## Lesson 12 Magnetostatics in Materials

# **12.1 Static Magnetic Field in the Presence of Magnetic Materials**

### ■ Concept of induced magnetic dipoles

Any material has many microscopic magnetic dipoles (i.e., tiny current loops) arising from: (1) orbiting electrons, (2) electrons and nucleus of an atom spinning on their own axes. However, a material bulk made up of a large number of randomly oriented molecules typically has no macroscopic dipole moment in the absence of external magnetic field.

As shown in Fig. 12-1a, consider an atom consisting of a nucleus of positive charge q and an electron of negative charge -e moving along a circular path of radius r with a constant angular velocity  $\omega_0$  in counterclockwise sense (corresponding to a velocity  $\bar{u}_0 = \bar{a}_{\phi} r \omega_0$ ). The Coulomb's force experienced by the electron  $\bar{F}_e = -e\bar{E}$  ( $\bar{E} = \bar{a}_r \frac{q}{4\pi\varepsilon_0 r^2}$ ) provides the centrifugal force  $\bar{a}_r m_e \omega_0^2 r$  ( $m_e$  means the mass of electron) to support the orbiting motion.



Fig. 12-1. Classic model to interpret diamagmetism.

When a magnetic field  $\vec{B} = \vec{a}_z B$  is applied, the orbiting electron experiences an extra force  $\vec{F}_m = -e\vec{u} \times \vec{B}$  [eq. (11.1)], where  $\vec{u} = \vec{a}_{\phi} r \omega$  (Fig. 12-1b). The total force  $\vec{F}_e + \vec{F}_m$  increases, providing a stronger centrifugal force  $\vec{a}_r m_e \omega^2 r$  and a larger angular velocity

 $\omega(>\omega_0)$ . Since the "loop current" equals  $I = \frac{ue}{2\pi r} = \frac{e\omega}{2\pi}$  (in clockwise sense), the presence of  $\vec{B}$  changes the magnetic dipole moment of each atom by an amount of  $\Delta \vec{m} = -\vec{a}_z (\Delta I \cdot \pi r^2)$ , where  $\Delta I = \frac{e(\omega - \omega_0)}{2\pi}$ . As a result, a net dipole moment emerges, contributing to a magnetic field in opposite direction with the applied one. This classical model can be used to interpret diamagnetism.

As shown in Fig. 12-2a, consider a circular current loop on the *xz*-plane with a magnetic dipole moment  $\vec{m} = \vec{a}_y m$ . When a magnetic field  $\vec{B} = \vec{a}_z B$  is applied, the positively charged particles located on the upper semicircle (x > 0) will experience a magnetic force  $\vec{F}_m$  in the  $-\vec{a}_y$  direction, while those located on the lower semicircle (x < 0) will experience a magnetic force  $\vec{F}_m$  in the  $+\vec{a}_y$  direction. The resulting torque will flip the current loop until the magnetic dipole moment is aligned with the applied magnetic field  $\vec{m} = \vec{a}_z m$  (Fig. 12-2b). Since all the microscopic magnetic dipole moments are aligned with  $\vec{B}$ , a net moment emerges, contributing to a magnetic field in the same direction with the applied one. This model can be used to interpret paramagnetism.



Fig. 12-1. Classic model to interpret paramagmetism.

#### <Comment>

Quantum mechanical model (spin) is required to properly interpret dia- and para-magnetism.

Magnetization vector and equivalent current densities

To analyze the effect of induced dipoles, we define a (microscopic) magnetization vector  $\overline{M}$  as the volume density of magnetic dipole moment:

$$\vec{M} \equiv \lim_{\Delta \nu \to 0} \frac{\sum \vec{m}_k}{\Delta \nu}$$
(12.1)

where  $\bar{m}_k$  denotes the k-th magnetic moment inside a differential volume  $\Delta v$ .

If the magnetization vector  $\vec{M}$  is inhomogeneous (i.e., varies with position) somewhere, there must exist net magnetization current at that position. This phenomenon can be illustrated in two cases.



Fig. 12-3. The model to deduce the magnetization surface current density.

1) The magnetization vector  $\overline{M}$  is discontinuous on the air-material interface, where there must exist net magnetization current (Fig. 12-3a). To quantitatively model this phenomenon, consider a differential volume  $\Delta V = dxdydz$  adjacent to the interface y = 0 (whose unit normal vector is  $\overline{a}_n = \overline{a}_y$ ). As shown in Fig. 12-3b, assume the volume has a rectangular current loop with area  $\Delta S = dydz$  and current I flowing in counterclockwise sense, corresponding to a magnetic dipole moment  $\Delta \overline{M} = \overline{a}_x I \Delta S$  and a magnetization vector  $\overline{M} = \frac{\Delta \overline{M}}{\Delta V} = \overline{a}_x \frac{I}{dx}$ . Since the currents from neighboring volumes will cancel with each other except for the components flowing on the interface, there is a current I flowing along  $\overline{a}_z$  over a cross-length dx. The corresponding surface current

density is  $\vec{J}_{ms} = \vec{a}_z \frac{I}{dx}$ , which can be generalized to:

$$\vec{J}_{ms} = \vec{M} \times \vec{a}_n \quad (A/m) \tag{12.2}$$

for  $\vec{M} \times \vec{a}_n = \vec{a}_x \frac{I}{dx} \times \vec{a}_y = \vec{a}_z \frac{I}{dx}$  in this particular case.

2) Consider two adjacent magnetic dipoles with magnetization vectors  $\vec{M} = \vec{a}_z M(x)$  and

$$\vec{a}_z M(x+dx)$$
, where  $M(x) = \frac{I(x)\Delta S}{\Delta S \cdot dz} = \frac{I(x)}{dz}$  (Fig. 12-4). The net current passing

through a surface bounded by contour C (dashed) between the two magnetic dipoles is:

$$I(x) - I(x + dx) = [M(x) - M(x + dx)]dz.$$

The corresponding volume current density is:

$$\vec{J}_m = \vec{a}_y \frac{I(x) - I(x + dx)}{dx dz} = \vec{a}_y \frac{M(x) - M(x + dx)}{dx} = -\vec{a}_y \frac{\partial M_z}{\partial x}$$

which can be generalized to:

$$\vec{J}_m = \nabla \times \vec{M} \quad (A/m^2) \tag{12.3}$$



Fig. 12-4. The model to deduce the magnetization volume current density.

### <Comment>

1) Eq. (12.2) can be regarded as a special case of eq. (12.3) on the surface, where the curl of

the magnetization vector  $\vec{M}$  is infinite.

2) Equivalent magnetization current densities  $\vec{J}_{ms}$ ,  $\vec{J}_{m}$  can be used in collaboration with eq. (11.11) to evaluate the vector potential (and then magnetic field) contributed by magnetized material.

Example 12-1: A uniformly magnetized cylinder (permanent magnet) has radius b, length L, and  $\overline{M} = \overline{a}_z M_0$  (Fig. 12-5a). Find  $\overline{B}$  along the z-axis.

Ans: By eq's (12.2), (12.3), there is uniform magnetization surface current density

$$J_{ms} = \vec{a}_z M_0 \times \vec{a}_r = \vec{a}_\phi M_0$$

on the side wall, and no magnetization current density elsewhere. Consider a circular ring of height dz' centered at (0,0,z'), where there is a differential current  $dI = J_{ms}dz' = M_0dz'$ flowing along  $\bar{a}_{\phi}$ . By Example 11-4, the current ring contributes to a differential magnetic flux density:

$$d\vec{B} = \vec{a}_{z} \frac{\mu_{0} M_{0} dz'}{2b} \left[ 1 + \left(\frac{z - z'}{b}\right)^{2} \right]^{-3/2}$$

at an observation point P(0,0,z). The total magnetic flux density becomes:

$$\vec{B} = \int_{z'=0}^{z'=L} d\vec{B} = \vec{a}_z \frac{\mu_0 M_0}{2} \left[ \frac{z}{\sqrt{z^2 + b^2}} - \frac{z - L}{\sqrt{(z - L)^2 + b^2}} \right].$$

Fig. 12-5b shows the normalized magnitude of magnetic flux density  $\overline{B}$  as a function of longitudinal position z. The two curves represent the results due to two cylinders of the same volume but different lengths and cross-sectional areas, respectively. It is evident that longer cylinders (with smaller cross section) can produce stronger axial magnetic field.



Fig. 12-5. (a) A uniformly magnetized cylinder (after DKC), and (b) the resulting  $B_z(z)$  versus axial position z for two different cylinders of the same volume.

■ Magnetic field intensity

In the presence of magnetic materials, the total magnetic field would be determined by both free and magnetization currents. The fundamental postulate eq. (11.3) is thus modified as:

 $\nabla \times \vec{R} - \mu \left( \vec{I} + \vec{I} \right)$ 

$$\nabla \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_m).$$
  
By eq. (12.3),  $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 (\nabla \times \vec{M}), \quad \nabla \times \vec{B} - \mu_0 (\nabla \times \vec{M}) = \mu_0 \vec{J}, \quad \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}, \Rightarrow$ 

$$\nabla \times \vec{H} = \vec{J} , \qquad (12.4)$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$
 (A/m) (12.5)

The term  $\overline{H}$  in eq. (12.4) is defined as the magnetic field intensity, totally determined by the "free" currents. The integral form of eq. (12.4) gives the Ampere's circuital law:

$$\oint_C \vec{H} \cdot d\vec{l} = I \tag{12.6}$$

### <Comment>

1) The magnetic field induced by a magnetic dipole (current loop) is in the same direction as the dipole moment  $\overline{M}$  (Fig. 12-6a). The total field (~ $\overline{B}$ ) is the summation of fields caused by free current  $(\sim \vec{H})$  and magnetization  $(\sim \vec{M})$ ,  $\Rightarrow \vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$ , i.e.,  $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$  [eq. (12.5)].

2) The electric field induced by an electric dipole (separated charges) is in opposite direction as the dipole moment  $\vec{P}$  (Fig. 12-6b). Total field  $(\sim \vec{E})$  is the summation of fields caused by free charge  $(\sim \vec{D})$  and polarization  $(\sim -\vec{P})$ ,  $\Rightarrow \varepsilon_0 \vec{E} = \vec{D} - \vec{P}$ , i.e.,  $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$  [eq. (7.9)].



Fig. 12-6. Comparison between (a) magnetic, and (b) electric dipoles.

For linear, homogeneous, and isotropic magnetic materials, the magnetization vector is proportional to the applied magnetic field:

$$\bar{M} = \chi_m \bar{H} , \qquad (12.7)$$

where  $\chi_m$  is a dimensionless quantity independent of the magnitude (linear), position (homogeneous), and direction (isotropic) of  $\vec{H}$ . Eq. (12.5) becomes  $\vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right)$ 

$$= \mu_0 \left( \vec{H} + \chi_m \vec{H} \right) = \mu_0 \left( 1 + \chi_m \right) \vec{H} , \Longrightarrow$$

$$\vec{B} = \mu \vec{H} , \qquad (12.8)$$

where the permeability of the medium  $\mu$  is defined as:

$$\mu = \mu_0 \left( 1 + \chi_m \right) \tag{12.9}$$

The strategy is using a single constant  $\mu$  to replace the tedious induced magnetic dipoles, magnetization vector, and equivalent current densities in determining total magnetic field.

Example 12-2: Consider a wound-coil inductor. (1)  $\vec{H}$  is related to the surface density of free current on the coil. (2)  $\vec{M}$  is related to the surface density of magnetization current (the direction is shown by assuming paramagnetic). (3)  $\vec{B}/\mu_0$  corresponds to the surface density

of total current or uncompensated free current.



Fig. 12-7. Physical meanings of magnetic flux density  $\vec{B}$ , magnetization vector  $\vec{M}$ , and magnetic field intensity  $\vec{H}$  illustrated in the example of a wound-coil inductor (after C. C. Su).

Example 12-3: A steady current  $I_0$  flows in N turns of wire wound around a ferromagnetic toroidal core of permeability  $\mu$  (magnetic source). The toroid has a large mean radius  $r_0$ , a small cross-sectional radius  $a(<< r_0)$ , and a narrow air gap of length  $l_g$ 

(Fig. 12-8a). Find  $\vec{B}$ ,  $\vec{H}$  in the core and air gap, respectively.

Ans: Assume the flux has no leakage and nor fringing effect in the air gap,  $\Rightarrow$  total flux (thus magnetic flux density  $\overline{B}$ ) is constant throughout the magnetic loop, i.e.,

$$\bar{B}_f = \bar{B}_g = \bar{a}_\phi B \, .$$

By eq's (12.6), (12.8),  $\Rightarrow$ 



Fig. 12-8. (a) Magnetic toroid with air gap. (b) The corresponding equivalent circuit (after DKC).

- 1)  $B_f = B_g$ , but  $H_f \ll H_g$ , for a small  $H_f$  can induce a strong  $\vec{M}$  in the ferromagnetic material  $(\mu \gg \mu_0)$ , providing a majority of magnetic flux in the ferromagnetic core.
- 2) Total flux  $\Phi = BS = \frac{NI_0}{(l_f / \mu S) + (l_g / \mu_0 S)}$ , which can be defined as:  $\Phi = \frac{\mathcal{V}_m}{R_f + R_g},$

where  $S = \pi a^2$  is the cross-sectional area,  $V_m = NI_0$  is the magnetomotive force (mmf),

 $R_i = \frac{l_i}{\mu S}$  (i = f, g) is the reluctance  $(R_f \text{ is the "internal" reluctance of the magnetic source, <math>R_g$  is the load reluctance). This is analogous to

$$I = \frac{V}{R}$$

in electric circuits (Fig. 12-8b). For a closed path with multiple magnetic sources and reluctances:

$$\sum_{j} N_{j} I_{j} = \sum_{k} R_{k} \Phi_{k}$$
(12.10)

 B-H curve of ferromagnetic material is usually nonlinear and depends on the "history" of Edited by: Shang-Da Yang magnetization,  $\Rightarrow \mu$  changes with *H* and its history (hysteresis), modification is required in finding *B* (DKC p225).

4) Due to very limited contrast of μ among different magnetic materials, confinement of magnetic field is usually much worse than that of current density, ⇒ flux leakage and fringing effect are normally non-negligible, making the model of magnetic circuit less accurate.

## **12.2 General Boundary Conditions for M-fields**

### Derivation

As in electrostatics, we apply integral forms of the two fundamental postulates on differential contour and thin pill box  $(\Delta h \rightarrow 0)$  across the interface of two magnetic media to derive the boundary conditions (BCs) for tangential and normal components of magnetic field.

1) Tangential BC: By eq. (12.6),

$$\oint_{abcda} \vec{H} \cdot d\vec{l} = \vec{H}_1 \cdot \Delta \vec{w} + \vec{H}_2 \cdot (-\Delta \vec{w}) = H_{1t} \cdot \Delta w - H_{2t} \cdot \Delta w = J_{sn} \Delta w ,$$

where  $H_{ii}(i=1, 2)$  means the  $\overline{H}_i$  component in the *ab*-direction,  $J_{sn}$  is the surface current density component in the right thumb-direction  $\overline{a}_{thumb}$  when the four fingers follow the direction of the path.  $\Rightarrow$ 

$$H_{1t} - H_{2t} = J_{sn}.$$

In general,

$$\bar{a}_{n2} \times \left(\bar{H}_1 - \bar{H}_2\right) = \bar{J}_s$$
 (12.11)

where  $\vec{a}_{n2}$  is the unit normal vector directed from medium 2 to medium 1.



Fig. 12-9. Differential contour used to derive tangential BC of static magnetic fields (after DKC).

Note that the projections of  $\vec{H}_1$  and  $\vec{H}_2$  on the interface are generally not in parallel. Fig. 12-10 illustrates a special example when the projections of  $\vec{H}_1$  and  $\vec{H}_2$  on the interface (*xy*-plane) are perpendicular with each other.



Fig. 12-10. Surface current density and boundary magnetic fields.

2) Normal BC: BC: By eq. (11.4),  $\oint_{S} \vec{B} \cdot d\vec{s} = (\vec{B}_{1} \cdot \vec{a}_{n2} - \vec{B}_{2} \cdot \vec{a}_{n2})(\Delta S) = 0, \Rightarrow$  $B_{1n} = B_{2n}$  (12.12)

#### <Comment>

1) Only "free" surface current  $\vec{J}_s$  counts in eq. (12.11). If none of the two interfacing

media is perfect conductor,  $\vec{J}_s = 0$ ,  $\Rightarrow H_{1t} = H_{2t}$ .

2) Eq's (12.11), (12.12) remain valid even the fields are time-varying (Lesson 14).

# 12.3 Behavior of Magnetic Materials (\*)

- Categorization
- Diamagnetic materials (μ < μ₀): As shown in Fig. 12-1, an external magnetic field can change the angular velocities of orbiting electrons, resulting in a net magnetic dipole moment in opposite direction of the applied field (Lenz's law). By eq's (12.7), (12.9), ⇒ χ<sub>m</sub> < 0, μ < μ₀. Diamagnetism is present in all materials, but is usually very weak (χ<sub>m</sub> ~ -10<sup>-5</sup>) and masked in paramagnetic/ferromagnetic materials. It disappears when the external field is withdrawn.
- 2) Paramagnetic materials (μ > μ₀): As shown in Fig. 12-2, an external magnetic field tends to align the dipole moments of orbiting and spinning electrons such that the net magnetic dipole moment is in the same direction of the applied field. ⇒ χ<sub>m</sub> > 0, μ > μ₀.
  Paramagnetism is usually very weak (χ<sub>m</sub> ~ 10<sup>-5</sup>), reduced by thermal vibration (randomizing the dipole moments), and disappears when the external field is withdrawn.
- 3) Ferromagnetic material (μ >> μ<sub>0</sub>): This type of materials consists of many domains with linear dimensions in the range between 1 μm and 1 mm. Each domain has fully aligned dipole moments due to strong coupling forces among spinning electrons (by quantum theory). Domain walls of ~100 atoms thick exist between adjacent domains. The domains are disorganized (material is demagnetized) if the temperature is above a critical value (curie temperature) when thermal energy exceeds the coupling energy. When an external magnetic field is applied, the domains with dipole moments aligned with the applied field will expand, largely increasing the total magnetic flux (iron has χ<sub>m</sub> ~ 4000). However, the relation between the total magnetic flux (~*B*-field) and the applied field (~*H*-field) is much more involved than a simple linear function like eq. (12.8).



Fig. 12-11. Domains of a ferromagnetic material (after DKC).

- Hysteresis curve of ferromagnetic materials
- 1) Reversible magnetization: If the external magnetic field is weak (up to  $P_1$  in Fig. 12-12), domain wall movement is reversible.  $\Rightarrow$  The *B*-*H* curve is a function, i.e., one *H*-value corresponds to a unique *B*-value.
- 2) Hystersis: If the external magnetic field is stronger (say  $P_2$  in Fig. 12-12), the *B*-*H* curve is not a function. For example, when the *H*-value is decreased from  $H_2(>0)$  to  $H'_2(<0)$ , the *B*-value will change along the upper branch of the broken lines connecting  $P_2$  and  $P'_2$ . When the *H*-value is increased from  $H'_2(<0)$  to  $H_2(>0)$ , the *B*-value will change along the broken lines. In other words, one *H*-value may correspond to two *B*-values, depending on the "history" of the external field.
- 3) Saturation: If the external magnetic field is very strong (say  $P_3$  in Fig. 12-12), all the domains are aligned, and further increasing external field does not increase the flux density.



Fig. 12-12. Hysteresis of a ferromagnetic material (after DKC).

- The hysterisis behavior of *B*-*H* curve makes that μ depends on: (1) magnitude of *H*,
   (2) history of the material's magnetization (same *H* may correspond to different *B*'s), seriously complicating the analysis of magnetic circuit.
- 2) Applications in generators, motors, transformers prefer tall, narrow hysteresis loop, while permanent magnets prefer fat hysteresis loop.
- 3) The area of hysteresis loop is the energy loss per unit volume per cycle in the form of heat in overcoming the friction of domain rotation (DKC, Problem P.6-29).