Lesson 08 Boundary-value Problems

■ Introduction

In Lesson 6 and Lesson 7, we have studied how to determine the electric field from the charge distribution. In many applications, however, we only know the "potentials of some conducting bodies" (boundary values) and it is difficult to evaluate the free charge distributions on the conducting surfaces. An alternative approach determines the potential distribution $V(\vec{r})$ by solving a differential equation with specified boundary values, then derives the electric field $\vec{E}$ and surface charge distributions $\rho_s(\vec{r})$ by gradient operation [eq. (6.11)] and the normal boundary condition [eq. (7.15)], respectively.

8.1 Poisson's and Laplace's Equations

■ Derivation

In a linear, homogeneous, and isotropic medium, $\vec{D} = \varepsilon\vec{E}$ [eq. (7.12)], where the permittivity of the medium ε is a scalar constant [eq. (7.13)]. The fundamental postulate of electrostatics $\nabla \cdot \vec{D} = \rho$ [eq. (7.8)] becomes $\nabla \cdot \vec{D} = \varepsilon\left(\nabla \cdot \vec{E}\right) = \rho$, $\Rightarrow$ $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon}$. By $\vec{E} = -\nabla V$ [eq. (6.11)], $\Rightarrow$ $\nabla \cdot \vec{E} = -\nabla \cdot \left(\nabla V\right) = \dfrac{\rho}{\varepsilon}$. By the definition of scalar Laplacian $\nabla^2 V \equiv \nabla \cdot \left(\nabla V\right)$ [eq. (5.24)], we arrive at the Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \tag{8.1}$$

In the regions of no "free" charge, eq. (8.1) is simplified to Laplace's equation:

$$\nabla^2 V = 0 \tag{8.2}$$

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1) Eq. (8.2) remains applicable for a charge-free region with "conducting boundaries", where free surface charges may exist.
2) By the physical meaning of scalar Laplacian operator (Lesson 5), eq. (8.2) implies that the potential at any point $V(\vec{r})$ is equal to the average of its neighboring potential values $\overline{V}(\vec{r})$ (dynamic equilibrium).

■ Uniqueness theorem

A function satisfying both the Poisson's equation and the given boundary conditions (BCs) is a unique solution, which is irrespective of the method by which the solution is obtained (even by guessing).

8.2 Boundary-value Problems

■ One-dimensional example in Cartesian coordinates

Example 8-1: Consider two parallel conducting plates separated by a dielectric material of permittivity ε and thickness d. The bottom plate ($y=0$) and top plate ($y=d$) are maintained at potentials 0 and V_0, respectively. Find V, $\vec{E}$, and ρ_s on the two plates.

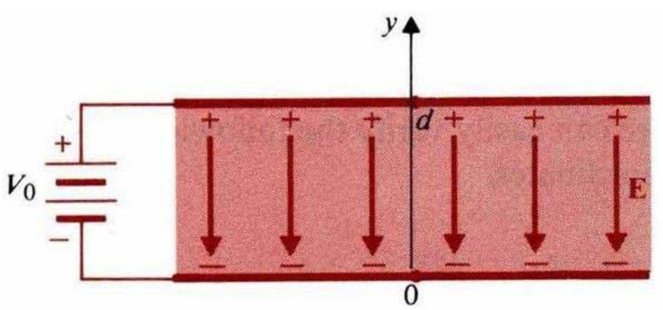

Fig. 8-1. Two parallel conducting plates spaced by a dielectric layer and biased by a dc voltage (after DKC).

Ans: (1) Eq. (8.2) is good for the dielectric region $0 < y < d$ where $\rho = 0$. Assume the plates are infinitely large (planar symmetry), $\Rightarrow$ $V = V(y)$, $\nabla^2 V = \dfrac{d^2 V}{dy^2} = 0$, which is an ordinary differential equation (ODE). The two BCs are: $\{V(0) = 0,\ V(d) = V_0\}$. $\Rightarrow$ $V(y) = \dfrac{V_0}{d} y$. (2) $\vec{E} = -\nabla V = -\vec{a}_y \dfrac{dV}{dy} = -\vec{a}_y \dfrac{V_0}{d}$, which is uniform. (3) For the upper plate, medium 1 and 2 represent the conductor and dielectric material, respectively. $\Rightarrow$ $\vec{D}_1 = 0$, $\vec{D}_2 = \varepsilon \vec{E}_2 = -\vec{a}_y \dfrac{\varepsilon V_0}{d}$, $\vec{a}_{n2} = \vec{a}_y$. By eq. (7.15), $\Rightarrow$ $\rho_{su} = \vec{a}_y \cdot \left(0 + \vec{a}_y \dfrac{\varepsilon V_0}{d}\right) = \dfrac{\varepsilon V_0}{d}$. For the lower plate, medium 1 and 2 represent the dielectric material and conductor, respectively. $\Rightarrow$ $\vec{D}_1 = \varepsilon \vec{E}_1 = -\vec{a}_y \dfrac{\varepsilon V_0}{d}$, $\vec{D}_2 = 0$, $\vec{a}_{n2} = \vec{a}_y$. By eq. (7.15), $\Rightarrow$ $\rho_{sl} = \vec{a}_y \cdot \left(-\vec{a}_y \dfrac{\varepsilon V_0}{d} - 0\right) = -\dfrac{\varepsilon V_0}{d}$.

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The assumption of infinitely large plates also implies uniform surface charges $\rho_s = \pm\rho$. By Gauss's law, $\Rightarrow$ $\vec{D} = -\vec{a}_y \rho$, $\vec{E} = -\vec{a}_y \dfrac{\rho}{\varepsilon}$, $V(y) = \dfrac{\rho}{\varepsilon} y$. By $V(d) = V_0$, $\Rightarrow$ $\rho = \dfrac{\varepsilon V_0}{d}$. We derive the same results.

■ One-dimensional example in cylindrical coordinates

<u>Example 8-2</u>: Consider a long coaxial cable with inner conductor of radius a (maintained at potential V_0), and outer conductor of radius b (grounded), respectively. Find the potential distribution $V(\vec{r})$ for the region in between.

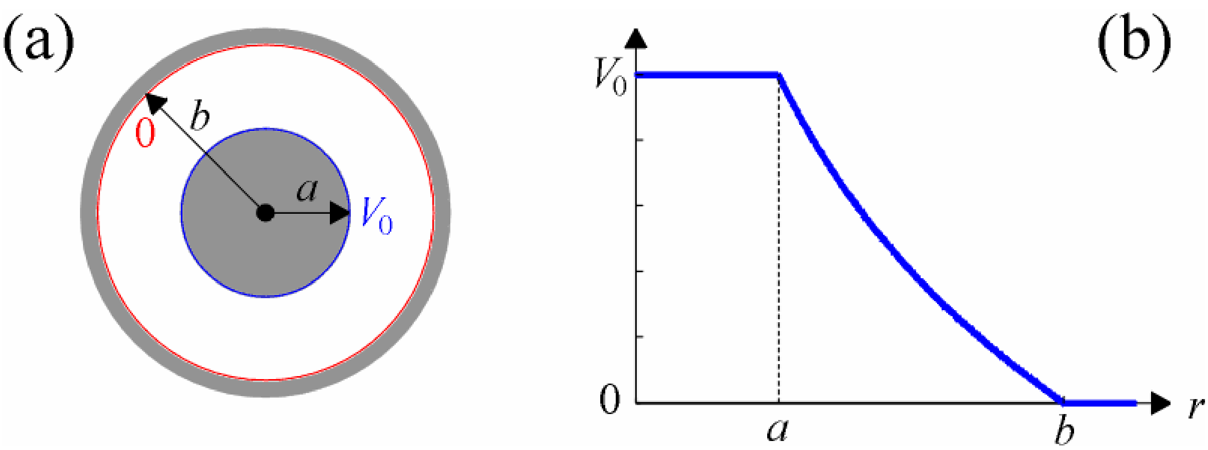

Fig. 8-1. (a) Cross-sectional view, and (b) the corresponding potential $V(r)$ of a long coaxial cable.

Ans: By cylindrical symmetry, $\Rightarrow$ $V=V(r)$, $\nabla^2 V=\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=0$ (DKC, inside of back cover), which is an ODE. The two BCs are: $\{V(b)=0,\ V(a)=V_0\}$. Let $U(r)=\frac{dV}{dr}$, $\Rightarrow$ $\frac{dU}{dr}+\frac{U}{r}=0$, $\frac{U'(r)}{U(r)}=-\frac{1}{r}$. Integration for both sides leads to: $\ln[U(r)]=\ln(r^{-1})+c'$, $U(r)=\frac{c}{r}$. By another integration, $V(r)=C_1\ln r+C_2$. By the BCs: $C_1=\frac{-V_0}{\ln(b/a)}$, $C_2=\frac{V_0\ln b}{\ln(b/a)}$, $\Rightarrow$ $V(r)=\frac{V_0}{\ln(b/a)}\ln\left(\frac{b}{r}\right)$.

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The cylindrical symmetry also implies uniform surface charges $\rho_s=\pm\rho$. By Gauss' law, $\Rightarrow$ $\vec{D}=\vec{a}_r\frac{\rho}{2\pi r}$, $\vec{E}=\vec{a}_r\frac{\rho}{2\pi\varepsilon r}$, $V(r)=V_0-\int_a^r \vec{E}(r')\cdot\vec{a}_r dr'=V_0-\frac{\rho}{2\pi\varepsilon}\ln\left(\frac{r}{a}\right)$. By BC $V(b)=0$, $\Rightarrow$ $V_0=\frac{\rho}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)$, $V(r)=\frac{V_0}{\ln(b/a)}\ln\left(\frac{b}{r}\right)$. We derive the same result.