Lesson 11
Magnetostatics in Free Space

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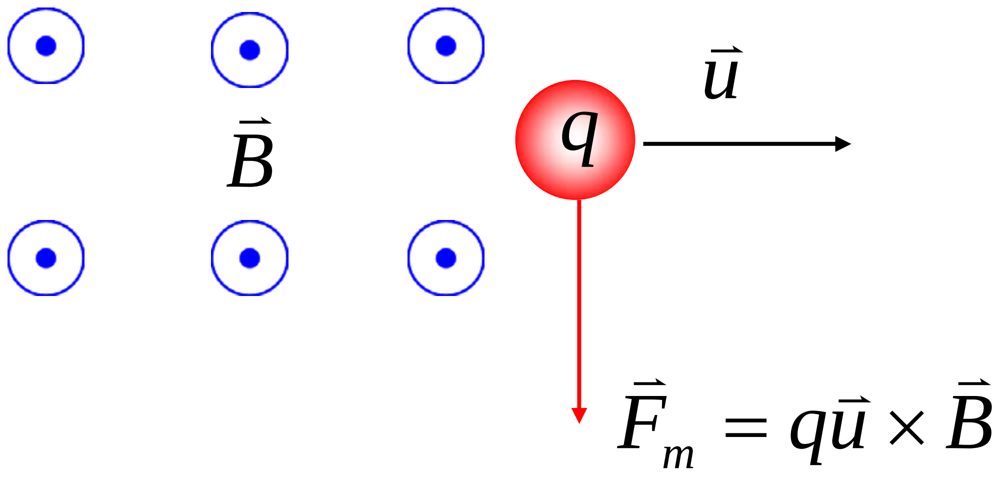
- Lorentz's force equation:

$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$



Introduction-2

- Historically, $\vec{B}$ is defined by measuring $\vec{u}$ and $\vec{F}_m$ experimentally.
- We will start with two fundamental postulates to define $\vec{B}$.
- Derive all experimental laws and the concept of vector magnetic potential.

Outline

- Fundamental postulates
- Ampere’s circuital law
- Vector magnetic potential
- Biot-Savart law
- Magnetic dipole

Sec. 11-1
Fundamental Postulates

1. Differential forms
2. Integral forms

Definition

Helmholtz's theorem: $\vec{B}$ can be uniquely defined by specifying its divergence and curl:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

permeability of vacuum

Physical meanings

$\nabla \cdot \vec{B} = 0$ …no flow source

$\nabla \times \vec{B} = \mu_0 \vec{J}$ … $\vec{J}$ acts as vortex source

Integral form-1

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \oint_S \vec{A} \cdot d\vec{s} = \int_V \left(\nabla \cdot \vec{A}\right) dv \end{cases} \longrightarrow \boxed{\oint_S \vec{B} \cdot d\vec{s} = 0}$$

The magnetic flux lines always close upon themselves, there is no isolated “magnetic pole” .

Integral form-2

$$\begin{cases} \nabla \times \vec{B} = \mu_0 \vec{J} \\ \oint_C \vec{A} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{A} \right) \cdot d\vec{s} \end{cases} \longrightarrow \boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I}$$

Ampere's circuital law

The counterpart of *Gauss's law* $\oint_S \vec{E} \cdot d\vec{s} = Q/\varepsilon_0$ in magnetostatics, relating the source (current I) and field ($\vec{B}$).

Sec. 11-2
Ampere’s Circuital Law

1. Definition
2. Examples

Definition

- If the current distribution has certain symmetry, s.t. the tangential component of $\vec{B}$ is constant over a contour C, $\Rightarrow$

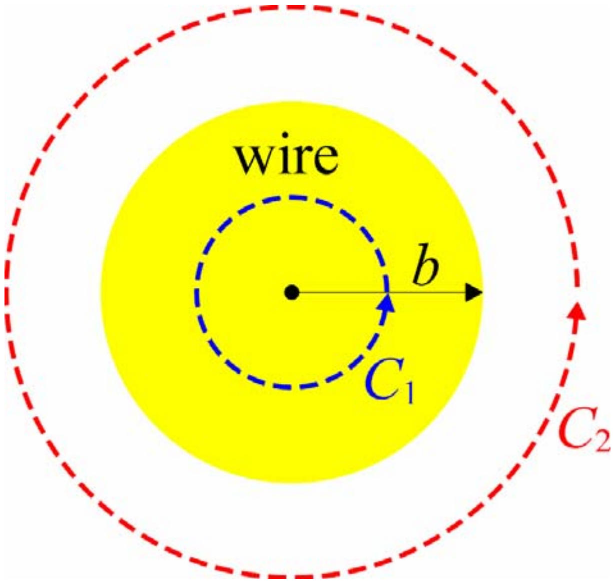
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

becomes convenient to determine $\vec{B}$

- If the charge distribution has certain symmetry, s.t. the normal component of $\vec{E}$ is constant over a Gaussian surface S, $\Rightarrow$ $\oint_S \vec{E} \cdot d\vec{s} = Q/\varepsilon_0$ becomes convenient to get $\vec{E}$

Example 11-1: Long conducting wire (1)

Consider an infinitely long conducting wire with circular cross section, and carrying a steady current I. Find $\vec{B}$



Cylindrical symmetry:

$$\vec{B} = \begin{cases} \vec{a}_\phi B_{\phi 1}(r), \text{if } r < b \\ \vec{a}_\phi B_{\phi 2}(r), \text{if } r > b \end{cases}$$

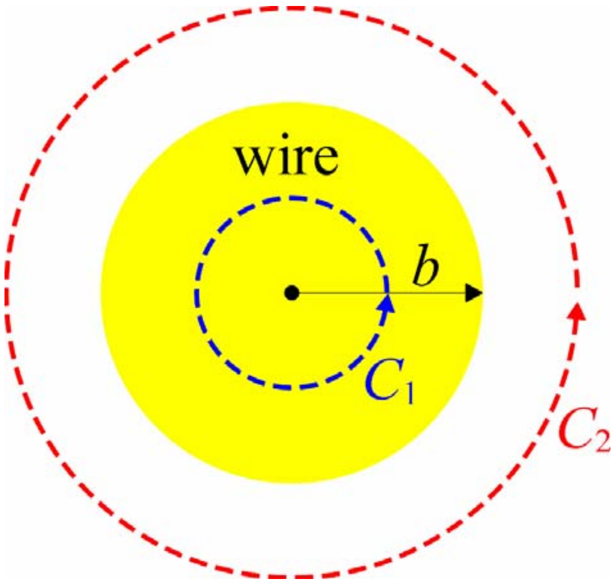
Choose a concentric circle of radius r as the integral path

Example 11-1: Long conducting wire (2)

(1) For $r < b$:

$$\oint_{C_1} \vec{B} \cdot d\vec{l} = 2\pi r B_{\phi 1}(r) = \mu_0 \left(\frac{r}{b} \right)^2 I$$

$$\Rightarrow B_{\phi 1}(r) = \frac{\mu_0 I}{2\pi b^2} r$$

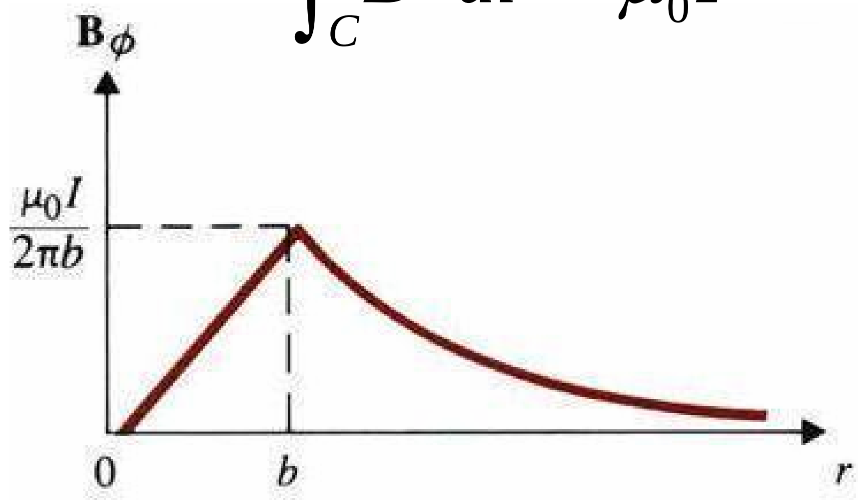


$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

(2) For $r > b$:

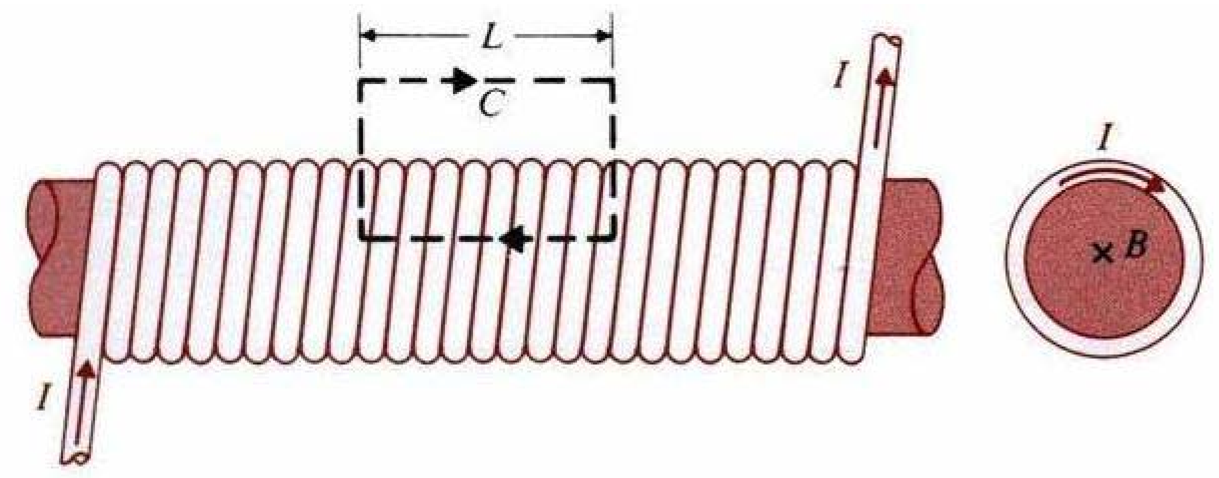
$$\oint_{C_2} \vec{B} \cdot d\vec{l} = 2\pi r B_{\phi 2}(r) = \mu_0 I$$

$$\Rightarrow \boxed{B_{\phi 2}(r) = \frac{\mu_0 I}{2\pi r}}$$



Example 11-2: Solenoid

Consider an infinitely long solenoid with air core, n turns per unit length, carrying a steady current I. Find $\vec{B}$ inside the solenoid



$$\begin{cases} \vec{B} = 0 & \text{outside} \\ \vec{B} = \vec{a}_z B & \text{inside} \end{cases} \longrightarrow \oint_C \vec{B} \cdot d\vec{l} = BL = \mu_0 (nL) I, \quad \boxed{B = \mu_0 n I}$$

Sec. 11-3
Vector Magnetic Potential

1. Definition
2. Vector Poisson equation
3. Vector magnetic potential due to current distributions
4. Finite current-carrying wires

Definition

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \cdot \left(\nabla \times \vec{A} \right) = 0 \end{cases} \longrightarrow \vec{B} = \nabla \times \vec{A}$$

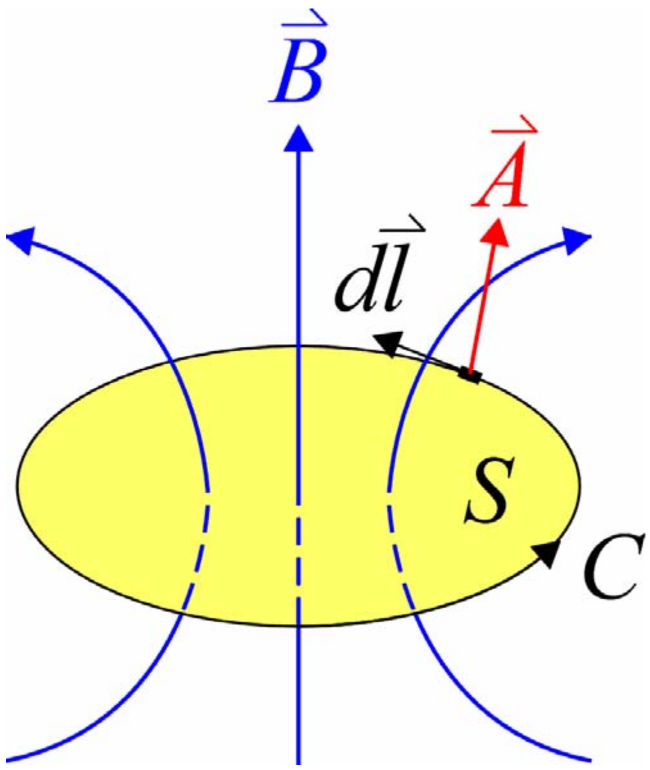
vector potential

Physical meaning

The flux Φ of $\vec{B}$ over a given area S bounded by contour C is:

$$\Phi(\text{Wb}) = \int_S \vec{B} \cdot d\vec{s}$$

$$= \int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$



Poisson's equation-1

$$\begin{cases} \nabla \times \vec{A} = \vec{B} \\ \\ \nabla \cdot \vec{A} = ? \end{cases}$$

Need to specify the divergence of $\vec{A}$ to uniquely define the vector field $\vec{A}$

In principle, specification of $\nabla \cdot \vec{A}$ is arbitrary (no physical constraint)

In reality, choose $\boxed{\nabla \cdot \vec{A} = 0}$ (Coulomb's gauge) to derive simple formula

Poisson's equation-2

$$\begin{cases} \nabla \cdot \vec{A} = 0 \\ \nabla \times \vec{A} = \vec{B} \\ \nabla \times \vec{B} = \mu_0 \vec{J} \end{cases} \longrightarrow \begin{aligned} \nabla^2 \vec{A} &\equiv \nabla\left(\nabla \cdot \vec{A}\right) - \nabla \times \nabla \times \vec{A} \\ &= 0 - \nabla \times \vec{B} = -\mu_0 \vec{J} \end{aligned}$$

Vector magnetic potential $\vec{A}$ satisfies vector Poisson's equation:

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

Poisson's equation-3

Scalar electric potential V satisfies scalar Poisson's equation:

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{D} = \varepsilon\left(\nabla \cdot \vec{E}\right) = \rho, \quad \Rightarrow \nabla \cdot \underline{\vec{E}} = \frac{\rho}{\varepsilon}$$

$$\nabla^2 V \equiv \nabla \cdot \left(\nabla V\right)$$

$$\Rightarrow \nabla \cdot \vec{E} = -\underline{\nabla \cdot \left(\nabla V\right)} = \frac{\rho}{\varepsilon}$$

$$\Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

Evaluation-1

In Cartesian coordinates:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$ …1 vector Poisson's equation

$$\Rightarrow \begin{cases} \vec{a}_x\left(\nabla^2 A_x\right) + \vec{a}_y\left(\nabla^2 A_y\right) + \vec{a}_z\left(\nabla^2 A_z\right) \\ = \vec{a}_x\left(\mu_0 J_x\right) + \vec{a}_y\left(\mu_0 J_y\right) + \vec{a}_z\left(\mu_0 J_z\right) \end{cases}$$

$$\Rightarrow \boxed{\nabla^2 A_i = -\mu_0 J_i} \; (i = x, y, z)$$

…3 independent scalar Poisson's equations

Evaluation-2

Solution to $\nabla^2 V = -\varepsilon_0^{-1}\rho_v$ is:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_v(\vec{r}\,')}{R(\vec{r},\vec{r}\,')} dv'$$

find potential by integration of source charge

Solution to $\nabla^2 A_i = -\mu_0 J_i$ is:

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_i(\vec{r}\,')}{R(\vec{r},\vec{r}\,')} dv'$$

Evaluation-3

Solution to $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ is:

$$\vec{A} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

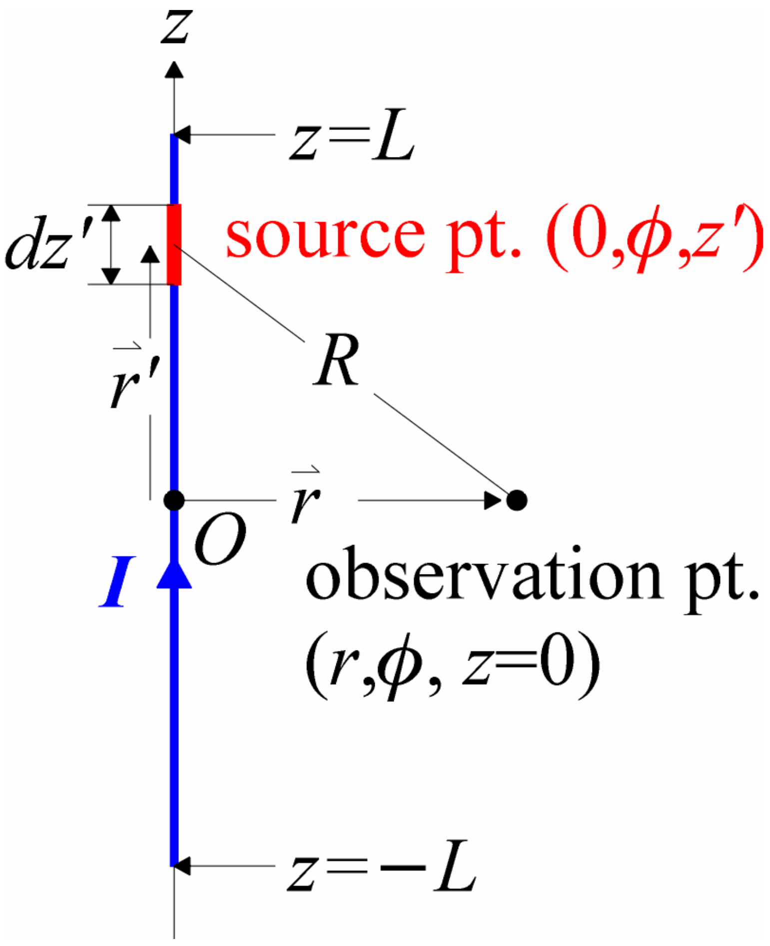
$$A_i(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_i(\vec{r}\,')}{R(\vec{r}, \vec{r}\,')} dv'$$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}\,')}{R(\vec{r}, \vec{r}\,')} dv'}$$

find potential by integration of source current

Example 11-3: Finite current-carrying wire (1)

Find $\vec{A}$ and $\vec{B}$ in the bisecting plane ($z=0$) due to a straight wire of length $2L$ carrying current I:



In cylindrical coordinates:

$$\vec{r} = (r, \phi, 0) = \vec{a}_r r \quad \phi: 0 \sim 2\pi$$

$$\vec{r}' = (0, \phi, z') = \vec{a}_z z' \quad z': -L \sim L$$

$$R(\vec{r}, \vec{r}') = |\vec{r} - \vec{r}'| = \sqrt{r^2 + z'^2}$$

$$\vec{J}(\vec{r}') = \vec{a}_z \frac{I}{S}, \quad dv' = Sdz'$$

Example 11-3: Finite current-carrying wire (2)

By $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}\,')dv'}{R(\vec{r},\vec{r}\,')}$

$\vec{J}(\vec{r}\,')dv' \rightarrow \vec{a}_z I dz'$

$R(\vec{r},\vec{r}\,') \rightarrow \sqrt{r^2 + z'^2}$

$$\vec{A}(r,\phi,z=0) = \vec{a}_z \frac{\mu_0 I}{4\pi}\left(\int_{-L}^{L} \frac{dz'}{\sqrt{r^2+z'^2}}\right)$$

$$= \vec{a}_z \frac{\mu_0 I}{2\pi} \ln\left[\frac{1+\sqrt{1+(r/L)^2}}{(r/L)}\right]$$

$$A_z(r,\phi,z=0)$$

Example 11-3: Finite current-carrying wire (3)

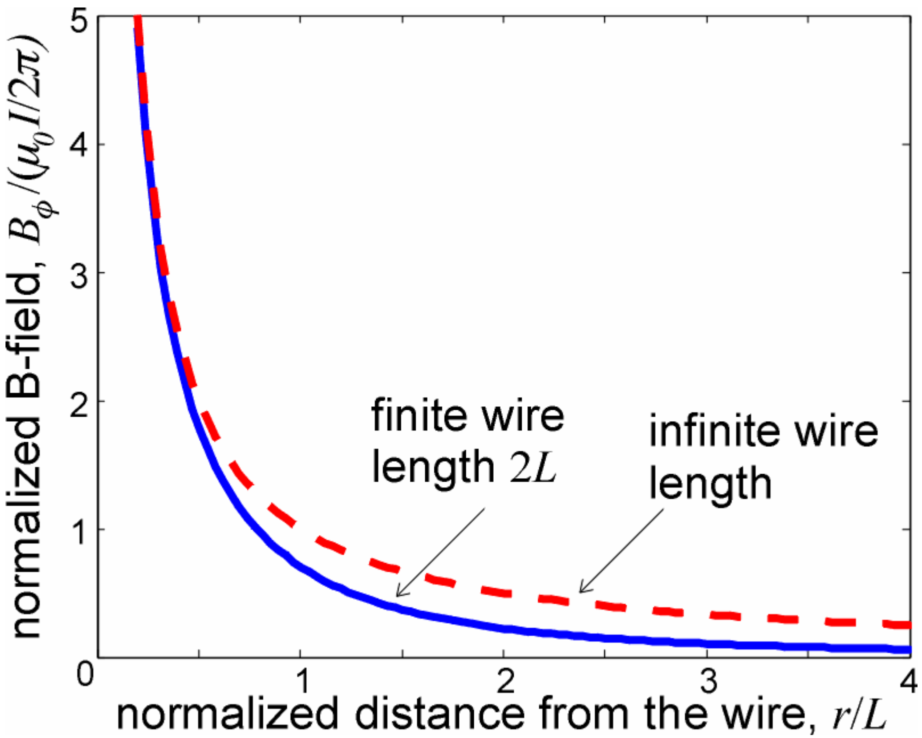
By $\vec{B} = \nabla \times \vec{A}$

$$\Rightarrow \vec{B}(r,\phi,0) = \nabla \times \left(\vec{a}_z A_z\right) = \vec{a}_r \frac{\partial A_z}{\partial \phi} - \vec{a}_\phi \frac{\partial A_z}{\partial r}$$

$$= \vec{a}_\phi \boxed{\frac{\mu_0 I}{2\pi r} \frac{1}{\sqrt{1+\left(r/L\right)^2}}}$$

$B_\phi(r)$

magnetic field due to infinite wire



Comment

1. Since $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}')dv'}{R(\vec{r},\vec{r}')}$ is also a vector integral, determine $\vec{B}$ by way of $\vec{A}$ is less helpful (vs. determine $\vec{E}$ by V)

2. $\vec{B}(r,\phi,0) \rightarrow \vec{a}_\phi \frac{\mu_0 I}{2\pi r}$ if $r << L$

Sec. 11-4
Biot-Savart Law

1. Magnetic field due to current loops
2. On-axis magnetic field due to a circular current loop

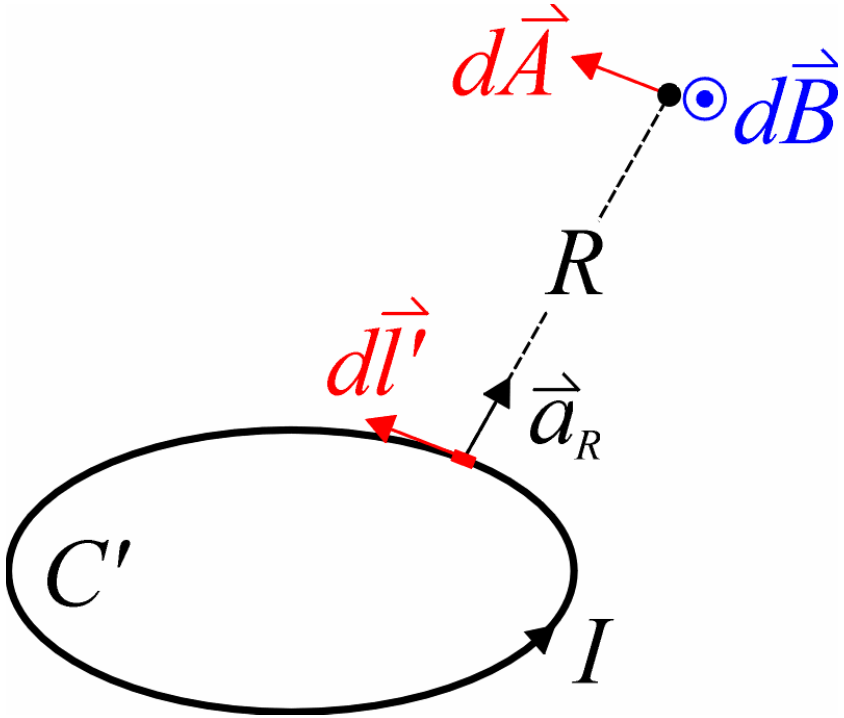
Magnetic field created by current loops-1

For a closed loop C' carrying a current I, $\Rightarrow$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}\,')dv'}{R(\vec{r},\vec{r}\,')} \longrightarrow \vec{J}\left(Sdl'\right) = Id\vec{l}\,'$$

$$= \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')}$$

$$\vec{B} = \oint_{C'} \underbrace{\frac{\mu_0 I}{4\pi}\frac{d\vec{l}\,'\times\vec{a}_R}{R^2}}_{d\vec{B}}$$



Proof:

(1) $\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')}, \quad \Rightarrow \vec{B} = \nabla \times \vec{A}$

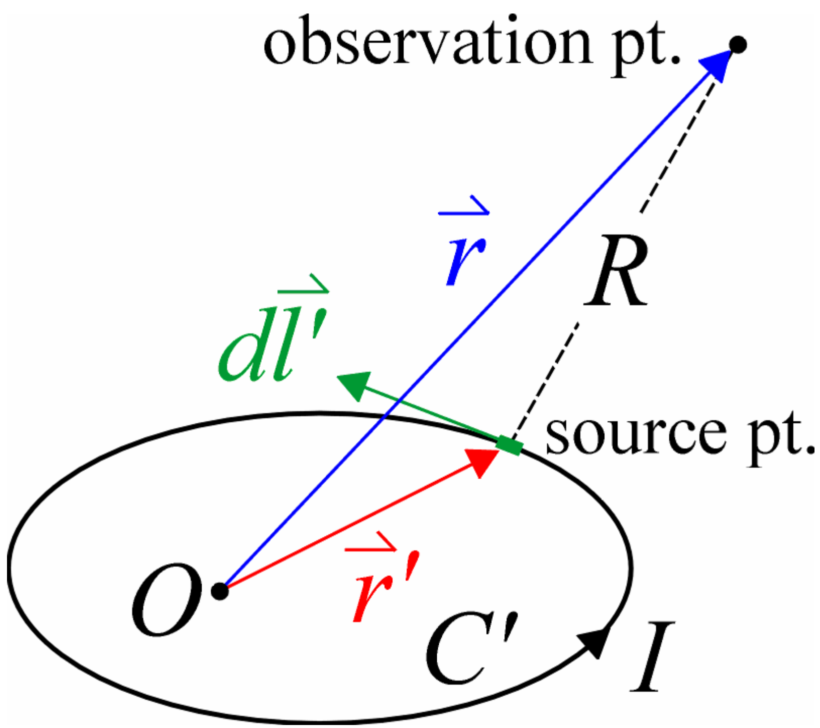
$$= \frac{\mu_0 I}{4\pi} \nabla \times \left[\oint_{C'} \frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')} \right] = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left[\frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')} \right]$$

(2) $\nabla \times \left(R^{-1} d\vec{l}\,' \right) = R^{-1} \left(\nabla \times d\vec{l}\,' \right) + \nabla \left(R^{-1} \right) \times d\vec{l}\,'$

Magnetic field created by current loops-3

(3) $\underset{\vec{r}}{\nabla} \times \underset{\vec{r}\,'}{d\vec{l}\,'} = 0$

$$\left(\text{as if } \frac{d}{dy} f(x) = 0 \right)$$



(4) $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$

$$\Rightarrow \underset{\vec{r}}{\nabla}\left(R^{-1}\right) = \vec{a}_x \frac{\partial R^{-1}}{\partial x} + \vec{a}_y \frac{\partial R^{-1}}{\partial y} + \vec{a}_z \frac{\partial R^{-1}}{\partial z} = -\vec{a}_R R^{-2}$$

Magnetic field created by current loops-4

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left[\frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')} \right]$$

$$-\vec{a}_R R^{-2}$$

$$= R^{-1}\left(\nabla \times d\vec{l}\,'\right) + \nabla\left(R^{-1}\right) \times d\vec{l}\,'$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{-\vec{a}_R \times d\vec{l}\,'}{R^2} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}$$

…Biot-Savart law

Comment

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}$$ …Biot-Savart law

Directly find magnetic field from current distribution

without first calculating vector potential $\vec{A}$

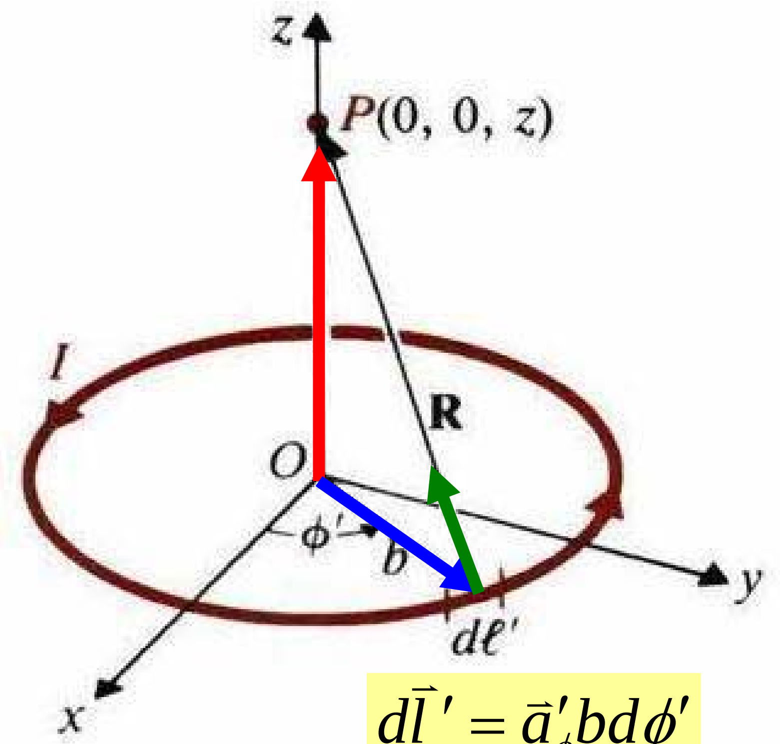
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \vec{a}_R \frac{\rho_v(\vec{r}\,')}{R^2} dv'$$ …Coulomb's law

Directly find electric field from charge distribution

without first calculating scalar potential V

Example 11-4: Circular current-carrying loop (1)

Find $\vec{B}$ on the z-axis due to a circular loop of radius b carrying current I:



$d\vec{l}' = \vec{a}'_\phi b d\phi'$

In cylindrical coordinates:

$$\vec{r} = (0,0,z) = \vec{a}_z z$$

$-\infty \sim \infty$

$$\vec{r}' = (b,\phi',0) = \vec{a}'_r b$$

$0 \sim 2\pi$

$$R = \sqrt{z^2 + b^2}, \quad \vec{a}_R = \frac{\vec{a}_z z - \vec{a}'_r b}{R}$$

Example 11-4: Circular current-carrying loop (2)

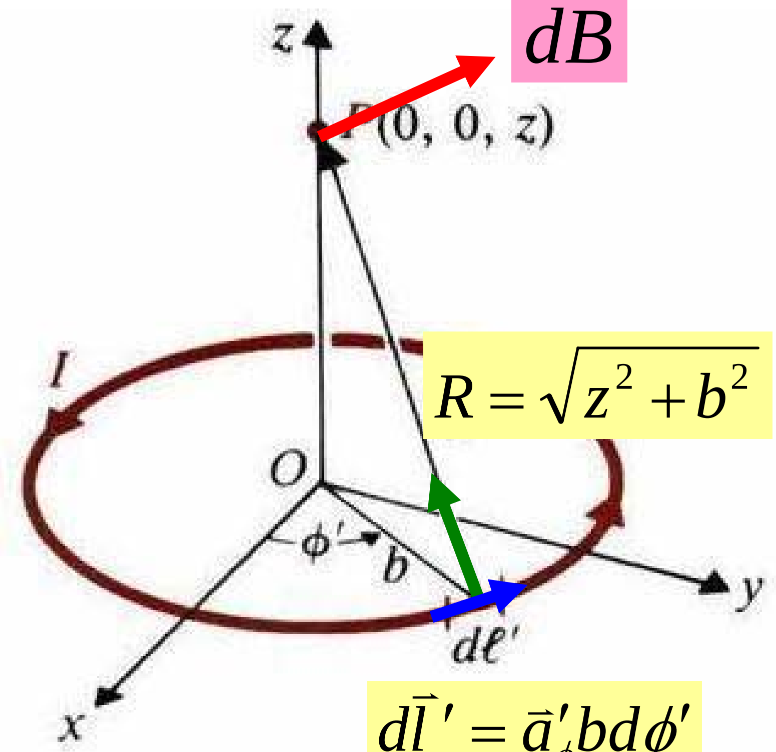
$$\vec{B} = \oint_{C'} \frac{\mu_0 I}{4\pi} \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}$$

$$= \left(\vec{a}_\phi' b\, d\phi'\right) \times \frac{\vec{a}_z z - \vec{a}_r' b}{R}$$

$$= \frac{b\, d\phi'}{R}\left[\left(\vec{a}_\phi' \times \vec{a}_z\right) z - \left(\vec{a}_\phi' \times \vec{a}_r'\right) b\right]$$

$$= \frac{b\, d\phi'}{R}\left(\vec{a}_r' z + \vec{a}_z b\right)$$

$d\vec{B}$



$$R = \sqrt{z^2 + b^2}$$

$$d\vec{l}\,' = \vec{a}_\phi' b\, d\phi'$$

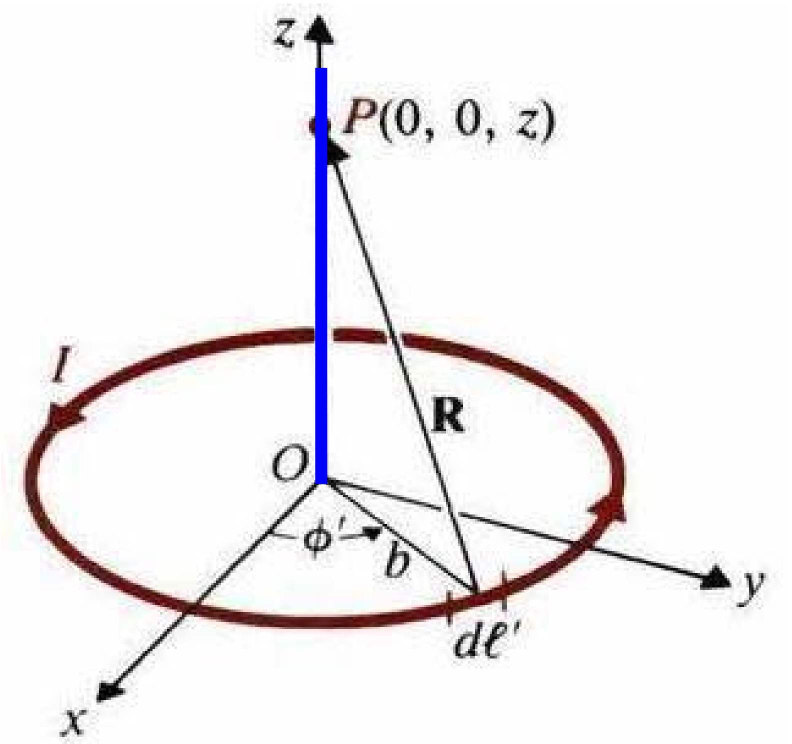
$$\vec{B}(0,0,z)$$

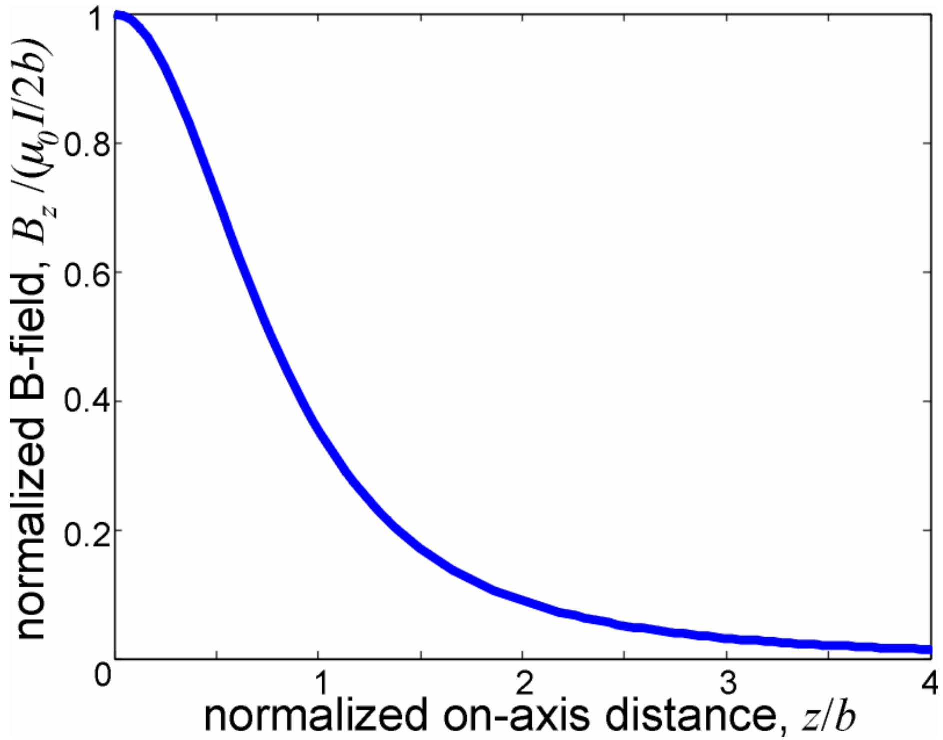
$$= \frac{\mu_0 I b}{4\pi R^3}\left[z \int_0^{2\pi} \vec{a}_r'(\phi')\, d\phi' + b\vec{a}_z \int_0^{2\pi} d\phi'\right]$$

$$= \vec{a}_z \frac{\mu_0 I}{2b}\left[1 + (z/b)^2\right]^{-3/2}$$

Example 11-4: Circular current-carrying loop (3)

$$\vec{B}(0,0,z) = \vec{a}_z \frac{\mu_0 I}{2b}\left[1+\left(z/b\right)^2\right]^{-3/2}$$



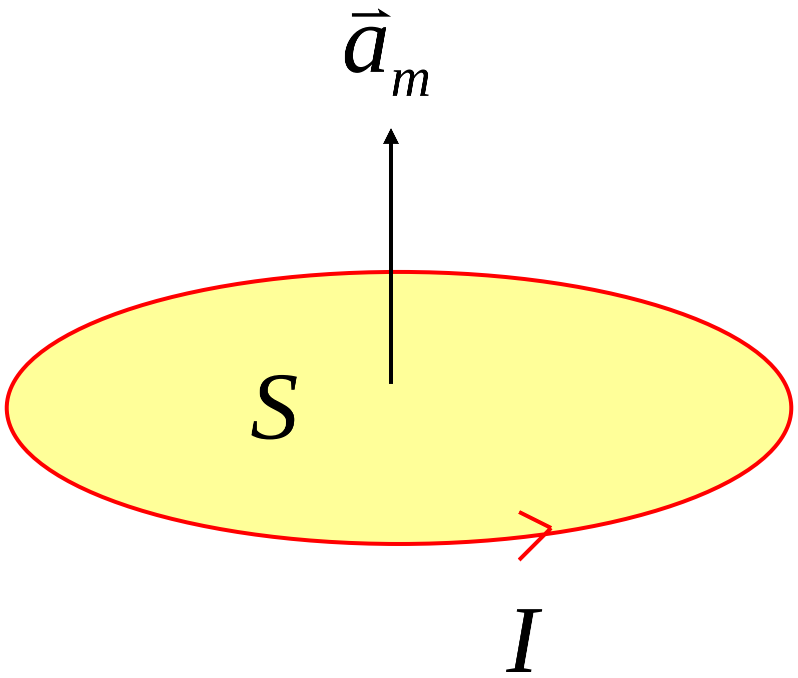


Sec. 11-5
Magnetic Dipole

1. Definition
2. Distant magnetic vector potential due to a dipole
3. Distant magnetic field due to a dipole

Definition

Essential in modeling the interaction between magnetic materials and external magnetic field (Lesson 12)



Dipole moment:

$$\vec{m} = \vec{a}_m I S$$

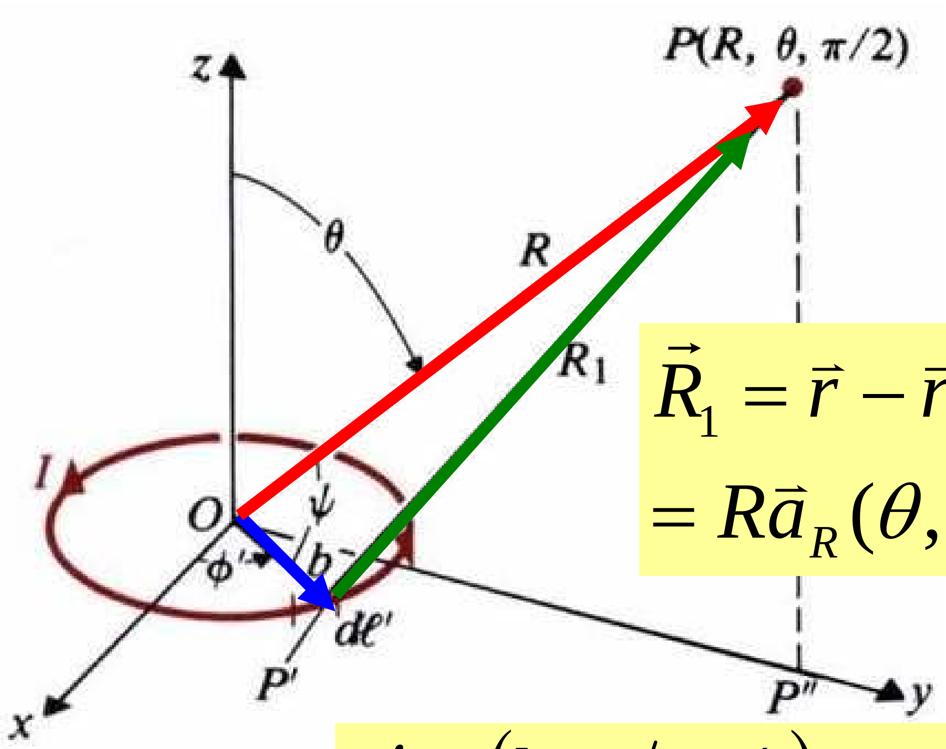
Far magnetic field-1

ϕ-symmetry

observation pt. $\vec{r} = (R, \theta, \pi/2)$

$= R\vec{a}_R(\theta, \pi/2)$

$= R(\vec{a}_y \sin\theta + \vec{a}_z \cos\theta)$



$\vec{R}_1 = \vec{r} - \vec{r}'$

$= R\vec{a}_R(\theta, \pi/2) - b\vec{a}'_R(\pi/2, \phi')$

$\vec{r}' = (b, \pi/2, \phi')$

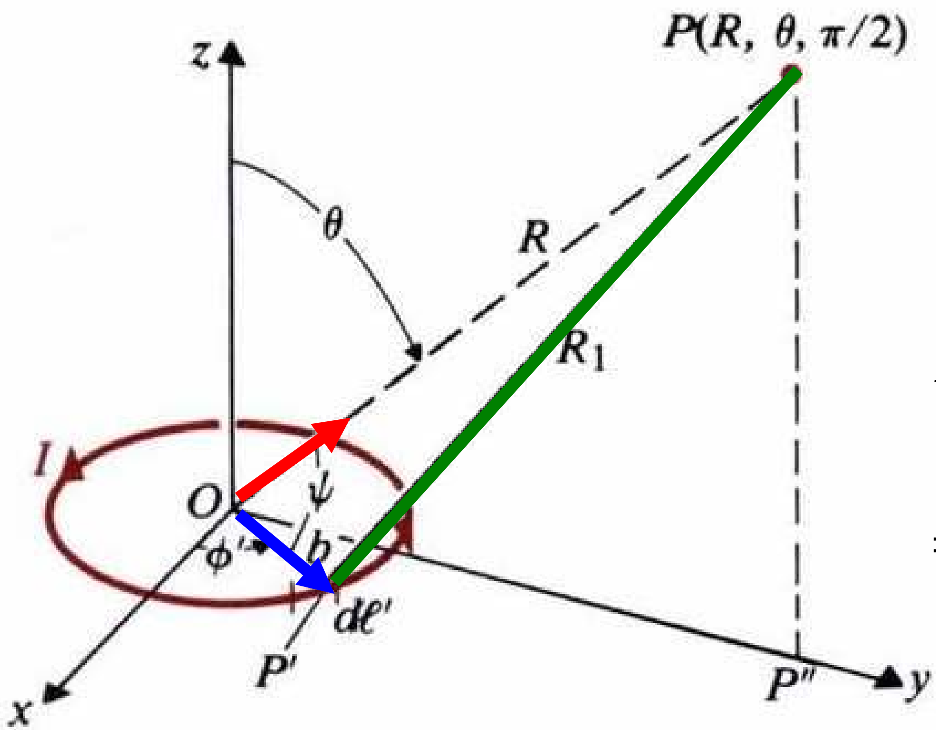
$= b\vec{a}'_R(\pi/2, \phi')$

$= b(\vec{a}_x \cos\phi' + \vec{a}_y \sin\phi')$

Far magnetic field-2

$$\vec{R}_1 = R\vec{a}_R(\theta, \pi/2) - b\vec{a}'_R(\pi/2, \phi')$$

$$\Rightarrow R_1^2 = \left(\vec{R}_1 \cdot \vec{R}_1\right) = R^2 + b^2 - 2Rb\left(\vec{a}_R \cdot \vec{a}'_R\right)$$



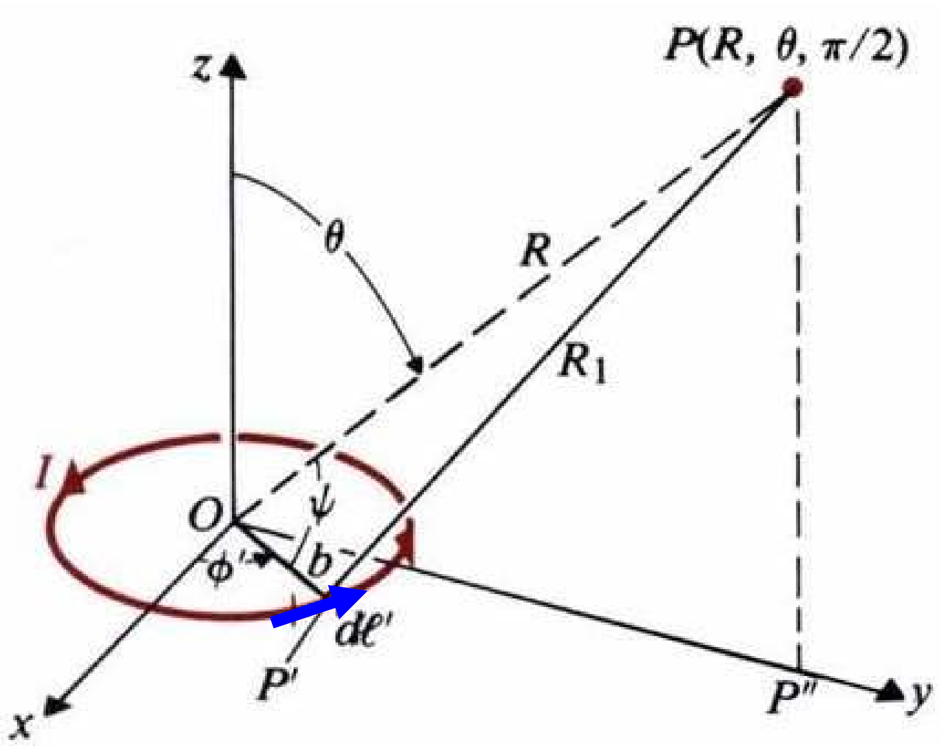
$$\cos\psi = \left(\vec{a}_y \sin\theta + \vec{a}_z \cos\theta\right) \cdot \left(\vec{a}_x \cos\phi' + \vec{a}_y \sin\phi'\right) = \sin\theta \sin\phi'$$

$$R_1^{-1} = \left(R^2 + b^2 - 2Rb\sin\theta\sin\phi'\right)^{-1/2}$$

$$= \left\{\left(R^2\right)\left[1 + \left(\frac{b}{R}\right)^2 - \frac{2b}{R}\sin\theta\sin\phi'\right]\right\}^{-1/2}$$

$$\approx \frac{1}{R}\left(1 + \frac{b}{R}\sin\theta\sin\phi'\right), \quad \text{if } R >> b$$

Far magnetic field-3



$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,'}{R(\vec{r},\vec{r}\,')}$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_{C'} R_1^{-1} d\vec{l}\,'$$

$$\approx \frac{\mu_0 I b}{4\pi R} \oint_{C'} \left(1 + \frac{b}{R}\sin\theta \sin\phi'\right) \left(-\vec{a}_x \sin\phi' + \vec{a}_y \cos\phi'\right) d\phi'$$

$$d\vec{l}\,' = \vec{a}_\phi' b\, d\phi' =$$

$$b\, d\phi' \left(-\vec{a}_x \sin\phi' + \vec{a}_y \cos\phi'\right)$$

Far magnetic field-4

$$\frac{1+\cos 2\phi'}{2}$$

$$\vec{A}(\vec{r}) \approx \frac{\mu_0 Ib}{4\pi R}\left[-\vec{a}_x \int_0^{2\pi}\left(\sin\phi' + \frac{b}{R}\sin\theta \sin^2\phi'\right)d\phi'\right.$$

$$\left.+\vec{a}_y \int_0^{2\pi}\left(\cos\phi' + \frac{b}{R}\sin\theta \sin\phi' \cos\phi'\right)d\phi'\right]$$

$$2\sin 2\phi'$$

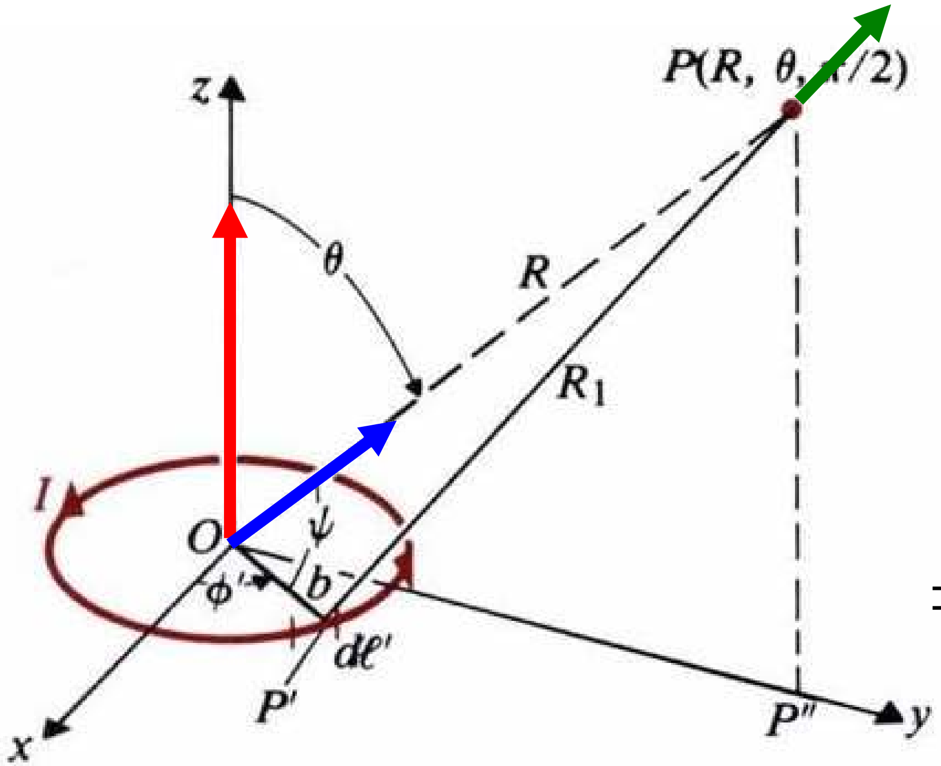
$$= \frac{\mu_0 Ib}{4\pi R}\left(-\vec{a}_x \frac{b}{R}\sin\theta \cdot \pi\right) = -\vec{a}_x \frac{\mu_0 I\pi b^2 \sin\theta}{4\pi R^2}$$

Far magnetic field-5

$$\vec{m}\times\vec{a}_R(\theta,\pi/2)=\left(\vec{a}_z I\pi b^2\right)\times\left(\vec{a}_y\sin\theta+\vec{a}_z\cos\theta\right)$$

$$=-\vec{a}_x I\pi b^2\sin\theta$$

$$\vec{A}(\vec{r})=\vec{a}_\phi A_\phi$$



$$\vec{A}(\vec{r})=-\vec{a}_x\frac{\mu_0 I\pi b^2\sin\theta}{4\pi R^2}$$

$$\Rightarrow\ \boxed{\vec{A}(\vec{r})=\frac{\mu_0\vec{m}\times\vec{a}_R}{4\pi R^2}}$$

$$\Rightarrow\vec{A}(\vec{r})=\vec{a}_\phi A_\phi,\ \ A_\phi(R,\theta)=\frac{\mu_0 m\sin\theta}{4\pi R^2}$$

Far magnetic field-6

$$A_\phi(R,\theta) = \frac{\mu_0 m \sin\theta}{4\pi R^2}$$

$$\vec{B}(\vec{r}) = \nabla \times \left(\vec{a}_\phi A_\phi\right) \approx \vec{a}_R \frac{1}{R\sin\theta} \frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \vec{a}_\theta \frac{1}{R} \frac{\partial}{\partial R}\left(A_\phi R\right)$$

$$\begin{cases} \dfrac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) = \dfrac{\mu_0 m}{4\pi R^2} \dfrac{\partial}{\partial\theta}\left(\sin^2\theta\right) = \dfrac{\mu_0 m}{4\pi R^2} \sin 2\theta \\ \dfrac{\partial}{\partial R}\left(A_\phi R\right) = \dfrac{\mu_0 m \sin\theta}{4\pi} \dfrac{\partial}{\partial R}\left(\dfrac{1}{R}\right) = -\dfrac{\mu_0 m \sin\theta}{4\pi R^2} \end{cases}$$

$$\vec{B}(\vec{r}) \approx \vec{a}_R \frac{1}{R\sin\theta} \frac{\mu_0 m}{4\pi R^2} \sin 2\theta + \vec{a}_\theta \frac{1}{R} \frac{\mu_0 m \sin\theta}{4\pi R^2}$$

$$\boxed{\vec{B}(\vec{r}) \approx \frac{\mu_0 m}{4\pi R^3}\left[\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta\right]}$$

Comment-1

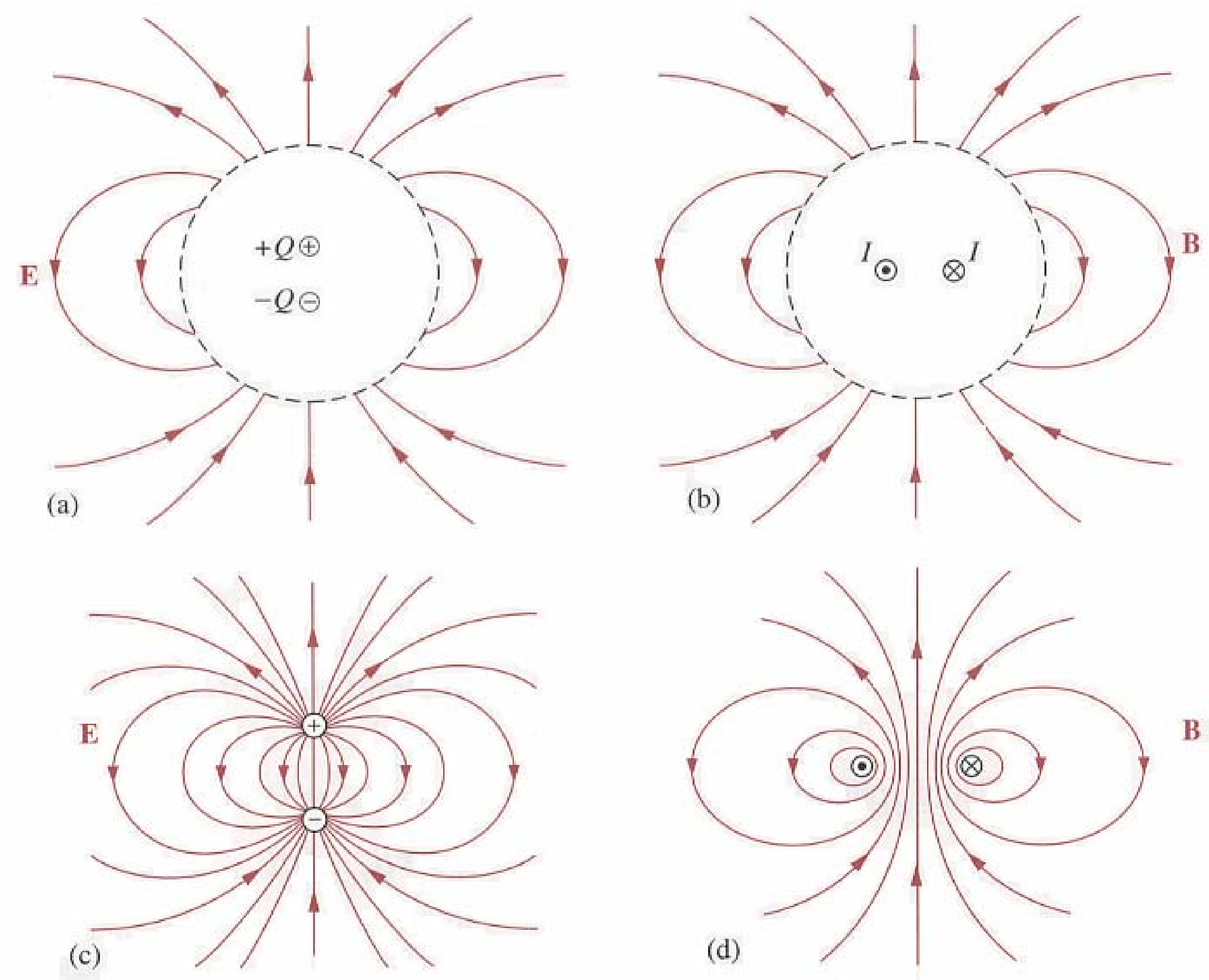
$$\begin{cases} \vec{A}(\vec{r}) \approx \dfrac{\mu_0 \vec{m} \times \vec{a}_R}{4\pi R^2} \\ V(\vec{r}) \approx \dfrac{\vec{p} \cdot \vec{a}_R}{4\pi\varepsilon_0 R^2} \end{cases}$$

$$\vec{p} \to \vec{m}$$

$$\varepsilon_0 \to \frac{1}{\mu_0}$$

$$\begin{cases} \vec{B}(\vec{r}) \approx \dfrac{\mu_0 m}{4\pi R^3}\left[\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta\right] \\ \vec{E}(\vec{r}) \approx \dfrac{p}{4\pi\varepsilon_0 R^3}\left(\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta\right) \end{cases}$$

Comment-2



Similar far-fields

Distinct near-fields