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- Materials consist of atoms, which are made up of nuclei (positive charges) and electrons (negative charges).
- The "total" E-field must be modified in the presence of materials.
- Discuss the field behaviors inside some material, and on the interface of different materials.

- Conductors: Electrons in the outermost shells of the atoms, very loosely held, can freely migrate due to thermal excitation $(\langle E_k \rangle = 3kT/2)$. These electrons are shared by all atoms.
- Dielectrics: Electrons are confined within the inner shells, can hardly migrate.
- Semiconductors: Electrons are moderately confined, movable when applying an external electric field. Free electron density (conductivity) can be changed by doping or bias voltage.

Classification of materials – Quantum view (1)



Electric properties of materials depend on the band structure, and how they are filled by the electrons (Pauli's exclusion principle).



Outline

- Static E-field in the presence of conductors
- Static E-field in the presence of dielectrics
- General boundary conditions



Sec. 7-1 Static E-field in the Presence of Conductors

- 1. Inside a conductor
- 2. Air-conductor interfaces
- 3. Example: Conducting shell



 \vec{E} pushes free charges away from one another, modify \vec{E} by turns. The process continues until steady state is reached. Why no E-field inside a conductor? (1)



 $\Rightarrow V_1 > V_2,$ $V(\bar{r})$ is nonuniform, \Rightarrow work has to be done to move free charges (contradiction!) Why no E-field inside a conductor? (2)



Correspond to the lowest system energy, state of equilibrium

Air-conductor interface-1



Charge is movable, not in equilibrium (contradiction!)

$$\Rightarrow E_t = 0, \ \vec{E} = \vec{a}_n E_n$$

Air-conductor interface-2



Air-conductor interface-3



Example 7-1: Conducting shell (1)



Spherical symmetry:

$$\Rightarrow \vec{E} = \vec{a}_R E_R(R)$$
1. $R < R_i : \oint_{S_3} \vec{E} \cdot d\vec{s} =$

$$= E_R(R) \cdot (4\pi R^2) = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E_R(R) = \frac{Q}{4\pi \varepsilon_0 R^2}$$

Example 7-1: Conducting shell (2)



2. $R_i < R < R_o$: $\vec{E} = 0$, $\oint \vec{E} \cdot d\vec{s} = 0$ $\Rightarrow -Q$ induced on the inner surface $\Rightarrow +Q$ induced on the outer surface Example 7-1: Conducting shell (3)



3.
$$R > R_o$$
: $\oint_{S_1} \vec{E} \cdot d\vec{s} =$
= $4\pi R^2 E_R(R) = \frac{Q - Q + Q}{\varepsilon_0}$
 $\Rightarrow E_R(R) = \frac{Q}{4\pi \varepsilon_0 R^2}$

Example 7-1: Conducting shell (4)



Conclusion

A "floating" conducting shell cannot isolate the effect of a charge.





Sec. 7-2 Static E-field in the Presence of Dielectrics

- 1. Microscopic and macroscopic dipoles
- 2. Polarization charge densities
- 3. Electric flux density
- 4. Example: Parallel-plate capacitor
- 5. Example: Dielectric shell

Microscopic electric dipoles – Nonpolar molecules

 H_2

Non-polar covalent bond (electrons are equally shared by the atoms).



Symmetric arrangement of polar bonds.

Microscopic electric dipoles – Polar molecules



Asymmetric arrangement of polar bonds

A dielectric bulk made up of a large number of randomly oriented molecules (polar or nonpolar) typically has no macroscopic dipole moment in the absence of external E-field. Macroscopic electric dipoles-2

In the presence of external E-field:

- 1. Non-polar molecule is polarized
- 2. Individual dipole moments are aligned



Macroscopic electric dipoles-3





(physics.udel.edu)

Each dipole (though electrically neutral) provides nonzero field, modifying the total E-field Strategy of analysis

It is too tedious to directly superpose the elementary fields:

$$\vec{E}(\vec{r}) \approx \frac{p}{4\pi\varepsilon_0 R^3} \left(\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta \right)$$

Instead, we define polarization vector as:

$$\vec{P} \equiv \lim_{\Delta \nu \to 0} \frac{\sum \vec{p}_k}{\Delta \nu}$$

- the *k*th dipole moment inside a differential volume Δv

- \Rightarrow Polarization charge ρ_p ,
- \Rightarrow Electric flux density \overline{D}

Polarization surface charge density-1

On the air-dielectric interface, \vec{P} is discontinuous, net polarization exists.



Polarization surface charge density-2

Consider a differential parallelepiped V bounded by S_b . The net polarization charge is:

$$\Delta Q = nq \left(\vec{d} \cdot \vec{a}_n \right) \Delta S = \left(\vec{P} \cdot \vec{a}_n \right) \Delta S$$
Differential
volume



Surface charge density:

$$\rho_{ps} = \frac{\Delta Q}{\Delta S}, \Rightarrow$$

$$\rho_{ps} = \vec{P} \cdot \vec{a}_n \, (\mathrm{C/m^2})$$

Polarization volume charge density-1

In the interior of a dielectric, net polarization charge exists where \vec{P} is inhomogeneous:



Polarization volume charge density-2

Consider two polarization vectors \vec{P}_1 , \vec{P}_2 in a differential volume V bounded by S.



Polarization charges on top and bottom surfaces:

$$\begin{array}{l}
\rho_{ps} \\
Q_{ps1} + Q_{ps2} = \oint_{S} \underbrace{\left(\vec{P} \cdot \vec{a}_{n}\right)} ds \\
= \oint_{S} \vec{P} \cdot d\vec{s} = \int_{V} \left(\nabla \cdot \vec{P}\right) dv \\
\underbrace{\text{divergence}}_{\text{theorem}}
\end{array}$$

Polarization volume charge density-3

By definition:

$$Q_p = \int_V \rho_p dv$$

To maintain the electric neutrality:

$$Q_{p} = -(Q_{ps1} + Q_{ps2}) = -\int_{V} (\nabla \cdot \vec{P}) dv = \int_{V} (-\nabla \cdot \vec{P}) dv$$

$$\Rightarrow \rho_p = -\left(\nabla \cdot \vec{P}\right) \left(C/m^3\right)$$

 $\rho_{ps} = \vec{P} \cdot \vec{a}_n$ can be regarded as a special case of $\rho_p = -(\nabla \cdot \vec{P})$, where $\nabla \cdot \vec{P} \to \infty$ $\vec{P}=0$ air $\Big) \nabla \cdot \vec{P} \to \infty$ dielectric $\overrightarrow{P}\neq 0$

Equivalent charge densities:

$$\rho_{ps} = \vec{P} \cdot \vec{a}_n, \quad \rho_p = -(\nabla \cdot \vec{P})$$

can be substituted into the formulas:

$$\begin{cases} \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \vec{a}_R \frac{\rho_v(\vec{r}')}{R(\vec{r},\vec{r}')^2} dv' \\ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho_v(\vec{r}')}{R(\vec{r},\vec{r}')} dv' \end{cases}$$

to evaluate the influence of polarized dielectrics.

Electric flux density-definition

Total E-field is created by free & polarization charges: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \longrightarrow \nabla \cdot \vec{E} = \frac{\rho + \rho_p}{\varepsilon_0}$ $\Rightarrow \left(\nabla \cdot \varepsilon_0 \vec{E} \right) = \rho - \left(\nabla \cdot \vec{P} \right)$ only free charge $\begin{cases} \nabla \cdot \vec{D} = \vec{\rho} & \longrightarrow \oint_{S} \vec{D} \cdot d\vec{s} = Q & \dots \text{Gauss's law} \\ \\ \vec{D} = \varepsilon_{0} \vec{E} + \vec{P} & (C/m^{2}) \end{cases}$ Electric flux density - Application (1)

For linear, homogeneous, and isotropic dielectrics, the polarization vector is proportional to the external electric field:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

- Susceptibility, independent of magnitude, position, direction of \vec{E}



$$\vec{E} = \vec{e}E_0$$

$$\vec{p} = \vec{e}qd$$



Electric flux density-application (2)

$$\varepsilon_{0} \chi_{e} \overline{E}$$

$$\overline{D} = \varepsilon_{0} \overline{E} + \overline{P} = \varepsilon_{0} (1 + \chi_{e}) \overline{E}$$

$$\Rightarrow \begin{cases} \overline{D} = \varepsilon \overline{E} \\ \varepsilon = (1 + \chi_{e}) \varepsilon_{0} \dots \text{permittivity of the dielectric} \end{cases}$$

In this way, a single constant ε replaces the tedious induced dipoles, polarization vector, equivalent polarization charges in determining the total electric field.

Example 7-2: Physical meanings of D, E, P



- \vec{D} : free charge
- \vec{P} : polarization charge
- $\varepsilon_0 \vec{E}$: total charge

Example 7-3: Dielectric shell (1)



Spherical symmetry:

$$\Rightarrow \vec{D} = \vec{a}_R D_R(R)$$

For all $R > 0$: $\oint_S \vec{D} \cdot d\vec{s} =$
$$= D_R(R) \cdot (4\pi R^2) = Q \qquad \text{free} \\ \text{charge} \end{cases}$$

$$\Rightarrow D_R(R) = \frac{Q}{4\pi R^2}$$

Example 7-3: Dielectric shell (2)



By
$$\vec{D} = \varepsilon \vec{E}$$
,
 $\Rightarrow E_R(R) = \frac{Q}{4\pi \varepsilon R^2}$,
 $\varepsilon = \begin{cases} \varepsilon_r \varepsilon_0, \text{ for } R_i < R < R_0 \\ \varepsilon_0, \text{ otherwise} \end{cases}$

Example 7-3: Dielectric shell (3)





Example 7-3: Dielectric shell (4)

 $\vec{D} = \epsilon \vec{E}$





Example 7-3: Dielectric shell (5)



Example 7-3: Dielectric shell (6)

By
$$Q_{ps} = \oint_{S} \rho_{ps} ds = \rho_{ps} S$$
,
 $Q_{ps} = \int_{-1}^{-1} Q(<0), \text{ for } R = R_{i};$
 $(1 - \varepsilon_{r}^{-1})Q(>0), \text{ for } R = R_{o}$

Polarization charges create a radially inward E-field to reduce the total field.

- 1. Tangential boundary condition
- 2. Normal boundary condition

General interface-1

$$\begin{split} \oint_C \vec{E} \cdot d\vec{l} &= 0, \\ \oint_{abcda} \vec{E} \cdot d\vec{l} \Big|_{\Delta h \to 0} \\ &= E_{1t} \cdot (-\Delta w) + E_{2t} \cdot (\Delta w) \\ &= 0, \end{split}$$

$$\Rightarrow E_{1t} = E_{2t}$$

General interface-2

$$\begin{split} \oint_{S} \vec{D} \cdot d\vec{s} &= Q \\ \oint_{S} \vec{D} \cdot d\vec{s} \Big|_{\Delta h \to 0} \\ &= \left(\vec{D}_{1} \cdot \vec{a}_{n2} + \vec{D}_{2} \cdot \vec{a}_{n1} \right) (\Delta S) \\ &= \rho_{s} \cdot \Delta S \\ \Rightarrow \overline{a}_{n2} \cdot \left(\vec{D}_{1} - \vec{D}_{2} \right) = \rho_{s} \\ & \text{only free charge} \\ \Rightarrow D_{1n} - D_{2n} = \rho_{s} \end{split}$$

$$E_{1t} = E_{2t}$$

$$\vec{a}_{n2} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s$$

$$If M_2 \text{ is a conductor:}$$

$$conductor:$$

$$\left\{\vec{E}_2, \vec{D}_2\right\} = 0$$

$$E_t = 0$$

$$E_t = 0$$

$$E_t = 0$$

$$E_t = 0$$

Only free surface charge density counts in: $\vec{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$

If the two interfacing media are dielectrics:

$$\rho_s = 0, \implies D_{1n} = D_{2n}$$

$$\begin{cases} E_{1t} = E_{2t} \\ \vec{a}_{n2} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s \end{cases}$$

remain valid even the E-fields are time-varying.