Lesson 07 Electrostatics in Materials

Introduction

Materials consist of atoms, which are made up of nuclei (positive charges) and electrons (negative charges). As a result, the "total" electric field must be modified in the presence of materials. We usually classify materials into three types:

- Conductors: The electrons in the outermost shells of the atoms are very loosely held and can freely migrate among atoms due to thermal excitation at room temperatures. These electrons are "shared" by all atoms.
- Dielectrics: All electrons are confined within the "inner" shells of atoms, and can hardly migrate even by applying a strong excitation.
- 3) Semiconductors: Outermost electrons are "moderately" confined, which may not migrate due to thermal excitation but become movable when applying an external electric field.





In terms of quantum theory, the electrons of an atom can only "stay" at discrete energy levels (Fig. 7-1a, left). When a large number of atoms aggregate in ordered manner (crystalline solid), each energy level splits into densely spaced levels (energy "band"). The allowed energy bands can be overlapped or separated by forbidden energy gap (Fig. 7-1a, right). Electric characteristics of materials depend on the band structure and how they are filled by the electrons at the temperature of 0 K (Fig. 7-1b):

2) Dielectrics: Energy gap is large, and the conduction band is empty.

3) Semiconductors: Energy gap is small, and the conduction band is empty.

In this lesson, we will discuss the field behaviors both inside some material and on the interface of different materials.

7.1 Static Electric Field in the Presence of Conductors

■ Charge and electric field inside a conductor

Assume that some positive (or negative) charges are introduced in the interior of a conductor. Initially an electric field will be set up, which will push the "free" charges away from one another by Coulomb's force and modify the electric field distribution by turns. This charge-field interaction process will continue until: (1) all the charges reach the conductor surface (no way to leave):

$$\rho = 0, \tag{7.1}$$

and (2) the surface charges redistribute themselves such that no electric field exists inside the conductor:

$$\vec{E} = 0 \tag{7.2}$$

Eq. (7.2) can be justified by:

- 1) If $\overline{E} \neq 0$ somewhere inside the conductor, the electric potential $V(\overline{r})$ is non-uniform. This will result in a contradiction that work has to be done to move "free" charges.
- 2) The free charge distribution that leads to $\vec{E} = 0$ actually corresponds to the lowest system energy, i.e., a state of equilibrium (Lesson 9).

For good conductors like copper, the time required to reach the equilibrium is only in the order of 10^{-19} sec (Lesson 10).

■ Air-conductor interface

Since the charges on the conductor surface will not be at rest if there is tangential electric field component, the surface electric field only has normal component in the equilibrium. Rigorous boundary conditions (BCs) are derived by applying eq's (6.3), (6.4) with respect to the differential contour "*abcda*" and the thin pill box (Fig. 7-2) across the air-conductor interface, respectively.



Fig. 7-2. Differential contour and pill box used to derive BCs on the air-conductor interface (after DKC).

- 1) Tangential BC: By eq's (6.4), (7.2), $\oint_{abcda} \vec{E} \cdot d\vec{l} \Big|_{\Delta h \to 0} = E_t \cdot \Delta w + 0 \cdot \Delta w = 0, \implies E_t = 0$ (7.3)
- 2) Normal BC: BC: By eq's (6.3), (7.1-3), $\oint_{S} \vec{E} \cdot d\vec{s} \Big|_{\Delta h \to 0} = E_{n} \cdot \Delta S = \frac{\rho_{s} \Delta S}{\varepsilon_{0}}, \Rightarrow E_{n} = \frac{\rho_{s}}{\varepsilon_{0}}, \qquad (7.4)$

where E_n is in the normal direction from the conductor to the air.

Example 7-1: Consider a point charge +Q located at the center of a spherical conducting shell of finite thickness (Fig. 7-3a). Find \vec{E} , V inside and outside the shell.



Fig. 7-3. (a) The geometry of a conducting sphere. The corresponding (b) radial components of \vec{E} (dashed), and (c) electric potential V.

Ans: Because of the spherical symmetry of the source, we have $\vec{E} = \vec{a}_R E_R(R)$ everywhere, and the Gaussian surfaces are concentric spherical surfaces centered at the origin.

(1) For $R < R_i$: $\oint_{S_3} \vec{E} \cdot d\vec{s} = E_R(R) \cdot (4\pi R^2) = \frac{Q}{\varepsilon_0}$, $\Rightarrow E_R(R) = \frac{Q}{4\pi\varepsilon_0 R^2}$. (2) For $R_i < R < R_o$: By eq. (7.2), $\vec{E} = 0$, $\Rightarrow \oint_{S_2} \vec{E} \cdot d\vec{s} = 0$ for any enclosed surface S_2 inside the shell. This implies that "free" charges of -Q must be induced on the inner surface of the shell. By the conservation of charges and eq. (7.1), free charges of +Q must be induced on the outer surface of the shell.

(3) For $R > R_o$: The total charge enclosed by a Gaussian surface S_1 is +Q, \Rightarrow

$$E_R(R) = \frac{Q}{4\pi\varepsilon_0 R^2} \,.$$

The curve of $|\vec{E}| = E_R(R)$ is shown in Fig. 7-3b. One can derive the electric potential distribution V(R) by line integral of \vec{E} (Fig. 7-3c).

7.2 Static Electric Field in the Presence of Dielectrics

■ Concept of induced dipoles

A dielectric molecule could be non-polar (e.g., H_2 , CH_4) or polar (e.g., H_2O , HCl, NH_3 , O_3), depending on whether there is nonzero electric dipole moment [eq. (6.17)]. However, a dielectric bulk made up of a large number of randomly oriented molecules (polar or non-polar) typically has no macroscopic dipole moment in the absence of external electric field.

When an electric field is applied, the "bound" charges of the dielectrics cannot freely migrate to the surface. Instead, (1) each non-polar molecule is polarized for the positively charged nuclei and negatively charged electrons are slightly displaced in opposite directions (distortion of electron cloud). (2) Most of the individual dipole moments (vectors) are aligned in the same direction due to the electric torque. In either case, a macroscopic dipole moment emerges.



Fig. 7-4. A cross section of a polarized dielectric medium (after DKC).

Although an electric dipole is neutral in ensemble, it provides nonzero potential and electric field [eq's (6.18), (6.16)], which will modify the "total" field inside and outside dielectrics.

Polarization vector and equivalent charge densities

To analyze the effect of induced dipoles, we define a (microscopic) polarization vector \vec{P} as

the volume density of electric dipole moment:

$$\vec{P} = \lim_{\Delta \nu \to 0} \frac{\sum \vec{p}_k}{\Delta \nu}$$
(7.5)

where \vec{p}_k denotes the *k*th dipole moment inside a differential volume Δv .

If the polarization vector \vec{P} is inhomogeneous (i.e., varies with position) somewhere, there must exist net (bound) polarization charge at that position. This phenomenon can be illustrated in two cases.



Fig. 7-5. (a) The model to deduce the polarization surface charge density. (b) Net polarization volume charge exists where the polarization vector \vec{P} is inhomogeneous.

1) The polarization vector \vec{P} is discontinuous on the air-dielectric interface, where there must exist net polarization charge (Fig. 7-5a). To quantitatively model this phenomenon, consider a differential parallelepiped V bounded by a closed surface S_b , where the dipole moment of each molecule is $\vec{p} = q\vec{d}$. The top surface of S_b has a unit outward normal vector \vec{a}_n and an area of ΔS . The effective height of the parallelepiped V is $\vec{d} \cdot \vec{a}_n$. The net polarization charge within V is:

$$\Delta Q = nq \left(\vec{d} \cdot \vec{a}_n \right) \Delta S = \left(\vec{P} \cdot \vec{a}_n \right) \Delta S ,$$

where $n (1/m^3)$ is the number density of molecules. The corresponding polarization

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surface charge density $\Delta Q/\Delta S$ is:

$$\rho_{ps} = \bar{P} \cdot \bar{a}_n \quad (C/m^2) \tag{7.6}$$

2) Consider two dipoles with different polarization vectors \vec{P}_1 , \vec{P}_2 in the interior of a differential volume V bounded by a closed surface S (Fig. 7-5b). Although each individual dipole is electrically neutral, some net charge Q_p can exist on the interface due to incomplete cancellation of the polarization charge between adjacent dipoles. To maintain the electric neutrality, Q_p must be equal in magnitude but opposite in sign to the total surface polarization charge $Q_{ps1} + Q_{ps2}$. Let ρ_p represent the polarization volume charge density , $\Rightarrow Q_p = \int_V \rho_p dv$. By eq's (7.6), (5.24), $\Rightarrow Q_{ps1} + Q_{ps2} = \oint_S (\vec{P} \cdot \vec{a}_n) ds = \oint_S \vec{P} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{P}) dv$. The requirement of $Q_p = -(Q_{ps1} + Q_{ps2})$ results in: $\rho_p = -(\nabla \cdot \vec{P})$ (C/m³) (7.7)

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- 1) Eq. (7.6) can be regarded as a special case of eq. (7.7) on the air-dielectric interface, where the divergence of the polarization vector \vec{P} is infinite.
- 2) Equivalent polarization charge densities ρ_{ps}, ρ_p can be used in collaboration with eq's (6.10), (6.15) to evaluate the electric field and potential contributed by polarized dielectrics.

■ Electric flux density

In the presence of dielectric materials, the total electric field would be created by both the free and polarization charges. Fundamental postulate eq. (6.1) is thus modified as:

$$\nabla \cdot \vec{E} = \frac{\rho + \rho_p}{\varepsilon_0}.$$

By eq. (7.7), we have:

$$\nabla \cdot \vec{D} = \rho, \tag{7.8}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad (C/m^2), \tag{7.9}$$

where the electric flux density \vec{D} characterizes the contribution from the "free" charges. The integral form of eq. (7.8) (Gauss's law) becomes:

$$\oint_{S} \vec{D} \cdot d\vec{s} = Q \tag{7.10}$$

For linear, homogeneous, and isotropic dielectrics, the polarization vector is proportional to the electric field (by the elastic spring model):

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \,, \tag{7.11}$$

where the electric susceptibility χ_e is a dimensionless quantity independent of magnitude (linear), position (homogeneous), and direction (isotropic) of \vec{E} . By eq's (7.9), (7.11),

$$\vec{D} = \varepsilon \vec{E} , \qquad (7.12)$$

where the permittivity of the medium ε is defined as:

$$\varepsilon = (1 + \chi_e)\varepsilon_0 \tag{7.13}$$

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The strategy is using a single constant ε to replace the tedious induced dipoles, polarization vector, and equivalent polarization charges in determining the total electric field.

Example 7-2: Consider a parallel-plate capacitor. (1) \vec{D} describes the surface density of free charges on the conducting plates. (2) \vec{P} describes the surface density of polarization charges.



(3) $\varepsilon_0 \vec{E}$ describes the surface density of total charge or uncompensated free charge.

Fig. 7-6. Physical meanings of electric flux density \vec{D} , polarization vector \vec{P} , and electric field intensity \vec{E} illustrated in the example of a parallel-plate capacitor (after C. C. Su).

Example 7-3: A point charge +Q at the center of a spherical dielectric shell of permittivity $\varepsilon_r \varepsilon_0$ (Fig. 7-7a), where $\varepsilon_r = 1 + \chi_e$ is the relative permittivity. Find \overline{D} , \overline{E} , V, \overline{P} , and ρ_{ps} .



Fig. 7-7. (a) The geometry of a dielectric sphere. The corresponding (b) radial components of \vec{D} (solid), $\varepsilon_0 \vec{E}$ (dashed), \vec{P} (dash-dot), and (c) electric potential V.

Ans: (1) By spherical symmetry, $\Rightarrow \vec{D} = \vec{a}_R D_R(R)$. By eq. (7.10) and the fact that dielectric materials contribute to no "free" charge: $\oint_S \vec{D} \cdot d\vec{s} = D_R(R) \cdot (4\pi R^2) = Q$, \Rightarrow

$$D_R(R) = \frac{Q}{4\pi R^2}$$
, for all $R > 0$.

(2) By spherical symmetry, $\vec{E} = \vec{a}_R E_R(R)$. By eq. (7.12), \Rightarrow

$$E_R(R) = \frac{Q}{4\pi\epsilon R^2}$$
, where $\varepsilon = \begin{cases} \varepsilon_r \varepsilon_0, \text{ for } R_i < R < R_0 \\ \varepsilon_0, \text{ otherwise} \end{cases}$

(3) V(R) is derived by integration of $E_R(R)$.

(4) By eq. (7.9), $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \vec{a}_R P_R(R)$, where $P_R(R) = D_R(R) - \varepsilon_0 E_R(R)$, \Rightarrow

$$P_{R}(R) = \begin{cases} \left(1 - \varepsilon_{r}^{-1}\right) \frac{Q}{4\pi R^{2}}, \text{ for } R_{i} < R < R_{0}\\ 0, \text{ otherwise} \end{cases}.$$

(5) By eq (7.6), the surface polarization charge density becomes:

$$\rho_{ps} = \begin{cases} \left[\vec{a}_{R} \left(1 - \varepsilon_{r}^{-1} \right) \frac{Q}{4\pi R_{i}^{2}} \right] \cdot \left(-\vec{a}_{R} \right) = -\left(1 - \varepsilon_{r}^{-1} \right) \frac{Q}{4\pi R_{i}^{2}} (<0), \text{ for } R = R_{i}; \\ \left(1 - \varepsilon_{r}^{-1} \right) \frac{Q}{4\pi R_{o}^{2}} (>0), \text{ for } R = R_{o} \end{cases}$$

The total polarization surface charge is:

$$Q_{ps} = \rho_{ps} S = \begin{cases} -\left(1 - \varepsilon_r^{-1}\right) \frac{Q}{4\pi R_i^2} \cdot \left(4\pi R_i^2\right) = -\left(1 - \varepsilon_r^{-1}\right) Q(<0), \text{ for } R = R_i; \\ \left(1 - \varepsilon_r^{-1}\right) Q(>0), \text{ for } R = R_o \end{cases}$$

They create a radially inward \vec{E} to "reduce" the total electric field in the dielectrics.

7.3 General Boundary Conditions for Electric Fields

Derivation

As in air-conductor interface, we apply the integral forms of the two fundamental postulates on the differential contour "*abcda*" and the thin pill box with $\Delta h \rightarrow 0$ across the interface of two dielectric media (Fig. 7-8) to derive the BCs for the tangential and normal components of the electric field.



Fig. 7-8. Differential contour and pill box used to derive general BCs (after DKC).

1) Tangential BC: By eq. (6.4), $\oint_{abcda} \vec{E} \cdot d\vec{l} = E_{1t} \cdot (-\Delta w) + E_{2t} \cdot (\Delta w) = 0, \Rightarrow$

$$E_{1t} = E_{2t} (7.14)$$

2) Normal BC: BC: By eq. (7.10), $\oint_{S} \vec{D} \cdot d\vec{s} = Q$, $(\vec{D}_{1} \cdot \vec{a}_{n2} + \vec{D}_{2} \cdot \vec{a}_{n1})(\Delta S) = \rho_{s} \cdot \Delta S$, \Rightarrow

$$\bar{a}_{n2} \cdot \left(\bar{D}_1 - \bar{D}_2 \right) = \rho_s \tag{7.15}$$

Eq. (7.15) can also be written as: $D_{1n} - D_{2n} = \rho_s$, where D_{in} (i = 1, 2) denotes component of \overline{D}_i in the direction of \overline{a}_{n2} (unit normal vector directed from medium 2 to medium 1).

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- 1) Eq's (7.3), (7.4) are special cases of eq's (7.14), (7.15), where $\{\bar{E}_2, \bar{D}_2\} = 0$ are used.
- 2) Only "free" surface charge density ρ_s counts in eq. (7.15). If the two interfacing media are both dielectrics, $\rho_s = 0$, $\Rightarrow D_{1n} = D_{2n}$.
- Eq's (7.14), (7.15) remain valid even the fields are time-varying (Lesson 14, or Ch 7 of the textbook). BCs explain why most optical components (with material interfaces) are "polarization"-dependent.