Lesson 10
Steady Electric Currents

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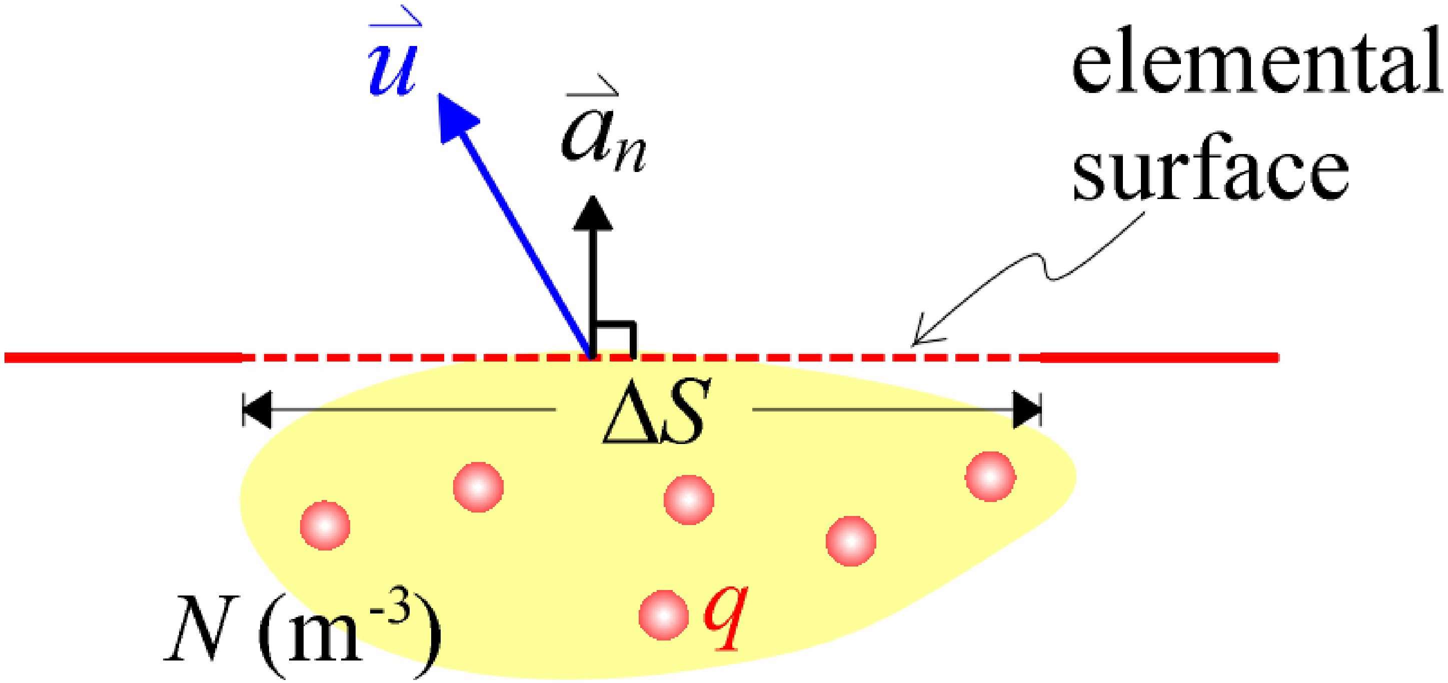
Outline

- Current density
- Current laws
- Boundary conditions
- Evaluation of resistance

Sec. 10-1
Current Density

1. Definition
2. Convection currents
3. Conduction currents

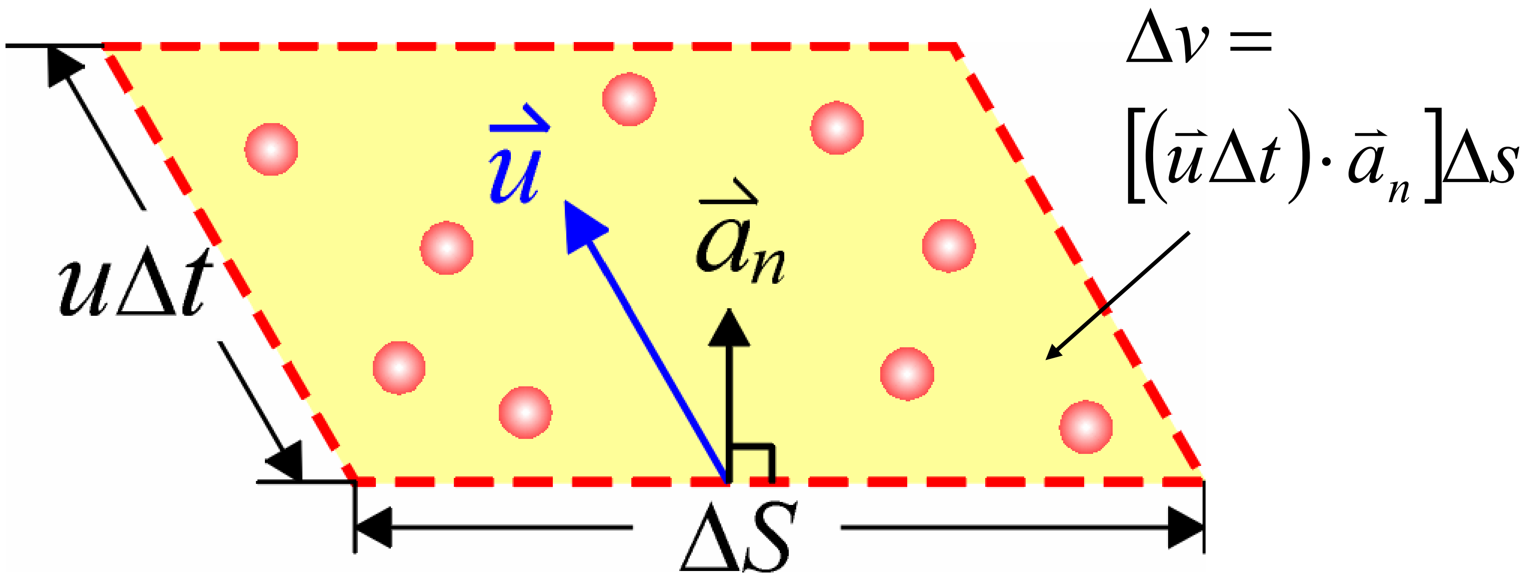
Definition-1



Volume charge density: $\rho = qN \left(\mathrm{C/m^3}\right)$

Definition-2

Within a time interval Δt, the amount of charge ΔQ enclosed by a volume Δv will pass through the elemental surface



$$\Delta Q = \rho \Delta v = \rho\left(\vec{u}\Delta t\right)\cdot\left(\vec{a}_n \Delta s\right) = \rho \Delta t\left(\vec{u}\cdot \vec{a}_n \Delta s\right)$$

Definition-3

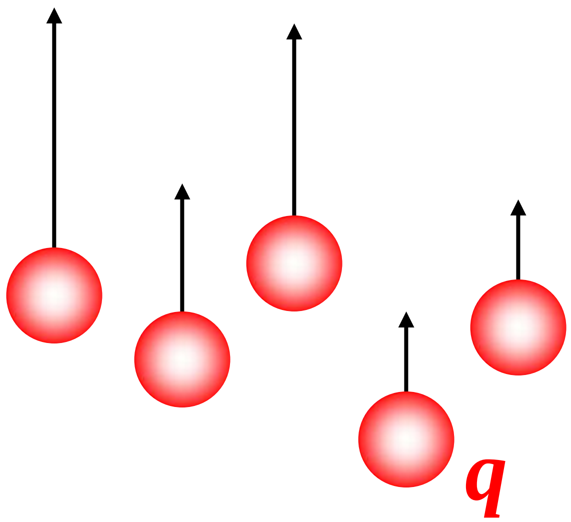
$$\Delta Q = \rho \Delta t \left(\vec{u} \cdot \vec{a}_n \Delta s \right), \quad \Rightarrow I = \frac{\Delta Q}{\Delta t} = \rho \left(\vec{u} \right) \cdot \vec{a}_n \Delta s \,)$$

Current I can be regarded as the "flux" of a volume current density $\vec{J}$, i.e., $I = \vec{J} \cdot \left(\vec{a}_n \Delta s \right)$,

$$\Rightarrow \boxed{\vec{J} = \rho \vec{u}} \quad (\text{A/m}^2)$$

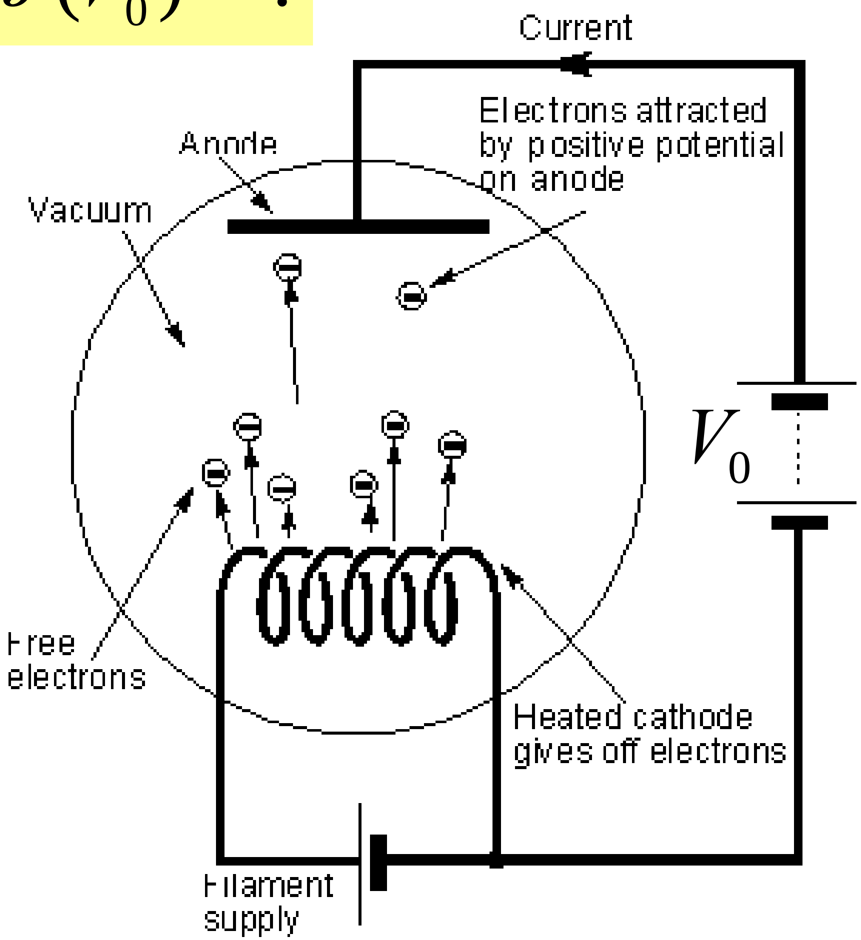
Convection currents

Convection currents result from motion of charged particles (e.g. electrons, ions) in vacuum (e.g. cathode ray tube), involving with mass transport but without collision.



Example 10-1: Vacuum tube diode (1)

$$\vec{J}(V_0) = ?$$



1. Electrons are boiled away from the incandescent cathode with zero initial velocity (space-charge limited condition).
2. Anode has positive potential, attracting the free electrons.

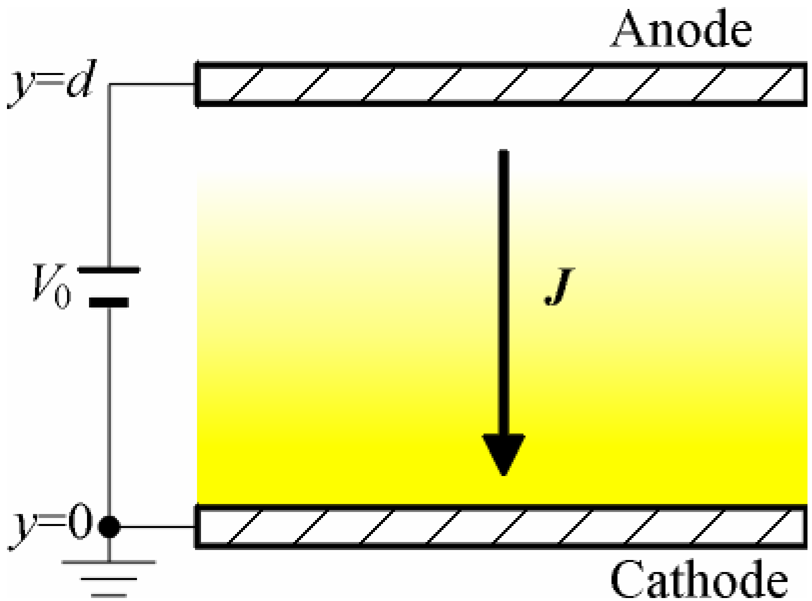
Example 10-1: Vacuum tube diode (2)

By planar symmetry, $\Rightarrow$

1. Potential: $V(y)$, BCs: $V(0)=0, V(d)=V_0$

2. Volume charge density: $\rho(y)$

3. Charge velocity: $\vec{u}=\vec{a}_y u(y)$

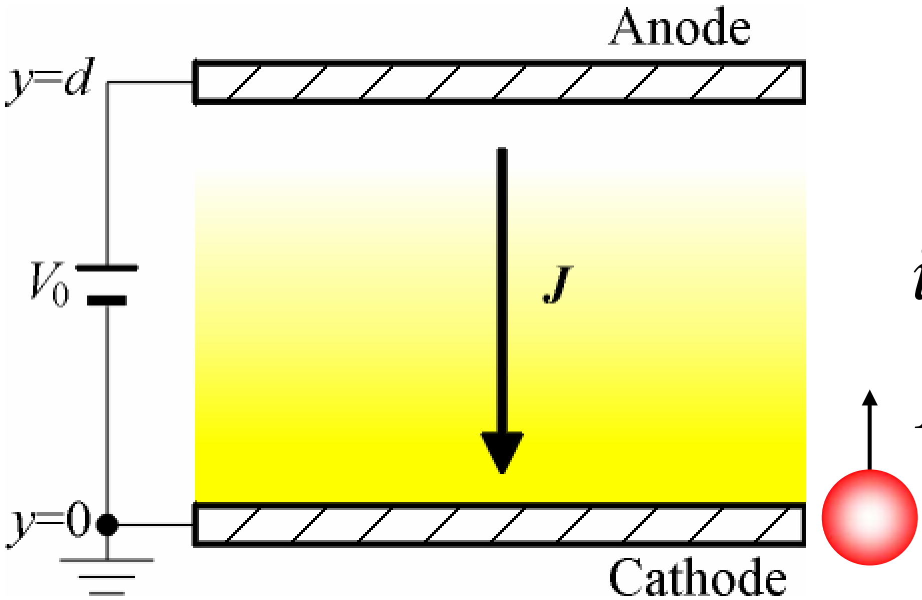


Example 10-1: Vacuum tube diode (3)

By space-charge limited condition, $\Rightarrow$

1. Zero initial charge velocity: $u(0) = 0$

2. Zero boundary E-field: $E_y(0) = 0, \Rightarrow V'(0) = 0$



$\vec{u} = 0,$

$\vec{E} = 0$

Example 10-1: Vacuum tube diode (4)

(1) In steady state, $\vec{J} = -\vec{a}_y J$

$$\because \vec{J} = \rho\vec{u}, \ \Rightarrow J = -\rho(y)u(y) = \text{constant},$$

$$\rho(y) = -\frac{J}{u(y)}$$

(2) By energy conservation, $eV(y) = \frac{1}{2}mu^2(y)$

$$\Rightarrow u(y) = \sqrt{\frac{2eV(y)}{m}}, \quad \rho(y) = -J\sqrt{\frac{m}{2eV(y)}}$$

Example 10-1: Vacuum tube diode (5)

(3) Since free charges exist inside the tube:

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \quad \text{...Poisson's equation}$$

$$\Rightarrow \frac{d^2 V}{dy^2} = -\frac{\rho(y)}{\varepsilon_0} = \frac{J}{\varepsilon_0}\sqrt{\frac{m}{2eV(y)}} \quad \text{...Nonlinear ODE}$$

(4) Shortcut to get $J(V_0)$:

Multiply both sides by $2\frac{dV}{dy}$

$$2\frac{dV}{dy}\frac{d^2 V}{dy^2} = 2\frac{J}{\varepsilon_0}\frac{dV}{dy}\sqrt{\frac{m}{2eV(y)}}$$

Example 10-1: Vacuum tube diode (6)

Integrate with respect to y:

$$2\frac{dV}{dy}\frac{d^2V}{dy^2} = 2\frac{J}{\varepsilon_0}\frac{dV}{dy}\sqrt{\frac{m}{2eV(y)}}$$

$$\int 2V'(y)\cdot V''(y)\underline{dy} = \frac{2J}{\varepsilon_0}\sqrt{\frac{m}{2e}}\left(\int V^{-1/2}\underline{dV}\right)$$

$$\left(\frac{dV}{dy}\right)^2 = \frac{4J}{\varepsilon_0}\sqrt{\frac{mV(y)}{2e}} + c$$

Example 10-1: Vacuum tube diode (7)

By BCs of the potential: $V(0)=0,\ V'(0)=0$

$$\Rightarrow c=0,\ \frac{dV}{dy}=2\sqrt{\frac{J}{\varepsilon_0}}\left[\frac{mV(y)}{2e}\right]^{1/4}$$

$$\Rightarrow V^{-1/4}dV=2\sqrt{\frac{J}{\varepsilon_0}}\left(\frac{m}{2e}\right)^{1/4}dy$$

$$\Rightarrow \int_0^{V_0} V^{-1/4}dV=2\sqrt{\frac{J}{\varepsilon_0}}\left(\frac{m}{2e}\right)^{1/4}\left(\int_0^{d}dy\right)$$

Example 10-1: Vacuum tube diode (8)

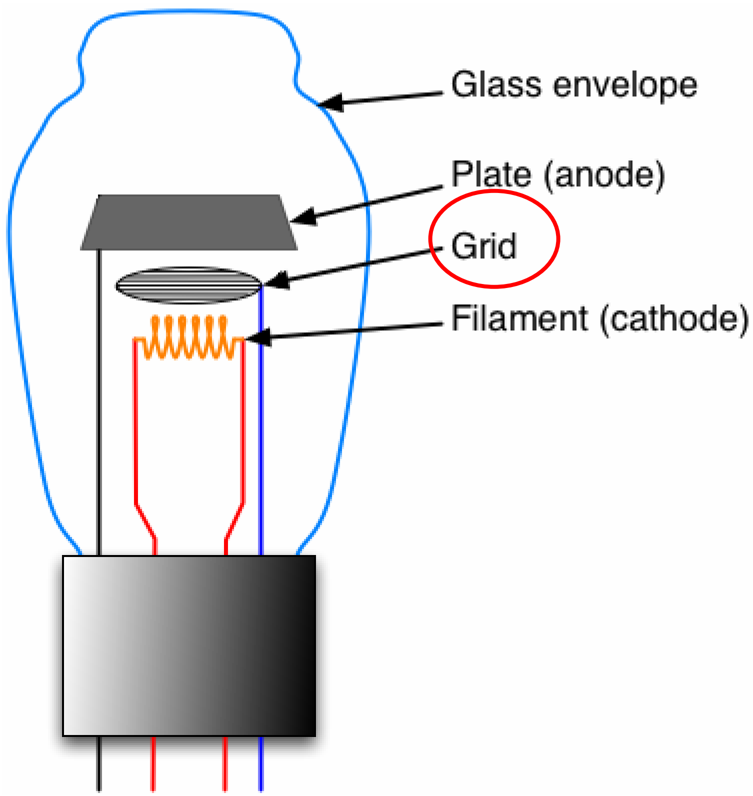
Child-Langmuir law:

$$J = \frac{4\varepsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V_0^{3/2}}{d^2}$$

Differ from Ohm's law ($I \propto V$) of conduction current.

Differ from forward-biased semiconductor diode ($I \propto e^{\alpha V}$).

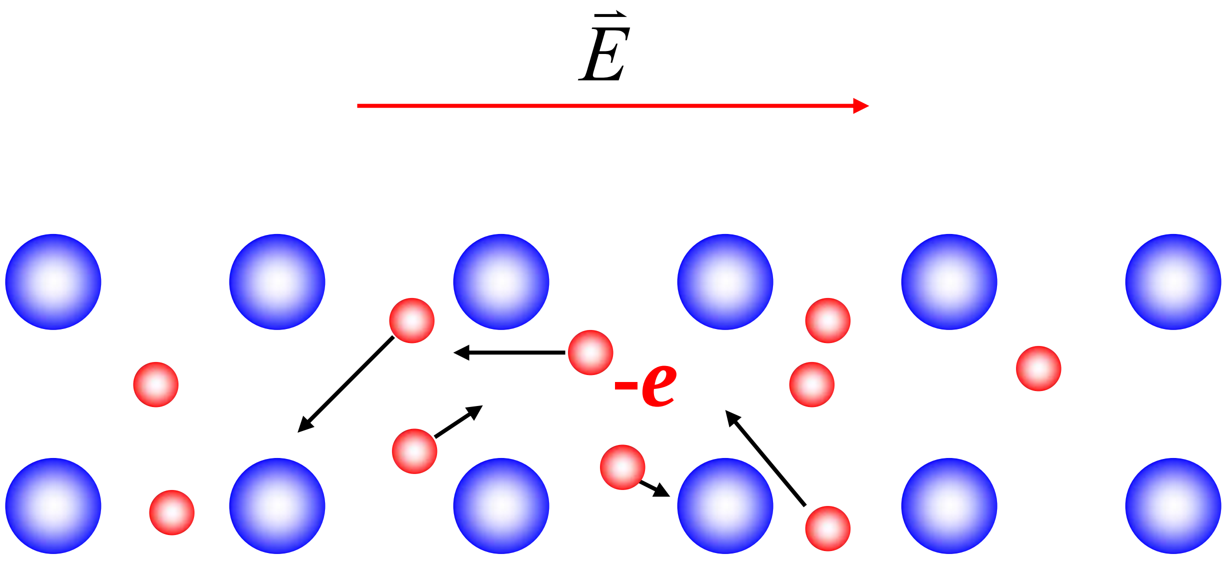
Example 10-1: Vacuum tube diode (9)



Triode (transistor)

Conduction (drift) currents-1

Conduction currents result from drift motion of free electrons due to applied E-field in conductors, involving with frequent collisions with immobile ions.



$$\frac{1}{2} m_n v_{th}^2 = \frac{3}{2} kT$$

Effective mass of e^-, depending on the lattice.

E.g. At T =300K, v_{th} ~ 10^5 m/sec

Conduction (drift) currents-2

In steady state, two forces are balanced with each other (Drude model):

1. Electric force: $-e\vec{E}$

2. Frictional force: $-\dfrac{m_n \vec{u}_d}{\tau}$

- $m_n \vec{u}_d$ → average momentum change in each collision
- τ → mean scattering time between collisions

$$\Rightarrow \vec{u}_d = -\frac{e\tau}{m_n}\vec{E} = -\mu_e \vec{E}$$ …drift (average) velocity

For typical fields, u_d ~ mm/sec

Conduction (drift) currents-3

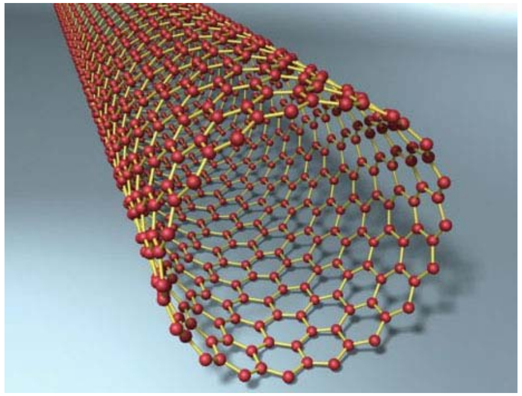
Electron mobility $\mu_e = \frac{e\tau}{m_n} \left(\frac{\mathrm{m}^2}{\mathrm{V \cdot sec}} \right)$ describes how easy an external E-field can influence the motion of conduction electrons.

E.g.

Silicon: $\mu_e = 1.35 \times 10^{-1}, \ \ \mu_h = 4.8 \times 10^{-2}$

1-D carbon nanotube: $\mu_e = 10$

Copper: $\mu_e = 2.3 \times 10^{-3}$



Conduction (drift) currents-4

By $\vec{J} = \rho\vec{u}, \quad \vec{u}_d = -\mu_e \vec{E}$,

$$\Rightarrow \vec{J} = \rho_e \vec{u}_d = -\underline{\rho_e}\mu_e \vec{E},$$

<0

$$\Rightarrow \boxed{\vec{J} = \sigma\vec{E}}$$ (A/m²) …conduction current density

$$\sigma = -\rho_e \mu_e$$ (S/m) …electric conductivity

For semiconductors:

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

holes

Sec. 10-2
Current Laws

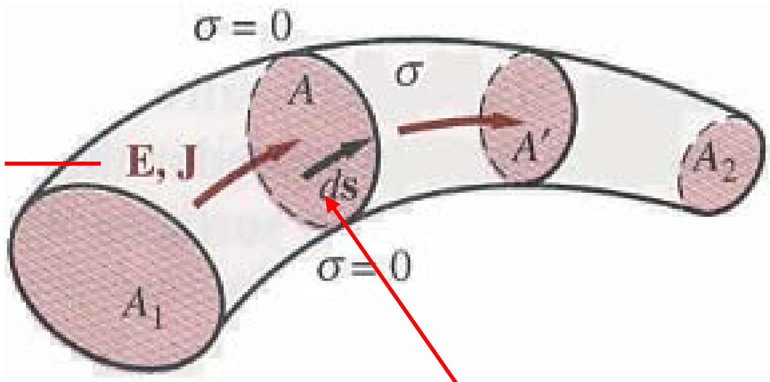
1. Ohm's law
2. Electromotive force & KVL
3. Equation of continuity & KCL
4. Joule's law

Ohm's law-1

Consider a piece of homogeneous, imperfect conductor ($\sigma < \infty$) with arbitrary shape:

microscopic law:

$$\vec{J} = \sigma \vec{E}$$



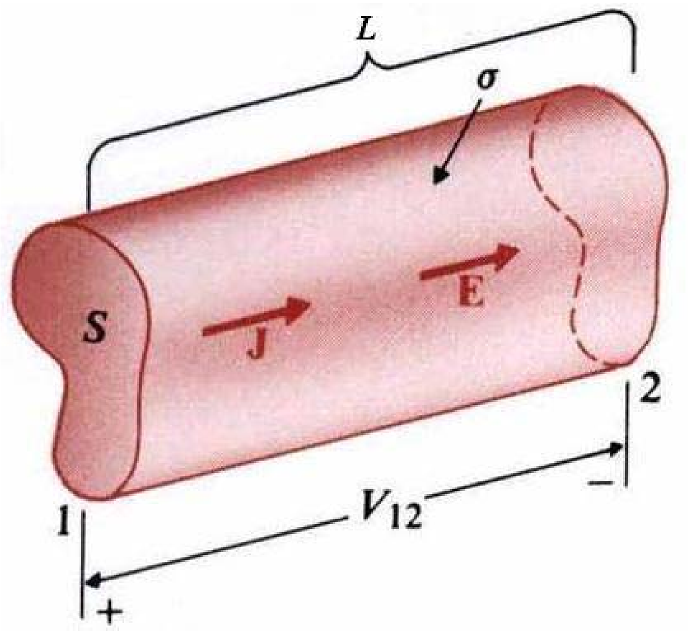
$$V_{12} = V_1 - V_2 = \int_L \vec{E} \cdot d\vec{l}\,, \quad I = \int_A \vec{J} \cdot d\vec{s} = \int_A \sigma \vec{E} \cdot d\vec{s}$$

Since the spatial distribution of $\vec{E}$ is independent of V_{12}

$$\Rightarrow R = \frac{V_{12}}{I} = \frac{\int_L \vec{E} \cdot d\vec{l}}{\sigma \int_A \vec{E} \cdot d\vec{s}} = \text{constant}$$

Ohm's law-2

Consider a piece of homogeneous, imperfect conductor ($\sigma < \infty$) with uniform cross-section:



$$\Rightarrow R = \frac{\int_L \vec{E} \cdot d\vec{l}}{\sigma \int_A \vec{E} \cdot d\vec{s}} = \frac{EL}{\sigma ES} = \frac{L}{\sigma S}$$

Electromotive force (emf)-1

If there is only conservative electric field $\vec{E}$ (created by charges):

$$\Rightarrow \vec{J} = \sigma \vec{E}, \quad \oint_C \vec{E} \cdot d\vec{l} = \oint_C \left(\vec{J} / \sigma \right) \cdot d\vec{l} = 0$$

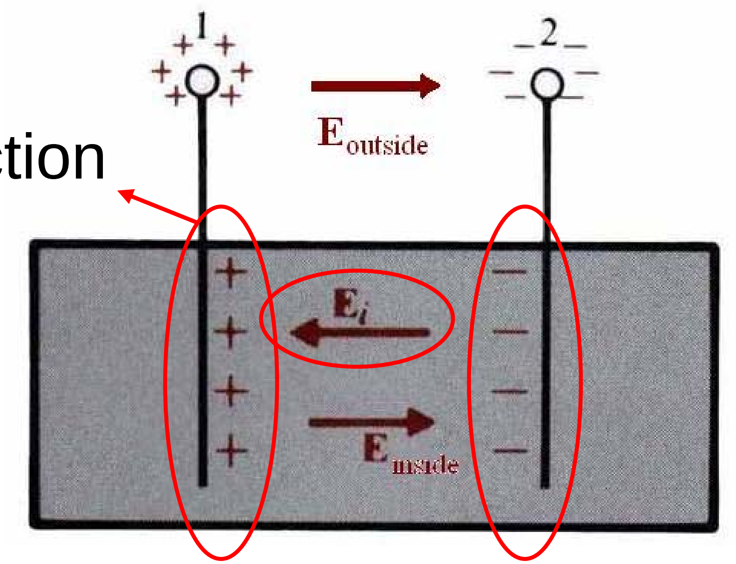
$\Rightarrow$ no steady loop current!

$\Rightarrow$ Non-conservative field is required to drive charges in a closed loop

Electromotive force (emf)-2

Consider an open-circuited battery:

chemical reaction



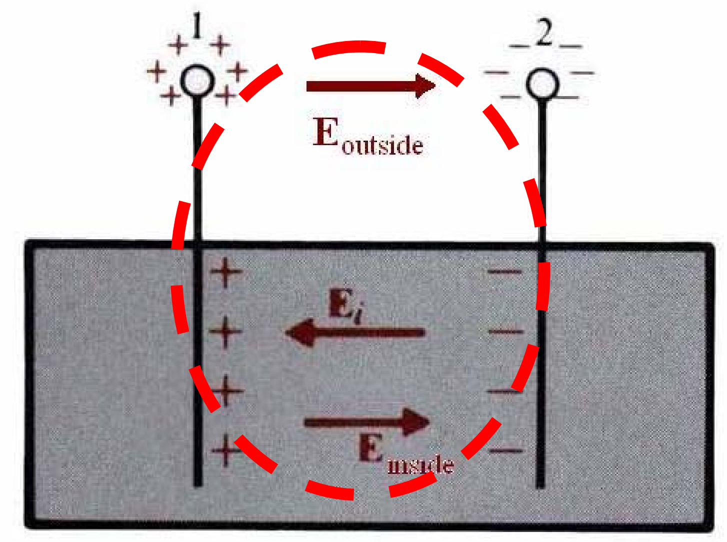
impressed field

$$\mathcal{V} \equiv \int_2^1 \vec{E}_i \cdot d\vec{l}$$

emf: the strength of non-conservative force

Electromotive force (emf)-3

By $\vec{E}_{\text{inside}} = -\vec{E}_i, \ \oint_C \vec{E}\cdot d\vec{l} = 0$:



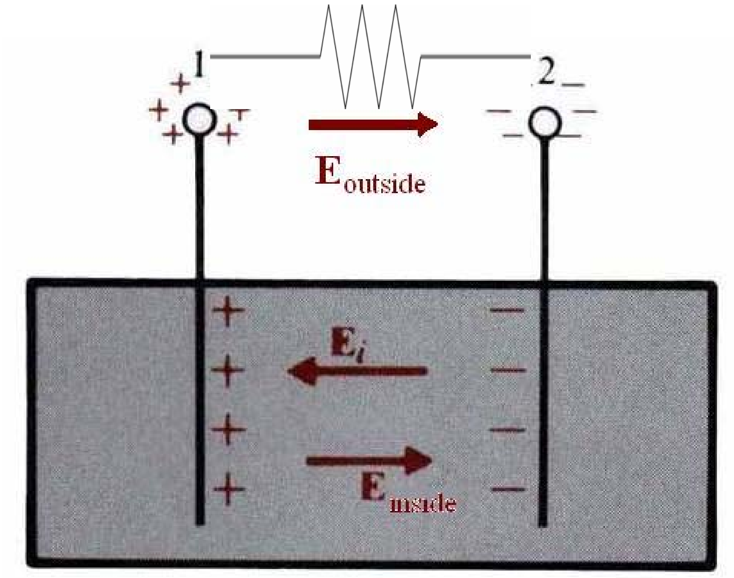
$\mathcal{V}$

$$\oint_C \vec{E}\cdot d\vec{l} = \int_1^2 \vec{E}_{\text{outside}}\cdot d\vec{l} + \int_2^1 \vec{E}_{\text{inside}}\cdot d\vec{l} = \int_1^2 \vec{E}_{\text{outside}}\cdot d\vec{l} - \underline{\int_2^1 \vec{E}_i\cdot d\vec{l}} = 0$$

$$\mathcal{V} = \int_1^2 \vec{E}_{\text{outside}}\cdot d\vec{l} = V_1 - V_2$$

Kirchhoff's voltage law (KVL)

If the two terminals are connected by a uniform conducting wire of resistance: $R = \dfrac{L}{\sigma S}$



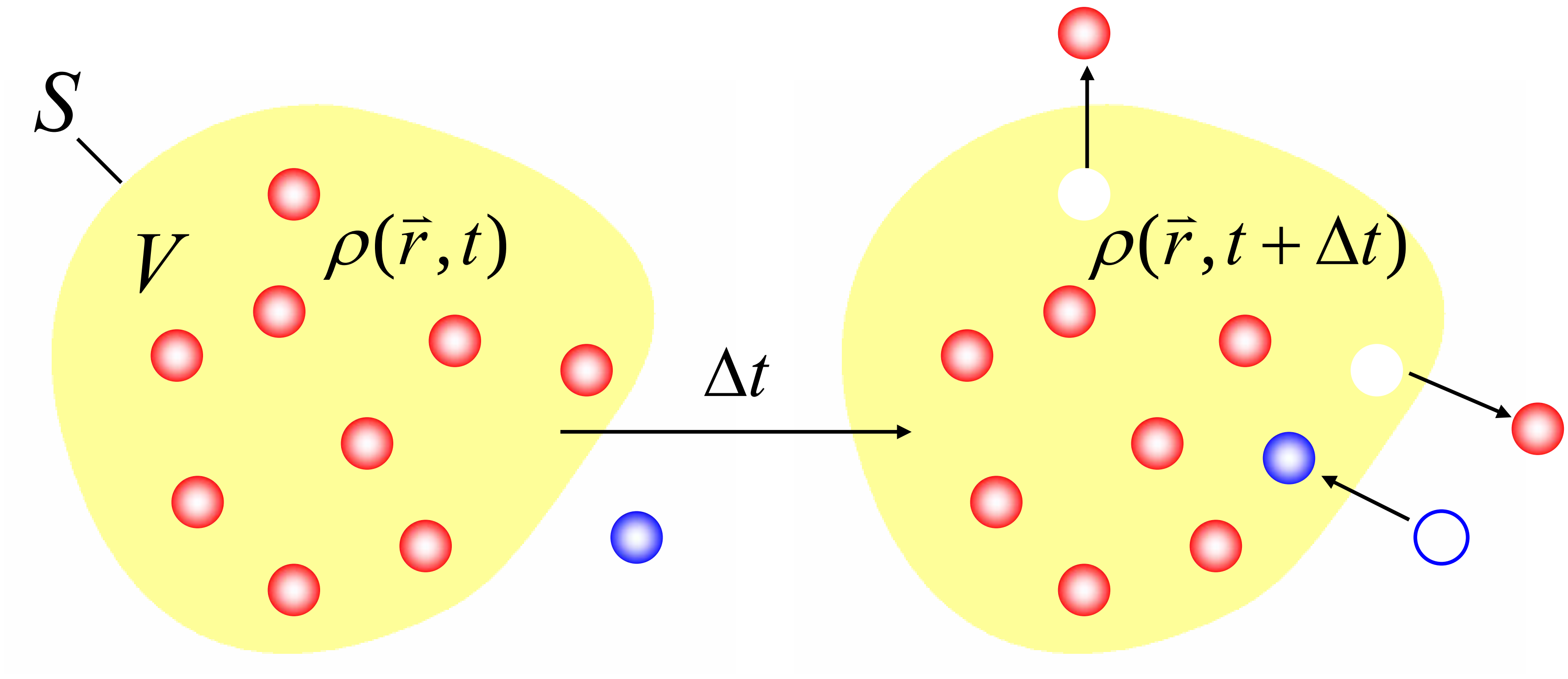
$$\vec{J} = \sigma \vec{E}_{\text{outside}}, \quad J = I/S,$$

$$\mathcal{V} = \int_1^2 \vec{E}_{\text{outside}} \cdot d\vec{l}$$

$$= \oint_C \left(\frac{\vec{J}}{\sigma} \right) \cdot d\vec{l} = \frac{IL}{\sigma S} = IR$$

Multi-sources: $\sum_j \mathcal{V}_j = \sum_k R_k I_k$ …KVL

Equation of continuity-1



$$Q = \int_V \rho(\vec{r},t)\,dv \qquad Q' = \int_V \rho(\vec{r},t+\Delta t)\,dv$$

Equation of continuity-2

Principle of conservation of charge, $\Rightarrow$ a net current I flowing out of V must be due to the decrease of the enclosed charge:

$$I = \lim_{\Delta t \to 0} \frac{Q - Q'}{\Delta t}$$

$Q - Q'$ → decrease of change within V during Δt

$$\oint_S \vec{J} \cdot d\vec{s} = \int_V \frac{\rho(\vec{r},t) - \rho(\vec{r},t+\Delta t)}{\Delta t} dv = \int_V \frac{-\partial \rho(\vec{r},t)}{\partial t} dv$$

↓ Divergence theorem

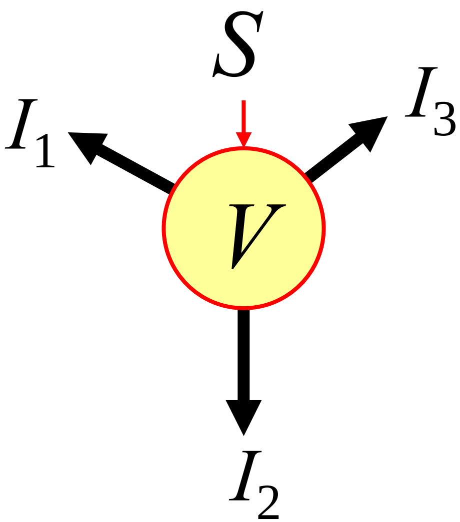
$$\int_V \left(\nabla \cdot \vec{J}\right) dv = -\int_V \frac{\partial \rho}{\partial t} dv \quad \Rightarrow \quad \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Kirchhoff's current law (KCL)

For steady currents, $\frac{\partial \rho}{\partial t} = 0$

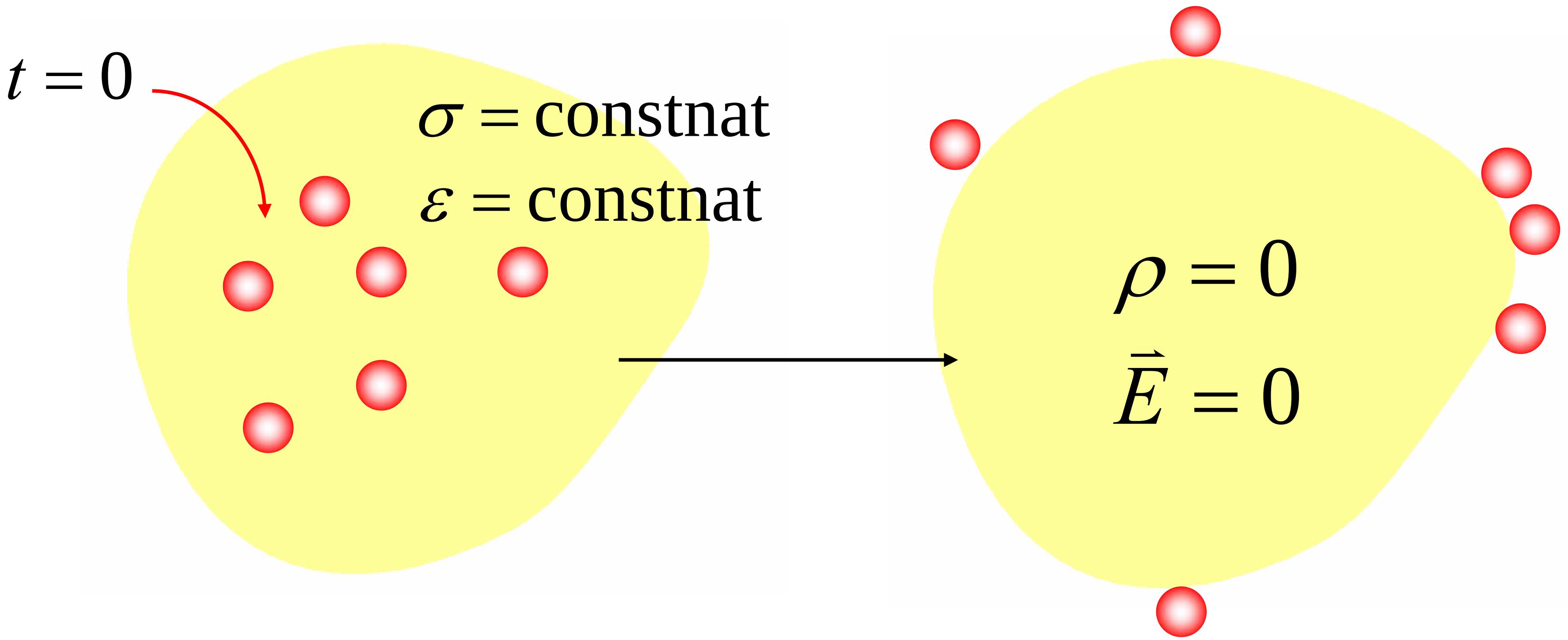
$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \longrightarrow \nabla \cdot \vec{J} = 0$$

…no current source/sink



$$\sum_j I_j = \oint_S \vec{J} \cdot d\vec{s} = \int_V \left(\nabla \cdot \vec{J}\right) dv = 0$$

Example 10-2: Dynamics of charge density (1)



$$\left.\begin{aligned} \nabla \cdot \vec{J} &= -\frac{\partial \rho}{\partial t} \\ \vec{J} &= \sigma \vec{E} \end{aligned}\right\} \quad \nabla \cdot \vec{J} = \nabla \cdot \left(\sigma \vec{E}\right) = \sigma \underline{\left(\nabla \cdot \vec{E}\right)} = -\frac{\partial \rho}{\partial t}$$

Example 10-2: Dynamics of charge density (2)

$$\left.\begin{array}{l}\nabla\cdot\vec{D}=\rho\\ \vec{D}=\varepsilon\vec{E}\end{array}\right\}\quad \nabla\cdot\vec{D}=\nabla\cdot\left(\varepsilon\vec{E}\right)=\varepsilon\underline{\left(\nabla\cdot\vec{E}\right)}=\rho$$

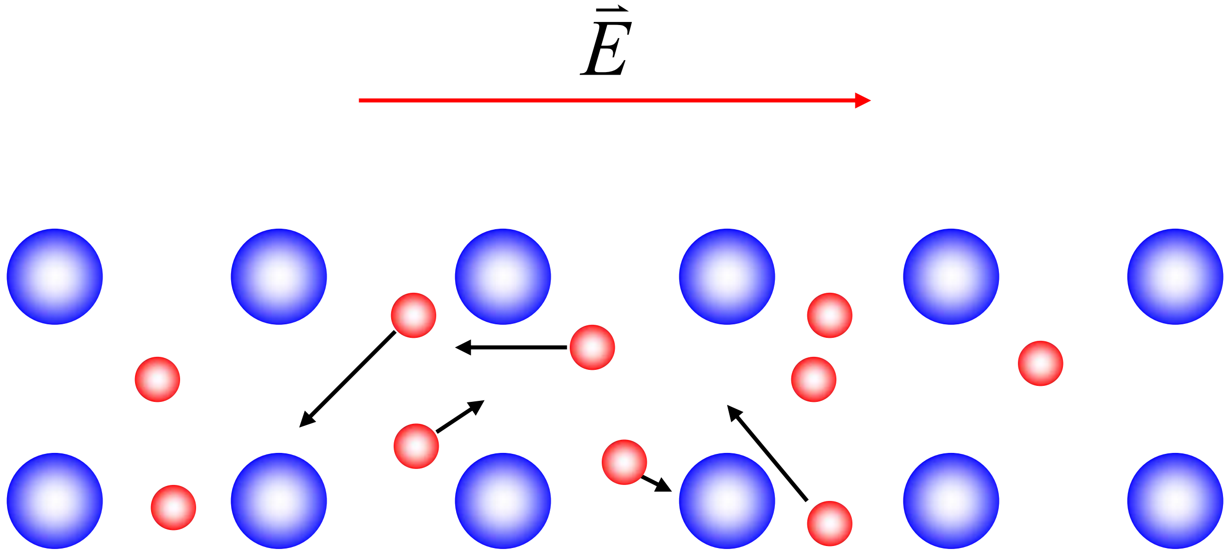
$$\left.\begin{array}{l}\nabla\cdot\vec{E}=-\dfrac{1}{\sigma}\dfrac{\partial\rho}{\partial t}\\ \nabla\cdot\vec{E}=\dfrac{\rho}{\varepsilon}\end{array}\right\}\quad \boxed{\frac{\partial\rho}{\partial t}+\frac{\sigma}{\varepsilon}\rho=0}$$

$$\Rightarrow \rho(t)=\rho_0 e^{-t/\tau},\quad \boxed{\tau=\frac{\varepsilon}{\sigma}}$$

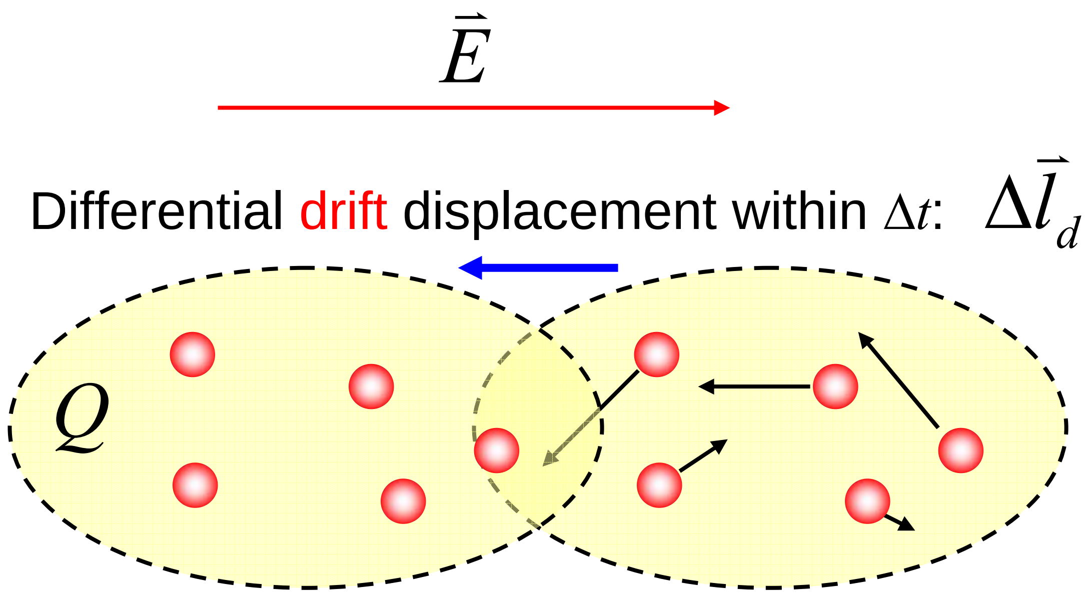
…Lifetime of free charge in a conductor

Joule's law-1

With $\vec{E}$, collisions among free electrons (with drift velocity $\vec{u}_d$) & immobile atoms transfer energy from $\vec{E}$ to thermal vibration.



Joule's law-2



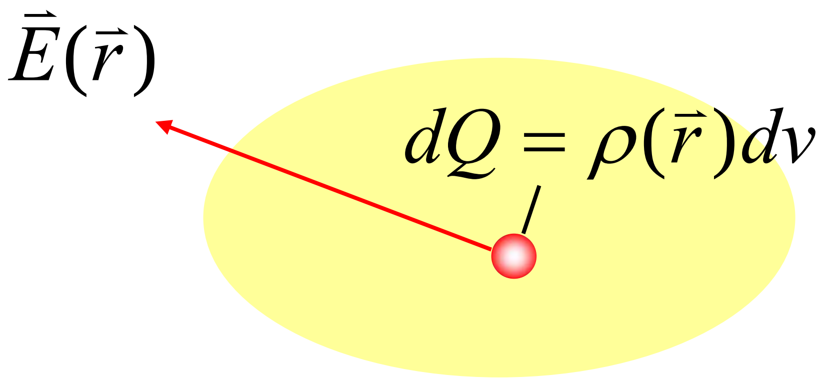
Work done by $\vec{E}$ to move charge Q for $\Delta\vec{l}_d$ is:

$$\Delta w = Q\vec{E}\cdot\Delta\vec{l}_d,$$

corresponding to a power dissipation:

$$p = \lim_{\Delta t\to 0}\frac{\Delta w}{\Delta t} = Q\vec{E}\cdot\vec{u}_d$$

Joule's law-3



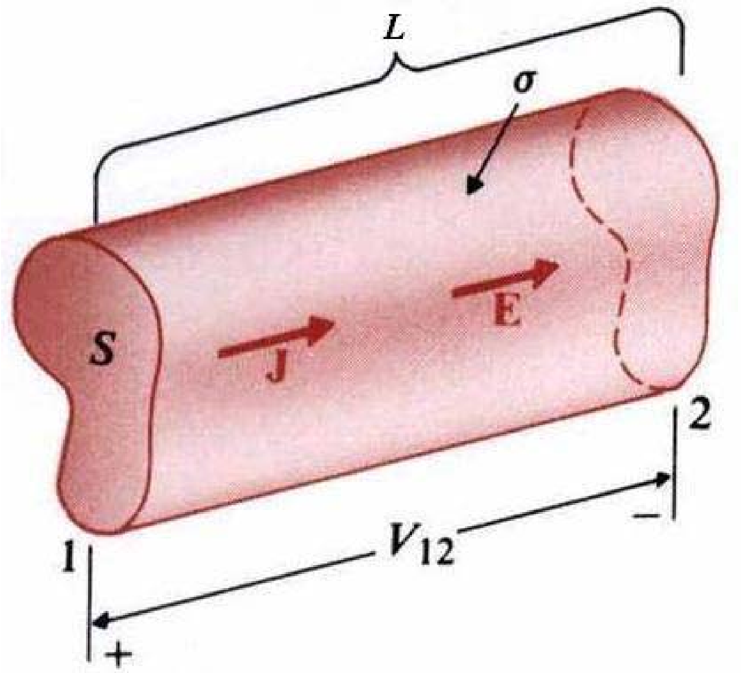
Power dissipated in a differential volume dv is:

$$\Delta p = \rho dv\left(\vec{E}\cdot\vec{u}_d\right) = \left(\vec{E}\cdot\rho\vec{u}_d\right)dv = \left(\vec{E}\cdot\vec{J}\right)dv$$

$\Rightarrow$ $\boxed{\vec{E}\cdot\vec{J}}$ (W/m^3) is power density,

$\Rightarrow$ $\boxed{P = \int_V \left(\vec{E}\cdot\vec{J}\right)dv}$ … Joule's law

Joule's law-4



Power dissipated in a homogeneous conductor of uniform cross section:

$$P = \int_V \left(\vec{E} \cdot \vec{J}\right) dv = \int_L \left(\vec{E} \cdot \vec{J}\right) S dl = \int_L \vec{E} \cdot I d\vec{l} = V_{12} I$$

Sec. 10-3
Boundary Conditions

1. BCs of current density
2. Examples: Two lossy media connected in series

Derivation of BCs-1

1. Divergence & curl relations of $\vec{J}$:

(i) For steady currents: $\nabla \cdot \vec{J} = 0$

(ii) $\vec{J} = \sigma \vec{E}$

$\nabla \times \vec{E} = 0$

$$\left.\begin{array}{l} \vec{J} = \sigma \vec{E} \\ \nabla \times \vec{E} = 0 \end{array}\right\} \nabla \times \vec{E} = \nabla \times \left(\vec{J}/\sigma\right) = 0$$

2. Integral forms:

(i) $\oint_S \vec{J} \cdot d\vec{s} = 0$

(ii) $\oint_C \dfrac{\vec{J}}{\sigma} \cdot d\vec{l} = 0$

Derivation of BCs-2

3. Apply the integral relations to a

(i) differential pillbox:

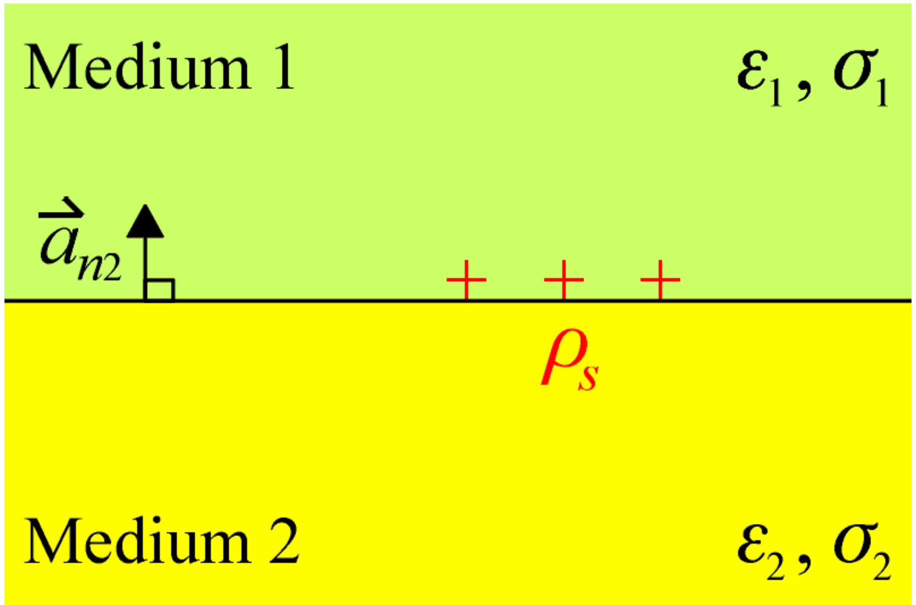
$$\oint_S \vec{J} \cdot d\vec{s} = 0 \longrightarrow \boxed{J_{1n} = J_{2n}}$$

(ii) differential loop:

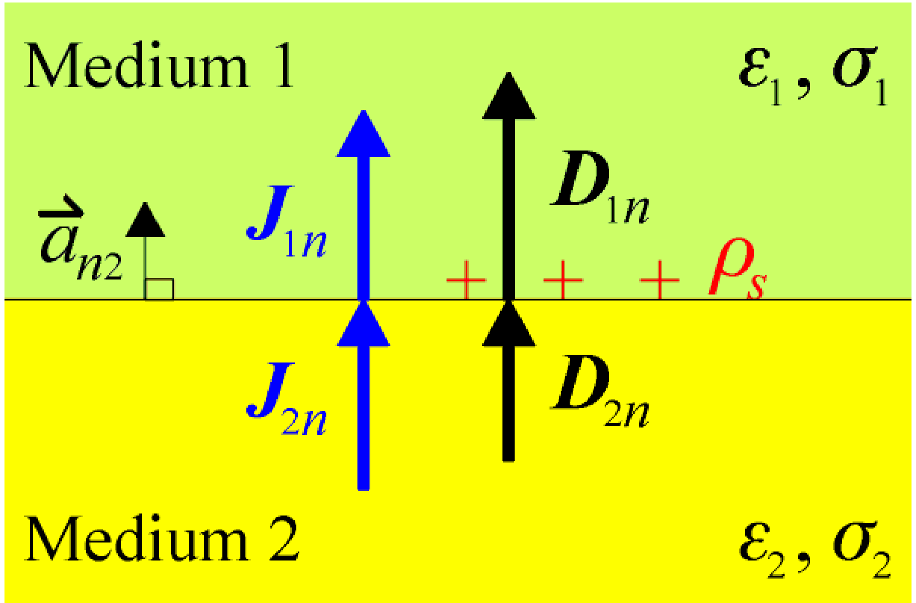
$$\oint_C \frac{\vec{J}}{\sigma} \cdot d\vec{l} = 0 \longrightarrow \boxed{\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}}$$

Example 10-3: BCs between two lossy media (1)

Represent the surface charge density ρ_s by either D_{1n} or D_{2n}



Example 10-3: BCs between two lossy media (2)



$$\begin{cases} \vec{a}_{n2} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s \\ \vec{D} = \varepsilon \vec{E} \end{cases}$$

$$\Rightarrow \rho_s = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}$$

$$\begin{cases} \vec{J} = \sigma \vec{E} \\ J_{1n} = J_{2n} \end{cases} \Rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \begin{cases} E_{1n} = \dfrac{\sigma_2}{\sigma_1} E_{2n} \\ E_{2n} = \dfrac{\sigma_1}{\sigma_2} E_{1n} \end{cases}$$

Example 10-3: BCs between two lossy media (3)

$$\rho_s = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \varepsilon_1\left(\frac{\sigma_2}{\sigma_1}E_{2n}\right) - \varepsilon_2 E_{2n} = \left(\varepsilon_1\frac{\sigma_2}{\sigma_1} - \varepsilon_2\right)\left(\frac{D_{2n}}{\varepsilon_2}\right),$$

$$\Rightarrow \quad \rho_s = \left(\frac{\sigma_2\varepsilon_1}{\sigma_1\varepsilon_2} - 1\right)D_{2n}; \quad \text{similarly,}$$

$$\Rightarrow \quad \rho_s = \left(1 - \frac{\sigma_1\varepsilon_2}{\sigma_2\varepsilon_1}\right)D_{1n}$$

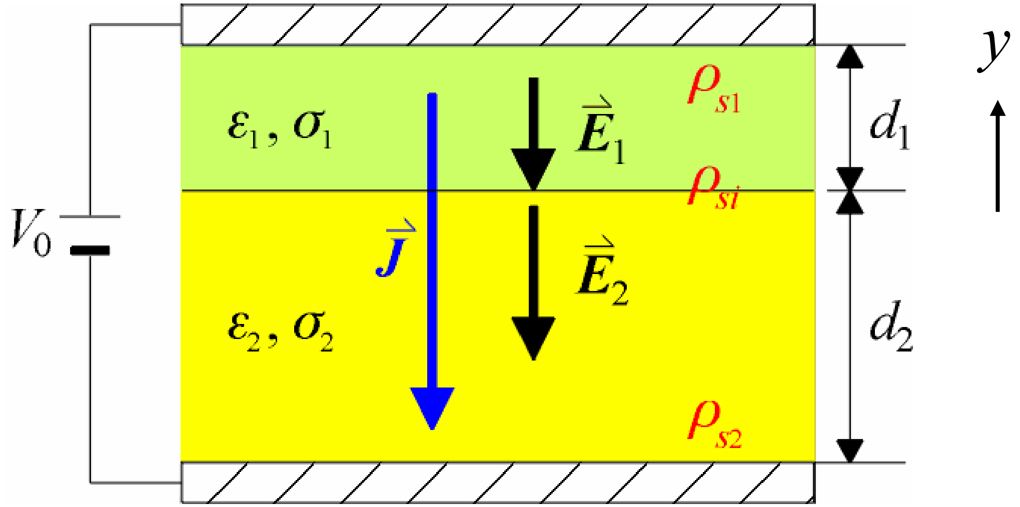
ρ_s=0 only if:

(i) $\dfrac{\sigma_1}{\varepsilon_1} = \dfrac{\sigma_2}{\varepsilon_2}$, rarely happens;

(ii) $\sigma_1 = \sigma_2 = 0$ …both media are lossless

Example 10-4: Electromagnetostatic field (1)

Find $\vec{J}$, $\vec{E}$, and surface charge densities ρ_s



By planar symmetry and $J_{1n} = J_{2n}, \quad \Rightarrow \vec{J} = -\vec{a}_y J$

By $\vec{J} = \sigma\vec{E}$, $\Rightarrow \vec{E}_i = \dfrac{\vec{J}}{\sigma_i} = -\vec{a}_y E_i, \quad E_i = \dfrac{J}{\sigma_i}$

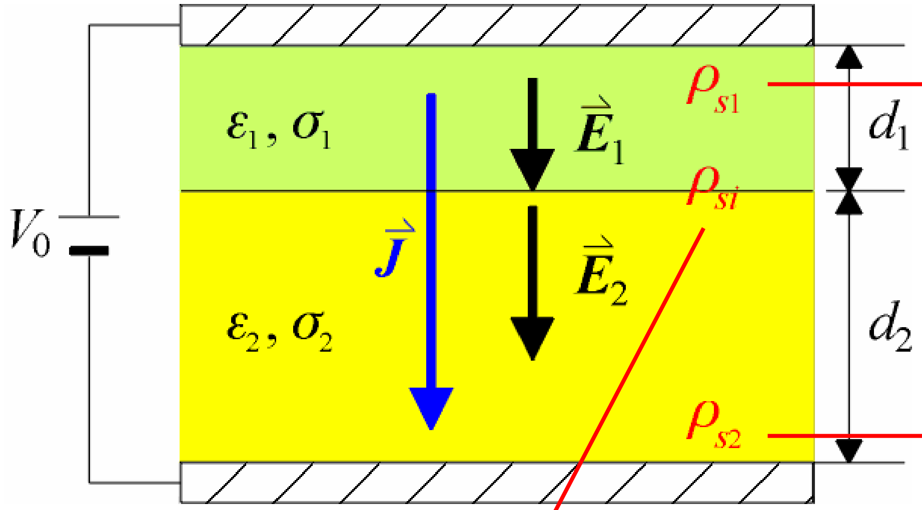
Example 10-4: Electromagnetostatic field (2)

By $$V_0 = E_1 d_1 + E_2 d_2 = J\left(\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2}\right),$$

$$\Rightarrow J = \frac{V_0}{(d_1/\sigma_1) + (d_2/\sigma_2)}$$

$$E_1 = \frac{J}{\sigma_1} = \frac{\sigma_2 V_0}{\sigma_2 d_1 + \sigma_1 d_2}, \quad E_2 = \frac{J}{\sigma_2} = \frac{\sigma_1 V_0}{\sigma_2 d_1 + \sigma_1 d_2}$$

Example 10-4: Electromagnetostatic field (3)



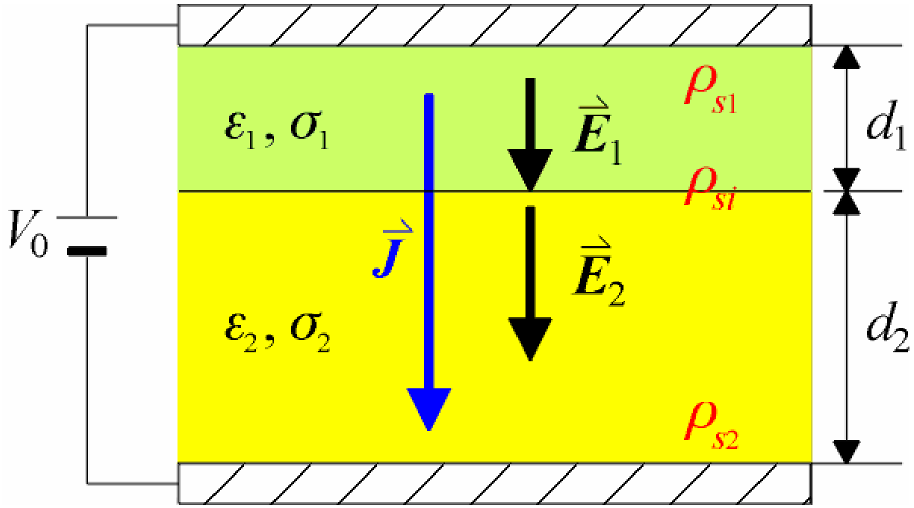
$$\rho_{s1} = D_1 = \varepsilon_1 E_1 = \frac{\varepsilon_1 \sigma_2 V_0}{\sigma_2 d_1 + \sigma_1 d_2}$$

$$\rho_{s2} = -D_2 = -\varepsilon_2 E_2 = -\frac{\varepsilon_2 \sigma_1 V_0}{\sigma_2 d_1 + \sigma_1 d_2}$$

By $\rho_s = \left(1 - \frac{\sigma_1 \varepsilon_2}{\sigma_2 \varepsilon_1}\right) D_{1n}$

$$\rho_{si} = \left(1 - \frac{\sigma_1 \varepsilon_2}{\sigma_2 \varepsilon_1}\right) \varepsilon_1 (-E_1) = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) V_0}{\sigma_2 d_1 + \sigma_1 d_2}$$

Comments



$|\rho_{s1}| \neq |\rho_{s2}|$, but

$$\rho_{s1} + \rho_{s2} + \rho_{si} = 0$$

Gauss's law, $\Rightarrow$ no E-field outside the capacitor.

Static charge and steady current (i.e., static magnetic field) coexist, $\Rightarrow$ one of rare examples of electromagnetostatic field.

Sec. 10-4
Evaluation of Resistance

1. Standard procedures
2. Example
3. Relation between R and C

Evaluation of single-(imperfect)conductor resistance

1. Assume V_0 between 2 selected end faces

2. Find $V(\vec{r})$ by solving $\nabla^2 V = 0$ with BCs

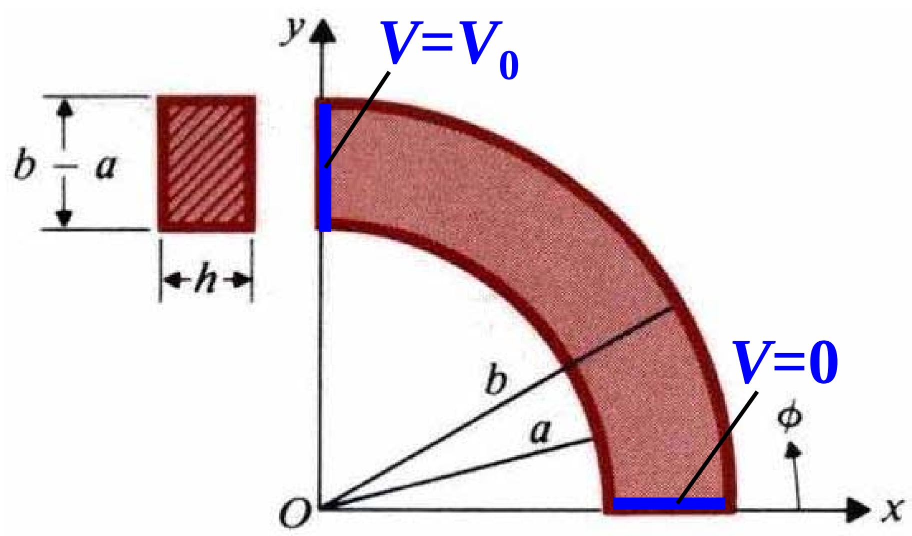
3. Find $\vec{E}$ by $\vec{E} = -\nabla V(\vec{r})$

4. Find total current by $I = \int_S \vec{J} \cdot d\vec{s} = \int_S \sigma\vec{E} \cdot d\vec{s}$

5. Find R by $R = \dfrac{V_0}{I}$, independent of V_0

Example 10-5: Resistance of a washer (1)

Find R of a quarter-circular washer of rectangular cross section and $\sigma < \infty$, if the two electrodes are located at $\phi = 0$, $\pi/2$, respectively

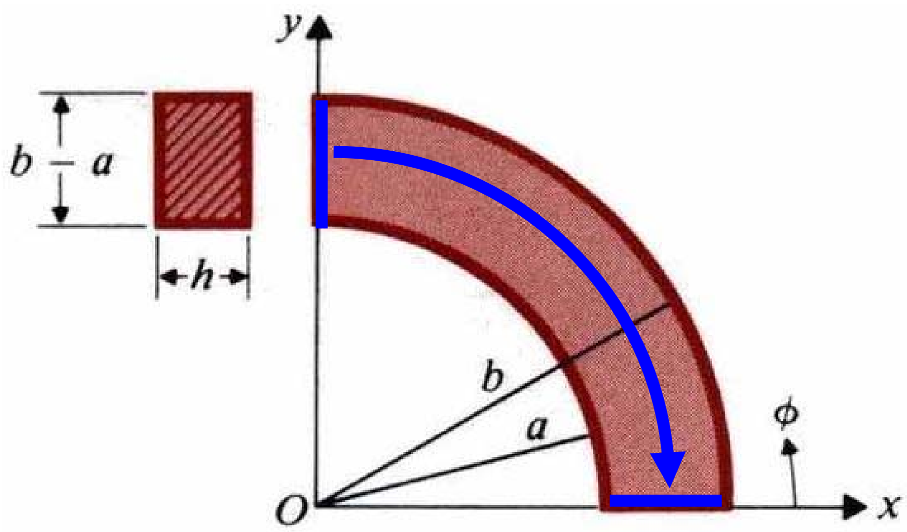


Assume: (1) $V(\phi = 0) = 0, \;\; V(\phi = \pi/2) = V_0$

Example 10-5: Resistance of a washer (2)

(2) Current flow and $\vec{E}$ are in $-\vec{a}_\phi$

$$\Rightarrow V(r,\phi,z) = V(\phi)$$



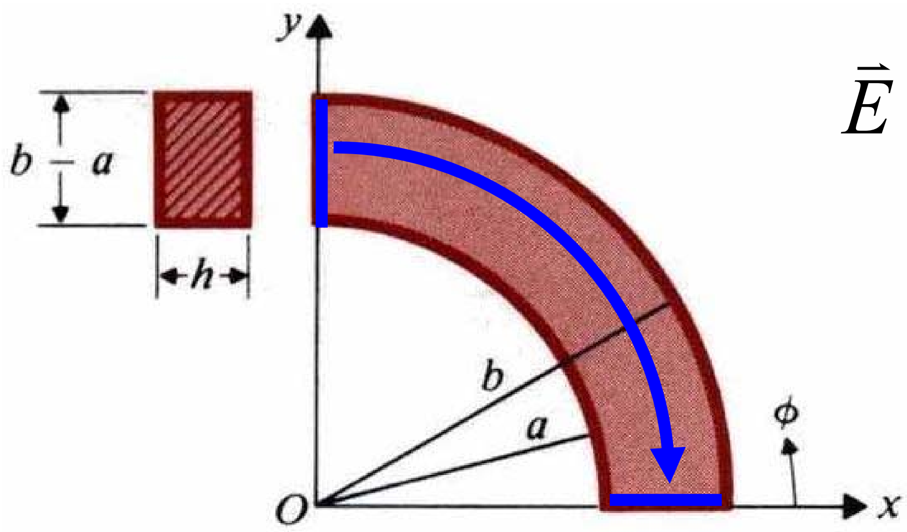
$$\nabla^2 V = 0, \ \Rightarrow V''(\phi) = 0$$

$$V(\phi) = c_1\phi + c_2$$

$$V(\phi = 0) = 0,$$
$$V(\phi = \pi/2) = V_0$$

$$\boxed{V(\phi) = \frac{2V_0}{\pi}\phi}$$

Example 10-5: Resistance of a washer (3)



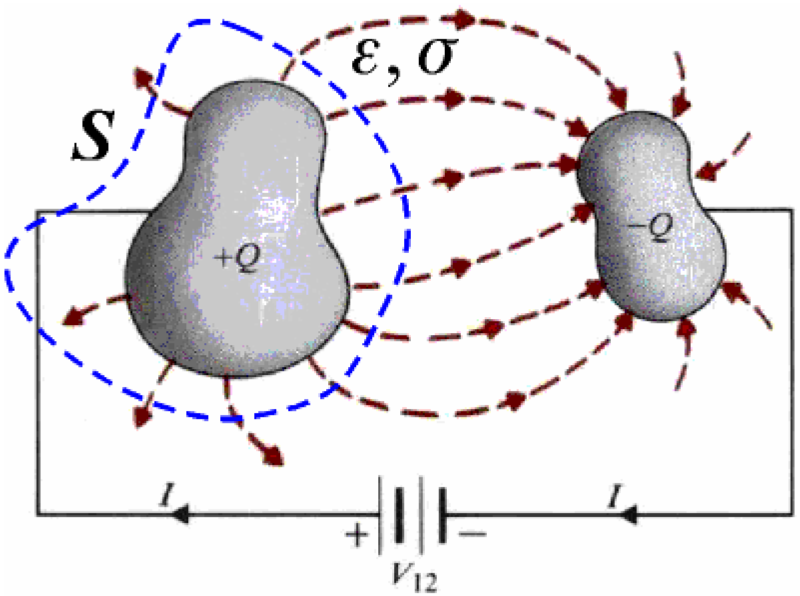
$$\vec{E} = -\nabla V = -\vec{a}_{\phi} \frac{2V_0}{\pi} \frac{1}{r} \left(\propto \frac{1}{r} \right)$$

$$\vec{J} = \sigma \vec{E} = -\vec{a}_{\phi} \frac{2V_0 \sigma}{\pi} \frac{1}{r}$$

$$I = \int_S \vec{J} \cdot d\vec{s} = \int_a^b \frac{2V_0 \sigma}{\pi \cdot r} \cdot h dr = \frac{2V_0 \sigma h}{\pi} \ln\left(\frac{b}{a} \right)$$

$$R = \frac{V_0}{I} = \frac{\pi}{2\sigma h \ln(b/a)}$$

Evaluation of resistance between two perfect conductors

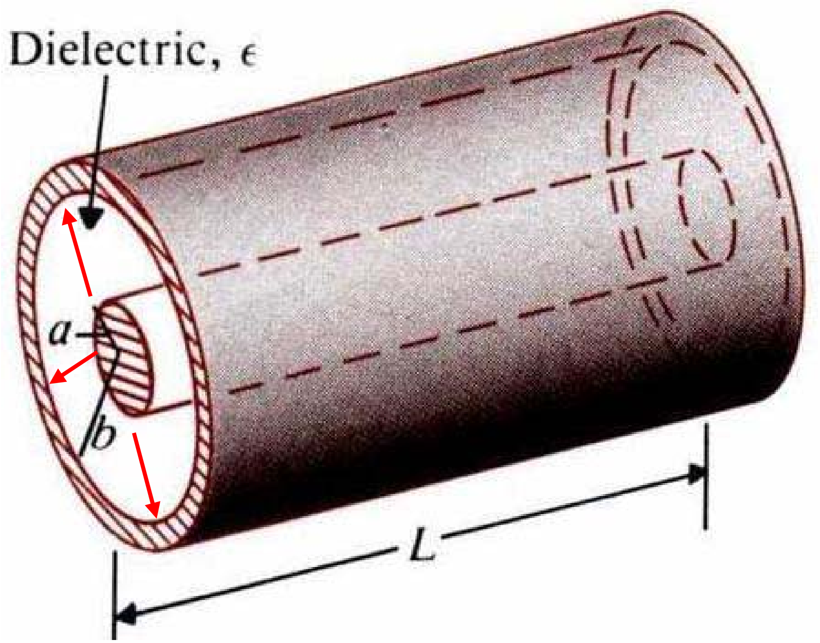


$$RC = \frac{V}{I} \cdot \frac{Q}{V} = \frac{Q}{I} = \frac{\oint_S \vec{D} \cdot d\vec{s}}{\oint_S \vec{J} \cdot d\vec{s}} = \frac{\oint_S \varepsilon \vec{E} \cdot d\vec{s}}{\oint_S \sigma \vec{E} \cdot d\vec{s}} = \frac{\varepsilon}{\sigma}$$

$\Rightarrow$ one can evaluate R by finding C first!

Example 10-6: Resistance of a coaxial cable

Find the leakage resistance between inner and outer conductors of a coaxial cable of length L, where a lossy medium of conductivity σ is filled



$$C = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

$$\Rightarrow R = \frac{\varepsilon}{\sigma C} = \frac{\ln(b/a)}{2\pi\sigma L}$$