Lesson 09 Capacitance, Electrostatic Energy

9.1 Capacitance

■ Definitions

If we deposit an amount of charge $Q=\sum_{k=1}^{n} q_k$ on one piece of conductor, the free charges will redistribute on the conducting surface such that the potential $V=\frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{R_k}$ (assuming $V=0$ at infinity) is constant for any observation point P inside and on the surface of the conductor (Fig. 9-1a), in agreement with eq's (7.2), (7.3). If we deposit a different amount of charge $Q'=rQ$ (r is a constant) on the same conductor, the surface charges will maintain the spatial distribution (Fig. 9-1b) such that the potential of the entire conductor V' can still be constant. As a result, the new potential value becomes $V'=\frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{rq_k}{R_k}=rV$, in proportional to the deposited charge. The constant ratio of deposited charge to resulting potential is defined as the capacitance of a single conductor:

$$C=\frac{Q}{V} \tag{9.1}$$

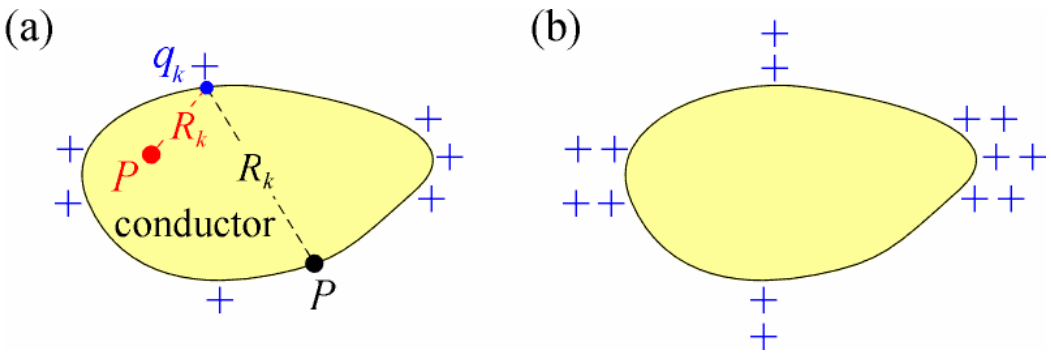

Fig. 9-1. A single-conductor capacitor deposited with an amount of charge (a) Q, and (b) $2Q$, respectively.

Example 9-1: Find the capacitance of a conducting sphere of radius b.

Ans: Assume an amount of charge Q is deposited on the conducting sphere. Due to spherical symmetry, the charges will distribute uniformly on the surface, creating an electric

field intensity (evaluated by Gauss's law):

$$\vec{E} = \begin{cases} \vec{a}_R \dfrac{Q}{4\pi\varepsilon_0 R^2}, \text{if } R \geq b \\ 0, \text{if } 0 < R < b \end{cases}.$$

The surface potential is: $V(R=b) = -\int_{\infty}^{b} \left(\vec{a}_R \dfrac{Q}{4\pi\varepsilon_0 R^2} \right) \cdot (\vec{a}_R dR) = \dfrac{Q}{4\pi\varepsilon_0 b}$. By eq. (6.1), $C = \dfrac{Q}{V} = 4\pi\varepsilon_0 b$.

If we connect two pieces of conductor by a dc voltage V_{12}, a proper amount of charge $+Q$ will be deposited on one piece and $-Q$ will be deposited on the other, creating an electric field distribution and supporting a potential difference V_{12} between them (Fig. 9-2). If the voltage changes to $V'_{12} = rV_{12}$, the spatial distributions of surface charges and electric field remain unchanged but the total amount of charge changes to $Q' = rQ$. The constant ratio of deposited charge to voltage difference is defined as the capacitance of the conducting pair:

$$C = \frac{Q}{V_{12}} \tag{9.2}$$

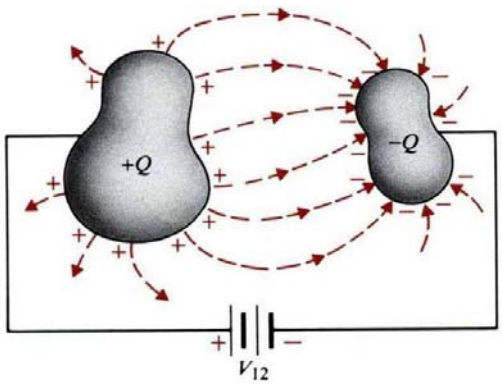

Fig. 9-2. A two-conductor capacitance (after DKC).

Note that the single-conductor capacitor can be regarded as a special case of two-conductor capacitor, where the second conductor with zero potential is located at infinity ($V_{12} = V - 0$).

■ Evaluation procedures

Three methods to evaluate the capacitance are summarized as follows:

1) (1) Assume charges $\pm Q$ are deposited on the conductors. (2) Find $\vec{E}$ by Gauss's law or vector integration [eq. (6.10)]. (3) Find V_{12} ($\propto Q$) by line integral: $V_{12} = -\int_1^2 \vec{E} \cdot d\vec{l}$. (4) Find C by eq. (9.2), which is independent of Q.
2) (1) Assume a potential difference $V = V_{12}$ between the two conductors. (2) Find the potential "distribution" $V(\vec{r})$ by solving the boundary-value problem (Lesson 8). (3) Find $\vec{E}$ by $\vec{E} = -\nabla V(\vec{r})$ [eq. (6.11)]. (4) Find the free surface charge density $\rho_s(\vec{r})$ of either conductor by the normal boundary condition $E_n = \frac{\rho_s}{\varepsilon_0}$ [eq. (7.4)]. (5) Find the total deposited free charge Q by scalar surface integral $Q = \oint_S \rho_s(\vec{r}) ds$ ($\propto V$), where S denotes the closed surface of one of the two conductors. (6) Find C by eq. (9.2), which is independent of V.
3) (1) Assume deposited charges $\pm Q$ or potential difference V for the two conductors. (2) Find $\vec{E}$ and $\vec{D}$ by the previous methods. (3) Find the stored electrostatic energy W_e ($\propto Q^2$ or V^2) by $W_e = \int_{V'} \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) dv$ [eq. (9.7)]. (4) Find C by $W_e = \frac{Q^2}{2C}$ or $W_e = \frac{CV^2}{2}$ [eq. (9.8)], which is independent of Q or V.

Example 9-2: Find the capacitance of a parallel-plate capacitor (Fig. 9-3).

Ans: (M1) Assume charges $\pm Q$ are deposited on the conducting plates. Because of planar symmetry, the charges are uniformly distributed on the plates. By Gauss's law, $\vec{E} = -\bar{a}_y \frac{Q}{\varepsilon S}$. The potential difference between the top and bottom plates is: $V_{12} = -\int_0^d \vec{E} \cdot (\bar{a}_y dy) = \frac{Qd}{\varepsilon S}$. By eq. (9.2), $C = \frac{Q}{V_{12}}, \Rightarrow$

$$C = \varepsilon \frac{S}{d} \tag{9.3}$$

(M2) Assume the potentials of the top and bottom plates are $V_2 = V_0$, $V_1 = 0$, respectively. $\Rightarrow V_{12} = V_2 - V_1 = V_0$. The solution to the corresponding boundary-value problem becomes: $V(\vec{r}) = \frac{V_0}{d} y$. $\Rightarrow$ $\vec{E} = -\nabla V(\vec{r}) = -\vec{a}_y \frac{V_0}{d}$. On the top surface ($y = d$), $E_{n2} = \vec{E} \cdot \vec{a}_{n2}$ $= \left(-\vec{a}_y \frac{V_0}{d}\right) \cdot \left(-\vec{a}_y\right) = \frac{V_0}{d}$. By the normal boundary condition $E_n = \frac{\rho_s}{\varepsilon_0}$, $\rho_{s2} = E_{n2}\varepsilon_0 = \frac{\varepsilon V_0}{d}$. $\Rightarrow Q = \rho_{s2} S = \frac{\varepsilon V_0 S}{d}$, $C = \frac{Q}{V_0} = \varepsilon \frac{S}{d}$, same as eq. (9.3).

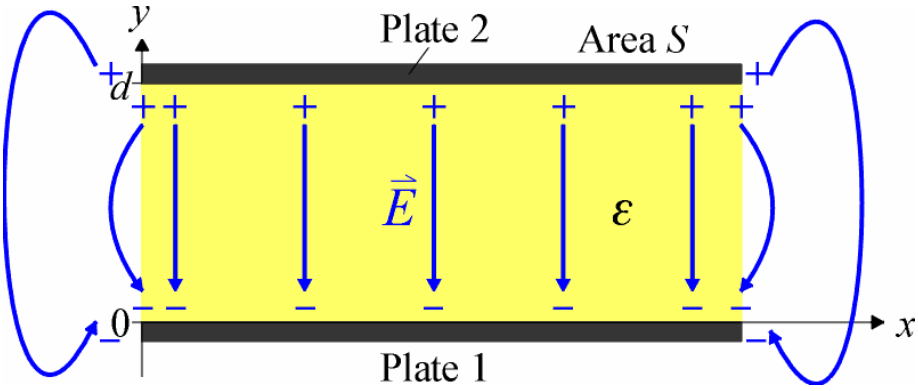

Fig. 9-3. A parallel-plate capacitor with two conducting plates of area *S* spaced by a dielectric material of thickness *d* and permittivity ε. Stray electric field lines exist at the edges.

Example 9-3: Find the capacitance of a coaxial cable of length L with inner and outer conducting radii of a and b, respectively (Fig. 9-4).

Ans: (M1) Assume charges $\pm Q$ are deposited on the conducting plates. Because of cylindrical symmetry, the charges are uniformly distributed on the two conductors. By Gauss's law, $\vec{E} = \vec{a}_r \frac{Q}{2\pi\varepsilon r L}$. The potential difference between the conducing core and shell is: $V_{12} = -\int_b^a \vec{E} \cdot \left(\vec{a}_r dr\right) = \frac{Q}{2\pi\varepsilon L} \ln\left(\frac{b}{a}\right)$. By eq. (9.2), $C = \frac{Q}{V_{12}}$, $\Rightarrow$

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)} \tag{9.4}$$

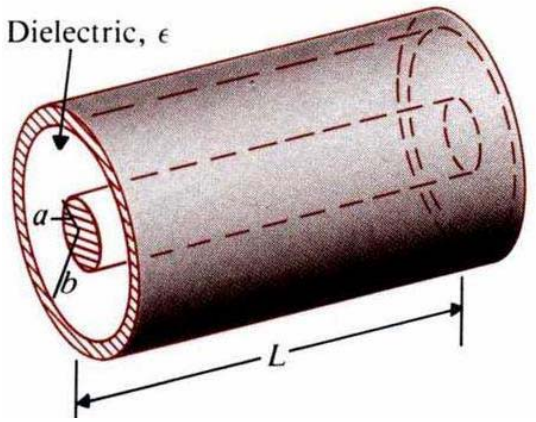

Fig. 9-4. A coaxial capacitor with inner conducting core of radius *a* and outer conducting shell of radius *b*, spaced by a dielectric material of permittivity ε. No stray electric field line exists. (After DKC.)

<Comment>

The capacitance is determined by: (1) The geometry of the conductors. E.g. $\frac{S}{d}$ in parallel plates, $\frac{L}{\ln(b/a)}$ in coaxial cable. (2) The permittivity ε of the dielectric material.

9.2 Electrostatic Energy

■ Electrostatic energy of assembling charges

By eq. (6.13), the potential field due to a point charge Q_1 at a position P_1 (the origin) is: $V(R)=\frac{Q_1}{4\pi\varepsilon_0 R}$ (assuming $V=0$ at infinity). To bring a charge Q_2 from infinity to a position P_2 spaced from P_1 by a distance R_{12}, the amount of work to be done is $W_2=Q_2V_2=\frac{Q_1Q_2}{4\pi\varepsilon_0 R_{12}}$, where $V_2=V(R_{12})$ denotes the potential at P_2 due to charge Q_1 (Fig. 9-5a).

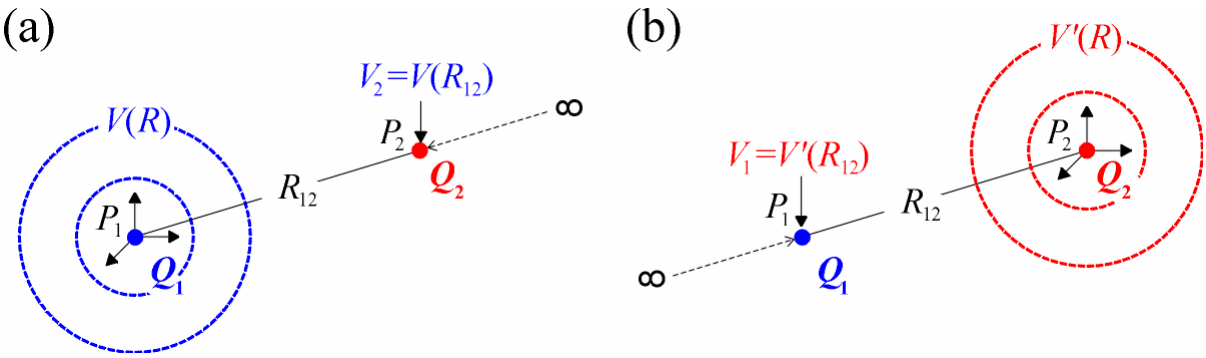

Fig. 9-5. A work W_2 has to be done to assemble a system of two point charges Q_1, Q_2 spaced by a distance of R_{12}.

Similarly, the potential field due to a point charge Q_2 at a position P_2 (the origin) is: $V'(R)=\dfrac{Q_2}{4\pi\varepsilon_0 R}$ (assuming $V'=0$ at infinity). To bring a charge Q_1 from infinity to a position P_1 spaced from P_2 by a distance R_{12}, the amount of required work is still $W_2=Q_1V_1=\dfrac{Q_1Q_2}{4\pi\varepsilon_0 R_{12}}$, where $V_1=V'(R_{12})$ denotes the potential at P_1 due to charge Q_2 (Fig. 9-5b).

If we bring a third point charge Q_3 from infinity to a position P_3 spaced from charges P_1 and P_2 by distances of R_{13} and R_{23}, respectively; an additional work of:

$$\Delta W=Q_3V_3=Q_3\left[V(R_{13})+V'(R_{23})\right]=Q_3\left(\frac{Q_1}{4\pi\varepsilon_0 R_{13}}+\frac{Q_2}{4\pi\varepsilon_0 R_{23}}\right),$$

has to be done. The total energy stored by the system of three charges is:

$$W_3=W_2+\Delta W=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1Q_2}{R_{12}}+\frac{Q_1Q_3}{R_{13}}+\frac{Q_2Q_3}{R_{23}}\right).$$

If V_k denotes the potential of charge Q_k at position P_k ($k=1, 2, 3$) due to the remaining charges, i.e. $V_1=\dfrac{Q_2}{4\pi\varepsilon_0 R_{12}}+\dfrac{Q_3}{4\pi\varepsilon_0 R_{13}}$, $V_2=\dfrac{Q_1}{4\pi\varepsilon_0 R_{12}}+\dfrac{Q_3}{4\pi\varepsilon_0 R_{23}}$, $V_3=\dfrac{Q_1}{4\pi\varepsilon_0 R_{13}}+\dfrac{Q_2}{4\pi\varepsilon_0 R_{23}}$, $\Rightarrow$

$$W_3=\frac{1}{2}\left(Q_1V_1+Q_2V_2+Q_3V_3\right).$$

The electrostatic energy stored by a system of N discrete charges is thus:

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \text{ , where } V_k = \frac{1}{4\pi\varepsilon_0}\sum_{j\neq k}\frac{Q_j}{R_{jk}} \tag{9.5}$$

For a system of continuous charge distribution with volume density $\rho(\vec{r})$ over a volume V', eq. (9.5) is modified as:

$$W_e = \frac{1}{2}\int_{V'} \rho(\vec{r})V(\vec{r})dv \tag{9.6}$$

where $V(\vec{r})$ represents the potential at source point $\vec{r}$ as a result of the "total" charge distribution.

Example 9-4: Find the energy stored in a sphere of radius b with uniform volume charge density ρ.

Ans: By spherical symmetry, $\vec{E} = \vec{a}_R E(R)$. By Gauss's law,

$$\vec{E} = \begin{cases} \vec{a}_R \dfrac{\rho R}{3\varepsilon_0}, \text{if } 0 < R < b \\ \vec{a}_R \dfrac{\rho b^3}{3\varepsilon_0 R^2}, \text{if } R \geq b \end{cases} . \Rightarrow$$

For $0 < R < b$, $V(R) = -\int_{\infty}^{R} \vec{E}(R')\cdot(\vec{a}_R dR') = -\left[\int_{\infty}^{b} \frac{\rho b^3}{3\varepsilon_0 R'^2}dR' + \int_{b}^{R}\frac{\rho R'}{3\varepsilon_0}dR'\right] = \frac{\rho}{6\varepsilon_0}\left(3b^2 - R^2\right)$.

$\Rightarrow$ By eq. (9.6), $W_e = \frac{1}{2}\int_0^b \rho V(R)(4\pi R^2 dR) = \frac{4\pi\rho^2 b^5}{15\varepsilon_0}$ ($\propto b^5$ if ρ is a constant). Since the total charge $Q = \frac{4\pi b^3\rho}{3}$, $\Rightarrow$ $W_e = \frac{3Q^2}{20\pi\varepsilon_0 b}$ ($\propto 1/b$ if Q is a constant).

Note that the potential outside the sphere ($R > b$) is: $V(R) = -\int_{\infty}^{R}\frac{\rho b^3}{3\varepsilon_0 R'^2}dR' = \frac{\rho b^3}{3\varepsilon_0 R}$.

However, this is not used in evaluating the stored energy for all the source points in eq. (9.6) are inside the sphere.

■ Electrostatic energy of electric fields

In terms of real applications of electromagnetism (especially electromagnetic waves), sources are usually far away from the region of interest and only the resulting fields are given. It becomes more convenient to express the electric energy W_e by the electric field quantities $\vec{E}$ and $\vec{D}$ in the absence of the charge distribution ρ.

(1) Substituting $\rho = \nabla \cdot \vec{D}$ [eq. (7.8)] into eq. (9.6), $\Rightarrow$

$$W_e = \frac{1}{2}\int_{V'} \left(\nabla \cdot \vec{D}\right) V dv ,$$

where V' is a volume containing all the source charges.

(2) Substituting $f = V$ and $\vec{A} = \vec{D}$ into the vector identity $\nabla \cdot \left(f\vec{A}\right) = f\left(\nabla \cdot \vec{A}\right) + \vec{A} \cdot \nabla f$, $\Rightarrow$

$$W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\vec{D}) dv - \frac{1}{2}\int_{V'} \vec{D} \cdot \left(\nabla V\right) dv .$$

(3) By divergence theorem [eq. (5.24)], $\Rightarrow$ $\int_{V'} \nabla \cdot (V\vec{D}) dv = \oint_{S'} V\vec{D} \cdot d\vec{s}$, where S' is the closed surface of V'. By $\vec{E} = -\nabla V$ [eq. (6.11)], $\Rightarrow$ $\int_{V'} \vec{D} \cdot \left(\nabla V\right) dv = -\int_{V'} \left(\vec{D} \cdot \vec{E}\right) dv$. $\Rightarrow$

$$W_e = I_1 + I_2 , \text{ where } I_1 = \frac{1}{2}\oint_{S'} V\vec{D} \cdot d\vec{s} , \quad I_2 = \frac{1}{2}\int_{V'} \left(\vec{D} \cdot \vec{E}\right) dv .$$

(4) One can choose S' as a spherical surface centered at the origin with an infinite radius $R \to \infty$, such that all the source charges are definitely enclosed. For an observation point (on S') far away from the source (at the origin), the potential $V \propto \frac{1}{R}$, and the field magnitude $\left|\vec{D}\right| \propto \frac{1}{R^2}$. $\Rightarrow$ $I_1 = \frac{1}{2}\oint_{S'} V\vec{D} \cdot d\vec{s} \approx \frac{1}{2} V(R) \left|\vec{D}(R)\right| \cdot 4\pi R^2 \propto \frac{1}{R} \cdot \frac{1}{R^2} \cdot R^2 \propto \frac{1}{R} \to 0$, $W_e = I_2$. $\Rightarrow$

$$W_e = \int_{V'} w_e(\vec{r}) dv , \quad w_e = \frac{1}{2}\vec{D} \cdot \vec{E} \ \ (\text{J/m}^3) \tag{9.7}$$

where V' has to cover everywhere with nonzero electric field, and w_e represents the electrostatic energy density. However, the physical justification has not been found to localized energy with an electric field.

Example 9-5: Find the energy stored in a parallel-plate capacitor with two conducting plates of area S spaced by a dielectric material of thickness d and permittivity ε (Fig. 9-3) and biased by a dc voltage V.

Ans: According to the solution to Example 9-2, the electric field between the plates is uniform: $\vec{E} = -\vec{a}_y \frac{V}{d}$, $\vec{D} = \varepsilon\vec{E}$. By eq. (9.7), the energy density is: $w_e = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2$. The total stored energy (assuming no field exists outside the capacitor) is:

$$W_e = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2 \cdot Sd = \frac{1}{2}\left(\varepsilon\frac{S}{d}\right)V^2.$$

By eq's (9.2), (9.3), $\Rightarrow$ $C = \frac{Q}{V} = \varepsilon\frac{S}{d}$, we got:

$$W_e = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{1}{2}QV \tag{9.8}$$

Eq. (9.8) is useful in evaluating electrostatic energy stored in general capacitors.

Example 9-6: Find the capacitance of a coaxial cable (Example 9-3) by eq. (9.8).

Ans: (M3) Assume charges $\pm Q$ are deposited on the conducting plates. Because of cylindrical symmetry, we can apply Gauss's law to get: $\vec{E} = \vec{a}_r \frac{Q}{2\pi\varepsilon rL}$, $w_e = \frac{1}{2}\varepsilon\left(\frac{Q}{2\pi\varepsilon Lr}\right)^2$,

$\Rightarrow$ $W_e = \frac{1}{2}\varepsilon\int_a^b\left(\frac{Q}{2\pi\varepsilon Lr}\right)^2(2\pi r dr L) = \frac{Q^2}{4\pi\varepsilon L}\left(\int_a^b\frac{dr}{r}\right) = \frac{Q^2}{4\pi\varepsilon L}\ln\left(\frac{b}{a}\right)$. By eq. (9.8), $\Rightarrow$

$$C = \frac{Q^2}{2W_e} = \frac{2\pi\varepsilon L}{\ln(b/a)},$$

same as eq. (9.4).

Example 9-7: Find the energy stored in a sphere of radius b with uniform volume charge density ρ (Example 9-4) by eq. (9.7).

Ans: By spherical symmetry, $\vec{E} = \bar{a}_R E(R)$. By Gauss's law,

$$\vec{E}=\begin{cases}\vec{a}_R\dfrac{\rho R}{3\varepsilon_0}, \text{if } 0<R<b\\ \vec{a}_R\dfrac{\rho b^3}{3\varepsilon_0 R^2}, \text{if } R\ge b\end{cases}.$$

For $0<R<b$, $W_{e1}=\int_0^b\frac{1}{2}\varepsilon_0\left(\frac{\rho R}{3\varepsilon_0}\right)^2(4\pi R^2dR)=\frac{2\pi\rho^2}{9\varepsilon_0}\left(\int_0^b R^4dR\right)=\frac{2\pi\rho^2b^5}{45\varepsilon_0}$.

For $R>b$, $W_{e2}=\int_b^\infty\frac{1}{2}\varepsilon_0\left(\frac{\rho b^3}{3\varepsilon R^2}\right)^2(4\pi R^2dR)=\frac{2\pi\rho^2b^6}{9\varepsilon_0}\left(\int_b^\infty\frac{1}{R^2}dR\right)=\frac{2\pi\rho^2b^5}{9\varepsilon_0}$.

The total stored energy is:

$$W_e=W_{e1}+W_{e2}=\frac{4\pi\rho^2b^5}{15\varepsilon_0}=\frac{3Q^2}{20\pi\varepsilon_0 b},$$

equal to that derived by eq. (9.6). The corresponding "capacitance" can be evaluated by eq. (9.8): $C=\frac{Q^2}{2W_e}=\frac{10\pi\varepsilon_0 b}{3}$.

Example 9-8: Find the energy stored in a conducting sphere of radius b with total charge Q by eq. (9.7). Find the corresponding capacitance (Example 9-1) by eq. (9.8).

Ans: By spherical symmetry and Gauss's law, $\vec{E}=\begin{cases}\vec{a}_R\dfrac{Q}{4\pi\varepsilon_0R^2}, \text{if } R\ge b\\ 0, \text{otherwise}\end{cases}\Rightarrow$

$$W_e=\int_b^\infty\frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0R^2}\right)^2(4\pi R^2dR)=\frac{Q^2}{8\pi\varepsilon_0 b}.$$

The corresponding capacitance can be evaluated by eq. (9.8): $C=\frac{Q^2}{2W_e}=4\pi\varepsilon_0 b$.