Lesson 12
Magnetostatics in Materials

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Outline

- Static magnetic field in materials
- Boundary conditions
- Properties of magnetic materials

Sec. 12-1
Static Magnetic Field in Materials

1. Classical models of induced magnetic dipoles
2. Magnetization vectors
3. Magnetization currents
4. Magnetic field intensities

Induced magnetic dipole-1

Any material has many small magnetic dipoles (current loops) arising from (1) orbiting electrons, (2) spinning nucleus and electrons

A material bulk made up of a large number of randomly oriented molecules typically has no macroscopic dipole moment in the absence of external magnetic field

What’s spin? (1)

Spin of elementary particles cannot be explained by assuming they are made up of even smaller particles rotating about a center

Spin is about angular momentum of elementary particles, quantized, cannot be altered

What's spin? (2)

A particle with charge q, mass m, spin S has an intrinsic magnetic dipole moment:

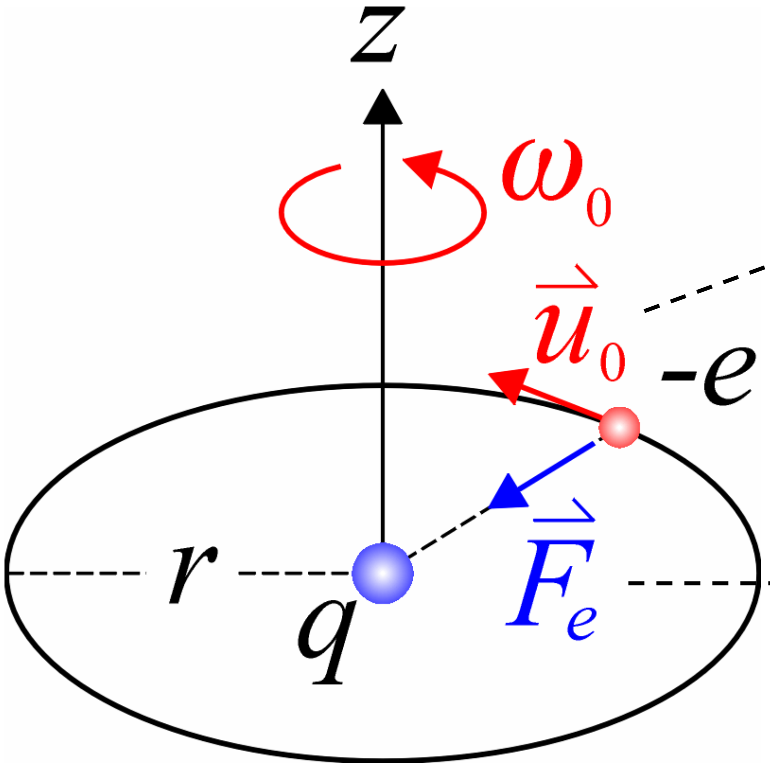
$$\boxed{g \approx 2}$$

$$\mu = g \frac{q}{2m} S$$

Angular momentum of an electron (spin-1/2) measured along any direction can only take on values of $\pm \hbar/2$

Photon (spin-1): $0, \pm \hbar$

Induced magnetic dipole due to orbiting electron-1

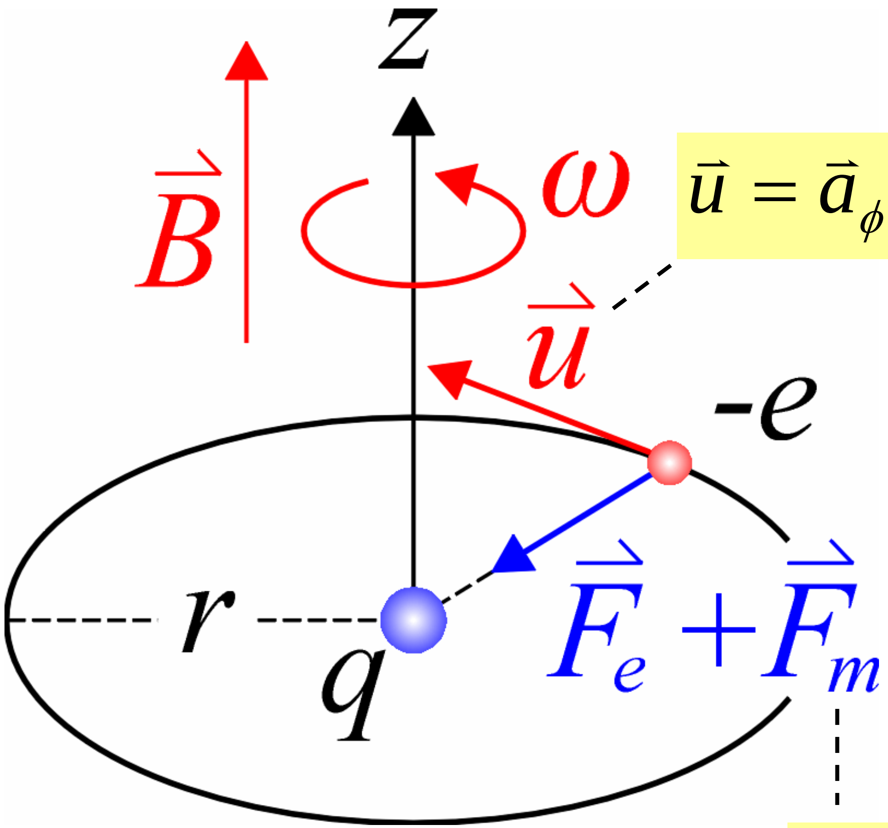


$$\vec{u}_0 = \vec{a}_\phi r \omega_0$$

$$\vec{F}_e = -e\vec{E} = \vec{a}_r m_e \omega_0^2 r$$

$$\vec{E} = \vec{a}_r \frac{q}{4\pi\varepsilon_0 r^2}$$

Induced magnetic dipole due to orbiting electron-2



$$\vec{u} = \vec{a}_\phi r\omega$$

$$\vec{F}_e + \vec{F}_m = \vec{a}_r m_e \omega^2 r$$

$$\vec{F}_m = -e\vec{u} \times \vec{B}$$

$$\omega > \omega_0, \Rightarrow$$

$$I = \frac{ue}{2\pi r} = \frac{e\omega}{2\pi} \uparrow$$

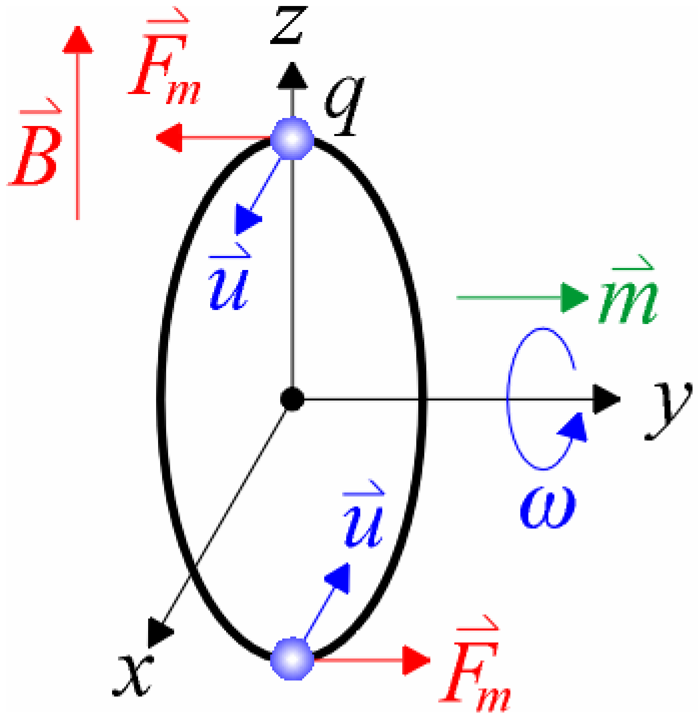
$$\Delta\vec{m} = -\vec{a}_z\left(\Delta I \cdot \pi r^2\right)$$

diamagnetism

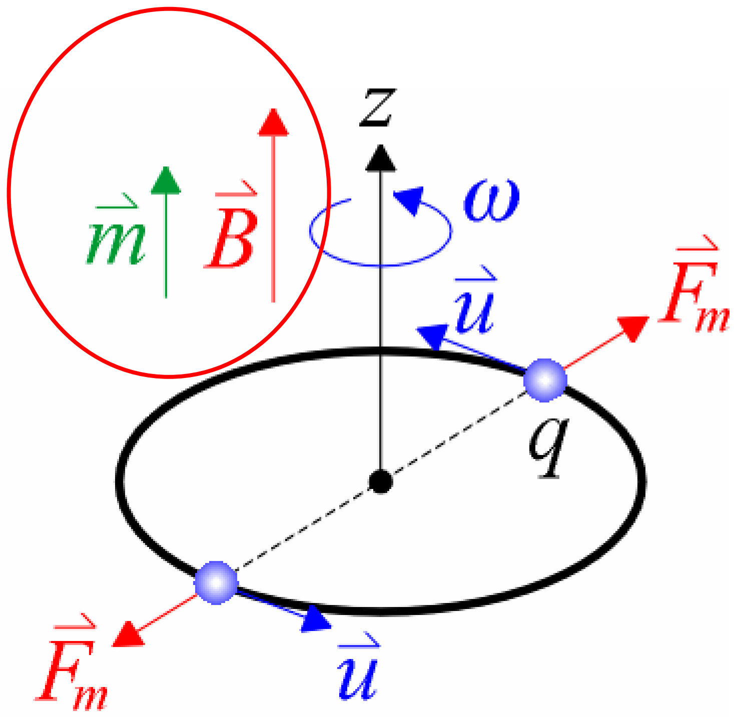
$$\Delta I = \frac{e(\omega - \omega_0)}{2\pi}$$

Induced magnetic dipole due to aligned moment

paramagnetism



torque



tension

Strategy of analysis

It is too tedious to directly superpose the elementary fields:

$$\vec{B}(\vec{r}) \approx \frac{\mu_0 m}{4\pi R^3}\left[\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta\right]$$

Instead, we define magnetization vector as:

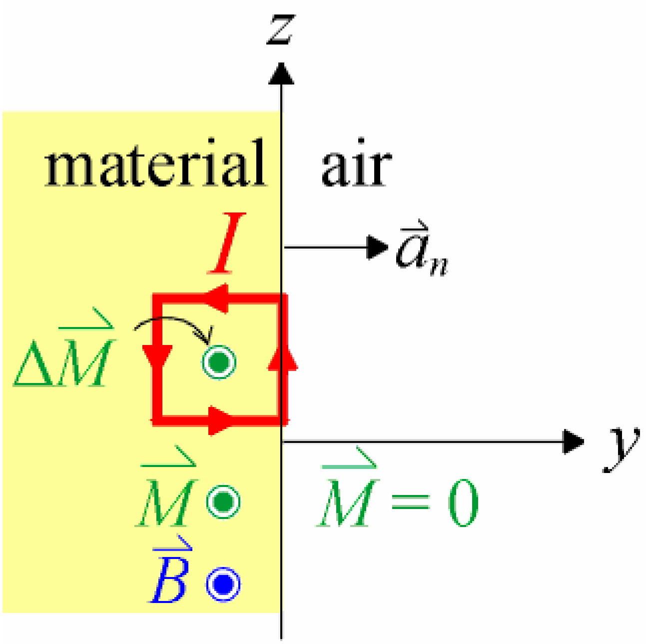
$$\boxed{\vec{M} \equiv \lim_{\Delta v \to 0} \frac{\sum \vec{m}_k}{\Delta v}}$$

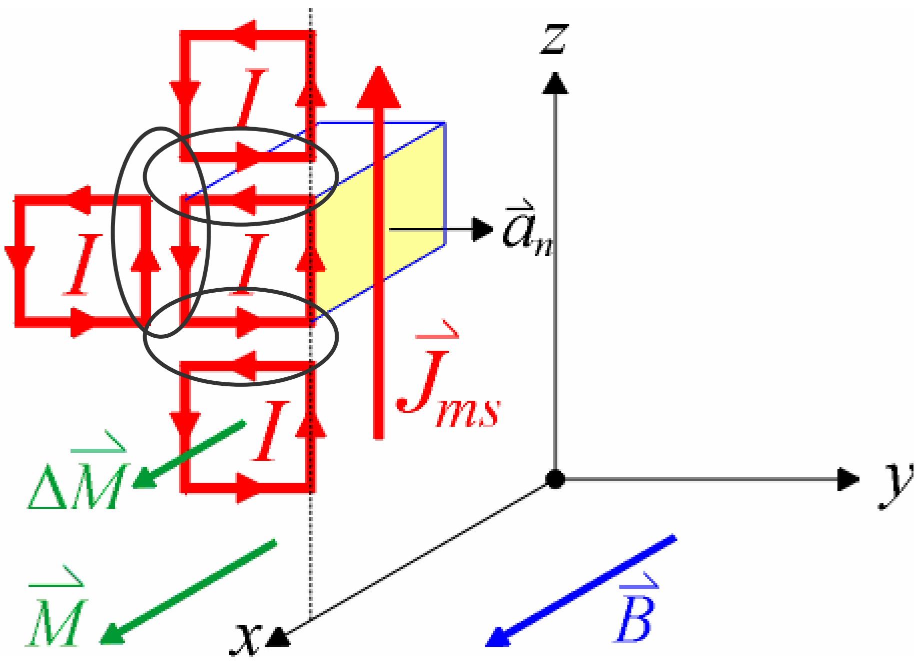
the kth dipole moment inside a differential volume Δv

$\Rightarrow$ Magnetization current $\vec{J}_m$, Magnetic field intensity $\vec{H}$

Magnetization surface current density-1

On the air-material interface, $\vec{M}$ is dis-continuous, net magnetization current exists

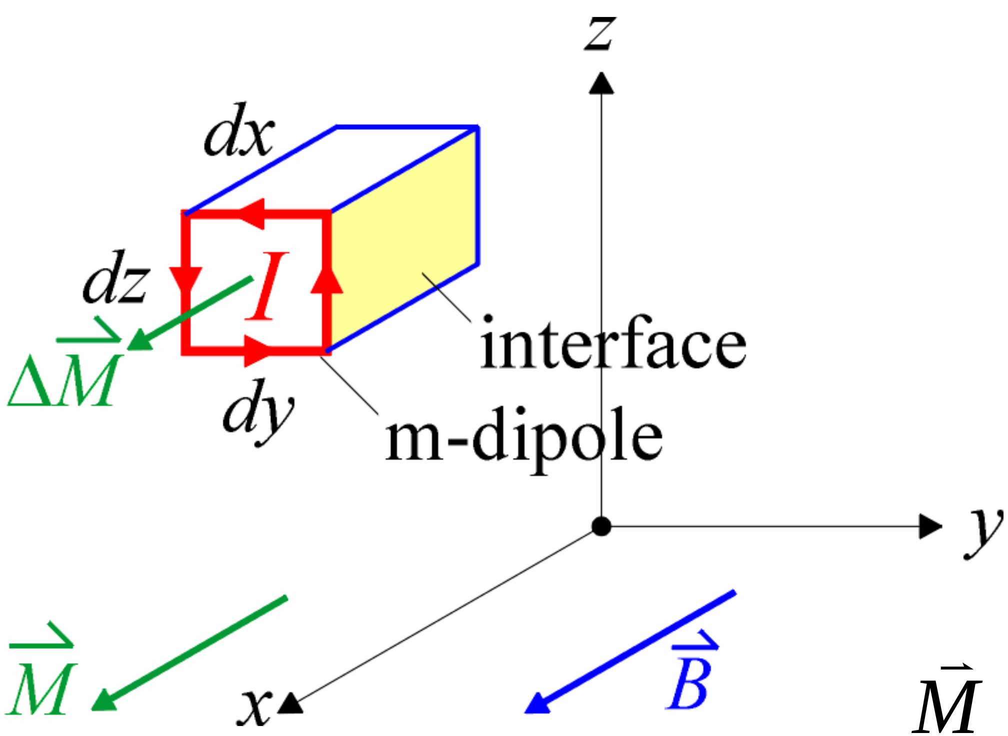




Magnetization surface current density-2

Consider a differential volume $\Delta V = dxdydz$

with rectangular current loop of $\Delta S = dydz,\ I$



Magnetic dipole moment:

$$\Delta\vec{M} = \vec{a}_x I\Delta S = \vec{a}_x I dydz$$

Magnetization vector:

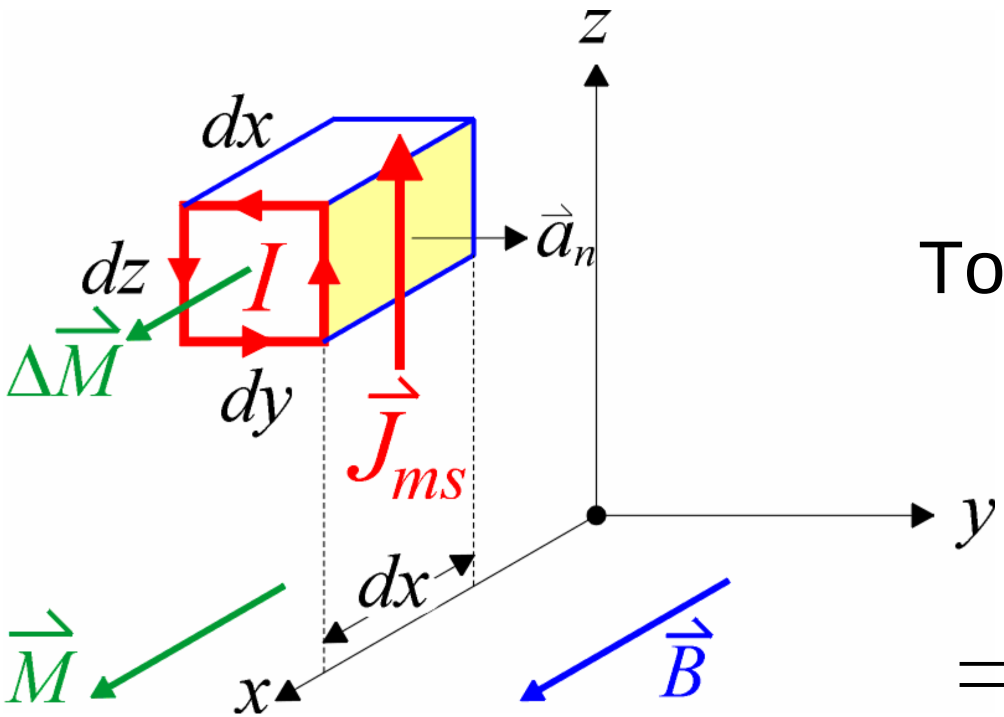
$$\vec{M} = \frac{\Delta\vec{M}}{\Delta V} = \vec{a}_x \frac{Idydz}{dxdydz} = \vec{a}_x \frac{I}{dx}$$

Magnetization surface current density-3

$$\vec{M} = \frac{\Delta\vec{M}}{\Delta V} = \vec{a}_x \frac{I}{dx}$$

By inspection, the surface current density:

$$\vec{J}_{ms} = \vec{a}_z \frac{I}{dx}\left(\frac{\mathrm{A}}{\mathrm{m}}\right)$$



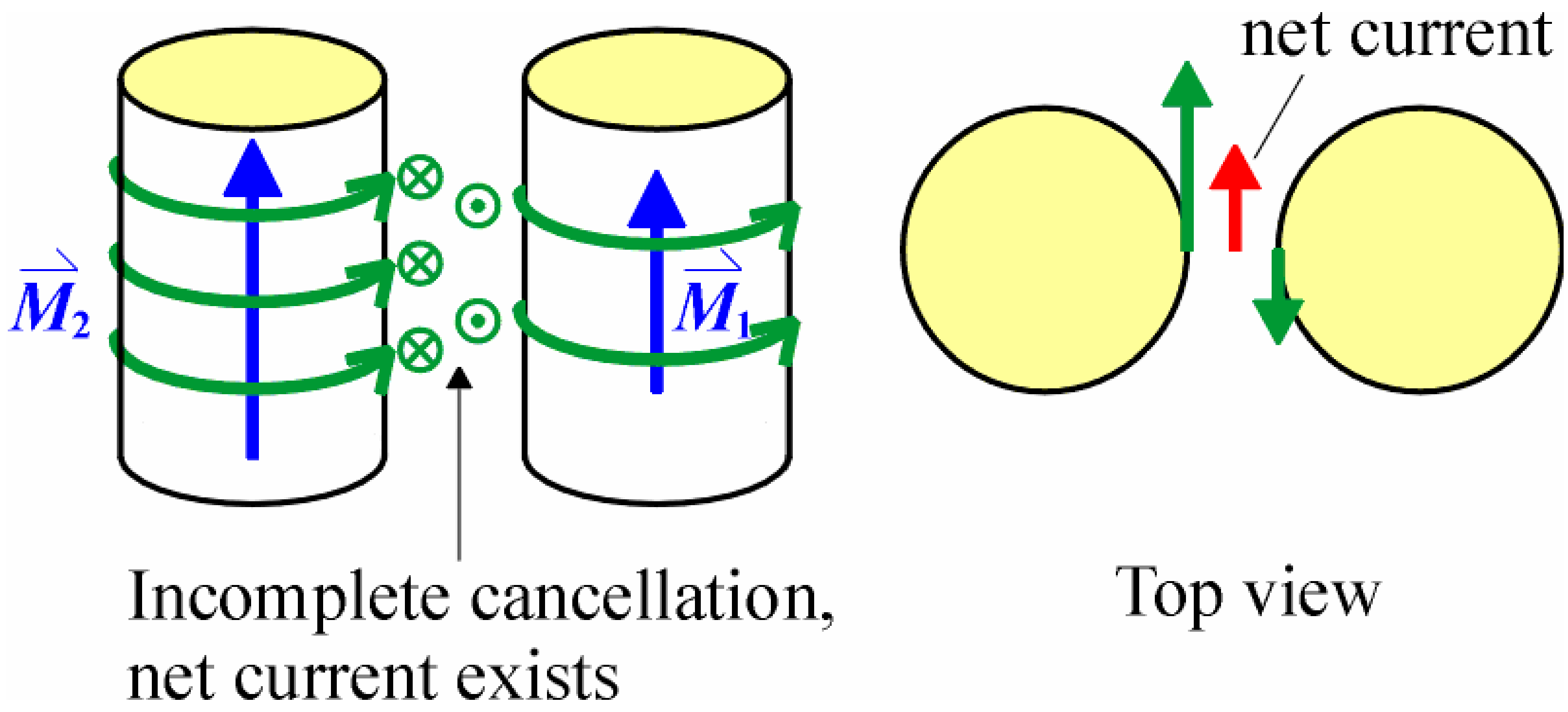
To generalize the result,

$$\vec{M} \times \vec{a}_n = \vec{a}_x \frac{I}{dx} \times \vec{a}_y = \vec{a}_z \frac{I}{dx},$$

$$\Rightarrow \boxed{\vec{J}_{ms} = \vec{M} \times \vec{a}_n \ (\mathrm{A/m})}$$

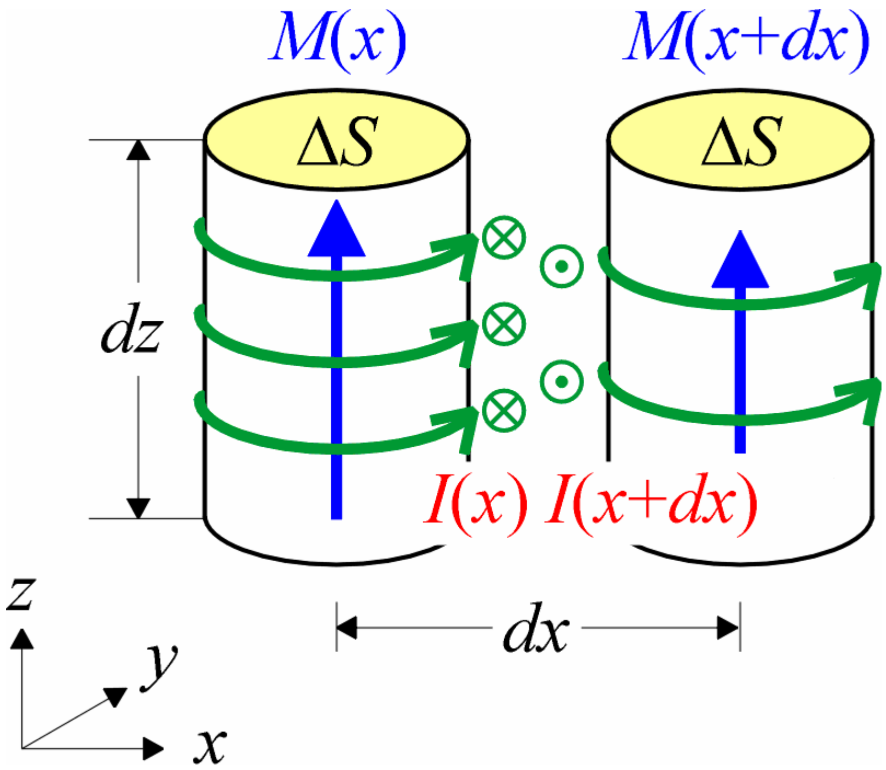
Magnetization volume current density-1

In the interior of a magnetic material, net magnetization current exists where $\vec{M}$ is inhomogeneous:



Magnetization volume current density-2

Consider two m-dipoles with x-dependent, z-direction magnetization vectors: $\vec{a}_z M(x)$, $\vec{a}_z M(x+dx)$



where

$$M(x) = \frac{I(x)\Delta S}{\Delta S \cdot dz} = \frac{I(x)}{dz}$$

$$\Rightarrow I(x) = M(x)dz$$

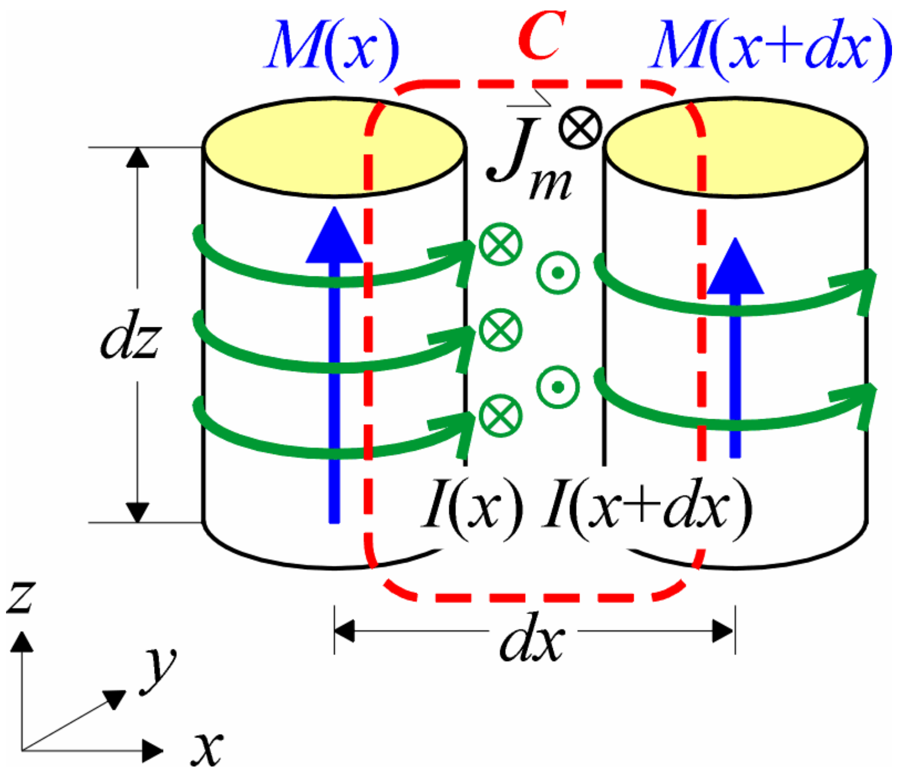
Similarly,

$$I(x+dx) = M(x+dx)dz$$

Magnetization volume current density-3

Net current passing through the interfacing surface bounded by C is: $I(x) - I(x+dx)$

$$= [M(x) - M(x+dx)]dz$$



The current density is:

$$\vec{J}_m = \vec{a}_y \frac{I(x) - I(x+dx)}{dxdz}$$

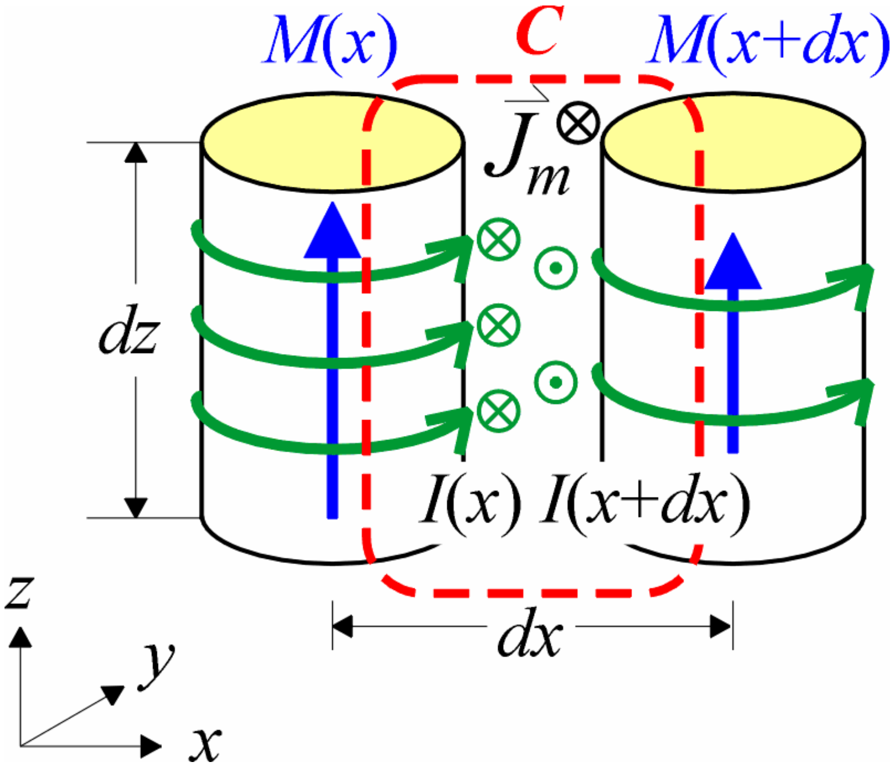
$$= \vec{a}_y \frac{M(x) - M(x+dx)}{dx}$$

$$= -\vec{a}_y \frac{\partial M_z}{\partial x}$$

Magnetization volume current density-4

To generalize the result:

$$\nabla \times \vec{M} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M_x & M_y & M_z \end{vmatrix}$$

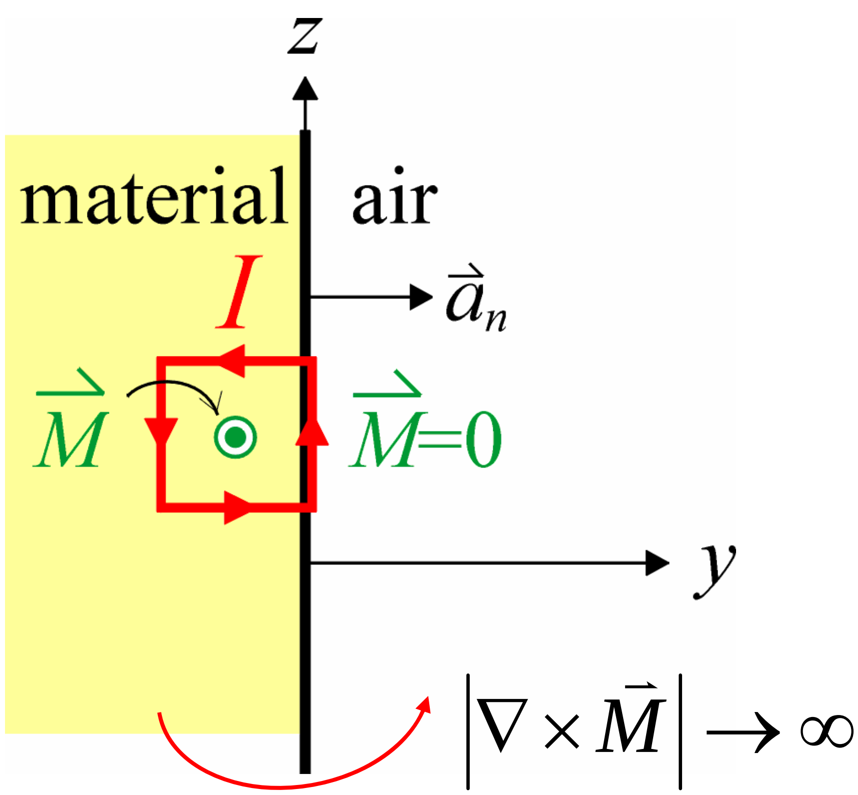


$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & 0 & 0 \\ 0 & 0 & M_z(x) \end{vmatrix} = -\vec{a}_y \frac{\partial M_z}{\partial x}$$

$$\Rightarrow \boxed{\vec{J}_m = \nabla \times \vec{M} \ \left(\mathrm{A/m^2}\right)}$$

Comment-1

$\vec{J}_{ms} = \vec{M} \times \vec{a}_n$ can be regarded as a special case of $\vec{J}_m = \nabla \times \vec{M}$, where $\left|\nabla \times \vec{M}\right| \to \infty$



Comment-2

Equivalent current densities:

$$\vec{J}_{ms} = \vec{M} \times \vec{a}_n, \quad \vec{J}_m = \nabla \times \vec{M}$$

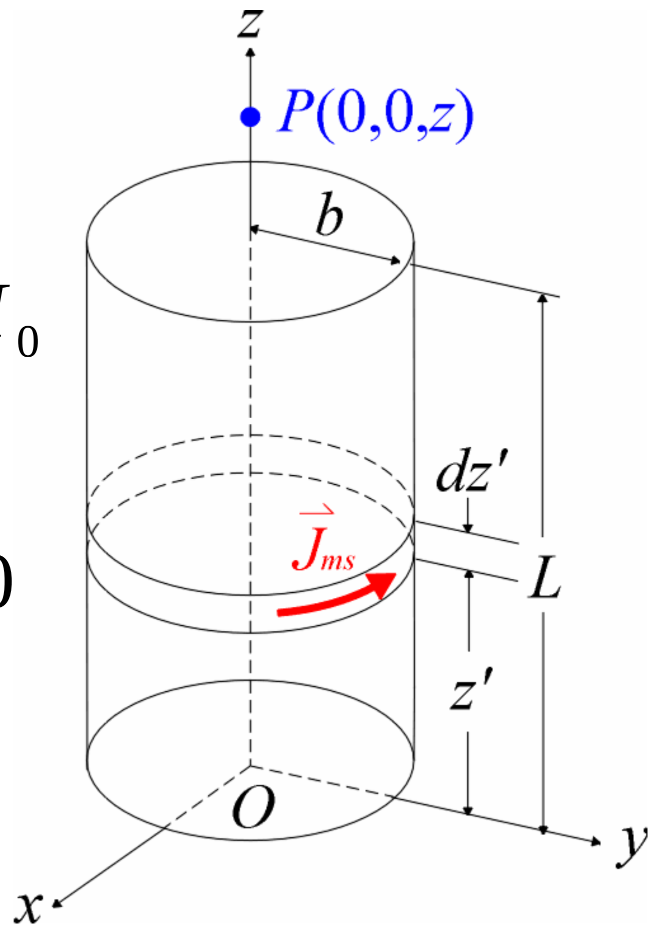
can be substituted into the formulas:

$$\begin{cases} \vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int_{V'} \dfrac{\vec{J}(\vec{r}\,')}{R(\vec{r}, \vec{r}\,')} dv' \\ \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$

to evaluate the contribution of m-field due to magnetized materials.

Example 12-1: Permanent magnet (1)

Consider a uniformly magnetized cylinder of radius ***b***, length ***L***, and $\vec{M} = \vec{a}_z M_0$. Find the on-axis $\vec{B}$



- On the side wall:

$$\vec{J}_{ms} = \vec{M} \times \vec{a}_n = \vec{a}_z M_0 \times \vec{a}_r = \underline{\vec{a}_\phi} M_0$$

- On the top & bottom walls:

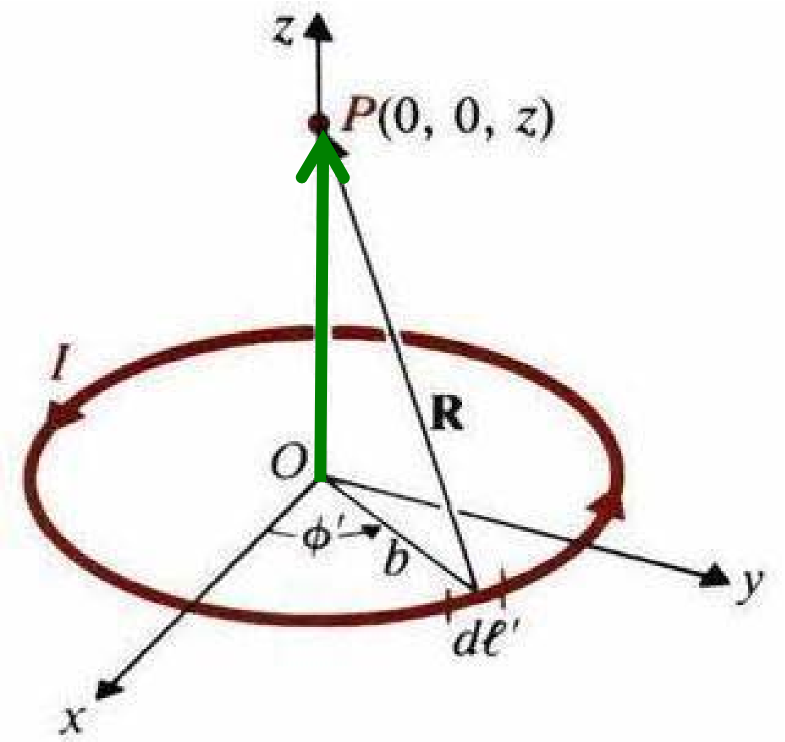
$$\vec{J}_{ms} = \vec{M} \times \vec{a}_n = \vec{a}_z M_0 \times \left(\pm \vec{a}_z\right) = 0$$

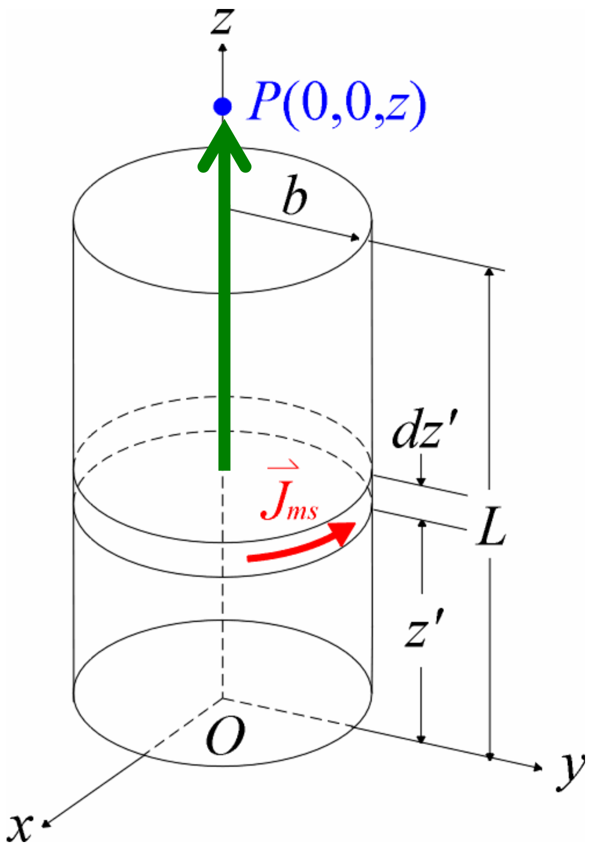
- In the interior:

$$\vec{J}_m = \nabla \times \vec{M} = \nabla \times \left(\vec{a}_z M_0\right) = 0$$

Example 12-1: Permanent magnet (2)

$$\vec{B}(0,0,z) = \vec{a}_z \frac{\mu_0 I}{2b}\left[1+(z/b)^2\right]^{-3/2} \longrightarrow d\vec{B} = \vec{a}_z \frac{\mu_0 M_0 dz'}{2b}\left[1+\left(\frac{z-z'}{b}\right)^2\right]^{-3/2}$$

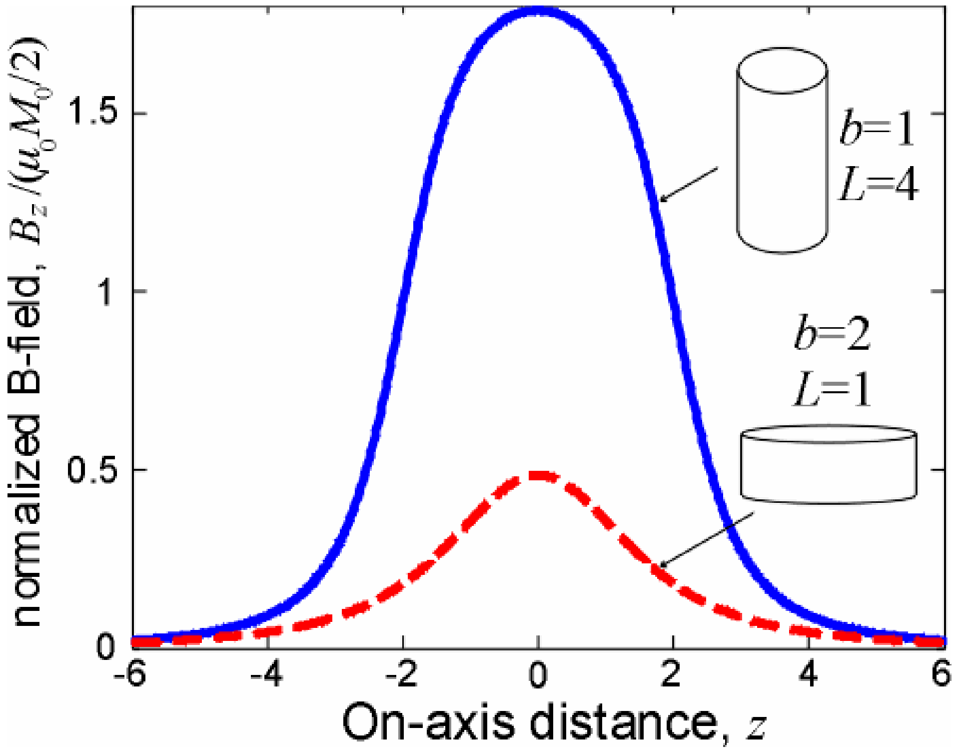




Example 12-1: Permanent magnet (3)

Total magnetic flux density:

$$\vec{B} = \int_{z'=0}^{z'=L} d\vec{B} = \vec{a}_z \frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{z^2+b^2}} - \frac{z-L}{\sqrt{(z-L)^2+b^2}}\right]$$



Magnetic field intensity - Definition (1)

Total M-field is created by free & magnetization currents:

$$\nabla \times \vec{B} = \mu_0 \vec{J} \longrightarrow \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \overset{\nabla \times \vec{M}}{\vec{J}_m} \right)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \left(\nabla \times \vec{M} \right),$$

$$\Rightarrow \nabla \times \vec{B} - \mu_0 \left(\nabla \times \vec{M} \right) = \mu_0 \vec{J},$$

$$\Rightarrow \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}$$

Magnetic field intensity - Definition (2)

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}$$

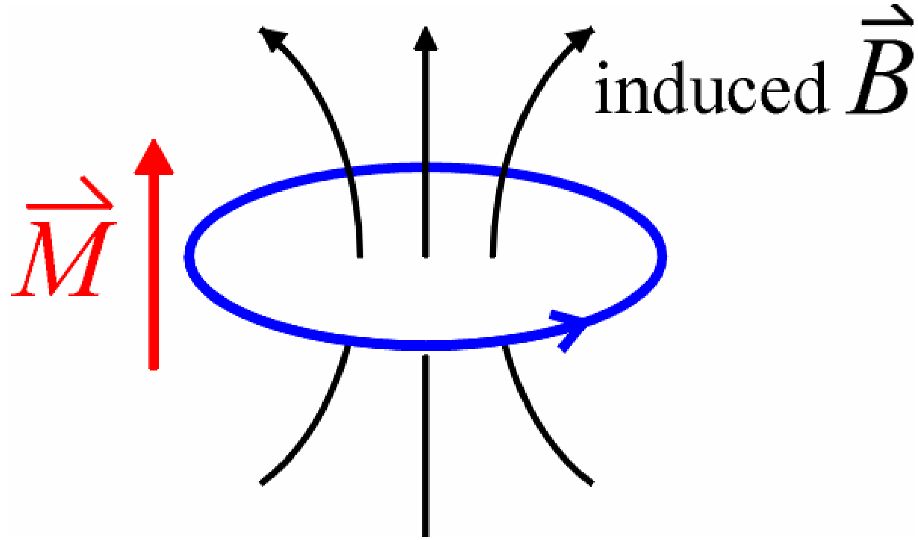
only free current

$$\Rightarrow \begin{cases} \boxed{\nabla \times \vec{H} = \vec{J}} \longrightarrow \oint_C \vec{H} \cdot d\vec{l} = I & \text{...Ampere's law} \\ \vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M} \quad (\mathrm{A/m}) & \text{...Magnetic field intensity} \end{cases}$$

Comment-1

Total M-field $\left(\sim \vec{B}\right)$ is the summation of fields due to free current $\left(\sim \vec{H}\right)$ & magnetization $\left(\sim +\vec{M}\right)$

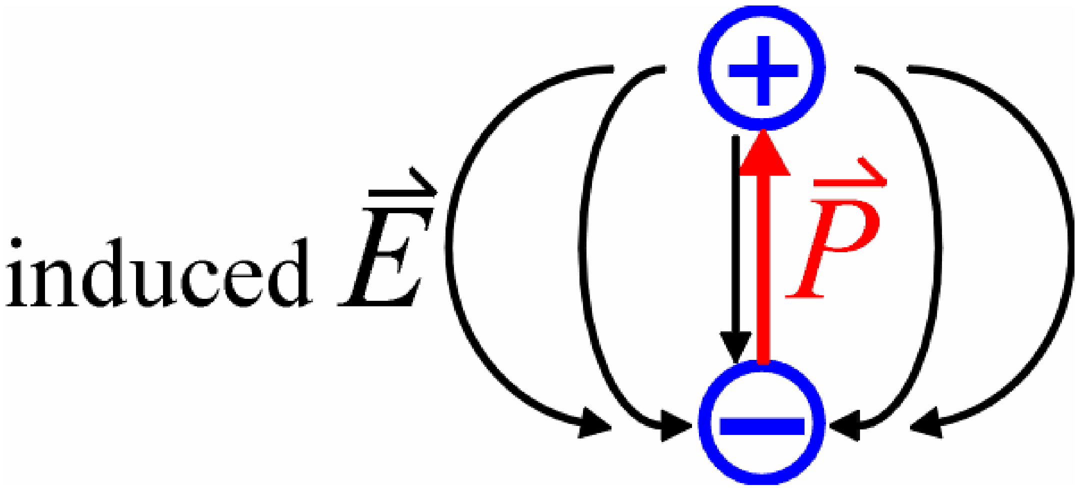
$$\Rightarrow \vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}, \quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$



Comment-2

Total E-field $\left(\sim \vec{E}\right)$ is the summation of fields due to free charge $\left(\sim \vec{D}\right)$ & polarization $\left(\sim -\vec{P}\right)$

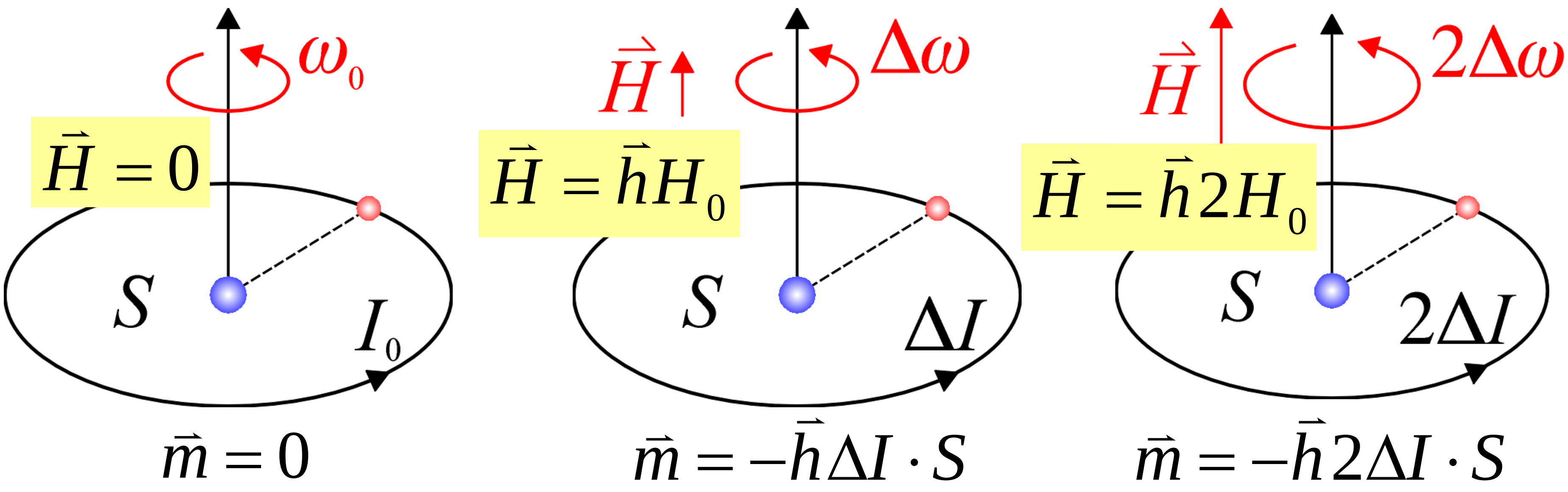
$$\Rightarrow \varepsilon_0 \vec{E} = \vec{D} - \vec{P}, \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$



For linear, homogeneous, and isotropic magnetic materials, the magnetization vector is proportional to the external magnetic field :

$$\vec{M} = \chi_m \vec{H}$$

Susceptibility, independent of magnitude, position, and direction of $\vec{H}$



Magnetic field intensity - Usefulness (2)

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}, \quad \Rightarrow \vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$$

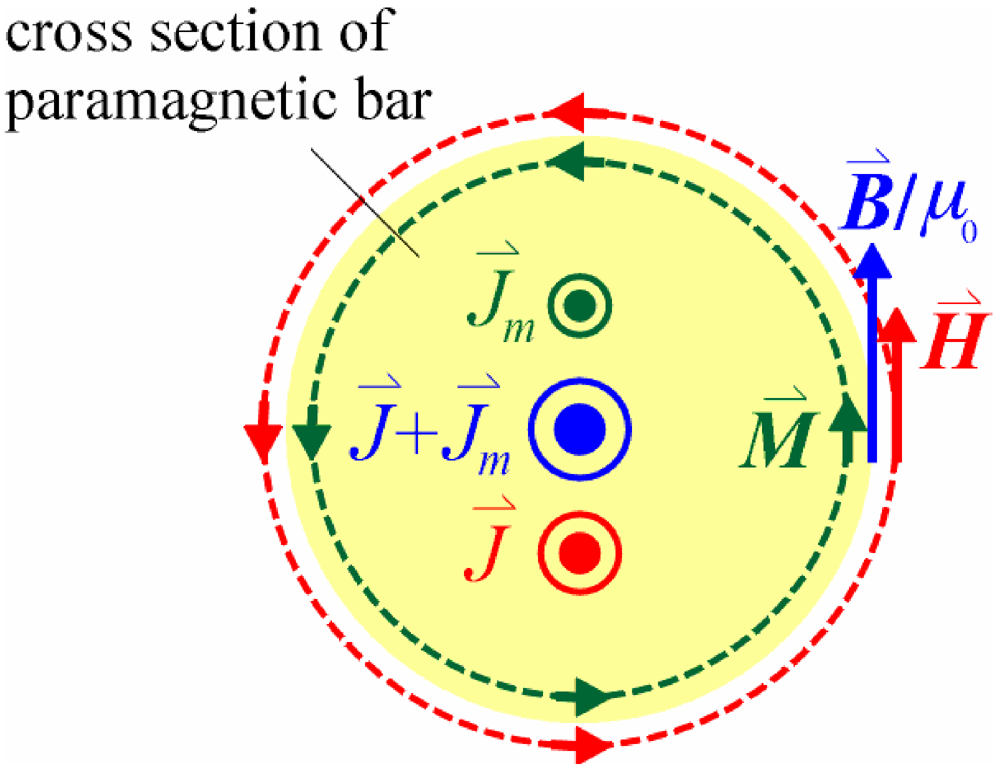
$$= \mu_0\left(\vec{H} + \chi_m \vec{H}\right) = \mu_0\left(1 + \chi_m\right)\vec{H}$$

$$\Rightarrow \begin{cases} \boxed{\vec{B} = \mu\vec{H}} \\ \mu = \mu_0\left(1 + \chi_m\right) \end{cases}$$

...Permeability of the medium

A single constant μ replaces the tedious induced m-dipoles, magnetization vector, equivalent magnetization currents in determining the total magnetic field.

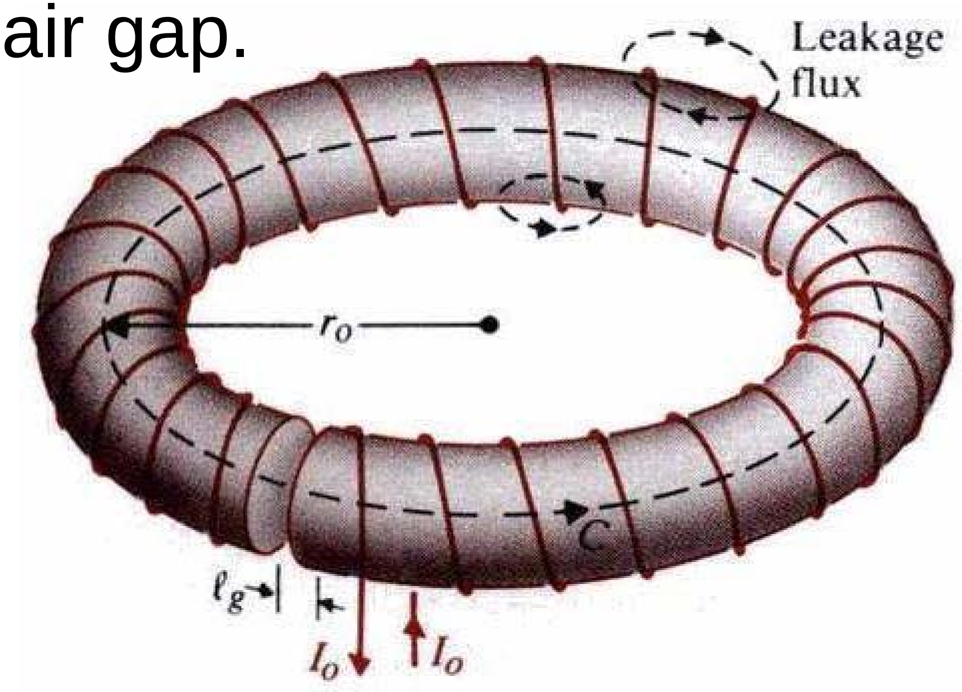
Example 12-2: Physical meanings of H, M, B



- $\vec{H}$: free current
- $\vec{M}$: magnetization current
- $\vec{B}/\mu_0$: total current

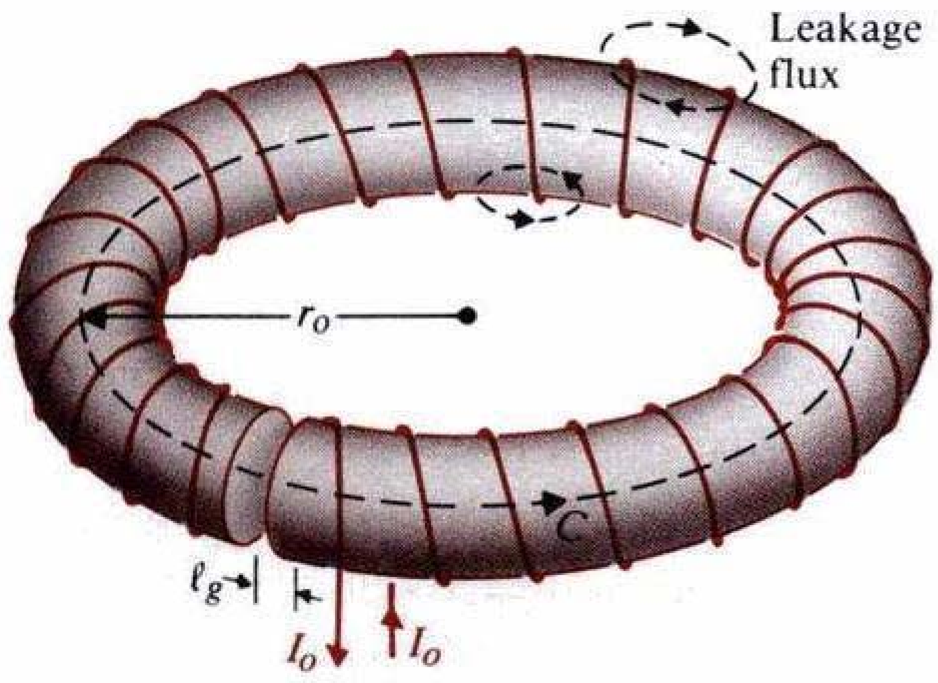
Example 12-3: Magnetic circuit (1)

A current I_0 flows in N turns of wire wound around a toroidal core of permeability μ, mean radius r_0, cross-sectional radius a, narrow air gap of length l_g. Find $\vec{B}$, $\vec{H}$ both in the core and the air gap.



Example 12-3: Magnetic circuit (2)

Assume the flux has no leakage and nor fringing effect in the air gap, $\Rightarrow$ total flux (thus $\vec{B}$) is constant throughout the loop:



gap

$$\vec{B}_f = \vec{B}_g = \vec{a}_\phi B$$

ferromagnetic core

$$\begin{cases} \oint_C \vec{H} \cdot d\vec{l} = I \\ \vec{B} = \mu \vec{H} \end{cases}$$

$$\Rightarrow \frac{B}{\mu} l_f + \frac{B}{\mu_0} l_g = N I_0$$

Example 12-3: Magnetic circuit (3)

$$\frac{B}{\mu} l_f + \frac{B}{\mu_0} l_g = NI_0, \Rightarrow \boxed{B = \frac{NI_0}{l_f/\mu + l_g/\mu_0}} \quad B_f = B_g$$

$$\Rightarrow \boxed{\vec{H}_f = \frac{\vec{B}_f}{\mu}, \quad \vec{H}_g = \frac{\vec{B}_g}{\mu_0}} \quad \because \mu >> \mu_0, \Rightarrow H_f << H_g$$

A small H_f can induce a strong $\vec{M}$ in the ferromagnetic material, providing a majority of magnetic flux in the ferromagnetic core.

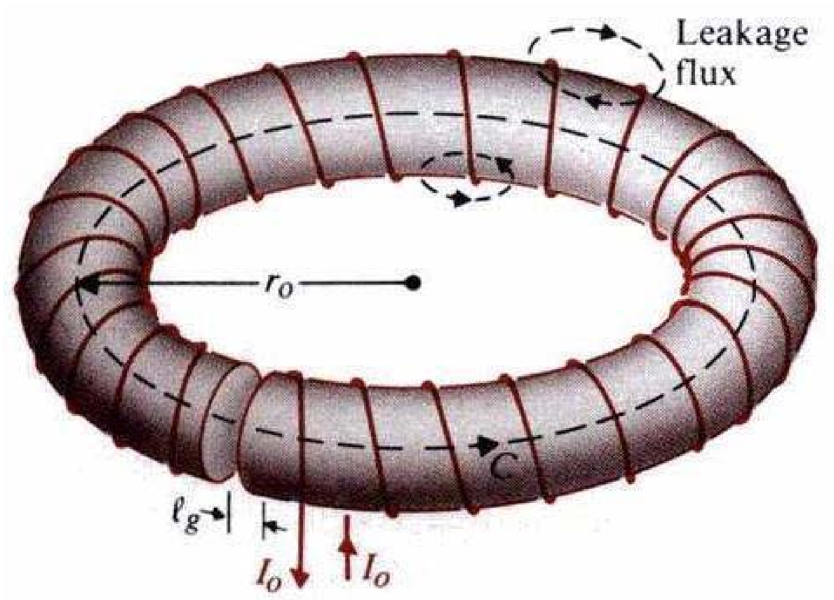
Example 12-3: Magnetic circuit (4)

$$S = \pi a^2$$

Total flux: $\Phi = BS = \dfrac{NI_0}{\left(l_f / \mu S\right) + \left(l_g / \mu_0 S\right)} \equiv \dfrac{\mathcal{V}_m}{R_f + R_g}$

$\mathcal{V}_m = NI_0$ …Magnetomotive force (mmf)

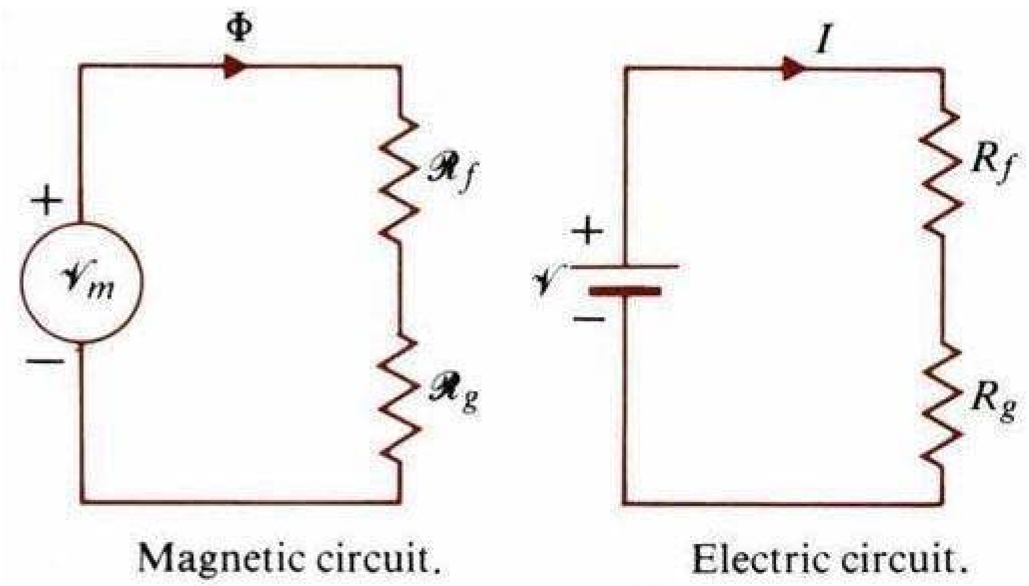
$R_i = \dfrac{l_i}{\mu_i S}$ $(i = f, g)$ …Reluctance



Example 12-3: Magnetic circuit (5)

Analogy between magnetic & electric circuits:

$$\Phi = \frac{\mathcal{V}_m}{R_f + R_g} \qquad I = \frac{V}{R_f + R_g}$$

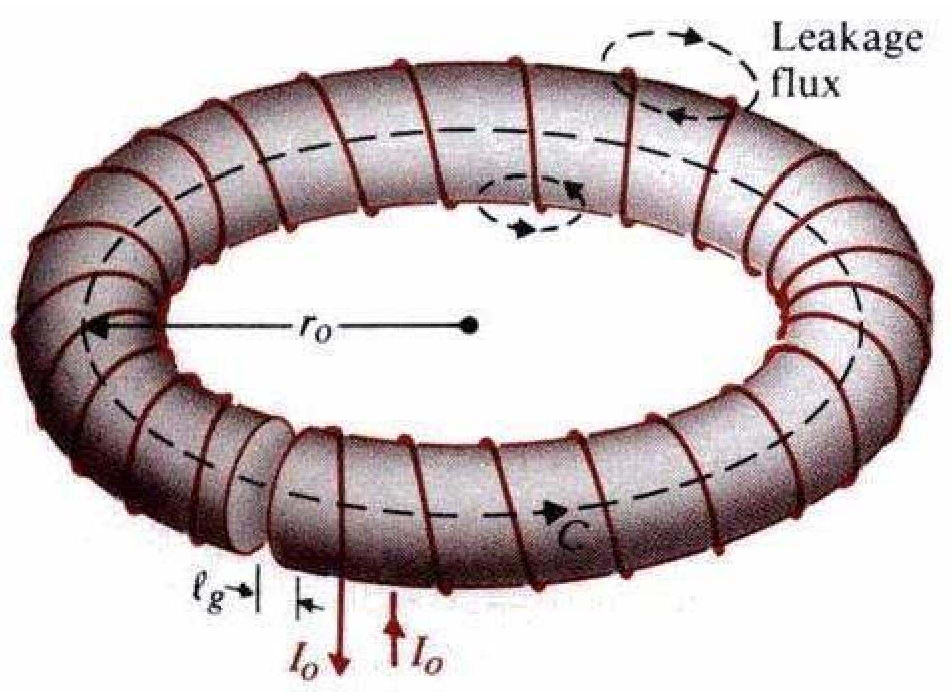


Multiple magnetic sources and reluctances:

$$\boxed{\sum_j N_j I_j = \sum_k R_k \Phi_k}$$

Comment

$$\begin{cases} \oint_C \vec{H} \cdot d\vec{l} = I \\ \vec{B} = \mu \vec{H} \end{cases} \Rightarrow \frac{B}{\mu} l_f + \frac{B}{\mu_0} l_g = NI_0, \ B = \ldots$$



For ferromagnetic materials, μ depends on the magnitude (nonlinear) and history (hysteresis) of H, $\Rightarrow$ modification is required to find B

Sec. 12-2
Boundary Conditions

1. Tangential boundary condition
2. Normal boundary condition

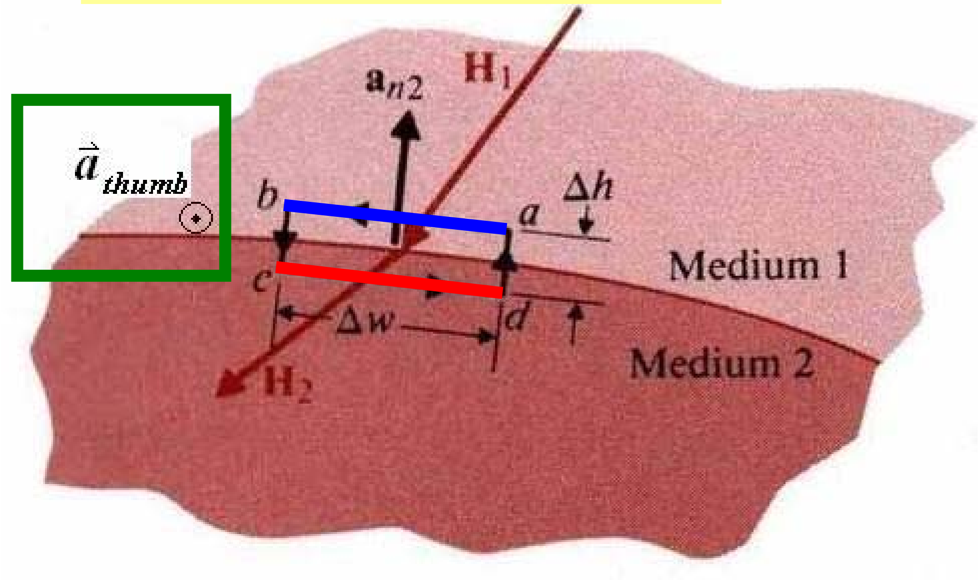
Tangential BC-1

$$\oint_C \vec{H}\cdot d\vec{l} = I, \;\; \Rightarrow \oint_{abcda} \vec{H}\cdot d\vec{l} = \vec{H}_1\cdot\Delta\vec{w} + \vec{H}_2\cdot(-\Delta\vec{w})$$

$$= \underline{H_{1t}}\cdot\Delta w - \underline{H_{2t}}\cdot\Delta w = \underline{J_{sn}}\Delta w$$

Component of $\vec{H}_i$ in the *ab*-direction

Component of $\vec{J}_s$ along $\vec{a}_{thumb}$



$$\Rightarrow \;\; H_{1t} - H_{2t} = J_{sn}$$

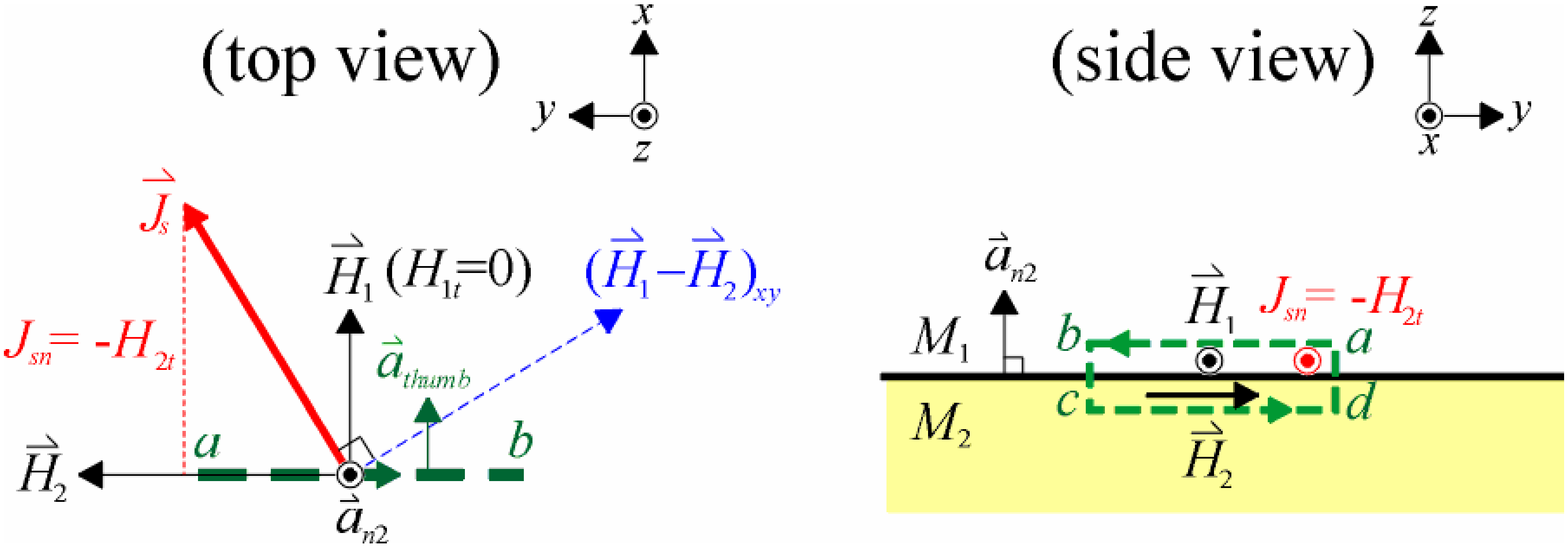
In general,

$$\boxed{\vec{a}_{n2}\times\left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s}$$

Comment

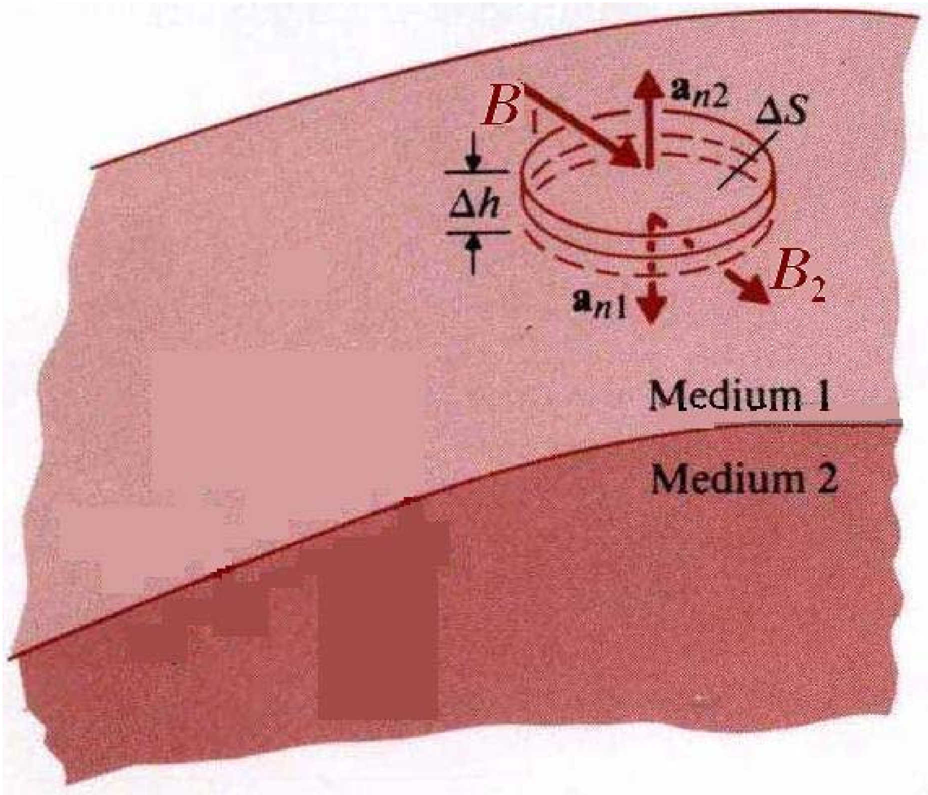
The projections of $\vec{H}_1$ and $\vec{H}_2$ on the interface are generally not collinear.

$$H_{1t} - H_{2t} = J_{sn}, \qquad \vec{a}_{n2} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s$$



Normal BC-1

$$\oint_S \vec{B}\cdot d\vec{s} = 0, \quad \Rightarrow \oint_S \vec{B}\cdot d\vec{s} = \left(\vec{B}_1\cdot\vec{a}_{n2} - \vec{B}_2\cdot\vec{a}_{n2}\right)\left(\Delta S\right) = 0$$



$$\Rightarrow \boxed{B_{1n} = B_{2n}}$$

Comment-1

Only free surface current density counts in:

$$\vec{a}_{n2} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s$$

If the two interfacing media are non-conducting:

$$\vec{J}_s = 0, \Rightarrow H_{1t} = H_{2t}$$

Comment-2

$$\begin{cases} \vec{a}_{n2} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s \\ B_{1n} = B_{2n} \end{cases}$$

remain valid even the M-fields are time-varying.

Comment-3

- BCs in electrostatics:

$$\begin{cases} E_{1t} = E_{2t} \\ \vec{a}_{n2} \cdot \left(\vec{D}_1 - \vec{D}_2 \right) = \rho_s \end{cases}$$

v.s.

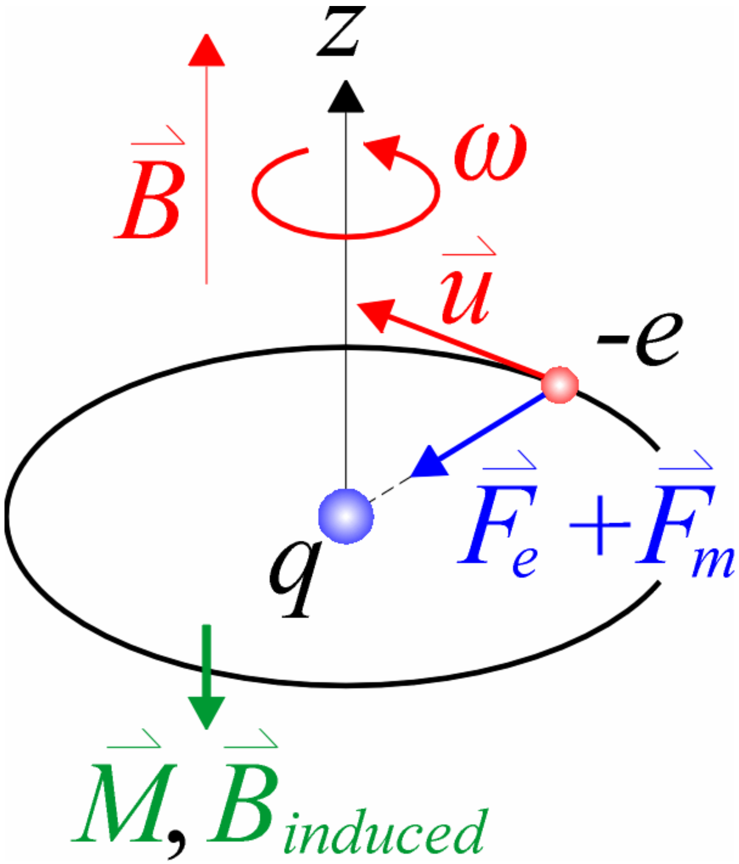
- BCs in magnetostatics:

$$\begin{cases} \vec{a}_{n2} \times \left(\vec{H}_1 - \vec{H}_2 \right) = \vec{J}_s \\ B_{1n} = B_{2n} \end{cases}$$

Sec. 12-3
Properties of Magnetic Materials

1. Diamagnetic materials
2. Paramagnetic materials
3. Quantum view of ferromagnetism
4. Hysteresis curve

Diamagnetic materials



$$\vec{M} = \chi_m \vec{H}, \Rightarrow \boxed{\chi_m < 0}$$

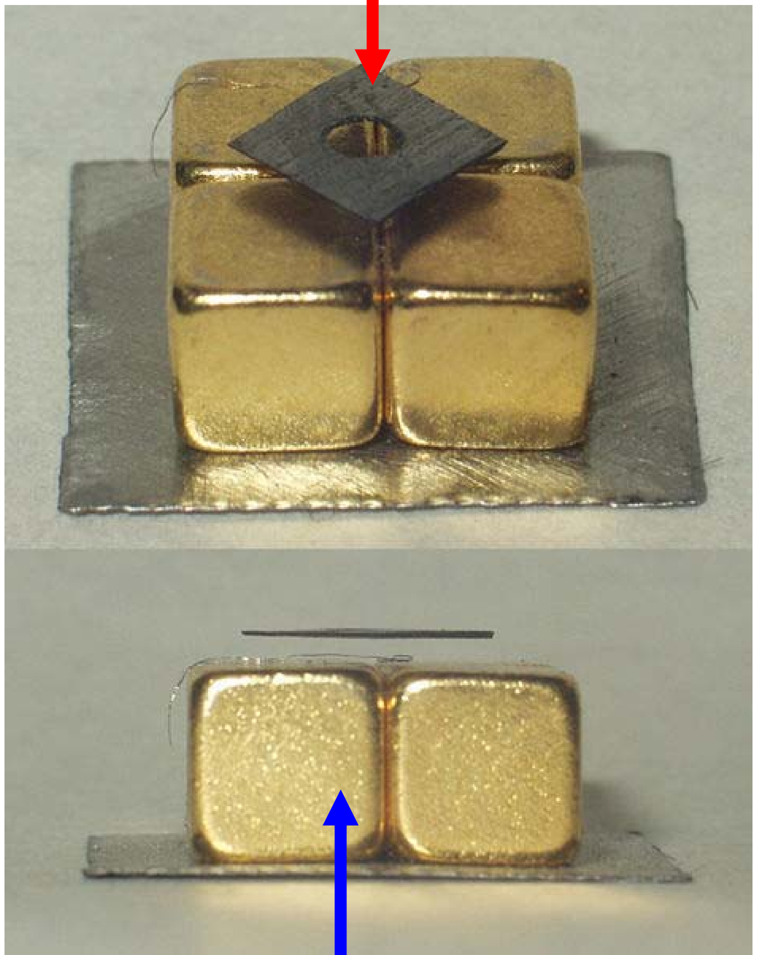
$$\mu = \mu_0 (1 + \chi_m), \boxed{\mu < \mu_0}$$

Present in all materials, usually weak $(\chi_m \sim -10^{-5})$ and masked in paramagnetic/ferromagnetic materials.

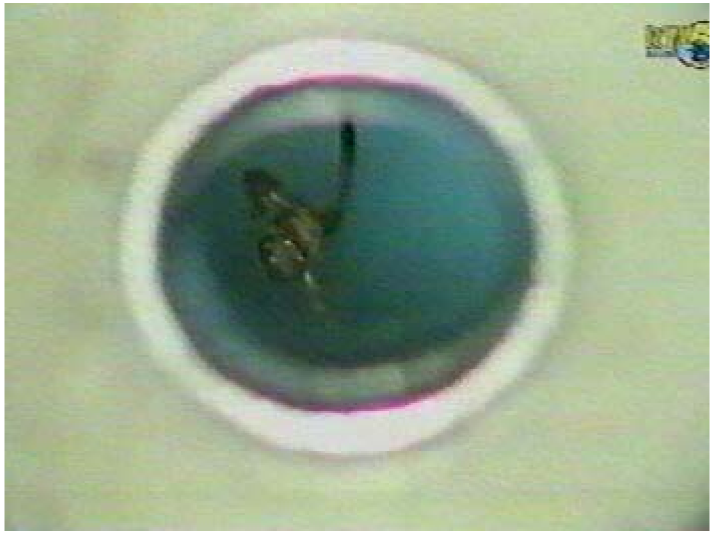
Disappears when the external field is off.

Diamagnetic materials-examples

pyrolytic graphite



permanent neodymium magnet



Frog flying in strong M-field

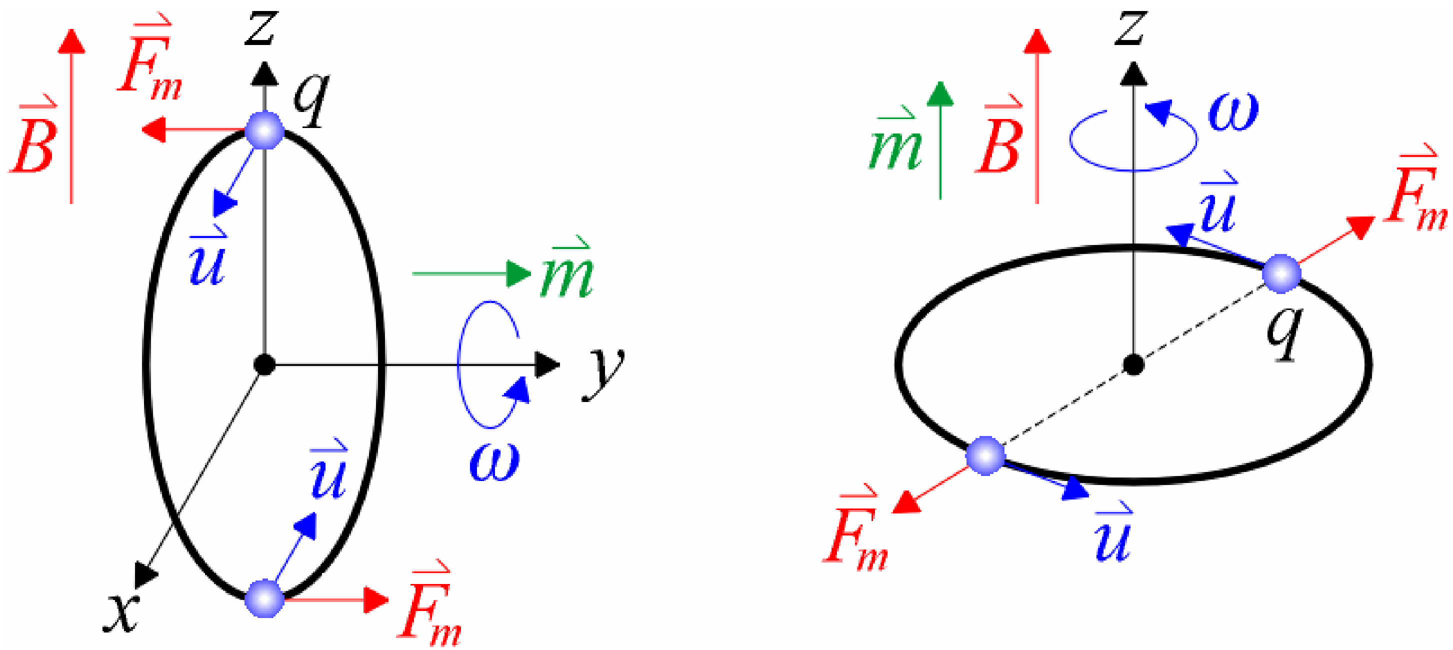
(D=32 mm, B=16T)

Paramagnetic materials

$$\vec{M} = \chi_m \vec{H}, \Rightarrow \boxed{\chi_m > 0}$$

$$\mu = \mu_0 (1 + \chi_m), \boxed{\mu > \mu_0}$$

Usually very weak $\left(\chi_m \sim 10^{-5}\right)$ reduced by thermal vibration (randomizing the dipole moments). Disappears when the external field is off.



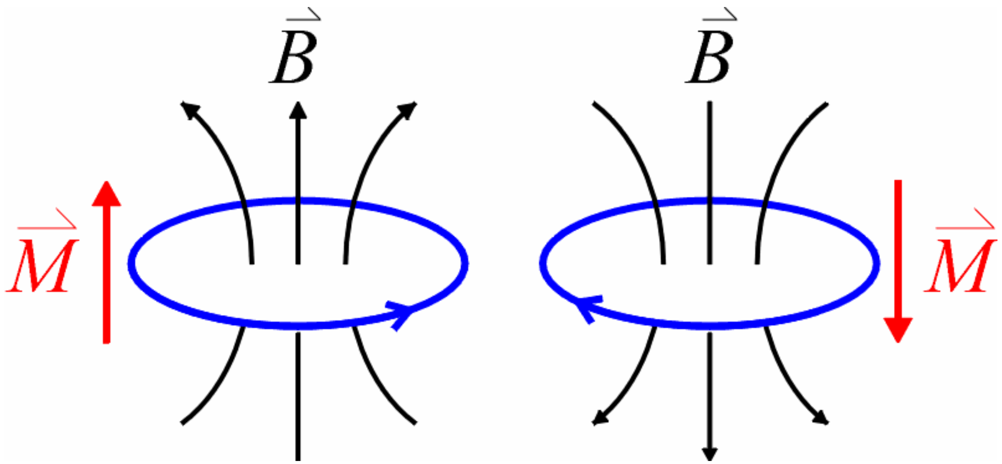
Non-classical, can only be explained by quantum mechanical view.

Ferromagnetism is determined by both the chemical makeup and the crystalline structure. E.g.

1. Heusler alloys: ferromagnetic, but constituents are not ferromagnetic.
2. Stainless steel: not ferromagnetic, but composed almost exclusively of ferromagnetic metals.

Main source: the spin of the electrons, Pauli's exclusion principle (quantum mechanical).

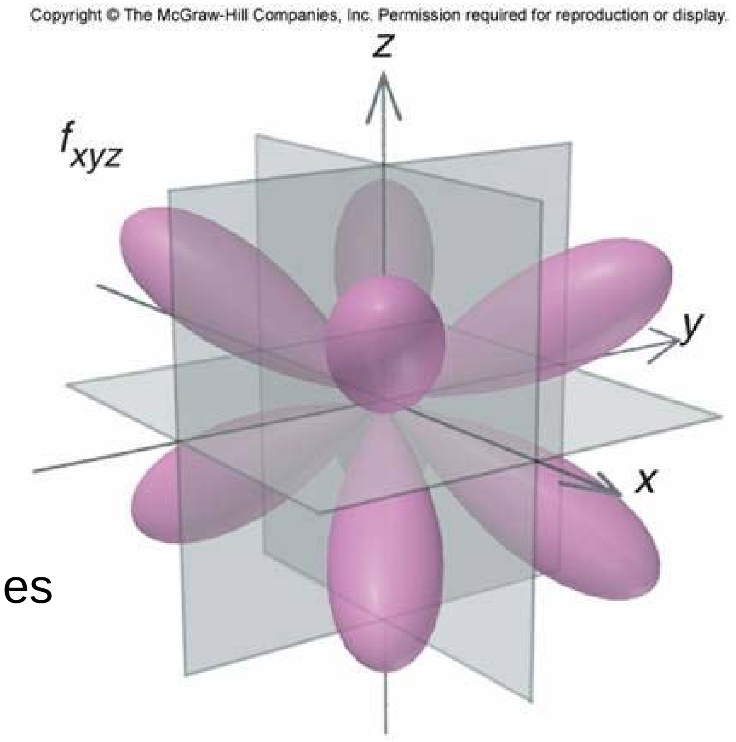
For atoms with fully filled shell (electrons are paired with up/down spins), no net dipole moment exists.



Ferromagnetic materials-3

For atoms with partially filled shell (unpaired electrons/spins exist), net dipole moment arise.

E.g. Lanthanide elements can carry up to 7 unpaired electrons in the 4f-orbitals



Quantum numbers:

n =1, 2, .., ~energy

l =0(s), 1(p), 2(d), 3(f), ..(n-1), .., ~angular momentum, orbital shape, # of nodal planes

m_l = -l, …, l, ~axis

m_s = 1/2, -1/2, ~spin

Unpaired dipoles tend to align in parallel to external M-field (classical model), $\Rightarrow$ paramagnetism

Unpaired dipoles tend to align spontaneously, (quantum mechanical effect), $\Rightarrow$ ferromagnetism

Exchange interaction:

Two electrons (fermions) from adjacent atoms with parallel spins will have lower system energy (more stable) than those with opposite spins.

Energy difference due to spin-spin interaction:

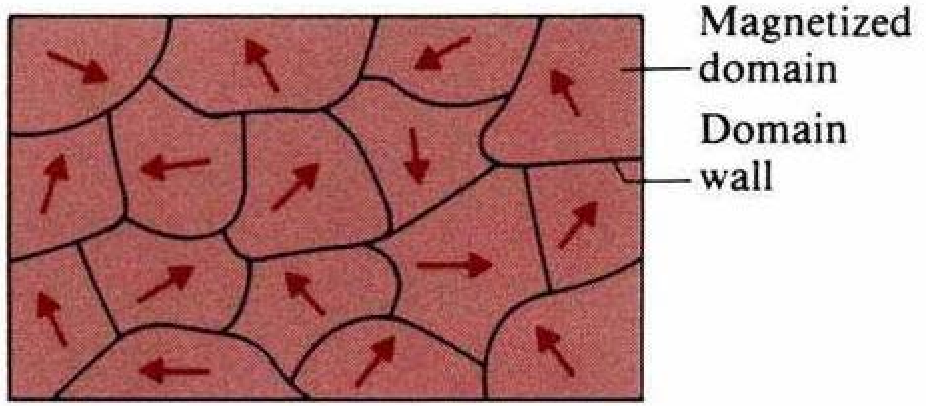
$$-J\underline{S_1 \cdot S_2} < 0$$

>0, if pure Coulomb interaction

>0, if parallel spins

Ferromagnetic materials-6

In iron, exchange (spin-spin) interaction is 1000 times stronger than dipole-dipole interaction, $\Rightarrow$ spontaneous alignment, magnetic domains.



At long distance, exchange interaction is overtaken by tendency of dipoles to anti-align, $\Rightarrow$ many randomly oriented domains.

Ferromagnetic materials-7

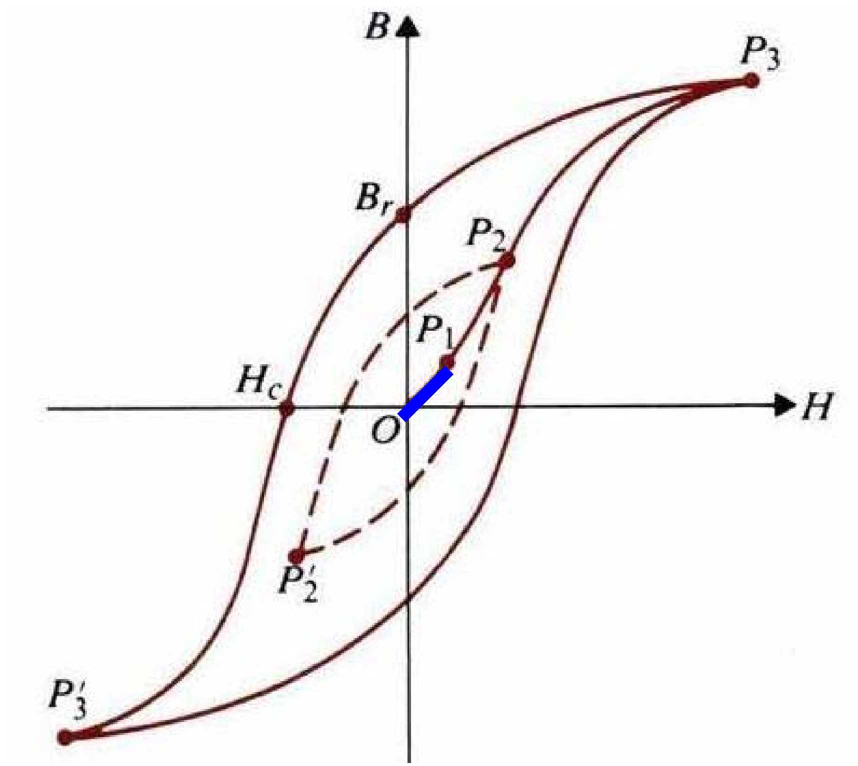
Placed in strong external M-field, domains will be aligned with the field.

The alignment remains after the field is turned off because the domain walls are pinned on defects in the lattice, $\Rightarrow$ permanent magnets.

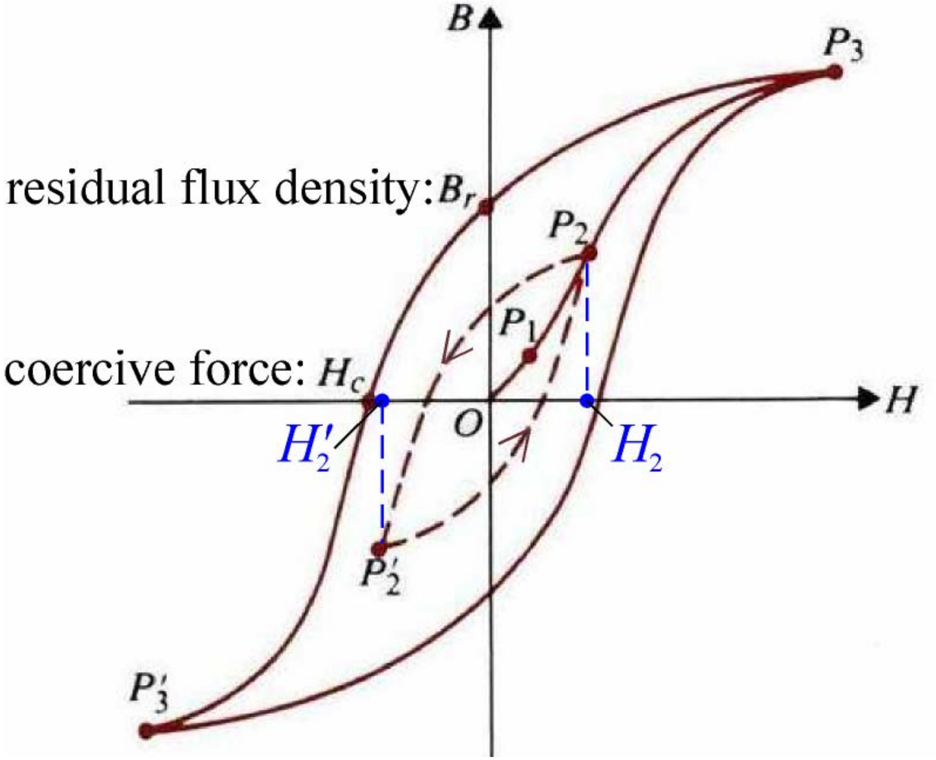
Domains are disorganized (demagnetized) if above Curie temperature (thermal energy > exchange interaction energy).

Hysteresis curve-1

1. Reversible magnetization: if external M-field is weak (up to P_1), domain wall movement is reversible, $\Rightarrow$ B-H curve is a function



2. Hysteresis: If external M-field is changed from H_2 to H'_2, B will change along the upper branch (from P_2 to P'_2). Conversely, along lower branch.



3. Saturation: If the external field is very strong (above P_3), all the domains are aligned.