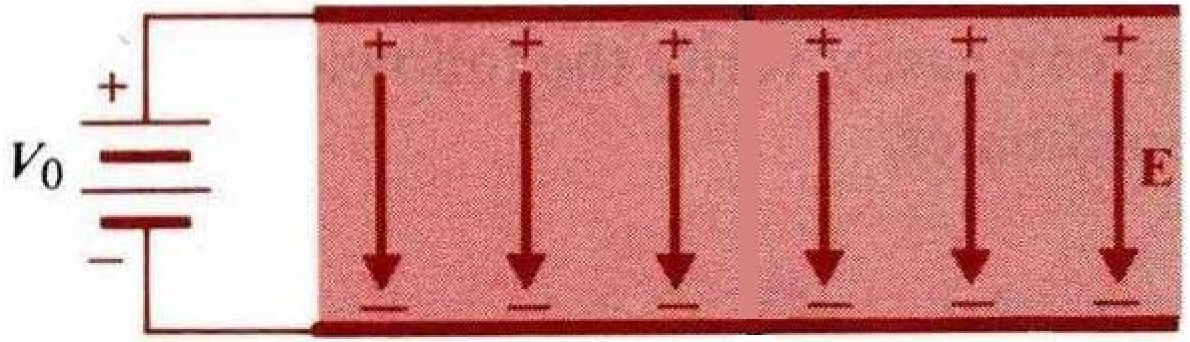
Lesson 8
Boundary-value problems

楊尚達 Shang-Da Yang

Institute of Photonics Technologies
Department of Electrical Engineering
National Tsing Hua University, Taiwan

Introduction

- In Lesson 6 and Lesson 7, we determined the E-field from charge distribution $\rho(\vec{r})$ (Gauss's law, integral of contribution from elementary charges).
- In practice, we might only know the electric potentials of some conducting bodies. It is difficult to evaluate the distribution of free charges $\rho_s(\vec{r})$ on the conducting surfaces.



Derivation

- In a simple (linear, homogeneous, isotropic) medium, $\vec{D} = \varepsilon\vec{E}$, where ε is a scalar constant.
- Fundamental postulate:

$$\nabla \cdot \vec{D} = \rho, \quad \Rightarrow \nabla \cdot \left(\varepsilon\vec{E}\right) = \varepsilon\left(\nabla \cdot \vec{E}\right) = \rho, \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

- By the relation $\vec{E} = -\nabla V$,

$$\Rightarrow \nabla \cdot \vec{E} = -\nabla \cdot \left(\nabla V\right) = \frac{\rho}{\varepsilon}$$

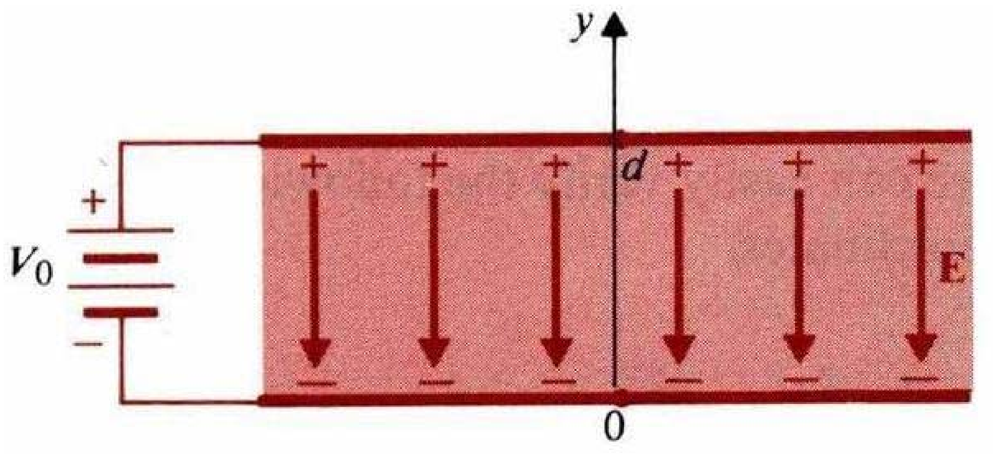
- By the definition of scalar Laplacian $\nabla^2 V \equiv \nabla \cdot \left(\nabla V\right)$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$ …Poisson's equation

In charge-free regions

$$\boxed{\nabla^2 V = 0}$$ …Laplace's equation

Example 8-1: Parallel-plate capacitor



Since free charges only exist at $y = 0^+, d^-$,

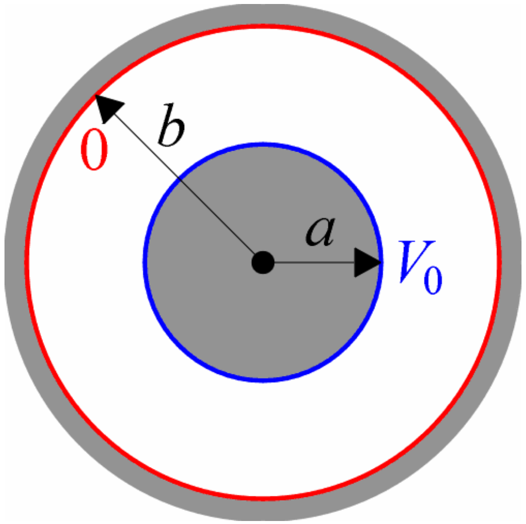
$\Rightarrow \nabla^2 V = 0$ is applicable for the dielectric region $0 < y < d$

Assume infinite plates, $\Rightarrow$ planar symmetry, $\Rightarrow$ $V = V(y)$

$\nabla^2 V = \dfrac{d^2 V}{dy^2} = 0$, with 2 BCs: $\{V(0) = 0,\ V(d) = V_0\}$

The solution is: $\boxed{V(y) = \dfrac{V_0}{d} y}$, $\vec{E} = -\nabla V = -\vec{a}_y \dfrac{dV}{dy} = \boxed{-\vec{a}_y \dfrac{V_0}{d}}$

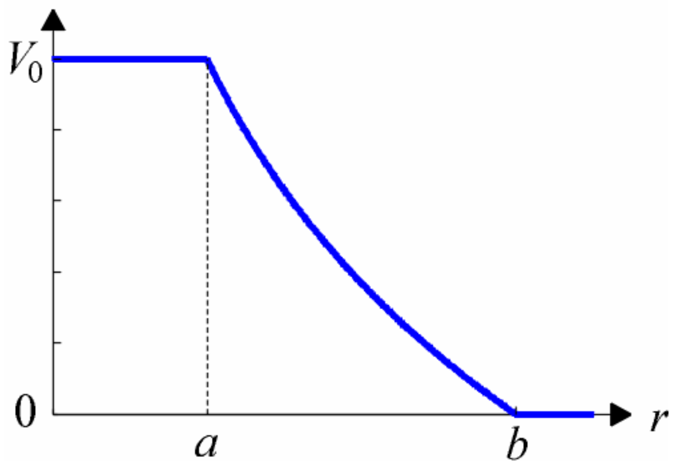
Example 8-2: Long coaxial cable



Cylindrical symmetry, $\Rightarrow\ V = V(r)$

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

with 2 BCs: $\{V(b) = 0,\ V(a) = V_0\}$



The solution is:

$$\boxed{V(r) = \frac{V_0}{\ln(b/a)}\ln\left(\frac{b}{r}\right)}$$