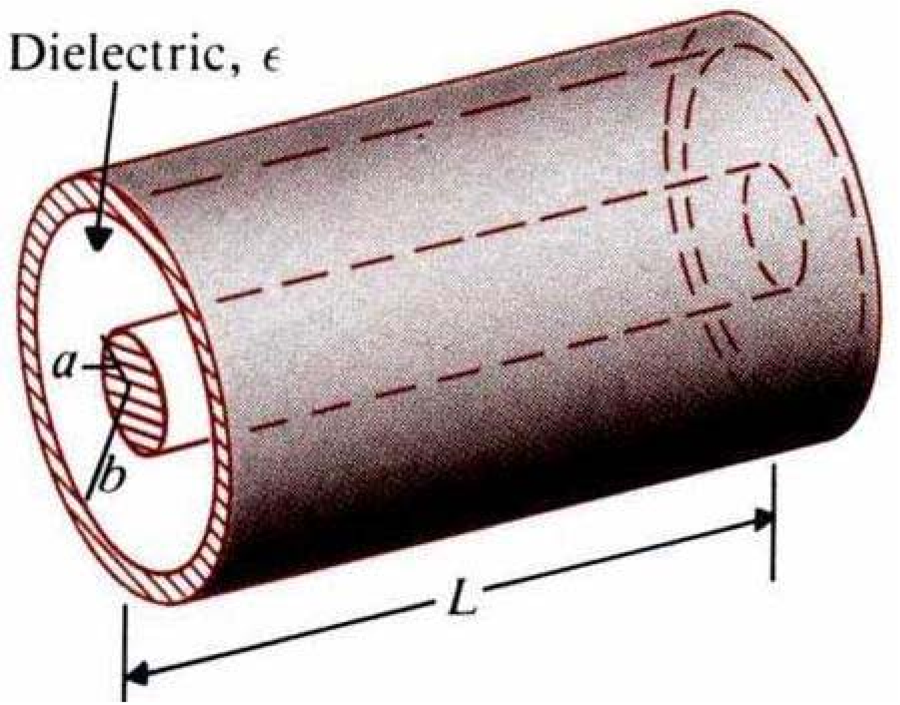
$$\vec{B} = \begin{cases} \vec{a}_\phi \dfrac{\mu_0 I}{2\pi a^2} r, \text{ if } r < a \\ \vec{a}_\phi \dfrac{\mu_0 I}{2\pi r}, \text{ if } a < r < b \end{cases}$$

$$\vec{H} = \vec{B} / \mu_0$$

Example 13-3: Coaxial cable inductor (2)

Energy density:

$$w_m = \frac{1}{2}\vec{H}\cdot\vec{B} = \begin{cases} \dfrac{\mu_0 I^2}{8\pi^2 a^4}\boxed{r^2}, r < a \\ \dfrac{\mu_0 I^2}{8\pi^2 \boxed{r^2}}, a < r < b \end{cases}$$



Differential volume

(L=1): $dv = 2\pi r \cdot dr$

Example 13-3: Coaxial cable inductor (3)

Total stored energy:

$$W_{m1} = \frac{\mu_0 I^2}{4\pi a^4} \int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi}, \ \ r < a$$

$$W_{m2} = \frac{\mu_0 I^2}{4\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi} \ln\left(\frac{b}{a}\right), \ \ a < r < b$$

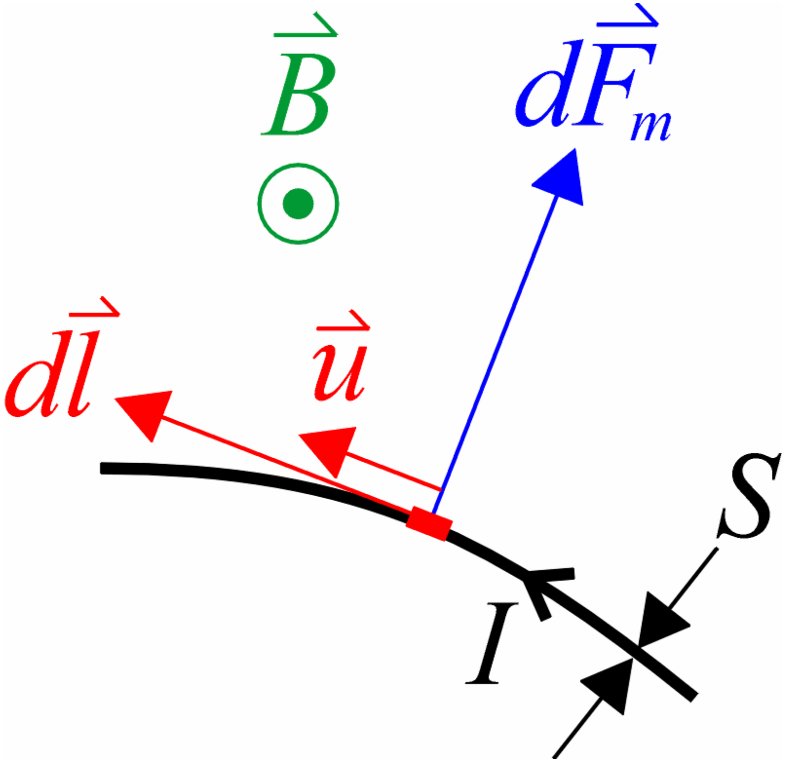
$$W_m = \frac{1}{2} L I^2, \quad L = \frac{2(W_{m1} + W_{m2})}{I^2} = \boxed{\underbrace{\frac{\mu_0}{8\pi}}_{\text{internal}} + \underbrace{\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)}_{\text{external}}}$$

Sec. 13-3
Magnetic Force

1. Force on current loops
2. Example: force between parallel wires

Force on current-carrying loops-1

Consider an elemental current-carrying wire of cross-sectional area S, represented by a differential displacement vector $d\vec{l}$

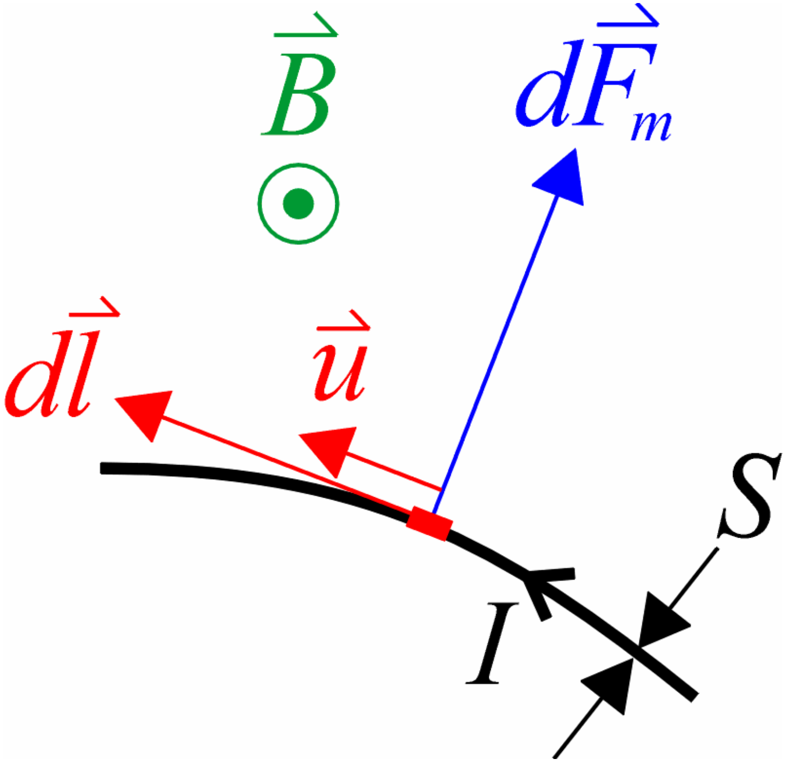


Free charges within the wire of charge density ρ move with velocity $\vec{u}\left(// d\vec{l}\right)$, experiencing a force of:

$$d\vec{F}_m = \underline{\rho S \left| d\vec{l} \right|} \left(\vec{u} \times \vec{B} \right)$$

Force on current-carrying loops-2

$$\begin{cases} \left|d\vec{l}\right|\vec{u} = \left|\vec{u}\right|d\vec{l} \\ \vec{J} = \rho\vec{u} \end{cases} \longrightarrow \begin{aligned} d\vec{F}_m &= \rho S \underline{\left|d\vec{l}\right|\left(\vec{u}\right.} \times \left.\vec{B}\right) \\ &= \underline{\rho} S \underline{\left|\vec{u}\right|} d\vec{l} \times \vec{B} = JS\left(d\vec{l} \times \vec{B}\right) \end{aligned}$$



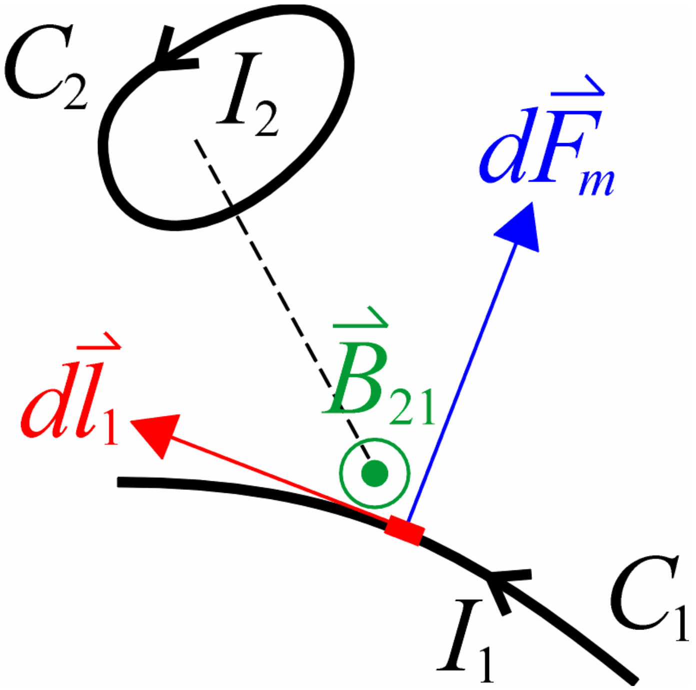
$$\Rightarrow d\vec{F}_m = I\left(d\vec{l} \times \vec{B}\right)$$

For a current loop C :

$$\boxed{\vec{F}_m = I\oint_C d\vec{l} \times \vec{B}}$$

Force on current-carrying loops-3

If $\vec{B}$ is created by another closed loop C_2 carrying a current I_2, the force exerted on the loop C_1 carrying a current I_1 is:



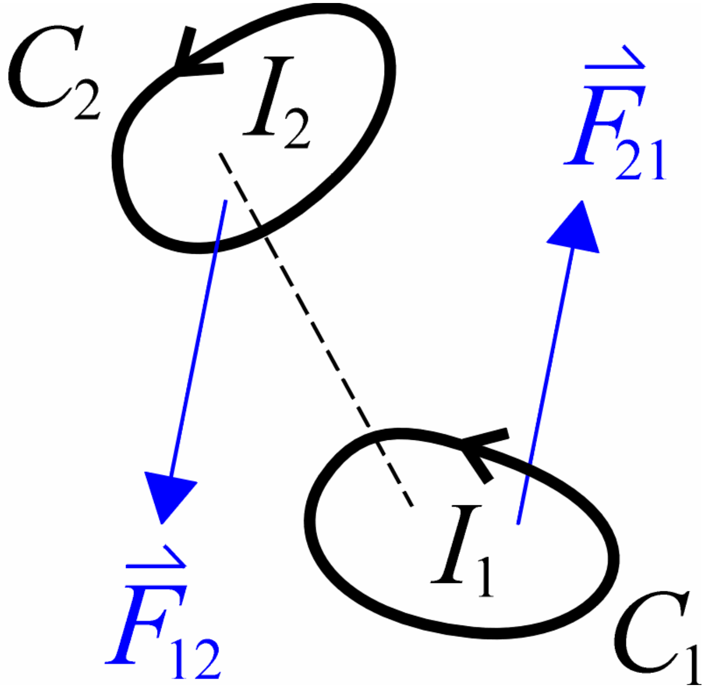
$$\vec{F}_{21} = I_1 \oint_{C_1} d\vec{l}_1 \times \underline{\vec{B}_{21}}$$

$$\vec{B} = \oint_{C'} \frac{\mu_0 I}{4\pi} \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}$$

$$\vec{B}_{21} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\vec{l}_2 \times \vec{a}_{R_{21}}}{R_{21}^2}$$

Force on current-carrying loops-4

$$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \times \left(d\vec{l}_2 \times \vec{a}_{R21} \right)}{R_{21}^2} = -\vec{F}_{12}$$

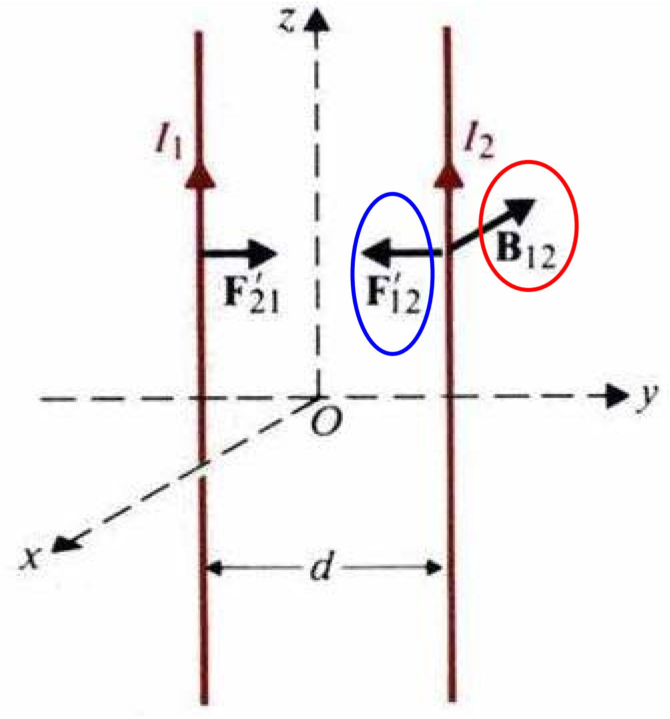


Counterpart in electrostatics: Coulomb’s force between two charges

$$\vec{F}_{12} = \vec{a}_{R_{12}} \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{R_{12}^2}$$

Example 13-4: Force between two long wires

Find the force per unit length between two infinitely long, parallel wires separated by d, carrying currents I_1, I_2 in the same direction.



$$\vec{B}_{12} = -\vec{a}_x \frac{\mu_0 I_1}{2\pi d}, \quad \vec{F}_{12} = I_2 \int_0^1 d\vec{l}_2 \times \vec{B}_{12}$$

$$= I_2 \int_0^1 (\vec{a}_z dz) \times \left(-\vec{a}_x \frac{\mu_0 I_1}{2\pi d} \right)$$

$$= \boxed{-\vec{a}_y \frac{\mu_0 I_1 I_2}{2\pi d}}$$ …attraction force

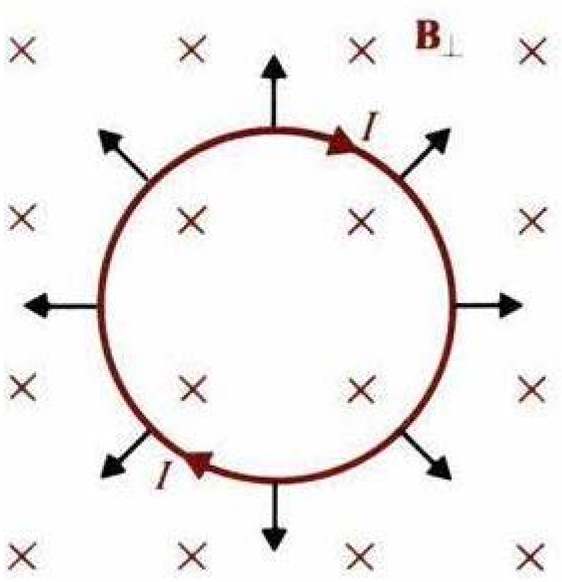
Sec. 13-4
Magnetic Torque

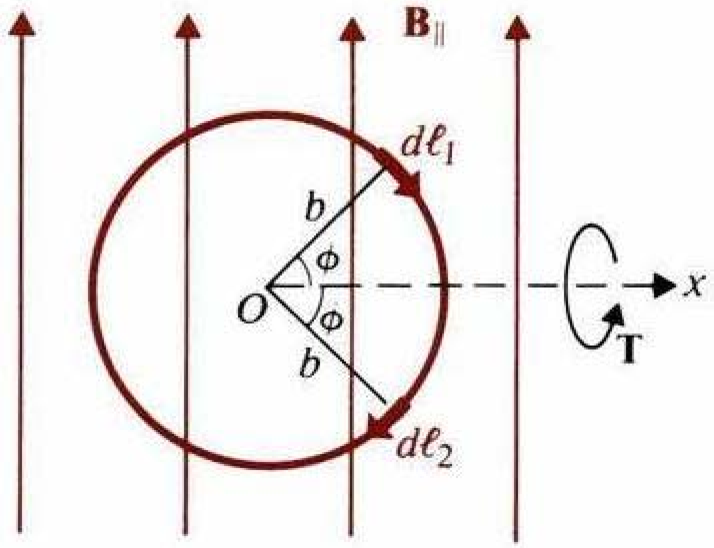
1. Example: magnetic force & torque exerted on a current loop

Example 13-5: Force & torque on current-carrying loops (1)

Consider a circular loop on the xy-plane with radius b, current I in clockwise sense, and is placed a “uniform” magnetic filed: $\vec{B} = \vec{B}_\perp + \vec{B}_\parallel$

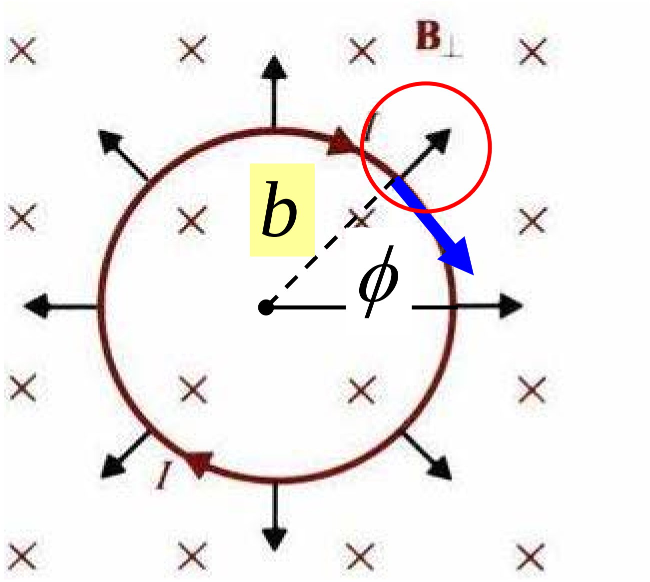
$$-\vec{a}_z B_\perp \qquad \vec{a}_y B_\parallel$$





Example 13-5: Force & torque on current-carrying loops (2)

The force exerted on a differential current element $d\vec{l} = -\vec{a}_\phi b d\phi$ on the loop due to $\vec{B}_\perp$:



$$d\vec{F}_m = I\left(d\vec{l} \times \vec{B}\right), \Rightarrow$$

$$d\vec{F}_\perp = I\left(-\vec{a}_\phi b d\phi\right)\times\left(-\vec{a}_z B_\perp\right)$$

$$= \vec{a}_r I b B_\perp d\phi$$

Example 13-5: Force & torque on current-carrying loops (3)

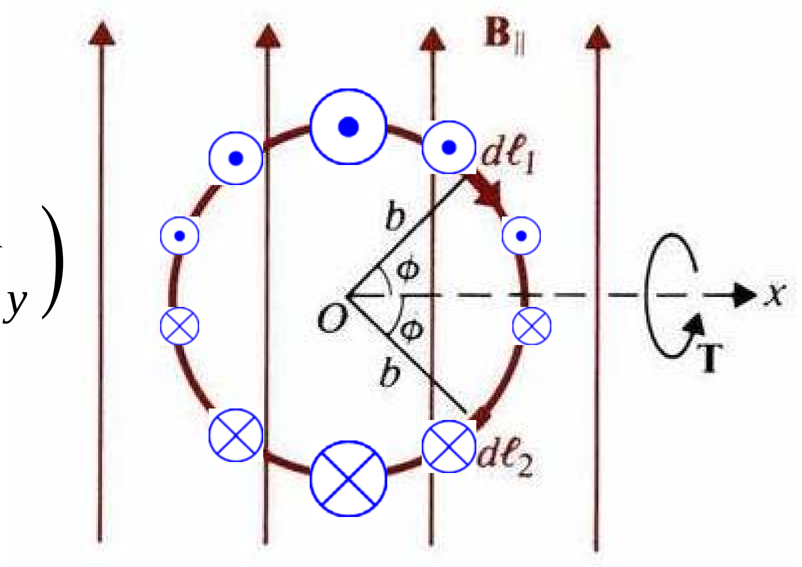
The force exerted on a differential current element $d\vec{l} = -\vec{a}_\phi b d\phi$ on the loop due to $\vec{B}_\parallel$:

$$d\vec{F}_m = I\left(d\vec{l} \times \vec{B}\right), \Rightarrow$$

$$d\vec{F}_\parallel = I\left(-\vec{a}_\phi b d\phi\right) \times \left(\vec{a}_y B_\parallel\right)$$

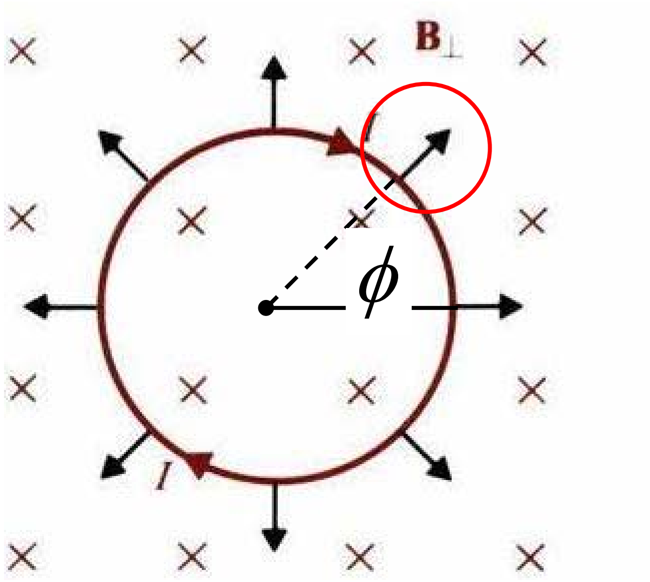
$$= IbB_\parallel d\phi\left(\vec{a}_x \sin\phi - \cancel{\vec{a}_y \cos\phi}\right) \times \left(\vec{a}_y\right)$$

$$= \vec{a}_z IbB_\parallel \underline{\sin\phi}\, d\phi$$



Example 13-5: Force & torque on current-carrying loops (4)

The total force exerted on the loop due to $\vec{B}_\perp$:



$$d\vec{F}_\perp = \vec{a}_r IbB_\perp d\phi, \Rightarrow$$

$$\vec{F}_\perp = \int_0^{2\pi} d\vec{F}_\perp$$

$$= IbB_\perp \left[\int_0^{2\pi} \underline{\vec{a}_r(\phi)d}\phi \right] = 0$$

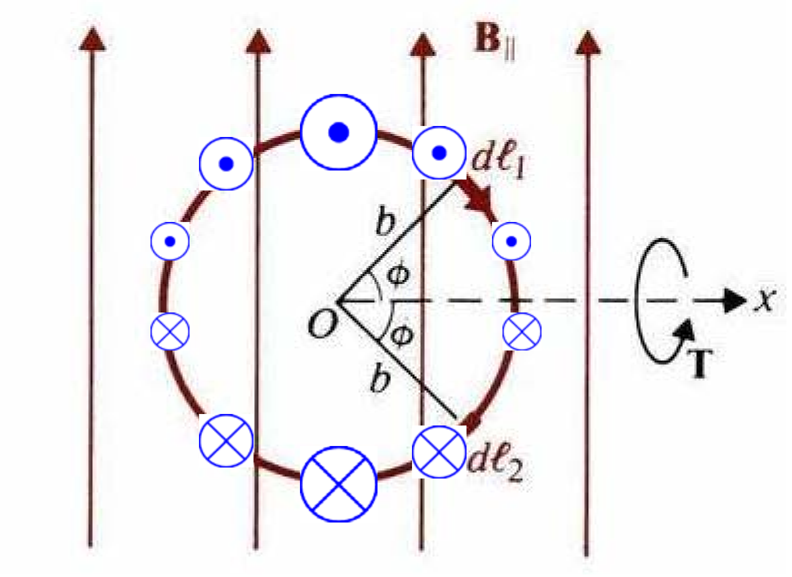
Example 13-5: Force & torque on current-carrying loops (5)

The total force exerted on the loop due to $\vec{B}_{\parallel}$:

$$d\vec{F}_{\parallel} = \vec{a}_z I b B_{\parallel} \sin\phi d\phi, \ \Rightarrow$$

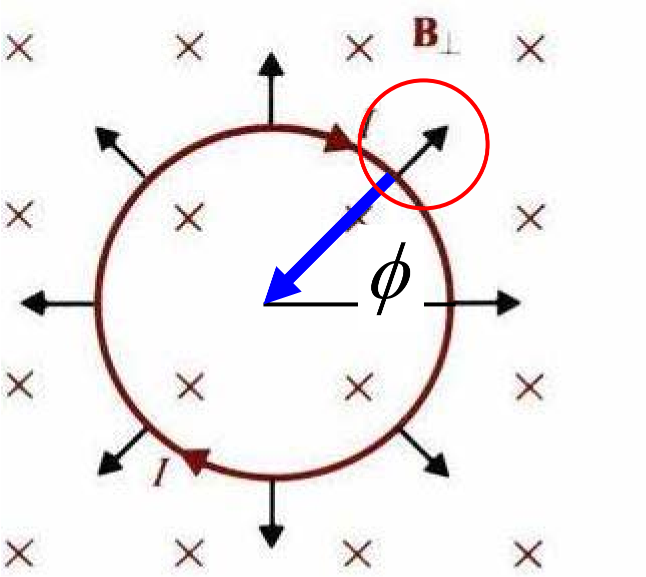
$$\vec{F}_{\parallel} = \int_0^{2\pi} d\vec{F}_{\parallel}$$

$$= \vec{a}_z I b B_{\parallel}\left(\int_0^{2\pi} \sin\phi \cdot d\phi\right) = 0$$



Example 13-5: Force & torque on current-carrying loops (6)

The total torque exerted on the loop due to $\vec{B}_{\perp}$:



$$d\vec{F}_{\perp} = \vec{a}_r IbB_{\perp} d\phi, \ \Rightarrow$$

$$\vec{T}_{\perp} = \int_0^{2\pi} d\vec{F}_{\perp} \times \underline{\left(-\vec{a}_r b\right)}$$

$$= -Ib^2 B_{\perp} \left(\int_0^{2\pi} \vec{a}_r \times \vec{a}_r \right) = 0$$

Example 13-5: Force & torque on current-carrying loops (7)

The total torque exerted on the loop due to $\vec{B}_{\|}$:

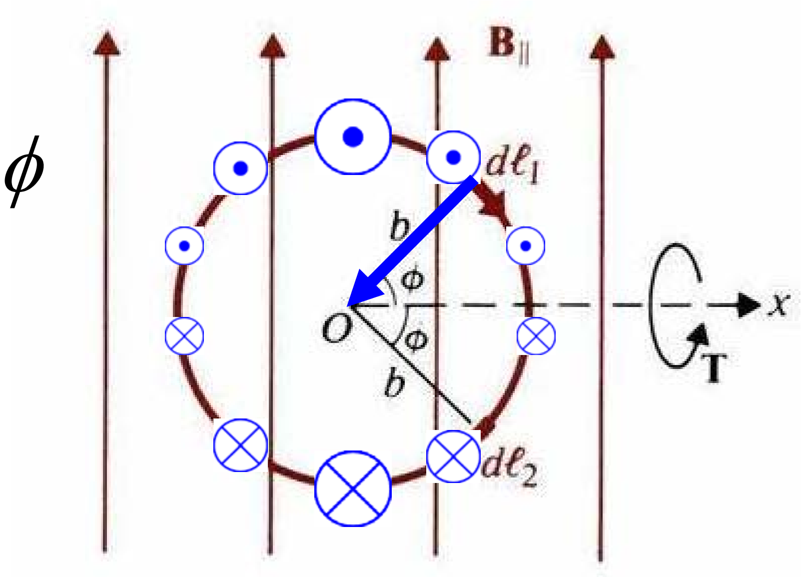
$$d\vec{F}_{\|} = \vec{a}_z I b B_{\|} \sin\phi d\phi, \;\Rightarrow$$

$$\vec{T}_{\|} = \int_0^{2\pi} d\vec{F}_{\|} \times \underline{\left(-\vec{a}_r b\right)}$$

$$= I b^2 B_{\|} \int_0^{2\pi} \left[\vec{a}_z \times \left(-\vec{a}_r\right)\right] \sin\phi d\phi$$

$$= I b^2 B_{\|} \int_0^{2\pi} \underline{\left(-\vec{a}_\phi\right)} \sin\phi d\phi$$

$$-\vec{a}_\phi = \vec{a}_x \sin\phi - \vec{a}_y \cos\phi$$

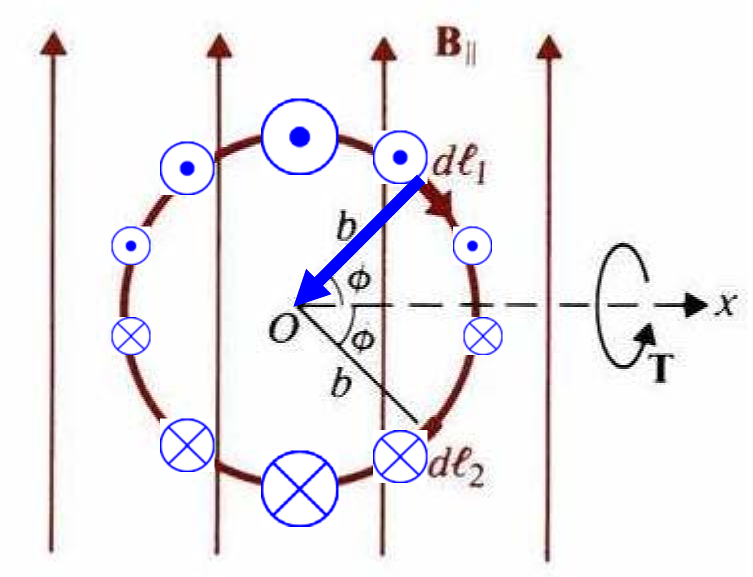


Example 13-5: Force & torque on current-carrying loops (7)

$$\vec{T}_{\parallel} = Ib^2 B_{\parallel} \cdot \left[\vec{a}_x \int_0^{2\pi} \overset{\frac{1-\cos 2\phi}{2}}{\sin^2 \phi}\, d\phi - \cancel{\vec{a}_y \int_0^{2\pi} \overset{\frac{\sin 2\phi}{2}}{\sin \phi \cdot \cos \phi} \cdot d\phi} \right]$$

$$= \vec{a}_x \left(I\pi b^2 \right) B_{\parallel} = \vec{a}_x m B_{\parallel}$$

$$\vec{T} = \vec{T}_{\perp} + \vec{T}_{\parallel} = \vec{T}_{\parallel} = \vec{a}_x (-m_z) B_y$$



In general, ⇒

$$\boxed{\vec{T} = \vec{m} \times \vec{B}}$$