Lesson 9
Capacitance, Electrostatic Energy

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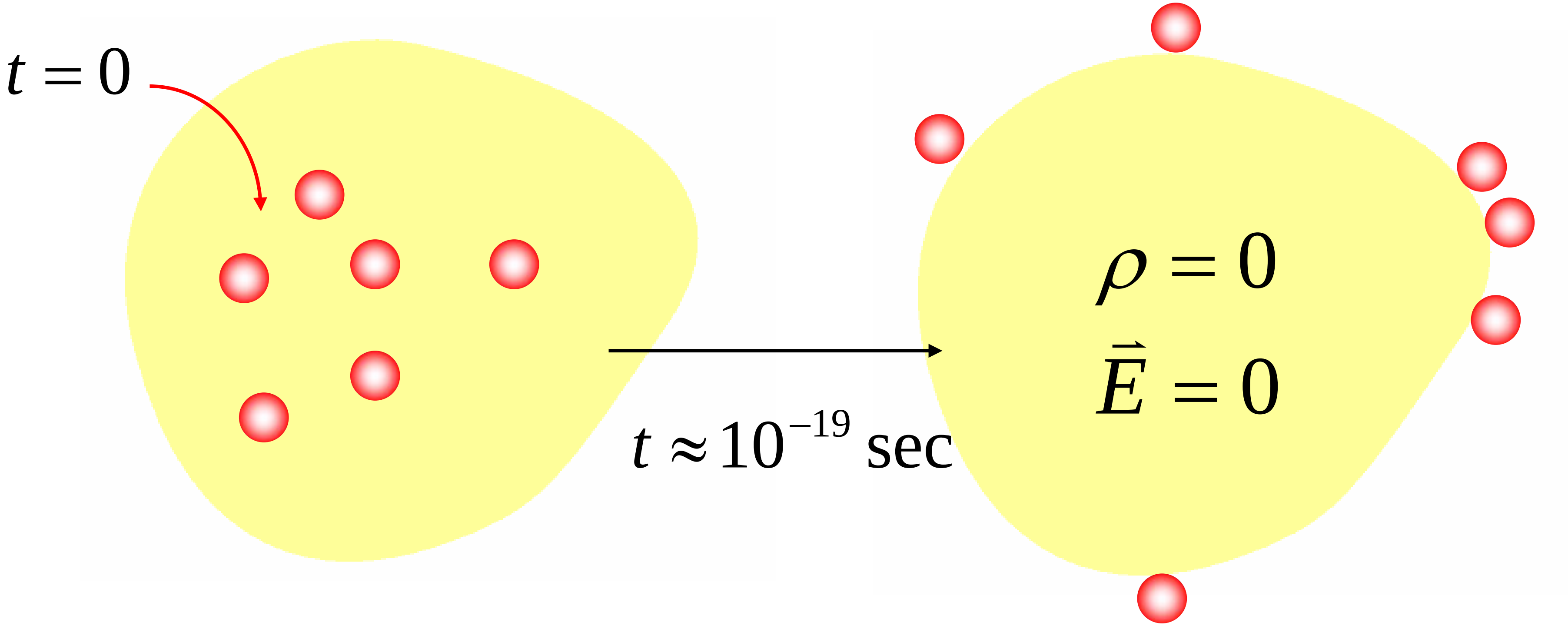
Outline

- Capacitance
- Electrostatic energy

Sec. 9-1 Capacitance

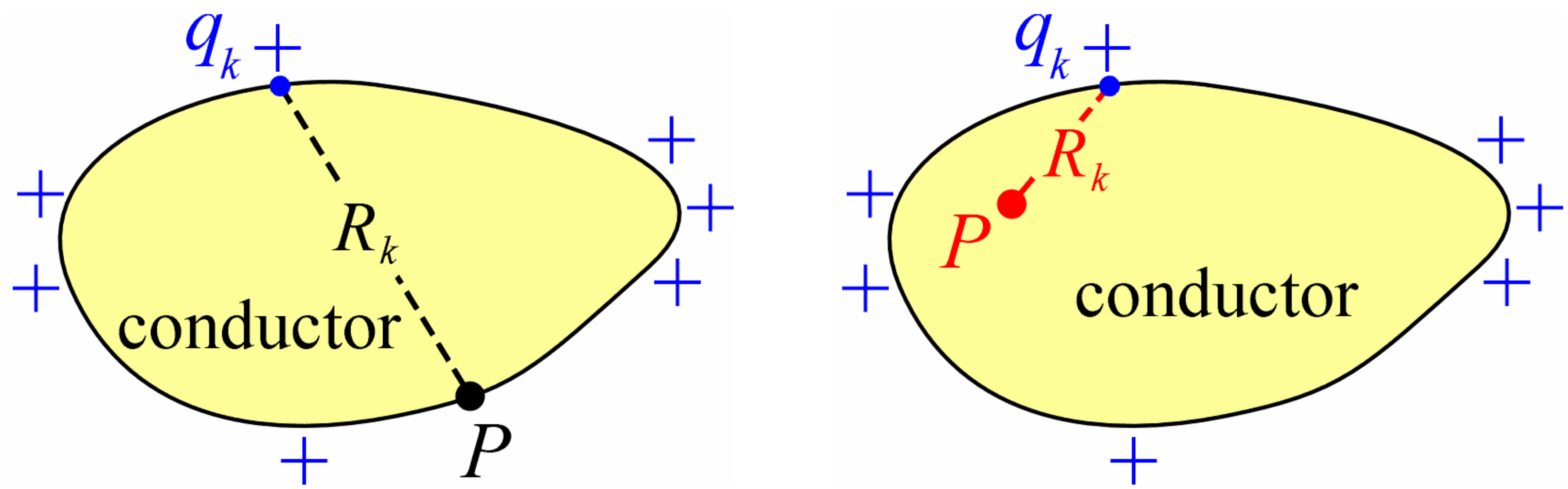
1. Single-conductor capacitors
2. Two-conductor capacitors
3. Methods to evaluate capacitance

Single-conductor capacitor-1



Single-conductor capacitor-2

$$Q = \sum_{k=1}^{n} q_k, \qquad V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{R_k}$$



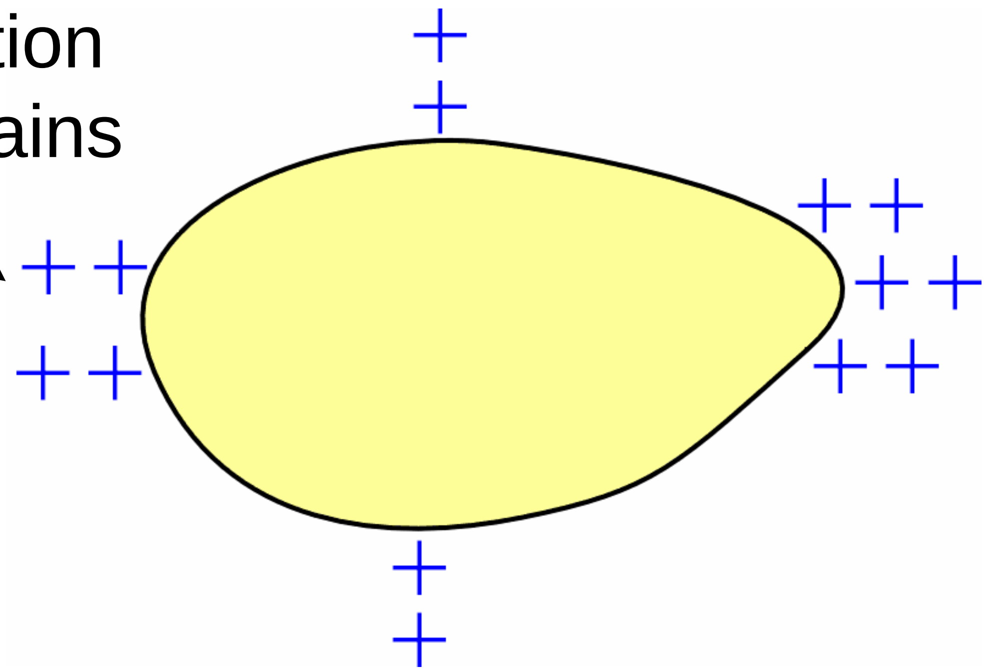
On the surface

Inside the conductor

Single-conductor capacitor-3

$$Q' = 2Q, \quad \Rightarrow V' = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{2q_k}{R_k} = 2V$$

Spatial distribution of $\rho_s(\vec{r})$ maintains



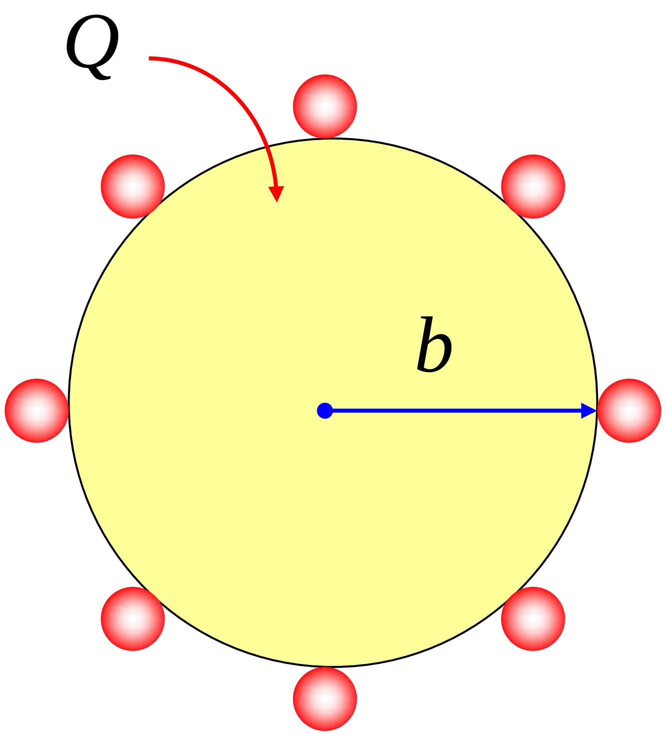
Single-conductor capacitor-4

The constant ratio of the deposited charge to the resulting potential is defined as the capacitance of a single conductor.

$$\boxed{C \equiv \frac{Q}{V}}$$

Example 9-1: Conducting sphere (1)

Find the capacitance of a conducting sphere of radius b



Assume charge Q is deposited

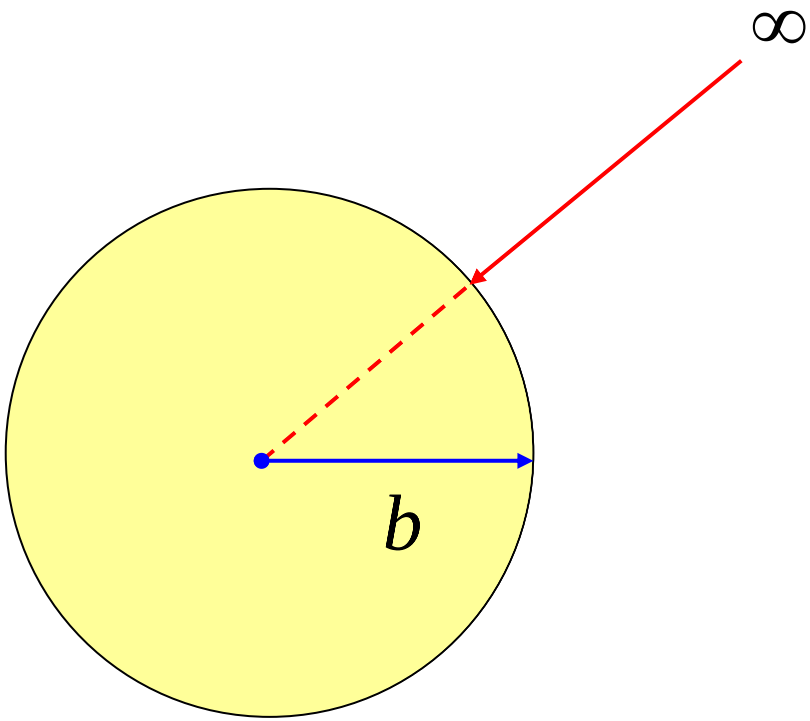
Spherical symmetry, $\Rightarrow \rho_s(\vec{r})$

is uniformly distributed, $\Rightarrow$

$$\vec{E} = \begin{cases} \vec{a}_R \dfrac{Q}{4\pi\varepsilon_0 R^2}, \text{if } R \geq b \\ 0, \text{if } 0 < R < b \end{cases}$$

Example 9-1: Conducting sphere (2)

The surface potential is:



$$V(R=b)$$

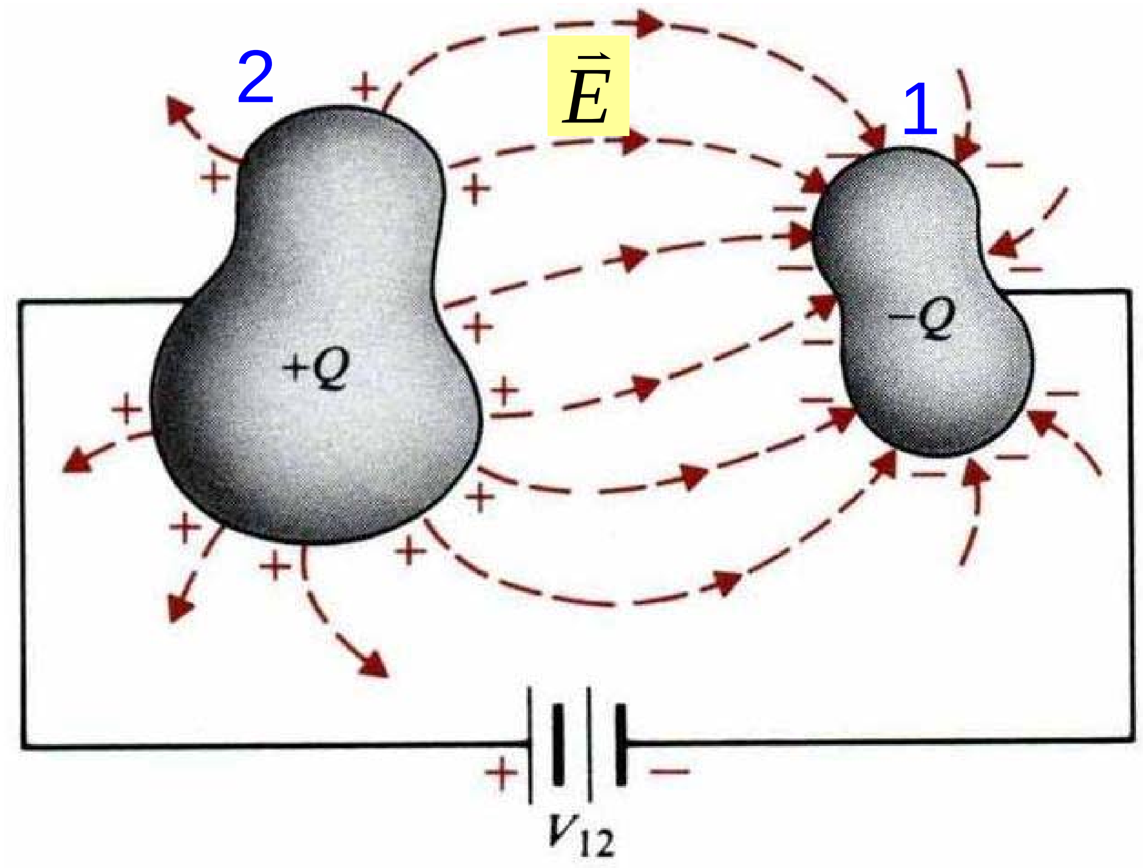
$$= -\int_{\infty}^{b}\left(\vec{a}_R \frac{Q}{4\pi\varepsilon_0 R^2}\right)\cdot\left(\vec{a}_R dR\right)$$

$$= \frac{Q}{4\pi\varepsilon_0 b} \propto Q$$

$$\Rightarrow C \equiv \frac{Q}{V} = 4\pi\varepsilon_0 b$$

Two-conductor capacitor-1

$$V_{12} \rightarrow \pm Q \rightarrow \vec{E} \rightarrow \underbrace{-\int_1^2 \vec{E}\cdot d\vec{l}}_{\text{any path}} = V_{12}$$



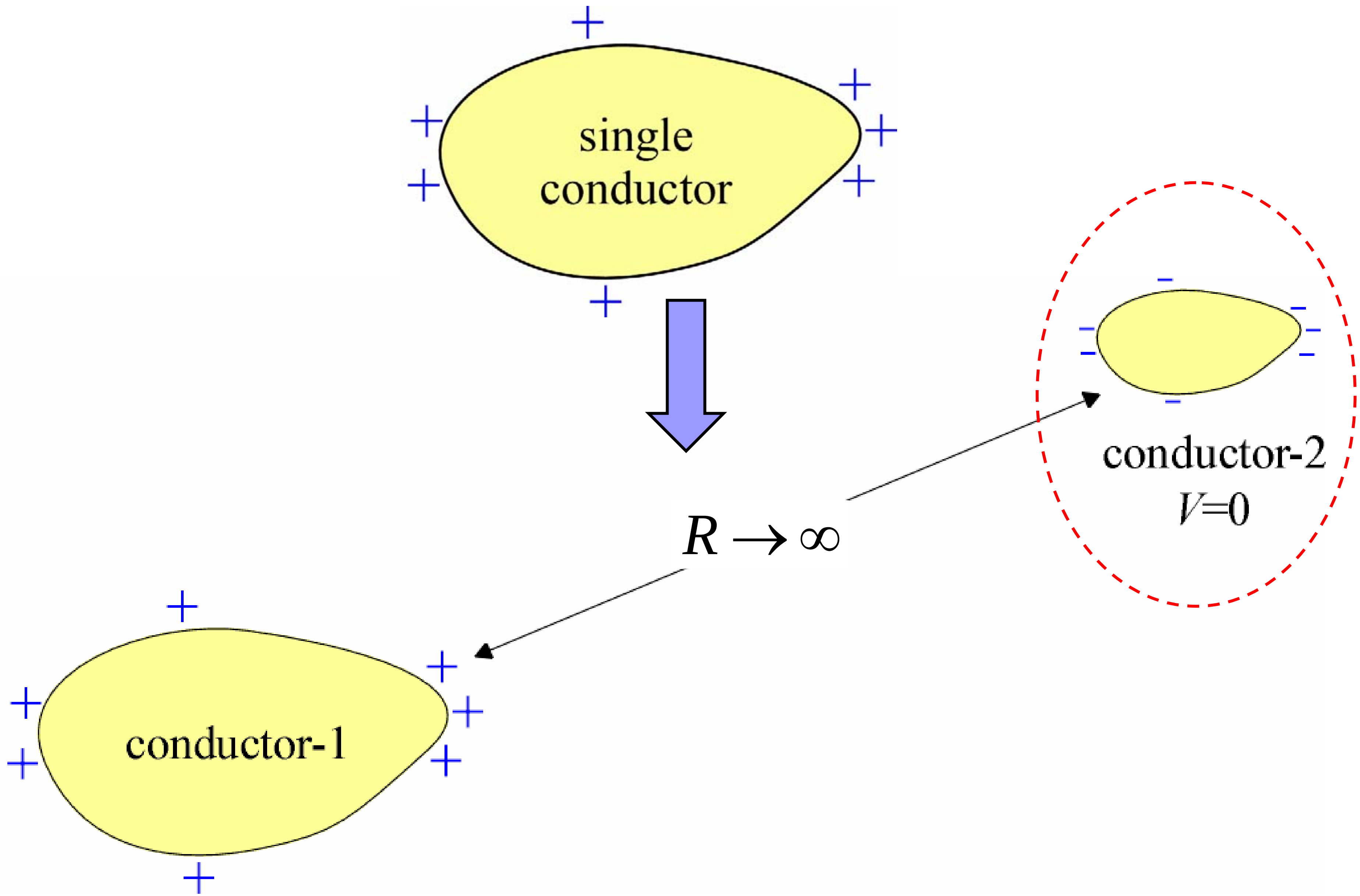
Two-conductor capacitor-2

$$V'_{12} = rV_{12} \rightarrow \pm rQ \rightarrow r\vec{E}$$

The constant ratio of the deposited charge to the voltage difference is defined as the capacitance of the conducting pair:

$$\boxed{C \equiv \frac{Q}{V_{12}}}$$

Comment



Evaluation of capacitance (Method 1)

1. Assume charges $\pm Q$ are deposited

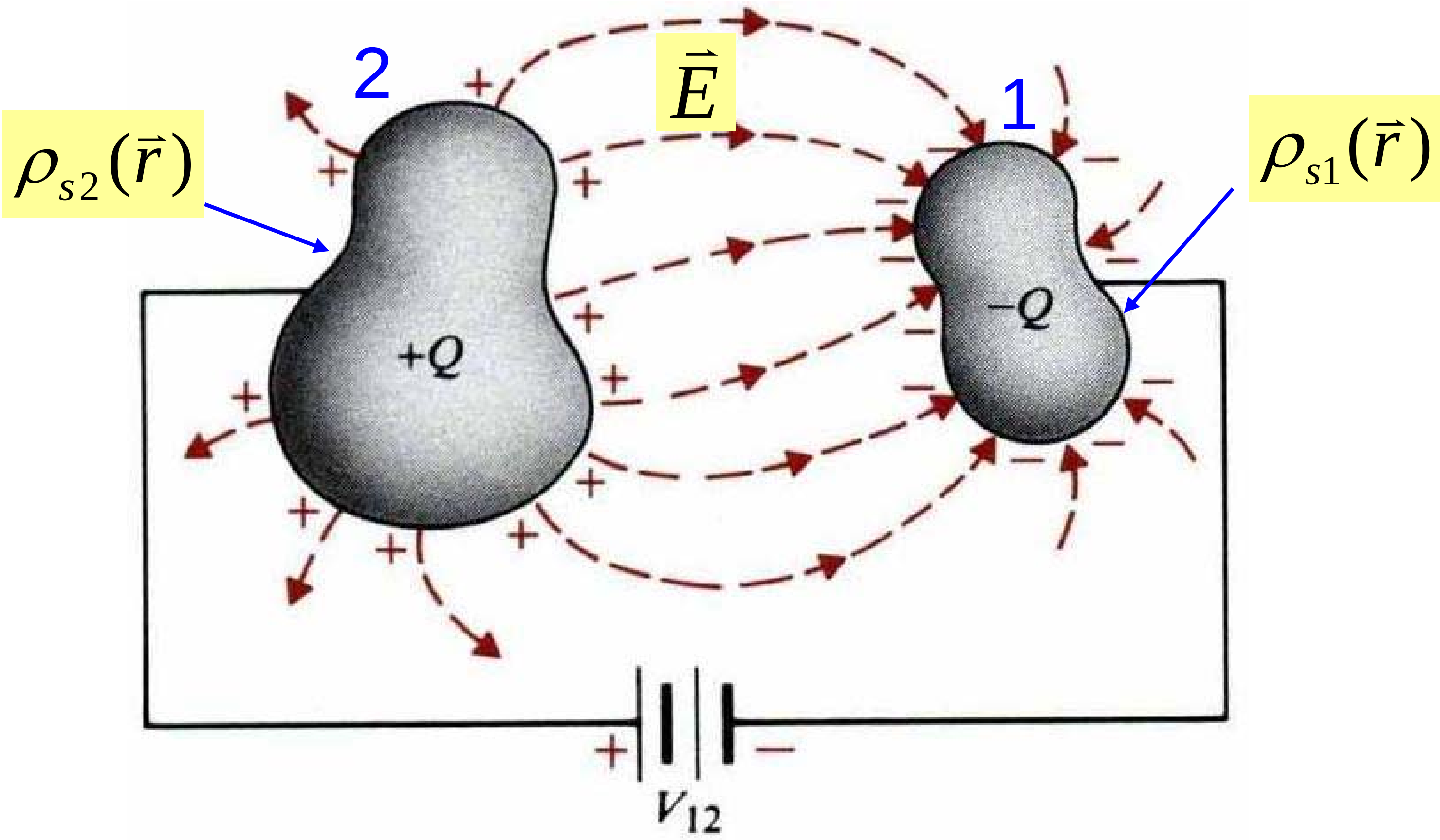
2. Find $\vec{E}$ by Gauss's law or

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_{S_1+S_2} \vec{a}_R \frac{\rho_s(\vec{r}\,')}{R(\vec{r},\vec{r}\,')^2} ds'$$

3. Find $V_{12}\left(\propto Q\right)$ by $V_{12} = -\int_1^2 \vec{E} \cdot d\vec{l}$

4. Find C by $C = \dfrac{Q}{V_{12}}$, independent of Q

Evaluation of capacitance-reference figure



Evaluation of capacitance (Method 2)

1. Assume V_{12} between the conductors

2. Find $V(\vec{r})$ by solving $\nabla^2 V = 0$ with BC

3. Find $\vec{E}$ by $\vec{E} = -\nabla V(\vec{r})$

4. Find $\rho_s(\vec{r})$ of either conductor by $E_n = \dfrac{\rho_s}{\varepsilon_0}$

5. Find deposited Q by $Q = \oint_S \rho_s(\vec{r}) ds \left(\propto V_{12}\right)$

6. Find C by $C = \dfrac{Q}{V_{12}}$, independent of V_{12}

Evaluation of capacitance (Method 3)

1. Assume $\pm Q$ or V for the two conductors

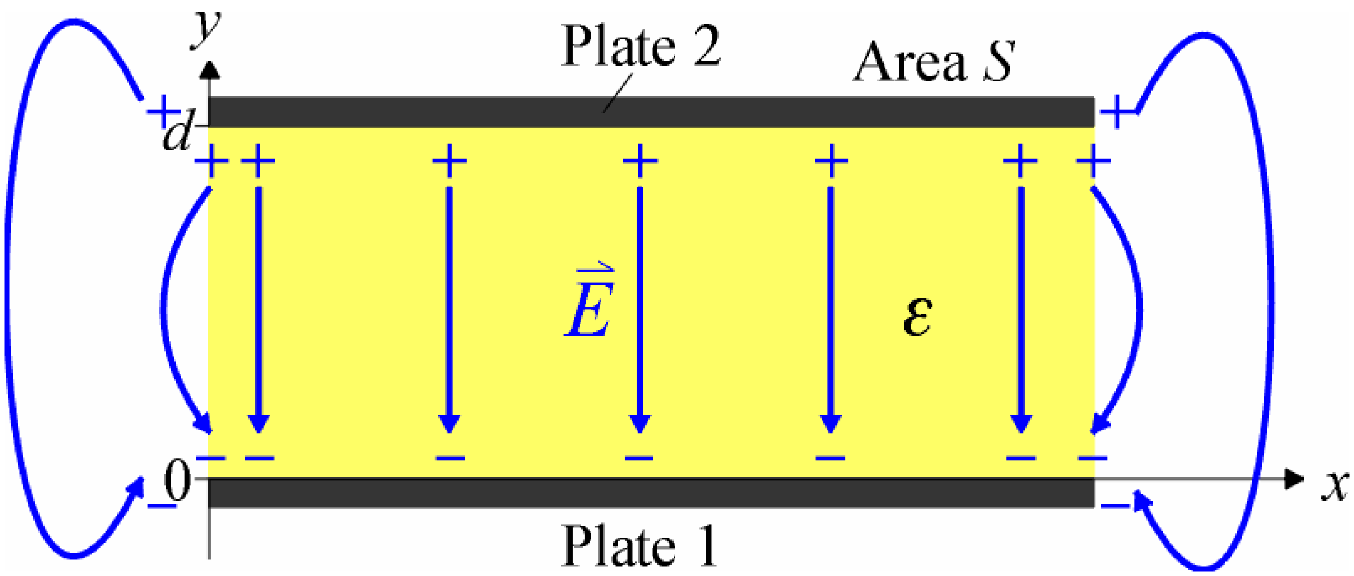
2. Find $\vec{E}$ and $\vec{D}$ by M1, M2

3. Find the stored energy

$$W_e = \int_{V'} \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) dv, \text{which is } \propto Q^2 \text{ or } V^2$$

4. Find C by $W_e = \dfrac{Q^2}{2C}$ or $W_e = \dfrac{CV^2}{2}$

Example 9-2: Parallel-plate capacitor (1)



1. Assume charges $\pm Q$ are deposited

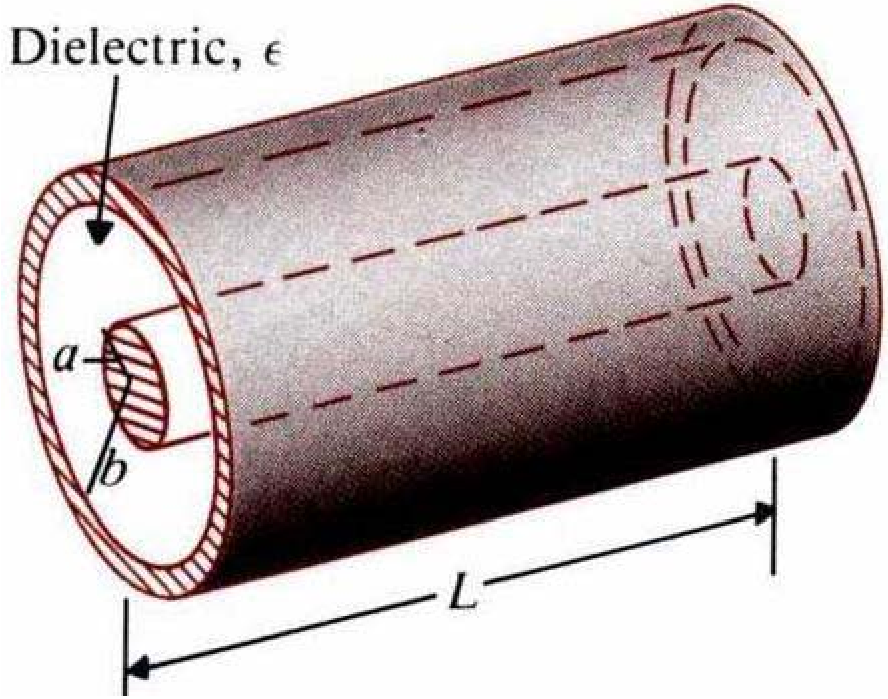
2. By Gauss's law (planar sym.): $\vec{E} = -\vec{a}_y \dfrac{Q}{\varepsilon S}$

Example 9-2: Parallel-plate capacitor (2)

3. $V_{12} = -\int_0^d \vec{E} \cdot \left(\vec{a}_y dy \right) = \frac{Qd}{\varepsilon S}$

4. $C = \frac{Q}{V_{12}} = \varepsilon \frac{S}{d}$

Example 9-3: Coaxial cable capacitor (1)



1. Assume charges $\pm Q$ are deposited

2. By Gauss's law (cylindrical sym.): $\vec{E} = \vec{a}_r \dfrac{Q}{2\pi\varepsilon r L}$

Example 9-3: Coaxial cable capacitor (2)

3. $V_{12} = -\int_b^a \vec{E} \cdot \left(\vec{a}_r dr\right) = \frac{Q}{2\pi\varepsilon L} \ln\left(\frac{b}{a}\right)$

4. $C = \frac{Q}{V_{12}} = \boxed{\frac{2\pi\varepsilon L}{\ln\left(b/a\right)}}$

Now you know how to evaluate the capacitance per unit length of coaxial TX lines!

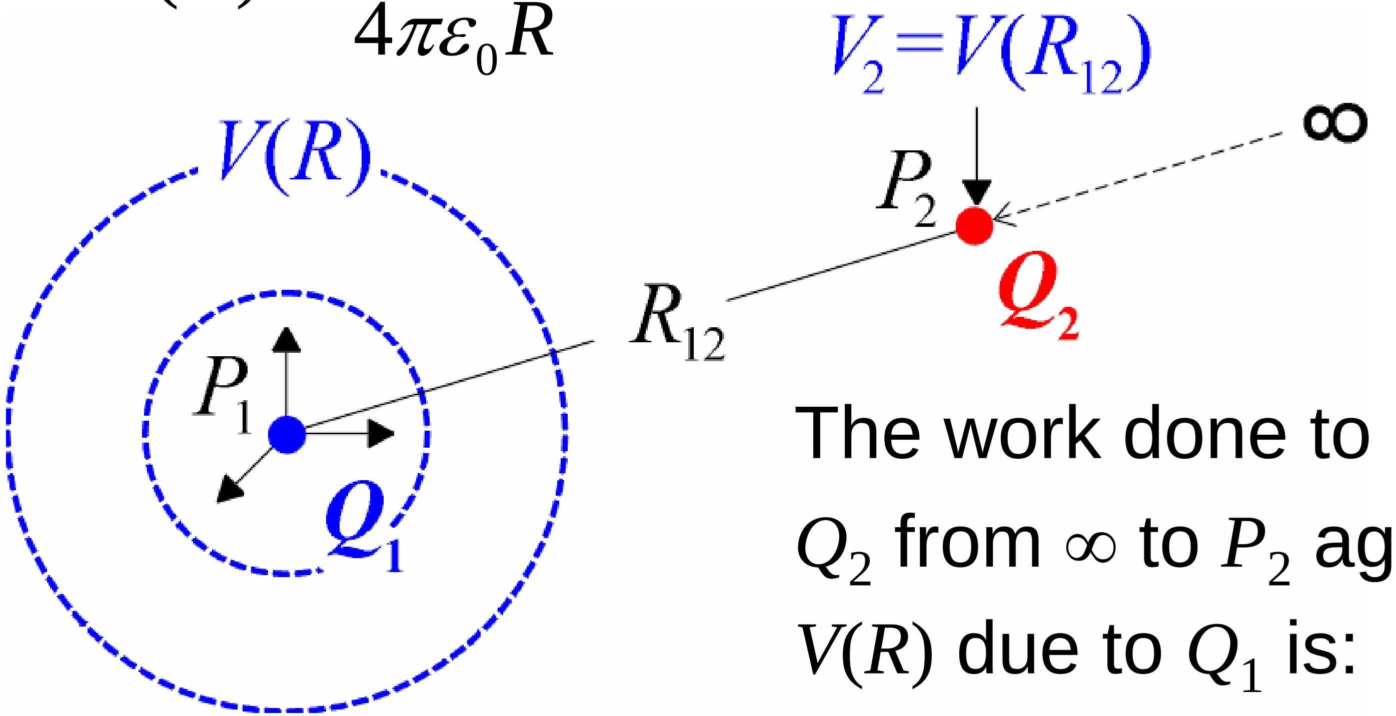
$$\frac{C}{L} = \frac{2\pi\varepsilon}{\ln\left(b/a\right)}$$

Sec. 9-2
Electrostatic Energy

1. Energy of charges
2. Energy of fields

Energy of two charges-1

$$V(R) = \frac{Q_1}{4\pi\varepsilon_0 R}$$



The work done to move Q_2 from ∞ to P_2 against $V(R)$ due to Q_1 is:

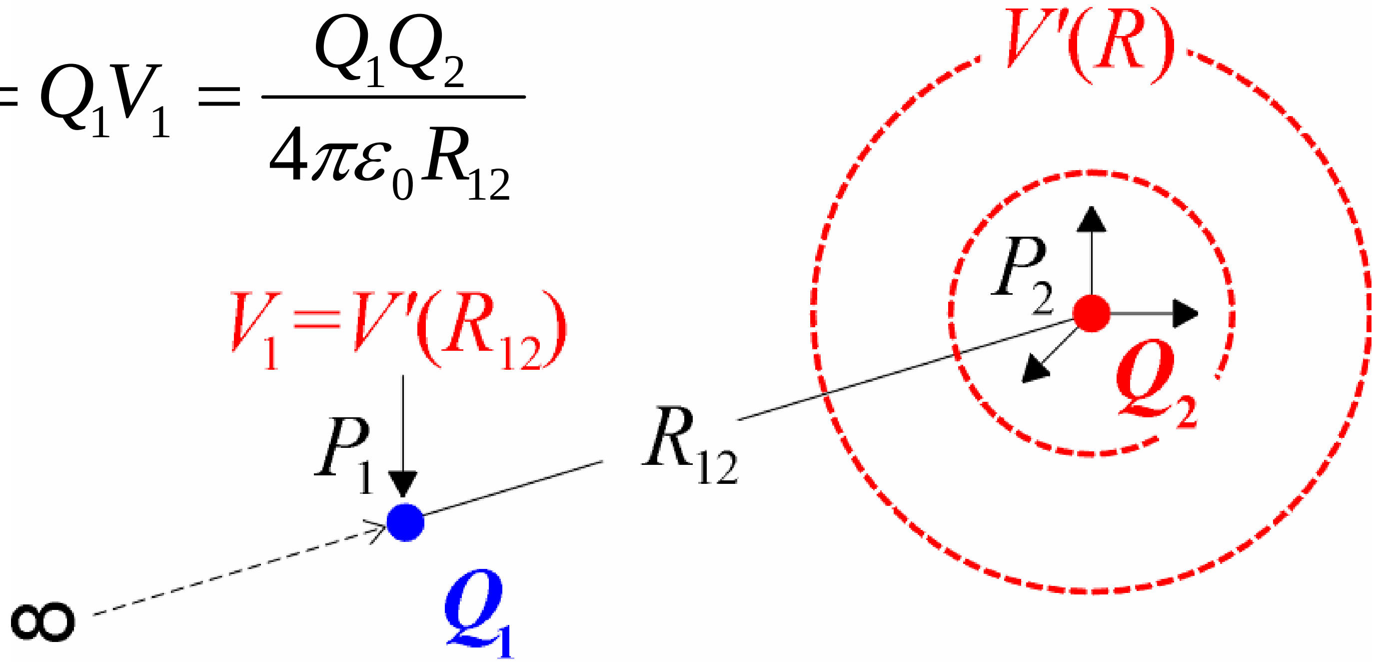
$$W_2 = Q_2 V_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}}$$

Energy of two charges-2

The work done to move Q_1 from ∞ to P_1 against $V'(R)$ due to Q_2 is:

$$V'(R) = \frac{Q_2}{4\pi\varepsilon_0 R}$$

$$W_2 = Q_1 V_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}}$$



Energy of two charges-3

The electrostatic energy stored by a system of two charges Q_1-Q_2 is:

$$W_2 = Q_1V_1 = Q_2V_2 = \frac{1}{2}\left(Q_1V_1 + Q_2V_2\right)$$

Potential of Q_1 at P_1 due to Q_2

Potential of Q_2 at P_2 due to Q_1

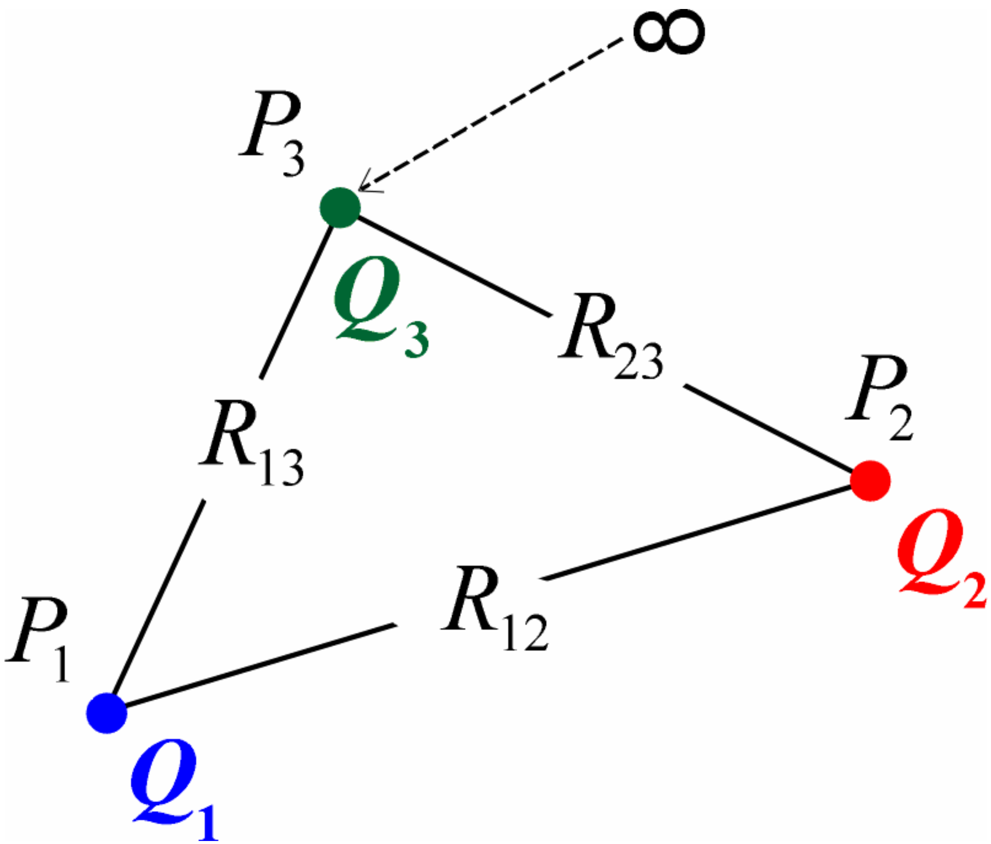
Energy of three charges-1

The extra work done to move Q_3 from ∞ to P_3 against $V(R)$, $V'(R)$ due to Q_1, Q_2 is:

$$V_3 = V(R_{13}) + V'(R_{23})$$

$$\Delta W = Q_3 V_3$$

$$= Q_3 \left(\frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}} \right)$$



Energy of three charges-2

The electrostatic energy stored by a system of three charges Q_1-Q_2-Q_3 is:

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1Q_2}{R_{12}} + \frac{Q_1Q_3}{R_{13}} + \frac{Q_2Q_3}{R_{23}}\right)$$

Energy of three charges-3

V_k denotes the potential of charge Q_k at position P_k (k =1, 2, 3) due to the remaining charges, i.e.,

$$V_1 = \frac{Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{13}}, \quad V_2 = \frac{Q_1}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{23}},$$

$$V_3 = \frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}}$$

$$\Rightarrow \quad W_3 = \frac{1}{2}\left(Q_1 V_1 + Q_2 V_2 + Q_3 V_3\right)$$

Energy of N charges

The electrostatic energy stored by a system of ***N*** discrete charges Q_1-Q_2-…-Q_N is:

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k}$$

$$V_k = \frac{1}{4\pi\varepsilon_0}\sum_{j\neq k}\frac{Q_j}{R_{jk}}$$

Energy of continuous charge distributions

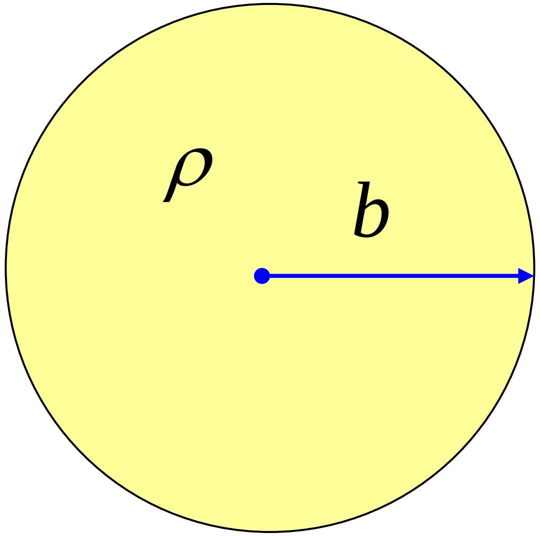
The electrostatic energy stored by a system of continuous charge distribution $\rho(\vec{r})$ over V' is:

$$\boxed{W_e = \frac{1}{2}\int_{V'} \rho(\vec{r})V(\vec{r})dv}$$

Potential at source point $\vec{r}$ due to the total charge distribution

Example 9-4: Sphere of uniform charge density (1)

Find the energy stored in a sphere of radius b with uniform volume charge density ρ



Spherical symmetry, $\Rightarrow$

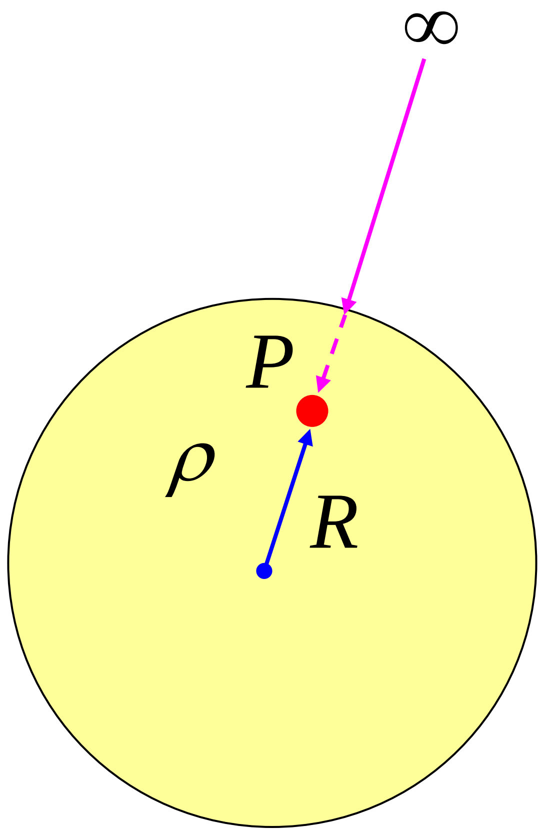
$$\vec{E} = \vec{a}_R E(R)$$

By Gauss's law, $\Rightarrow$

$$\vec{E} = \begin{cases} \vec{a}_R \dfrac{\rho R}{3\varepsilon_0}, \text{ if } 0 < R < b \\ \vec{a}_R \dfrac{\rho b^3}{3\varepsilon_0 R^2}, \text{ if } R \geq b \end{cases}$$

Example 9-4: Sphere of uniform charge density (2)

For any point P inside the sphere $\left(0<R<b\right)$



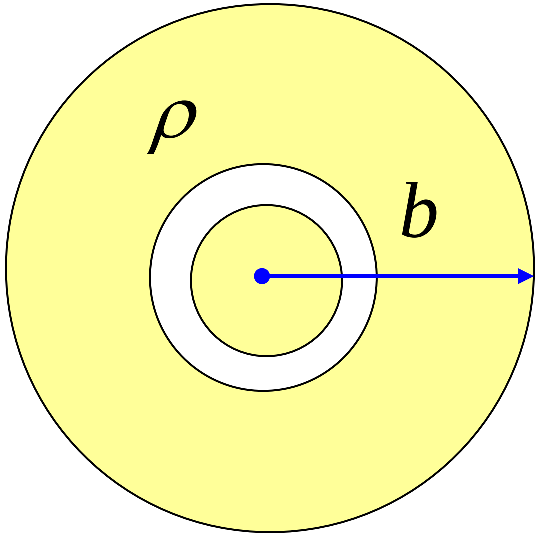
$$V(R) = -\int_{\infty}^{R} \vec{E}(R')\cdot\left(\vec{a}_R dR'\right)$$

$$= -\left[\int_{\infty}^{b} \frac{\rho b^3}{3\varepsilon_0 R'^2} dR' + \int_{b}^{R} \frac{\rho R'}{3\varepsilon_0} dR'\right]$$

$$= -\frac{\rho}{3\varepsilon_0}\left[\int_{\infty}^{b} \frac{b^3}{R'^2} dR' + \int_{b}^{R} R' dR'\right]$$

$$= -\frac{\rho}{3\varepsilon_0}\left[-\frac{b^3}{R'}\bigg|_{\infty}^{b} + \frac{R'^2}{2}\bigg|_{b}^{R}\right] = \frac{\rho}{6\varepsilon_0}\left(3b^2 - R^2\right)$$

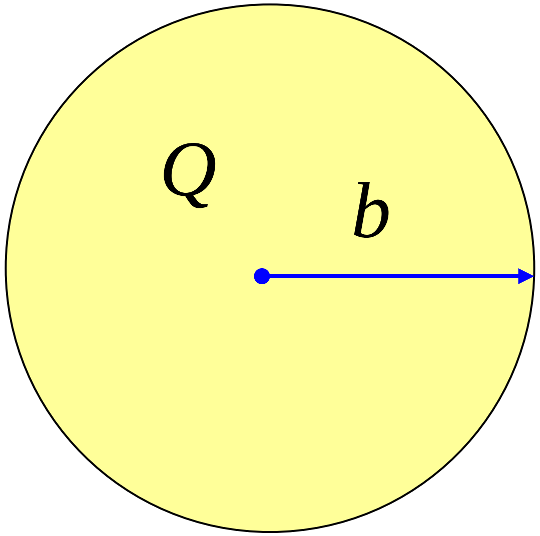
Example 9-4: Sphere of uniform charge density (3)



By $W_e = \frac{1}{2}\int_{V'} \rho(\vec{r})V(\vec{r})dv$

$$W_e = \frac{1}{2}\int_0^b \rho V(R)(4\pi R^2 dR)$$

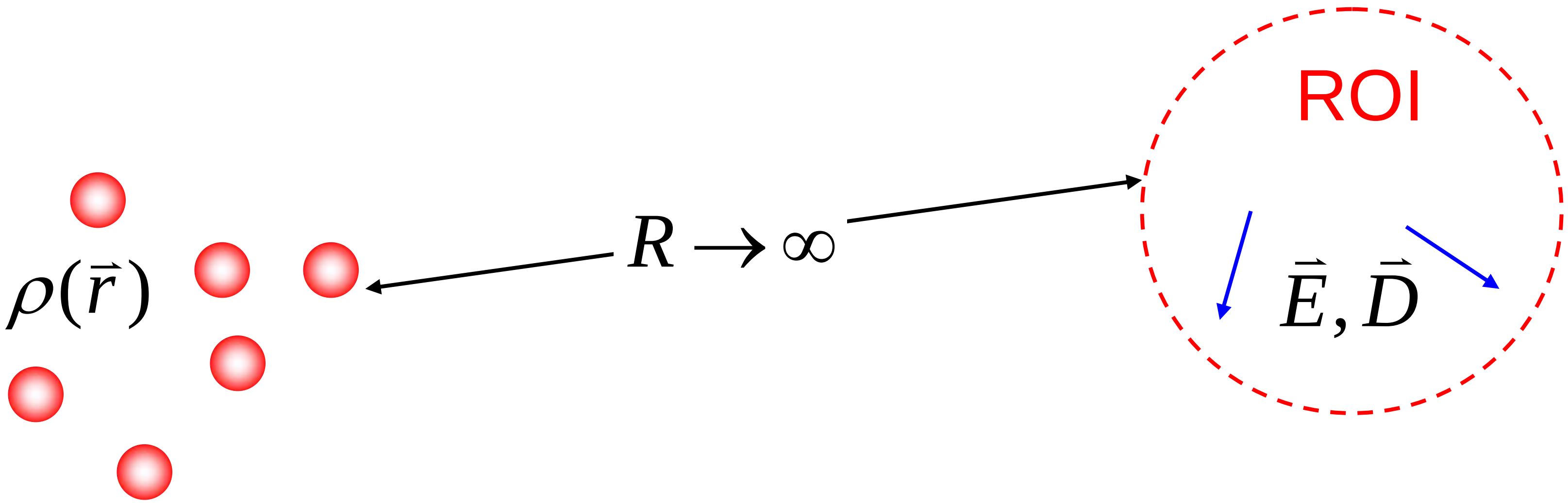
$$= \boxed{\frac{4\pi\rho^2 b^5}{15\varepsilon_0}} \propto b^5 \text{ (if } \rho = \text{constant)}$$



Total charge is: $Q = \frac{4\pi b^3 \rho}{3}$

$$W_e = \boxed{\frac{3Q^2}{20\pi\varepsilon_0 b}} \propto \frac{1}{b} \text{ (if } Q = \text{constant)}$$

In real applications (especially electromagnetic waves), sources are usually far away from the region of interest, only the fields are given.

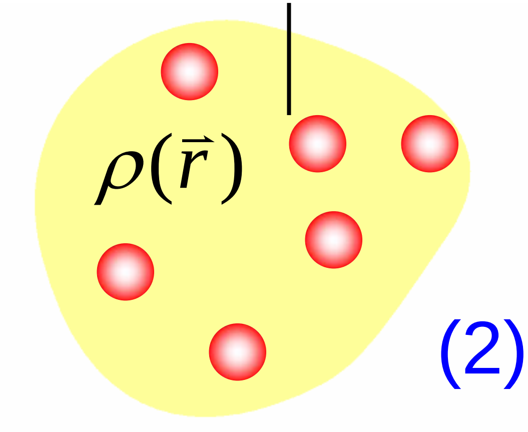


Energy of electric fields-2

$$(1)\quad W_e = \frac{1}{2}\int_{V'} \rho(\vec{r})V(\vec{r})dv = \frac{1}{2}\int_{V'} \left(\nabla \cdot \vec{D}\right)V dv$$

$$\rho = \nabla \cdot \vec{D} \qquad \vec{A} \quad f$$

contain all the source charges V'



By vector identity:

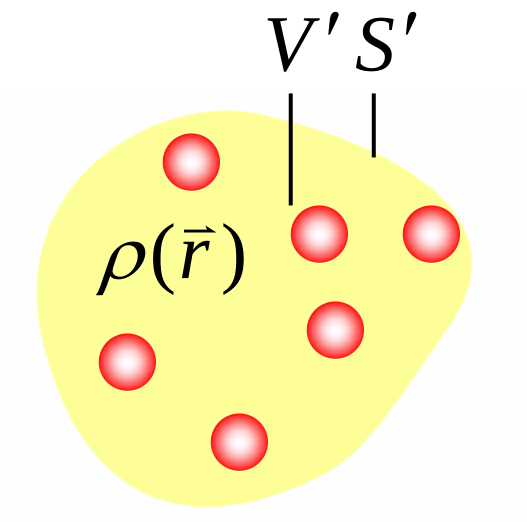
$$\nabla \cdot \left(f\vec{A}\right) = f\left(\nabla \cdot \vec{A}\right) + \vec{A} \cdot \nabla f$$

$$(2)\quad W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\vec{D})dv - \frac{1}{2}\int_{V'} \vec{D} \cdot \left(\nabla V\right)dv$$

Energy of electric fields-3

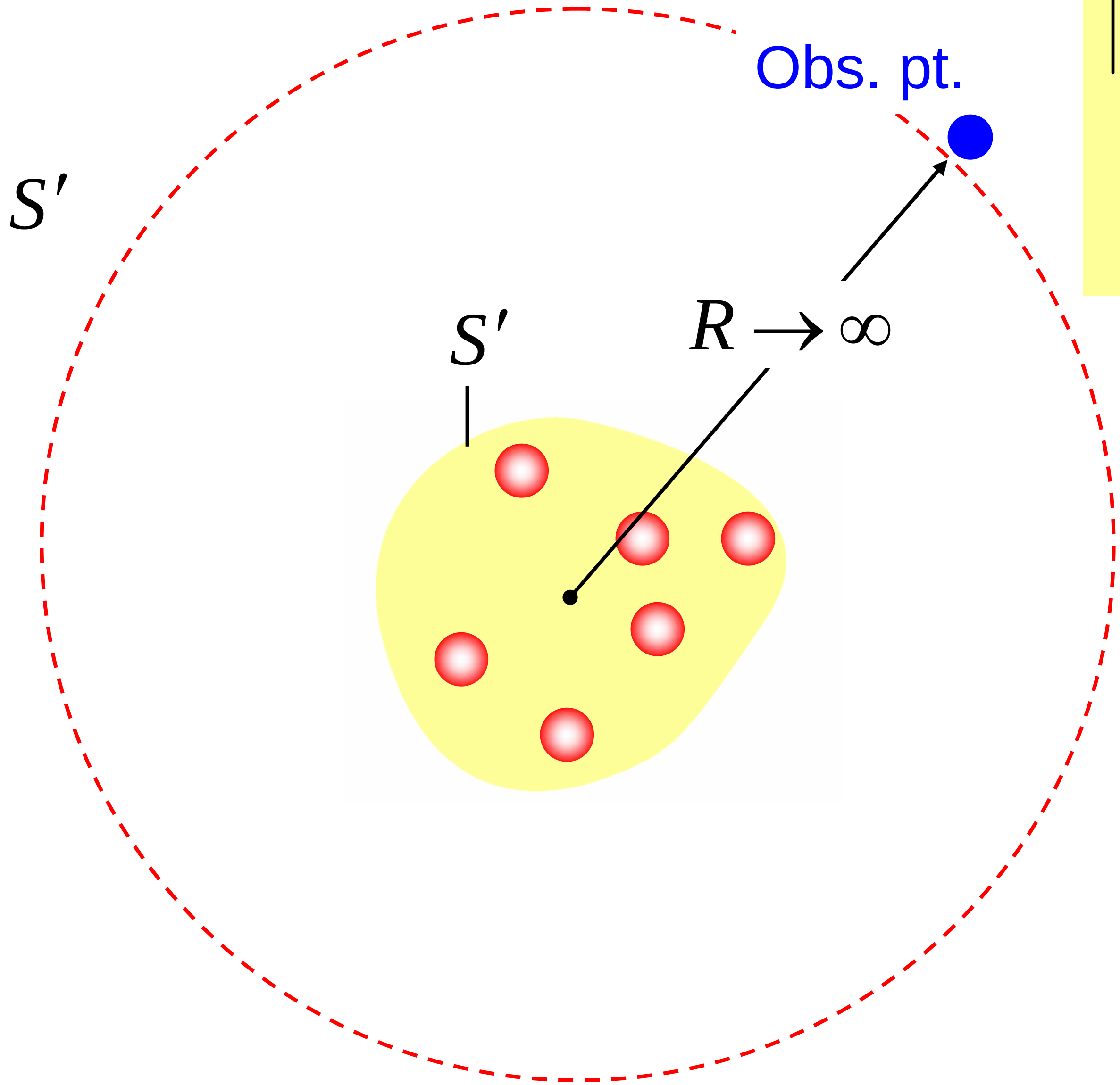
$$\because \oint_S \vec{A}\cdot d\vec{s} = \int_V \left(\nabla\cdot\vec{A}\right)dv, \quad \Rightarrow \int_{V'} \nabla\cdot(V\vec{D})dv = \oint_{S'} V\vec{D}\cdot d\vec{s}$$

$$\because \vec{E} = -\nabla V, \quad \Rightarrow \int_{V'} \vec{D}\cdot\left(\nabla V\right)dv = -\int_{V'}\left(\vec{D}\cdot\vec{E}\right)dv$$



(3) $$W_e = \underbrace{\frac{1}{2}\oint_{S'} V\vec{D}\cdot d\vec{s}}_{I_1} + \underbrace{\frac{1}{2}\int_{V'}\left(\vec{D}\cdot\vec{E}\right)dv}_{I_2}$$

Energy of electric fields-4



$$\left|\vec{D}\right| \propto \frac{1}{R^2}$$

$$V \propto \frac{1}{R}$$

Energy of electric fields-5

$$I_1 = \frac{1}{2}\oint_{S'} V\vec{D}\cdot d\vec{s} \approx \frac{1}{2}V(R)\left|\vec{D}(R)\right|\cdot 4\pi R^2$$

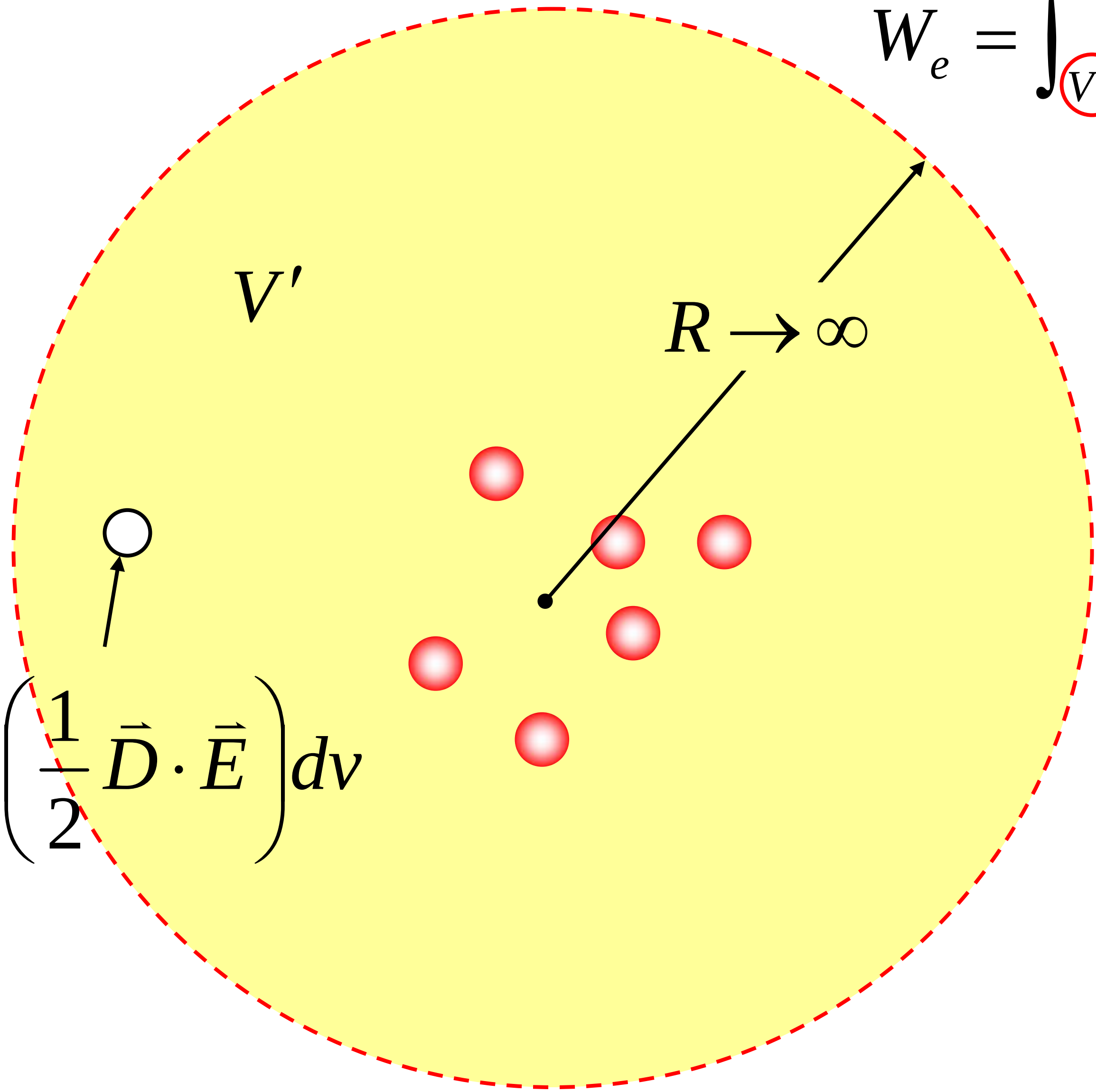
$$\propto \frac{1}{R}\cdot\frac{1}{R^2}\cdot R^2 \propto \frac{1}{R} \to 0$$

$$\Rightarrow W_e = I_2 = \int_{V'} \underline{w_e(\vec{r})} dv$$

$$\boxed{\frac{1}{2}\vec{D}\cdot\vec{E}\ (\mathrm{J/m^3})}$$ …energy density

Energy of electric fields-6

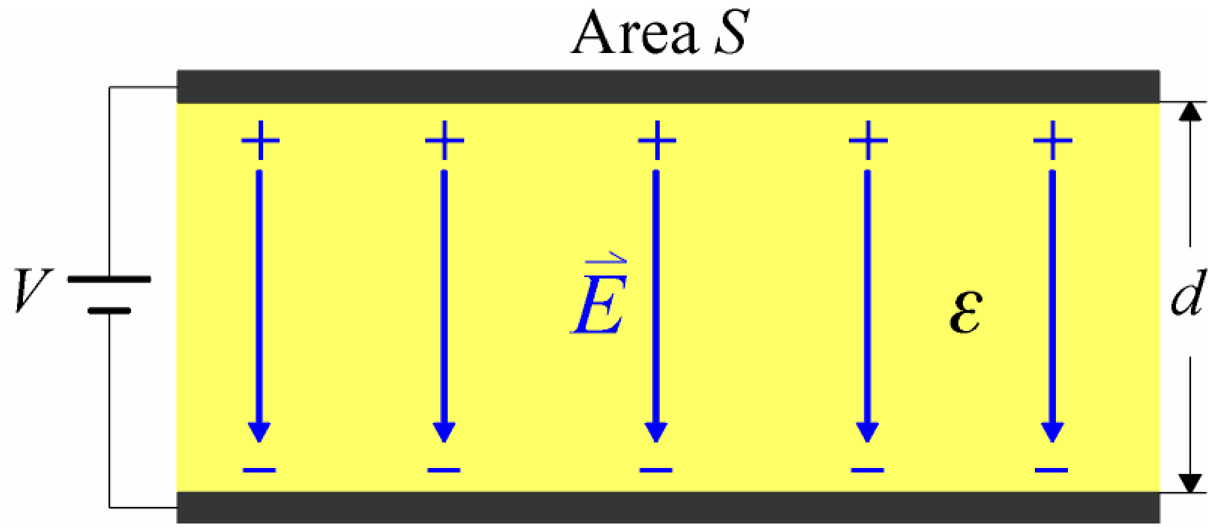
$$W_e = \int_{V'} \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) dv$$



$$dW_e = \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) dv$$

Example 9-5: Parallel-plate capacitor (1)

Find the stored electrostatic energy of:



Planar symmetry, $\Rightarrow$ uniform E-field:

$$\vec{E} = -\vec{a}_y \frac{V}{d}, \quad \vec{D} = \varepsilon \vec{E}$$

Example 9-5: Parallel-plate capacitor (2)

Energy density:

$$w_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon\left|\vec{E}\right|^2 = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2$$

Total stored energy:

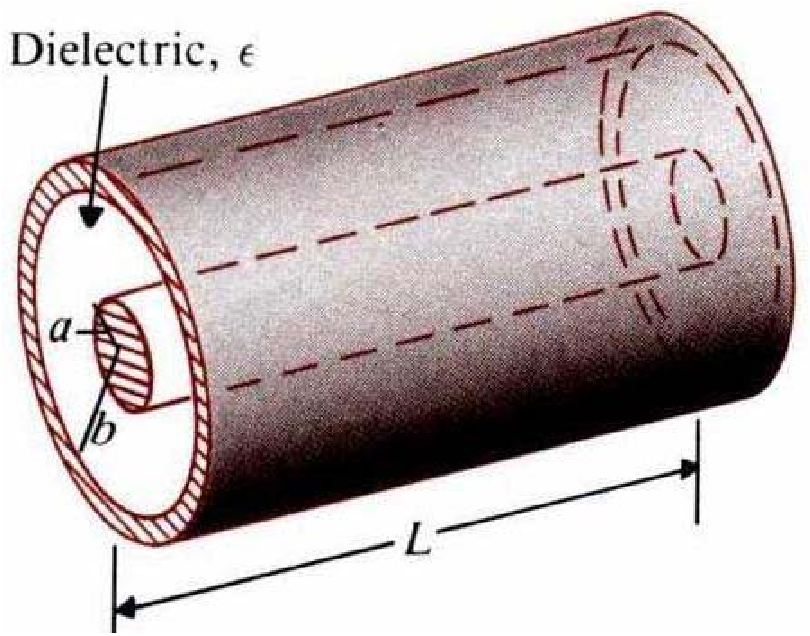
$$\boxed{C = \frac{Q}{V}}$$

$$W_e = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2 \cdot Sd = \frac{1}{2}\underline{\left(\varepsilon\frac{S}{d}\right)}V^2$$

$$\Rightarrow W_e = \boxed{\frac{1}{2}CV^2} = \boxed{\frac{Q^2}{2C}} = \frac{1}{2}QV$$

Example 9-6: Coaxial cable capacitor (1)

Find the capacitance of:



1. Assume charges $\pm Q$ are deposited

2. By Gauss's law (cylindrical sym.): $\vec{E} = \vec{a}_r \dfrac{Q}{2\pi\varepsilon rL}$

Example 9-6: Coaxial cable capacitor (2)

Instead of evaluating V by line integral of $\vec{E}$,

3. $w_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon\left|\vec{E}\right|^2$

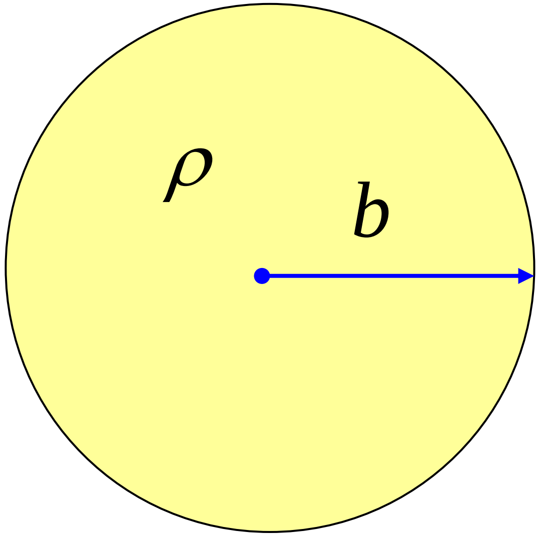
4. $W_e = \frac{1}{2}\varepsilon\left(\int\left|\vec{E}\right|^2 \underline{dv}\right) = \frac{1}{2}\varepsilon\int_a^b\left(\frac{Q}{2\pi\varepsilon Lr}\right)^2 \underline{(2\pi r drL)}$

$$= \frac{Q^2}{4\pi\varepsilon L}\left(\int_a^b \frac{dr}{r}\right) = \frac{Q^2}{4\pi\varepsilon L}\ln\left(\frac{b}{a}\right)$$

5. $C = \frac{Q^2}{2W_e} = \boxed{\frac{2\pi\varepsilon L}{\ln(b/a)}}$

Example 9-7: Sphere of uniform charge density (1)

Find the energy stored in a sphere of radius b with uniform volume charge density ρ



Spherical symmetry, $\Rightarrow$

$$\vec{E} = \vec{a}_R E(R)$$

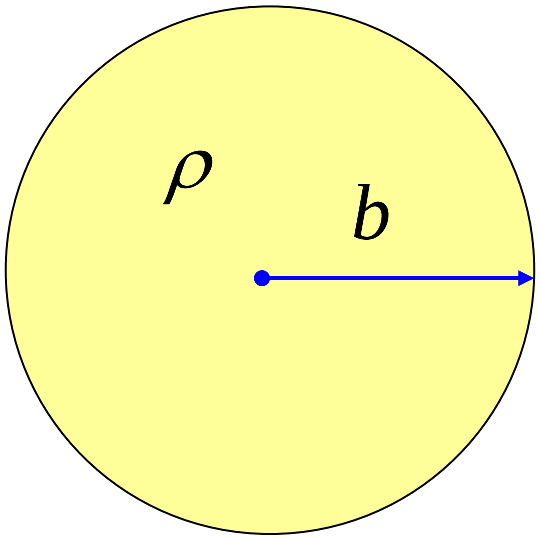
By Gauss's law, $\Rightarrow$

$$\vec{E} = \begin{cases} \vec{a}_R \dfrac{\rho R}{3\varepsilon_0}, \text{ if } 0 < R < b \\ \vec{a}_R \dfrac{\rho b^3}{3\varepsilon_0 R^2}, \text{ if } R \geq b \end{cases}$$

Example 9-7: Sphere of uniform charge density (2)

Instead of evaluating V by line integral of $\vec{E}$,

and arriving at $W_e = \dfrac{1}{2}\int_{V'} \rho(\vec{r})V(\vec{r})dv$

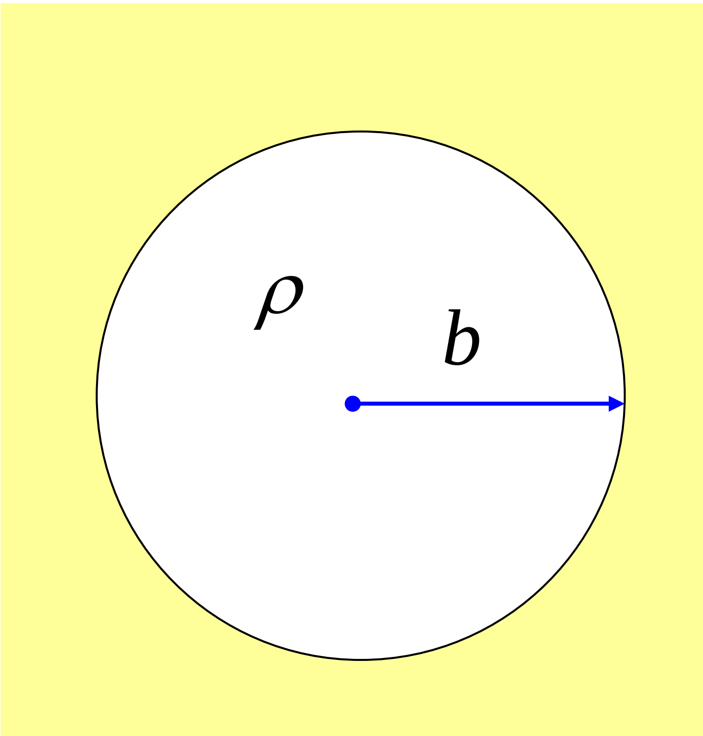


For $0 < R < b$:

$$W_{e1} = \int_0^b \frac{1}{2}\varepsilon_0 \overbrace{\left(\frac{\rho R}{3\varepsilon_0}\right)^2}^{\left|\vec{E}\right|^2} \overbrace{(4\pi R^2 dR)}^{dV}$$

$$= \frac{2\pi\rho^2}{9\varepsilon_0}\left(\int_0^b R^4 dR\right) = \frac{2\pi\rho^2 b^5}{45\varepsilon_0}$$

Example 9-7: Sphere of uniform charge density (3)



For $R > b$:

$$W_{e2} = \int_b^{\infty} \frac{1}{2}\varepsilon_0 \overbrace{\left(\frac{\rho b^3}{3\varepsilon R^2}\right)^2}^{\left|\vec{E}\right|^2} \overbrace{(4\pi R^2 dR)}^{dV}$$

$$= \frac{2\pi\rho^2 b^6}{9\varepsilon_0}\left(\int_b^{\infty} \frac{1}{R^2} dR\right) = \frac{2\pi\rho^2 b^5}{9\varepsilon_0}$$

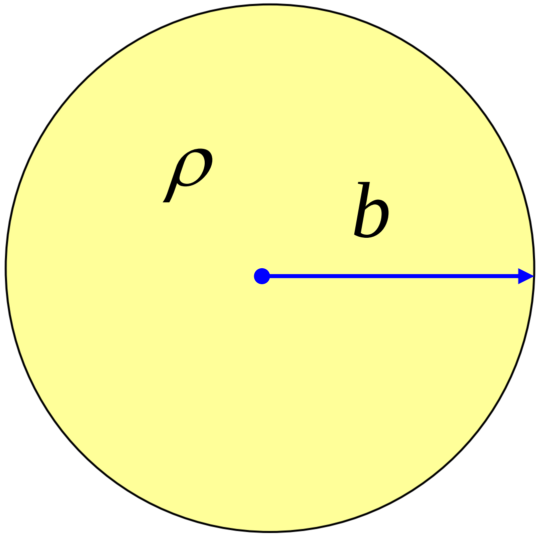
$$W_e = W_{e1} + W_{e2} = \frac{4\pi\rho^2 b^5}{15\varepsilon_0} = \frac{3Q^2}{20\pi\varepsilon_0 b}$$

(1:5)

Example 9-7: Sphere of uniform charge density (4)

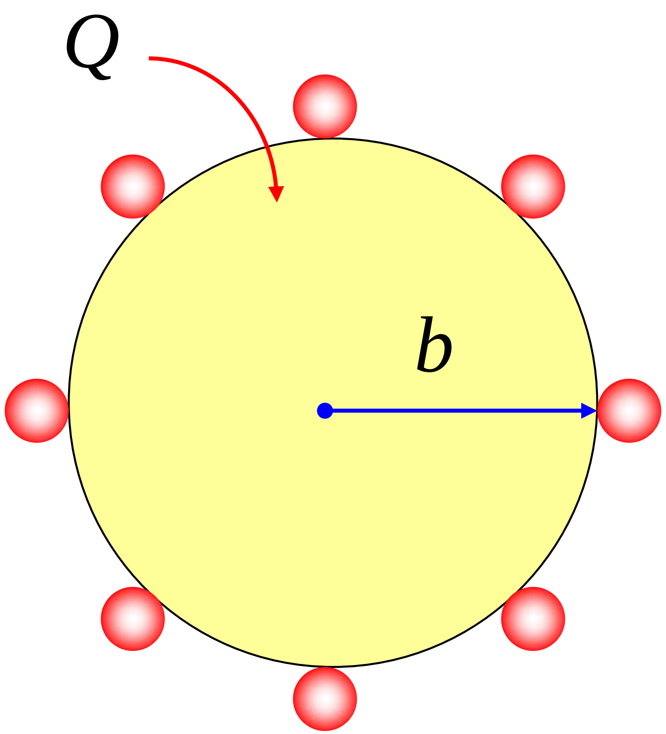
The corresponding "capacitance" is:

$$C = \frac{Q^2}{2W_e} = \frac{10\pi\varepsilon_0 b}{3}$$



Example 9-8: Conducting sphere (1)

Find the stored energy and the capacitance of a conducting sphere of total charge Q



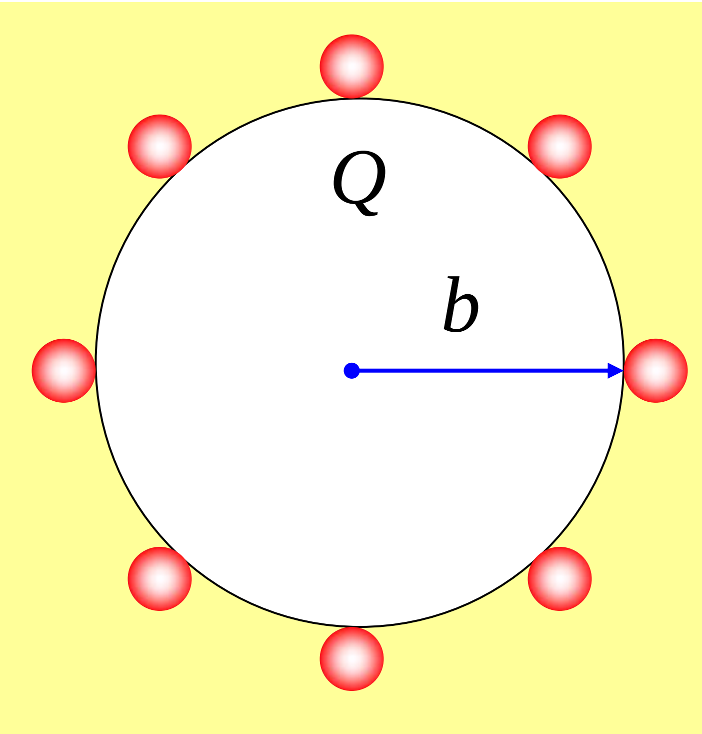
Spherical symmetry, $\Rightarrow \rho_s(\vec{r})$ is uniformly distributed, $\Rightarrow$

$$\vec{E} = \begin{cases} \vec{a}_R \dfrac{Q}{4\pi\varepsilon_0 R^2}, \text{if } R \ge b \\ 0, \text{otherwise} \end{cases}$$

Example 9-8: Conducting sphere (2)

Instead of evaluating V by line integral of $\vec{E}$,

and arriving at $W_e = \frac{1}{2}\int_{V'} \rho(\vec{r})V(\vec{r})dv$



For $R > b$:

$$W_e = \int_b^\infty \frac{1}{2}\varepsilon_0 \underbrace{\left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2}_{|\vec{E}|^2} \underbrace{(4\pi R^2 dR)}_{dV}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 b} \quad \Rightarrow C = \frac{Q^2}{2W_e} = \boxed{4\pi\varepsilon_0 b}$$