Lesson 11 Magnetostatics in Free Space

■ Introduction

When a small test charge q travels with velocity $\vec{u}$ in a magnetic field characterized by a magnetic flux density $\vec{B}$ (Wb/m^2), it will experience a magnetic force $\vec{F}_m = q\vec{u}\times\vec{B}$. If there is an electric field $\vec{E}$ as well, the total electromagnetic force is governed by the Lorentz's force equation:

$$\vec{F} = q\left(\vec{E} + \vec{u}\times\vec{B}\right) \tag{11.1}$$

Historically, $\vec{B}$ is defined by measuring $\vec{F}_m$ and $\vec{u}$ experimentally. However, we will start with two fundamental postulates of magnetostatics in free space to define $\vec{B}$, from which all experimental laws and the concept of magnetic "potential" can be derived.

11.1 Fundamental Postulates

■ Definition and physical meaning

By Helmholtz's theorem (Lesson 5), $\vec{B}$ can be uniquely specified if its divergence and curl are given:

$$\nabla\cdot\vec{B} = 0 \tag{11.2}$$

$$\nabla\times\vec{B} = \mu_0\vec{J} \tag{11.3}$$

where $\vec{J}$ is the volume current density (A/m^2), and μ_0 is the permeability of vacuum. Eq's (11.2), (11.3) indicate that: (1) there is no "flow source" of magnetic field, (2) $\vec{J}$ acts as the "vortex source" of magnetic field, respectively.

■ Integral forms

By integrating both sides of eq (11.2) over a volume enclosed by surface S and applying the divergence theorem [eq. (5.24)], we derive:

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \qquad (11.4)$$

This means that the magnetic flux lines always close upon themselves, and there is no isolated "magnetic pole".

By integrating both sides of eq. (11.3) over a surface bounded by contour C and applying the Stokes' theorem [eq. (5.29)], we derive the Ampère's circuital law:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \qquad (11.5)$$

As the Gauss's law in electrostatics, eq. (11.5) relates the magnetic source (I) and field $\vec{B}$.

11.2 Ampère's Circuital Law

■ Definition and applications

If the current distribution has certain symmetry, such that the tangential component of $\vec{B}$ is constant over a contour C, eq. (11.5) becomes convenient in determining $\vec{B}$.

Example 11-1: Consider an infinitely long conducting wire with circular cross section of radius b, and carrying a steady current I(A) in the +z direction. Find $\vec{B}$ inside and outside the conductor.

Ans: By cylindrical symmetry, $\Rightarrow$ $\vec{B} = \begin{cases} \vec{a}_\phi B_{\phi 1}(r), \text{if } r < b \\ \vec{a}_\phi B_{\phi 2}(r), \text{if } r > b \end{cases}$. Choose a circle of radius r as the integral path C (Fig. 11-1a).

(1) For $r < b$: $\oint_{C_1} \vec{B} \cdot d\vec{l} = 2\pi r B_{\phi 1}(r) = \mu_0 \left(\frac{r}{b}\right)^2 I$, $\Rightarrow$ $B_{\phi 1}(r) = \frac{\mu_0 I}{2\pi b^2} r \, (\propto r)$.

(2) For $r > b$: $\oint_{C_2} \vec{B} \cdot d\vec{l} = 2\pi r B_{\phi 2}(r) = \mu_0 I$, $\Rightarrow$ $B_{\phi 2}(r) = \frac{\mu_0 I}{2\pi r} \left(\propto r^{-1}\right)$.

This example shows that a conducting wire creates a "circulating" magnetic field outside the wire itself:

$$\vec{B} = \vec{a}_\phi \frac{\mu_0 I}{2\pi r} \tag{11.6}$$

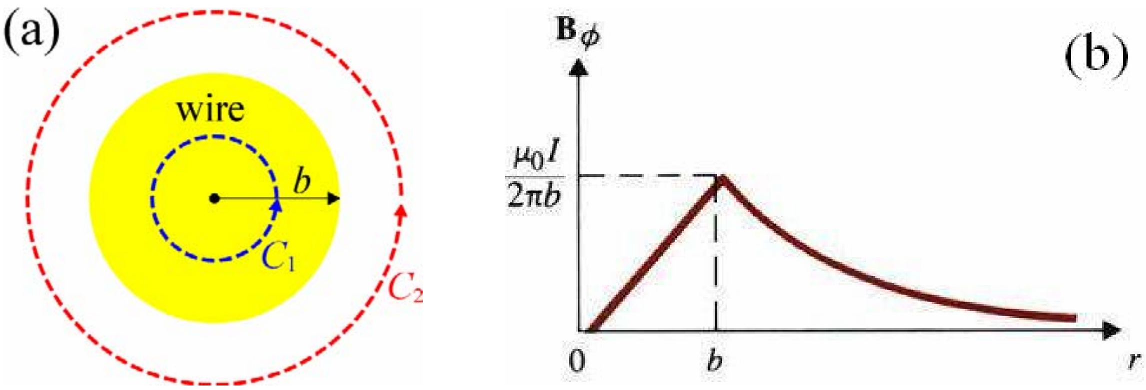

Fig. 11-1. (a) Cross-section of the conducting wire. (b) Magnitude of magnetic flux density (after DKC).

Example 11-2: Consider an infinitely long solenoid with air core, n turns per unit length, and carrying a steady current I. Find $\vec{B}$ inside the solenoid.

Ans: Since (1) $\vec{B} = 0$ outside the solenoid, (2) $\vec{B}$ is constant and in axial direction inside the solenoid, $\Rightarrow$ choose a rectangular contour C (Fig. 11-2). $\oint_C \vec{B} \cdot d\vec{l} = BL = \mu_0 (nL) I$, $\Rightarrow$

$$B = \mu_0 n I \tag{11.7}$$

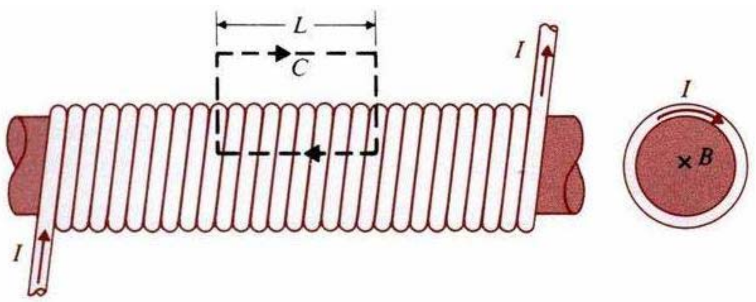

Fig. 11-2. A current-carrying solenoid (after DKC).

11.3 Vector Magnetic Potential

■ Definition and physical meaning

From the null identity of eq. (5.35) and the fundamental postulate of eq. (11.2), magnetic flux density $\vec{B}$ is divergence-free and can be expressed as the curl of some vector potential field:

$$\vec{B} = \nabla \times \vec{A} \tag{11.8}$$

The flux Φ of $\vec{B}$ over a given area S bounded by contour C is:

$$\Phi\left(\text{Wb}\right) = \int_S \vec{B} \cdot d\vec{s} = \int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l} \ .$$

The line integral of $\vec{A}$ over a contour C equals the total magnetic flux passing through the area bounded by C (physical meaning of $\vec{A}$).

Unlike scalar electric potential V, we need to specify the divergence of $\vec{A}$ (in addition to the curl of $\vec{A}$, specified by eq. (11.8)) to uniquely define the vector magnetic potential $\vec{A}$. In magnetostatics, we choose the Coulomb's gauge:

$$\nabla \cdot \vec{A} = 0 \tag{11.9}$$

Substitute eq's (11.3), (11.8), (11.9) into the definition of vector Laplacian eq. (5.31):

$$\nabla^2 \vec{A} \equiv \nabla\left(\nabla \cdot \vec{A}\right) - \nabla \times \nabla \times \vec{A} = 0 - \nabla \times \vec{B} = -\mu_0 \vec{J} \ ,$$

i.e., the vector magnetic potential $\vec{A}$ is satisfied with a vector Poisson's equation:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{11.10}$$

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Coulomb's gauge is chosen to simplify eq. (11.10). We will choose different gauges in time-varying fields (Lesson 15).

■ Evaluation

Eq. (11.10) can be solved by the following procedures:

(1) By eq. (5.33), eq. (11.10) is equivalent to three scalar Poisson's equations:

$$\nabla^2 A_i = -\mu_0 J_i \ (i = x, y, z)$$

in Cartesian coordinates.

(2) The solution to the scalar Poisson's equation $\nabla^2 V = -\dfrac{\rho_v}{\varepsilon_0}$ [eq. (8.1)] is:

$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_v(\vec r')}{R(\vec r, \vec r')} dv' \quad \text{[eq. (6.15)]}.$$

(3) By analogy, the solution to the scalar Poisson's equation $\nabla^2 A_i = -\mu_0 J_i$ is:

$$A_i(\vec r) = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_i(\vec r')}{R(\vec r, \vec r')} dv'.$$

(4) Combining *x*-, *y*-, *z*-components, ⇒

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec J(\vec r')}{R(\vec r, \vec r')} dv' \quad \text{(Wb/m)} \tag{11.11}$$

Example 11-3: Find the magnetic potential $\vec A$ and magnetic flux density $\vec B$ in the "bisecting plane $z = 0$" (not for the entire 3-D space) created by a straight current-carrying wire of length $2L$ and current I (Fig. 11-3a).

Ans: In cylindrical coordinates, an arbitrary observation point and a source point are located at $\vec r = (r, \phi, 0) = \vec a_r r$ ($\phi = 0 \sim 2\pi$), and $\vec r' = (0, \phi, z') = \vec a_z z'$ ($z' = -L \sim L$), respectively. ⇒ $R(\vec r, \vec r') = |\vec r - \vec r'| = \sqrt{r^2 + z'^2}$, $\vec J(\vec r') = \vec a_z \dfrac{I}{S}$ (S is the cross-sectional area of the wire), $dv' = S dz'$. By eq. (11.11), $\vec A(r, \phi, 0) = \vec a_z \dfrac{\mu_0 I}{4\pi}\left(\displaystyle\int_{-L}^{L} \frac{dz'}{\sqrt{r^2 + z'^2}}\right) = \vec a_z \dfrac{\mu_0 I}{2\pi} \ln\left[\dfrac{1 + \sqrt{1 + (r/L)^2}}{(r/L)}\right]$.

By eq. (11.8), $\vec B(r, \phi, 0) = \nabla \times (\vec a_z A_z) = \vec a_r \dfrac{\partial A_z}{\partial \phi} - \vec a_\phi \dfrac{\partial A_z}{\partial r} = \vec a_\phi \dfrac{\mu_0 I}{2\pi r\sqrt{1 + (r/L)^2}}$. Fig. 11-3b compares the magnetic fields caused by an infinitely wire [dashed, eq. (11.6)] and a finite wire (solid), respectively.

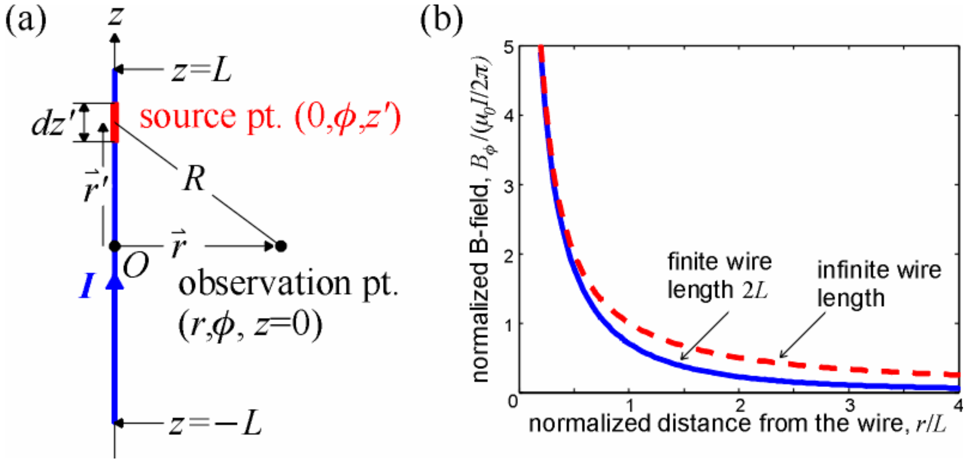

Fig. 11-3. (a) Geometry of a section of straight wire of length 2*L* carrying current *I*. (b) Normalized magnitude of magnetic flux density in the bisecting plane.

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1) Since eq. (11.11) is a vector integral, determining $\vec{B}$ by way of calculating $\vec{A}$ is less helpful in comparison with determining $\vec{E}$ by way of calculating scalar potential V .

2) $\vec{B}(r,\phi,0) \rightarrow \vec{a}_\phi \dfrac{\mu_0 I}{2\pi r}$ when $r << L$, consistent with that derived in Example 11-1.

11.4 Biot-Savart Law

■ Magnetic field created by current loops

For a closed current loop C' made by thin wire carrying a current I (typical magnetic source), the term $\vec{J}(\vec{r}\,')dv'$ in eq. (11.11) becomes $Id\vec{l}\,'$, $\Rightarrow$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')} \tag{11.12}$$

The resulting magnetic flux density $\vec{B}$ is governed by Biot-Savart law:

$$\vec{B} = \oint_{C'} d\vec{B}\,, \quad d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}, \tag{11.13}$$

where $\vec{a}_R(\vec{r},\vec{r}\,') = \dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$ is the unit vector in the direction from the source point $\vec{r}\,'$ to the observation point $\vec{r}$, $R(\vec{r},\vec{r}')=|\vec{r}-\vec{r}'|$, $d\vec{B}$ is the magnetic field contributed by a differential current segment.

Proof:

(1) By eq. (11.12), $\vec{B}=\nabla\times\vec{A}=\frac{\mu_0 I}{4\pi}\nabla\times\left[\oint_{C'}\frac{d\vec{l}\,'}{R(\vec{r},\vec{r}')}\right]=\frac{\mu_0 I}{4\pi}\oint_{C'}\nabla\times\left[\frac{d\vec{l}\,'}{R(\vec{r},\vec{r}')}\right]$.

(2) By vector identify: $\nabla\times\left(R^{-1}d\vec{l}\,'\right)=R^{-1}\left(\nabla\times d\vec{l}\,'\right)+\nabla\left(R^{-1}\right)\times d\vec{l}\,'$.

(3) $\nabla\times d\vec{l}\,'=0$, for $\nabla\times$ is performed with respect to $\vec{r}$ while $d\vec{l}\,'$ only changes with $\vec{r}\,'$ (analogous to the fact: $\frac{d}{dy}f(x)=0$).

(4) $R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, $\nabla\left(R^{-1}\right)=\vec{a}_x\frac{\partial R^{-1}}{\partial x}+\vec{a}_y\frac{\partial R^{-1}}{\partial y}+\vec{a}_z\frac{\partial R^{-1}}{\partial z}=-\vec{a}_R R^{-2}$. $\Rightarrow$ Eq. (11.13) is derived.

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1) Eq. (11.13) allows for direct evaluation of $\vec{B}$ from current distribution without first calculating $\vec{A}$. Its counterpart in electrostatics is the Coulomb's law [eq. (6.10)].

2) In non-Cartesian coordinates, pay attention to the "position-dependent" unit vectors (e.g. $\vec{a}_R\neq\vec{a}_R'$).

Example 11-4: Find the magnetic flux density $\vec{B}$ "on the axis" of a circular loop (not for the entire 3-D space) of radius b and carrying a current I (Fig. 11-4a).

Ans: (1) In cylindrical coordinates, an arbitrary on-axis observation point P and a source point on the loop C' are located at: $\vec{r}=(0,0,z)=\vec{a}_z z$ and $\vec{r}\,'=(b,\phi',0)=\vec{a}_r' b$, respectively.

$\Rightarrow$ The displacement vector from the source point to the observation point is: $\vec{r}-\vec{r}\,'=\vec{a}_R R$, where $R=\sqrt{z^2+b^2}$, $\vec{a}_R=\frac{\vec{a}_z z-\vec{a}_r' b}{R}$.

(2) The differential displacement of the source point is: $d\vec{l}\,'=\vec{a}_\phi' b\,d\phi'$, $\Rightarrow$

$$d\vec{l}\,'\times\vec{a}_R=\left(\vec{a}_\phi'bd\phi'\right)\times\frac{\vec{a}_z z-\vec{a}_r'b}{R}=\frac{bd\phi'}{R}\left(\vec{a}_r'z+\vec{a}_z b\right)$$

(3) By eq. (11.13),

$$\vec{B}(0,0,z)=\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\vec{l}\,'\times\vec{a}_R}{R^2}=\frac{\mu_0 Ib}{4\pi R^3}\left[z\int_0^{2\pi}\vec{a}_r'(\phi')d\phi'+b\vec{a}_z\int_0^{2\pi}d\phi'\right]=\vec{a}_z\frac{\mu_0 I}{2b}\left[1+(z/b)^2\right]^{-3/2}.$$

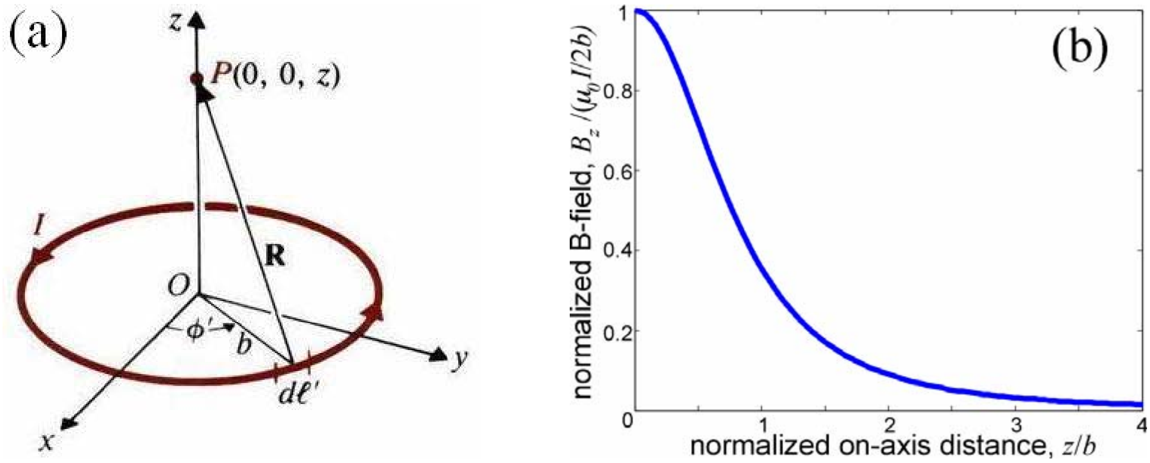

Fig. 11-4. (a) A circular loop of radius b carrying current I (after DKC). (b) Normalized magnitude of magnetic flux density on the z-axis.

11.5 Magnetic Dipole

A circular current-carrying loop forms a magnetic dipole (Fig. 11-5). The resulting $\vec{A}$ and $\vec{B}$ at any position $\vec{r}=\vec{a}_R R$ (not just on the loop axis) far away from the loop ($R>>b$) are:

$$\vec{A}(\vec{r})\approx\frac{\mu_0\vec{m}\times\vec{a}_R}{4\pi R^2} \qquad (11.14)$$

$$\vec{B}(\vec{r})\approx\frac{\mu_0 m}{4\pi R^3}\left[\vec{a}_R 2\cos\theta+\vec{a}_\theta\sin\theta\right] \qquad (11.15)$$

where $\vec{m}=\vec{a}_z I\pi b^2$ represents the magnetic dipole moment. In general, any current loop forms a magnetic dipole, where the dipole moment $\vec{m}$ has a magnitude equal to the product of current I and loop area S, and is in the direction of right thumb as the remaining four fingers follow the direction of current flow:

$$\vec{m}=\vec{a}_m IS \qquad (11.16)$$

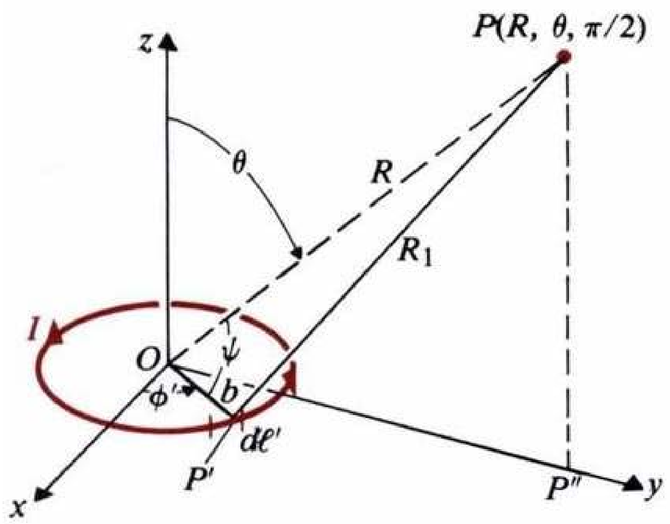

Fig. 11-5. Evaluation of far-fields generated by a magnetic dipole (after DKC).

Proof: (1) The geometry has $\phi-$symmetry in spherical coordinates. It is sufficient to consider an observation point on the *yz*-plane ($\phi=\pi/2$) with position vector:

$$\vec{r}=(R,\theta,\pi/2)=R\vec{a}_R(\theta,\pi/2)=R(\vec{a}_y\sin\theta+\vec{a}_z\cos\theta).$$

An arbitrary source point on the current loop is located at:

$$\vec{r}'=(b,\pi/2,\phi')=b\vec{a}'_R(\pi/2,\phi')=b(\vec{a}_x\cos\phi'+\vec{a}_y\sin\phi'),\quad \phi'=0\sim 2\pi.$$

The displacement vector from the source point to the observation point is:

$$\vec{R}_1=\vec{r}-\vec{r}'=R\vec{a}_R(\theta,\pi/2)-b\vec{a}'_R(\pi/2,\phi'),\Rightarrow\ R_1^2=\vec{R}_1\cdot\vec{R}_1=R^2+b^2-2Rb(\vec{a}_R\cdot\vec{a}'_R),$$

where $\vec{a}_R\cdot\vec{a}'_R=\cos\psi=(\vec{a}_y\sin\theta+\vec{a}_z\cos\theta)\cdot(\vec{a}_x\cos\phi'+\vec{a}_y\sin\phi')=\sin\theta\sin\phi'$, $\Rightarrow$

$$R_1^{-1}=(R^2+b^2-2Rb\sin\theta\sin\phi')^{-1/2}\approx\frac{1}{R}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right),\ \text{given}\ \ R>>b.$$

The differential displacement of the source point is:

$$d\vec{l}'=\vec{a}'_\phi bd\phi'=bd\phi'(-\vec{a}_x\sin\phi'+\vec{a}_y\cos\phi').$$

(2) By eq. (11.11), $\vec{A}(\vec{r})=\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\vec{l}'}{R_1}\approx\frac{\mu_0 Ib}{4\pi R}\oint_{C'}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right)(-\vec{a}_x\sin\phi'+\vec{a}_y\cos\phi')d\phi'$

$$\approx\frac{\mu_0 Ib}{4\pi R}\left[-\vec{a}_x\int_0^{2\pi}\left(\sin\phi'+\frac{b}{R}\sin\theta\sin^2\phi'\right)d\phi'+\vec{a}_y\int_0^{2\pi}\left(\cos\phi'+\frac{b}{R}\sin\theta\sin\phi'\cos\phi'\right)d\phi'\right]$$

$$=\frac{\mu_0 Ib}{4\pi R}\left(-\vec{a}_x\frac{b}{R}\sin\theta\cdot\pi\right)=-\vec{a}_x\frac{\mu_0 I\pi b^2\sin\theta}{4\pi R^2}.$$

Since $\vec{m} = \vec{a}_z I\pi b^2$, $\vec{m}\times\vec{a}_R(\theta,\pi/2) = (\vec{a}_z I\pi b^2)\times(\vec{a}_y \sin\theta + \vec{a}_z \cos\theta) = -\vec{a}_x I\pi b^2 \sin\theta$, $\Rightarrow$

$$\vec{A}(\vec{r}) = \frac{\mu_0 \vec{m}\times\vec{a}_R}{4\pi R^2},$$

as predicted by eq. (11.14). For a general observation point ($\phi \neq \pi/2$), $-\vec{a}_x = \vec{a}_\phi(\theta,\pi/2)$ is generalized to $\vec{a}_\phi(\theta,\phi)$, $\Rightarrow$ $\vec{A}(\vec{r}) = \vec{a}_\phi A_\phi$, where $A_\phi(R,\theta) = \frac{\mu_0 m \sin\theta}{4\pi R^2}$.

(3) By eq. (11.8) and the curl formula in spherical coordinates,

$$\vec{B}(\vec{r}) = \nabla\times(\vec{a}_\phi A_\phi) \approx \vec{a}_R \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \vec{a}_\theta \frac{1}{R}\frac{\partial}{\partial R}(A_\phi R).$$

(i) $$\frac{\partial}{\partial\theta}(A_\phi \sin\theta) = \frac{\mu_0 m}{4\pi R^2}\frac{\partial}{\partial\theta}(\sin^2\theta) = \frac{\mu_0 m}{4\pi R^2}\sin 2\theta$$

(ii) $$\frac{\partial}{\partial R}(A_\phi R) = \frac{\mu_0 m \sin\theta}{4\pi}\frac{\partial}{\partial R}\left(\frac{1}{R}\right) = -\frac{\mu_0 m \sin\theta}{4\pi R^2}$$

$$\Rightarrow \vec{B}(\vec{r}) \approx \vec{a}_R \frac{1}{R\sin\theta}\frac{\mu_0 m}{4\pi R^2}\sin 2\theta + \vec{a}_\theta \frac{1}{R}\frac{\mu_0 m \sin\theta}{4\pi R^2} = \frac{\mu_0 m}{4\pi R^3}[\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta],$$

as predicted by eq. (11.15).

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Resemblance between eq's (11.14), (6.18) and eq's (11.15), (6.16) implies that the "far-field" patterns established by electric and magnetic dipoles are similar if the electric dipole moment is replaced by the magnetic dipole moment ($\vec{p}\to\vec{m}$) and permittivity is replaced by the inverse of permeability ($\varepsilon_0 \to \frac{1}{\mu_0}$).

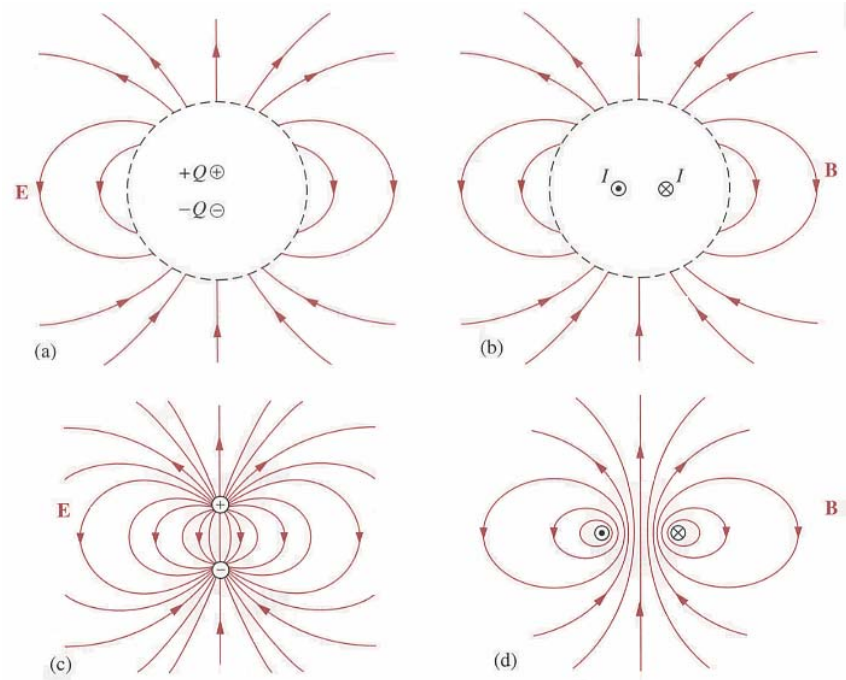

Fig. 11-6. (a) E-field lines far away from an electric dipole. (b) M-field lines far away from a magnetic dipole. (c) Close-up view of (a). (d) Close-up view of (b) (after Inans).