Lesson 06 Electrostatics in Free Space

■ Introduction

In electrostatics, electric charges are at rest and there is no magnetic field. We will start with two fundamental postulates of electrostatics in free space to define electric field intensity $\vec{E}$, from which all experimental laws and the concept of electric potential can be derived.

6.1 Fundamental Postulates

■ Definition and physical meaning

Electric field intensity $\vec{E}$ describes the electrostatic force experienced by a unit charge: $\vec{F}/q$ (N/C or V/m). From the Helmholtz's theorem (Lesson 5), $\vec{E}$ can be uniquely specified if its divergence and curl are given:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \tag{6.1}$$

$$\nabla \times \vec{E} = 0, \tag{6.2}$$

where ρ is the volume charge density (C/m^3), and ε_0 is the permittivity of vacuum. Eq's (6.1), (6.2) mean: (1) free charges are the "flow sources" of $\vec{E}$, (2) there is no "vortex source" of $\vec{E}$.

■ Integral forms

Integration of both sides of eq. (6.1) over a volume V enclosed by the surface S becomes:

$$\int_V \left(\nabla \cdot \vec{E}\right) dv = \int_V \frac{\rho}{\varepsilon_0} dv = \frac{Q}{\varepsilon_0},$$

where Q represents the total charge inside S. Applying the divergence theorem [eq. (5.24)] to the left hand side of the equality, we arrive at the Gauss's law:

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0} \tag{6.3}$$

Integration of both sides of eq. (6.2) over a surface S bounded by the contour C becomes:

$$\int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{s} = 0 .$$

Applying the Stokes' theorem [eq. (5.29)] to the left hand side of the equality, we arrive at the Kirchhoff's voltage law:

$$\oint_C \vec{E} \cdot d\vec{l} = 0 \tag{6.4}$$

Eq. (6.4) also implies that the scalar line integral of $\vec{E}$ (i.e., the voltage difference) depends only on the end points, not on the path.

6.2 Gauss's Law

■ Definition and applications

If the charge distribution has certain symmetry, such that the normal component of $\vec{E}$ is constant over an enclosed surface (Gaussian surface), eq. (6.3) becomes convenient in determining $\vec{E}$.

Example 6-1: Consider an infinite planar sheet located at $z = 0$ with constant surface charge density ρ_s (C/m^2) (Fig. 6-1).

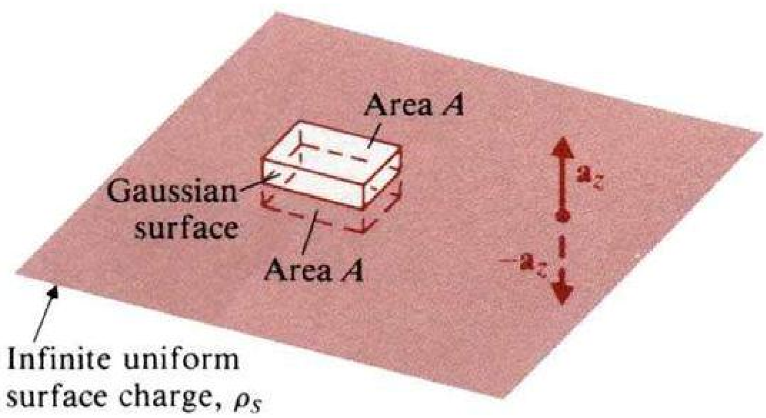

Fig. 6-1. An infinite planar sheet with constant surface charge density (after DKC).

The planar symmetry ensures that the electric field must be of the form: $\vec{E}=\begin{cases}\vec{a}_z E_z(z), \text{if } z>0\\ -\vec{a}_z E_z(-z), \text{if } z<0\end{cases}$. The corresponding Gaussian surface can be a rectangular box with top and bottom faces located at $z=\pm z_0$. $\oint_S \vec{E}\cdot d\vec{s} = 2E_z(z)A = \frac{\rho_s A}{\varepsilon_0}$, $\Rightarrow$ $E_z(z)=\frac{\rho_s}{2\varepsilon_0}$ (independent of z). $\Rightarrow$

$$\vec{E}=\begin{cases}\vec{a}_z \dfrac{\rho_s}{2\varepsilon_0}, \text{if } z>0\\[2ex] -\vec{a}_z \dfrac{\rho_s}{2\varepsilon_0}, \text{if } z<0\end{cases} \tag{6.5}$$

Example 6-2: Consider an infinitely long, straight line charge with constant line charge density ρ_l (C/m) (Fig. 6-2).

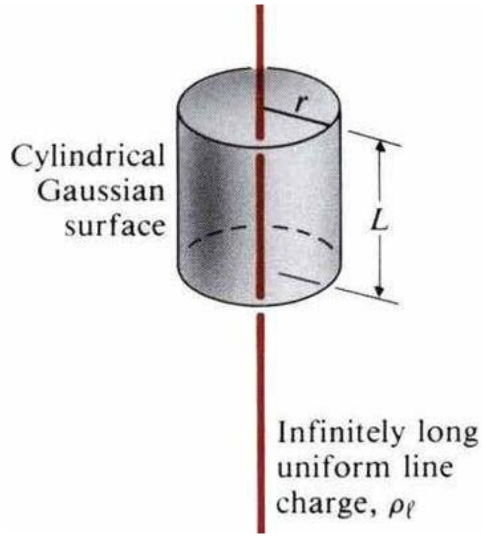

Fig. 6-2. An infinite line with constant line charge density (after DKC).

The cylindrical symmetry ensures that the electric field must be of the form: $\vec{E}=\vec{a}_r E_r(r)$. The corresponding Gaussian surface is a cylinder of radius r and length L. $\oint_S \vec{E}\cdot d\vec{s}$ $E_r(r)\cdot(2\pi rL)=\frac{\rho_l L}{\varepsilon_0}$, $\Rightarrow$ $E_r(r)=\frac{\rho_l}{2\pi\varepsilon_0 r}\propto\frac{1}{r}$. $\Rightarrow$

$$\vec{E}=\vec{a}_r \frac{\rho_l}{2\pi\varepsilon_0 r} \tag{6.6}$$

6.3 Coulomb's Law

■ Electric field created by point charge, discrete charges, and charge distribution

The electric field $\vec{E}$ created by a point charge q in a boundless free space must be of spherical symmetry, $\Rightarrow$ $\vec{E}=\vec{a}_R E_R(R)$. By eq. (6.3), $\oint_S \vec{E}\cdot d\vec{s}=E_R(R)\cdot\left(4\pi R^2\right)=\dfrac{q}{\varepsilon_0}$, $\Rightarrow$

$$\vec{E}=\vec{a}_R\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \tag{6.7}$$

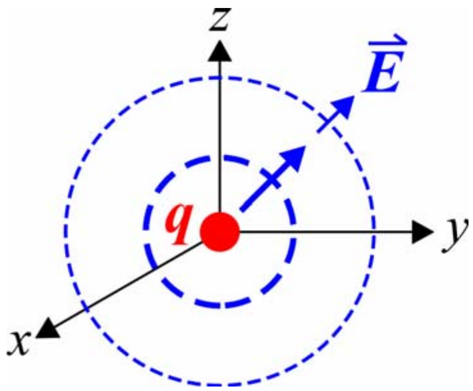

Fig. 6-3. A point charge constructs a spherically symmetric electric field.

By the relation $\vec{E}=\vec{F}/q$, we arrive at the Coulomb's law, i.e., the force exerted by charge q_1 on charge q_2 is:

$$\vec{F}_{12}=q_2\vec{E}_{12}=\vec{a}_{R_{12}}\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{R_{12}^2} \tag{6.8}$$

where $\vec{a}_{R_{12}}R_{12}$ is a vector directed from the "source point" q_1 to the "observation point" q_2.

Since the two fundamental postulates eq's (6.1), (6.2) are linear, the total electric field at position $\vec{r}$ as a result of a system of discrete charges $\{q_k, k=1,2,\ldots,n\}$ or a continuous charge distribution $\rho_v(\vec{r}\,')$ within the volume V' can be derived by the principle of superposition:

$$\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\vec{a}_{R_k}\frac{q_k}{R_k^2} \tag{6.9}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{a}_R \frac{\rho_v(\vec{r}\,')}{R(\vec{r},\vec{r}\,')^2}dv' \tag{6.10}$$

In eq. (6.9), $\vec{a}_{R_k} = \dfrac{\vec{r}-\vec{r}_k'}{|\vec{r}-\vec{r}_k'|}$ is a unit vector directed from the kth source charge q_k at position $\vec{r}_k'$ to the observation point $\vec{r}$, and $R_k = |\vec{r}-\vec{r}_k'|$ is the distance between them. In eq. (6.10), $\vec{a}_R = \dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$ is a unit vector directed from the source point $\vec{r}\,'$ to the observation point $\vec{r}$, and $R(\vec{r},\vec{r}\,') = |\vec{r}-\vec{r}\,'|$ is the distance between them. The differential charge at $\vec{r}\,'$, i.e., $\rho_v(\vec{r}\,')dv'$, is replaced by $\rho_s(\vec{r}\,')ds'$ or $\rho_l(\vec{r}\,')dl'$ if it is the surface or line charge density of interest, respectively.

<Comment>

If the point charge at the origin is regarded as an impulse source, eq. (6.7) standards for the electric field "impulse response" of the free space (a linear system).

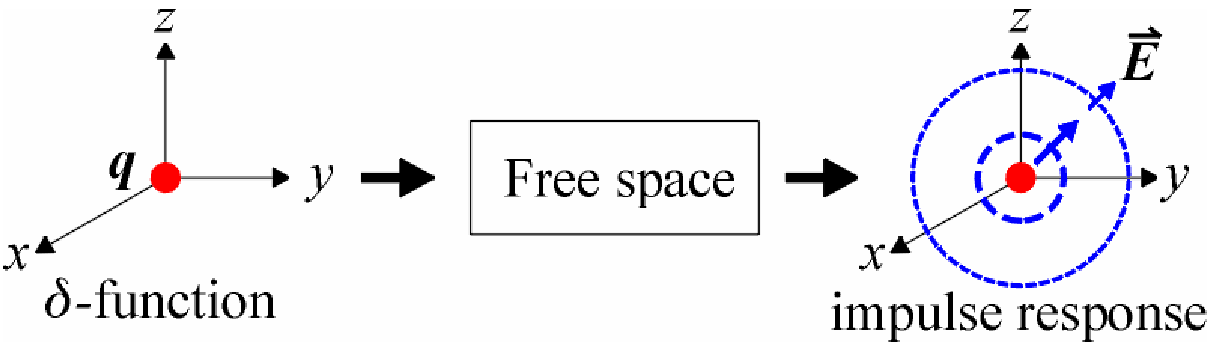

Fig. 6-4. The electric field impulse response of free space.

Example 6-3: For a thin spherical shell with uniform surface charge distribution, what is the electric field at an arbitrary point "inside" the shell?

Ans: (M1) Since no charge exists inside the shell, $\oint_S \vec{E}\cdot d\vec{s} = E_R(R)\cdot\left(4\pi R^2\right) = 0$ for any "Gaussian" surface S inside the shell. $\Rightarrow$ $\vec{E} = 0$ everywhere. (M2) By eq. (6.10), the contributions of $\vec{E}$ from a pair of elementary cones (DKC p80) will cancel with each other. By scanning the solid angle $d\Omega$ to cover the entire spherical surface, $\Rightarrow$ $\vec{E} = 0$

everywhere.

6.4 Electric Potential

■ Definition and physical meaning

From the null identity of eq. (5.34) and fundamental postulate eq. (6.2), vector E-field $\vec{E}$ is curl-free and can be expressed as the gradient of a scalar potential field V:

$$\vec{E} = -\nabla V \tag{6.11}$$

The work that has to be done to move a charge q from P_1 to P_2 in an electric field $\vec{E}$ is: $W_{12} = -q\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$. Substituting eq. (6.11) into the line integral, we have: $\frac{W_{12}}{q} = \int_{P_1}^{P_2} \nabla V \cdot d\vec{l}$.

By eq. (5.21),

$$\frac{W_{12}}{q} = \int_{P_1}^{P_2} \left[\vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z} \right] \cdot \left[\vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz \right]$$

$$= \int_{P_1}^{P_2} \Delta V_x + \Delta V_y + \Delta V_z = V_2 - V_1 , \Rightarrow$$

$$\frac{W_{12}}{q} = V_2 - V_1 \text{ (J/C, or Volt)} \tag{6.12}$$

i.e., the electric potential V defined by eq. (6.11) has a physical meaning of: the work done in moving a unit charge against the existing electric field.

<Comment>

1) The line integral $\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$ is path-independent, $\Rightarrow$ consistent with the Kirchhoff's voltage law eq. (6.4).
2) Since ∇V is in the direction of maximum rate of increase of V, eq. (6.11) states that the electric field is always normal to the "equi-potential" surfaces or lines.

■ Electric potential created by point charge, discrete charges, and charge distribution

Let the point P_1 be at infinity and $V_1 = 0$ (reference). The potential at a sphere of radius R due to a point charge q located at the origin can be evaluated by eq. (6.7):

$$V_2 - V_1 = V(R) - 0 = \int_{\infty}^{R}\left(-\vec{a}_R \frac{1}{4\pi\varepsilon_0}\frac{q}{R'^2}\right)\cdot(\vec{a}_R dR') = \frac{q}{4\pi\varepsilon_0}\int_R^{\infty}\frac{dR'}{R'^2}, \Rightarrow$$

$$V(R) = \frac{q}{4\pi\varepsilon_0 R} \tag{6.13}$$

The point charge at the origin is regarded as an impulse source, eq. (6.13) represents the potential “impulse response” of free space (a linear system). The potential due to discrete charges or continuous charge distribution can be derived by superposition:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{R_k} \tag{6.14}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_v(\vec{r}\,')}{R(\vec{r},\vec{r}\,')}dv' \tag{6.15}$$

where $R_k = |\vec{r} - \vec{r}_k'|$, $R(\vec{r},\vec{r}\,') = |\vec{r} - \vec{r}\,'|$ are the distance between the source and observation points.

<Comment>

Given charge distribution, we should derive $\vec{E}$ according to the following priority:

1) By Gauss’s law whenever possible.
2) Evaluate potential field V first (scalar integration), then determine $\vec{E}$ by eq. (6.11).
3) Directly determine $\vec{E}$ eq. (6.10) (vector integration).

6.5 Electric Dipole

An electric dipole consists of a pair of point charges $\pm q$ separated by distance d. The resulting electric field at a distant point $\vec{r} = \vec{a}_R R$ ($R >> d$) can be evaluated in two ways.

(M1) Refer to Fig. 6-5a. By eq. (6.9), the electric dipole forms an electric field of:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\vec{R}-\vec{d}/2}{\left|\vec{R}-\vec{d}/2\right|^3} - \frac{\vec{R}+\vec{d}/2}{\left|\vec{R}+\vec{d}/2\right|^3}\right\}.$$

Under the far-field approximation ($R >> d$) (DKC p83):

(1) $\left|\vec{R}-\vec{d}/2\right|^2 = \left(\vec{R}-\vec{d}/2\right)\cdot\left(\vec{R}-\vec{d}/2\right) = R^2 - \left(\vec{R}\cdot\vec{d}\right) + \frac{d^2}{4} = R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}+\frac{d^2}{4R^2}\right) \approx R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}\right),$

$\Rightarrow \left|\vec{R}-\vec{d}/2\right|^{-3} \approx \left[R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}\right)\right]^{-3/2} \approx R^{-3}\left(1+\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right)$. Also, $\left|\vec{R}+\vec{d}/2\right|^{-3} \approx R^{-3}\left(1-\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right)$.

(2) By $\vec{R} = \vec{a}_R R$, $\vec{d} = \vec{a}_z d$, $\Rightarrow$ $\vec{R}\cdot\vec{d} = Rd\cos\theta$, $\Rightarrow$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 R^3}\left\{\left(1+\frac{3}{2}\frac{Rd\cos\theta}{R^2}\right)\left(\vec{a}_R R - \vec{a}_z\frac{d}{2}\right) - \left(1-\frac{3}{2}\frac{Rd\cos\theta}{R^2}\right)\left(\vec{a}_R R + \vec{a}_z\frac{d}{2}\right)\right\}$$
$$\approx \frac{qd}{4\pi\varepsilon_0 R^3}\cdot\left(\vec{a}_R 3\cos\theta - \vec{a}_z\right).$$

(3) In the spherical coordinate system, $\vec{a}_z = \vec{a}_R\cos\theta - \vec{a}_\theta\sin\theta$, $\Rightarrow$

$$\vec{E}(\vec{r}) \approx \frac{p}{4\pi\varepsilon_0 R^3}\left(\vec{a}_R 2\cos\theta + \vec{a}_\theta\sin\theta\right) \tag{6.16}$$

where $p = \left|\vec{p}\right|$, and $\vec{p}$ is the electric dipole moment:

$$\vec{p} \equiv q\vec{d} \tag{6.17}$$

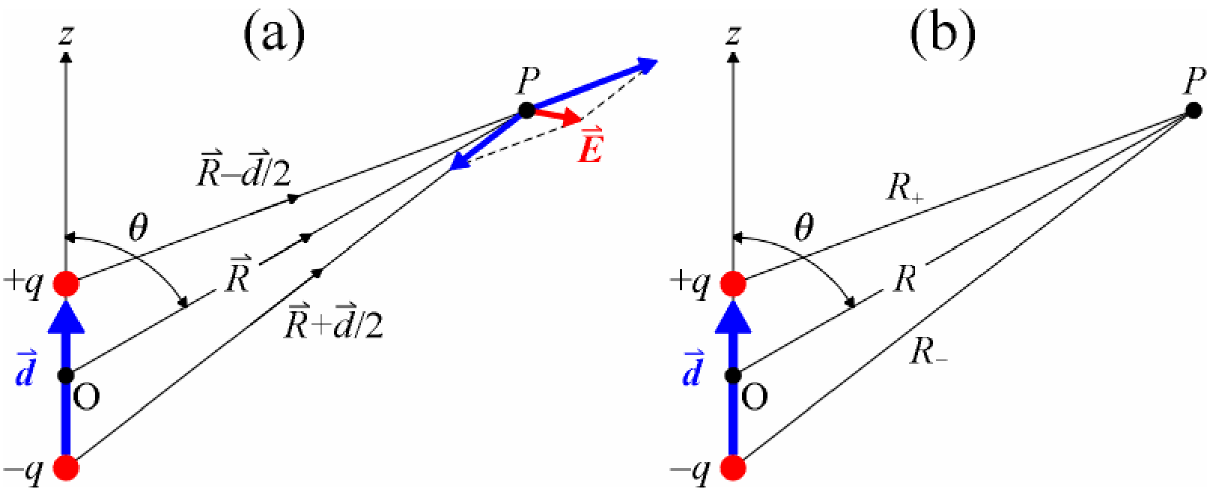

Fig. 6-5. (a) The electric field at P created by an electric dipole. (b) The geometry used to evaluate potential at P created by an electric dipole.

(M2) Refer to Fig. 6-5b. By eq. (6.14), the electric dipole forms an electric potential field of:

$$V(R)=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+}-\frac{1}{R_-}\right).$$

Under the far-field approximation ($R>>d$) (DKC p95):

$$R_+\approx R-\frac{d}{2}\cos\theta=R\left(1-\frac{d}{2R}\cos\theta\right),\Rightarrow\ \frac{1}{R_+}\approx R^{-1}\left(1+\frac{d}{2R}\cos\theta\right);$$

$$R_-\approx R+\frac{d}{2}\cos\theta=R\left(1+\frac{d}{2R}\cos\theta\right),\Rightarrow\ \frac{1}{R_-}\approx R^{-1}\left(1-\frac{d}{2R}\cos\theta\right);$$

$$\frac{1}{R_+}-\frac{1}{R_-}\approx\frac{d}{R^2}\cos\theta=\frac{(\vec{p}/q)\cdot\vec{a}_R}{R^2},\Rightarrow$$

$$V(R)\approx\frac{\vec{p}\cdot\vec{a}_R}{4\pi\varepsilon_0R^2} \tag{6.18}$$

Substitute eq. (6.18) into eq. (6.11) gives rise to the same formula as eq. (6.16).

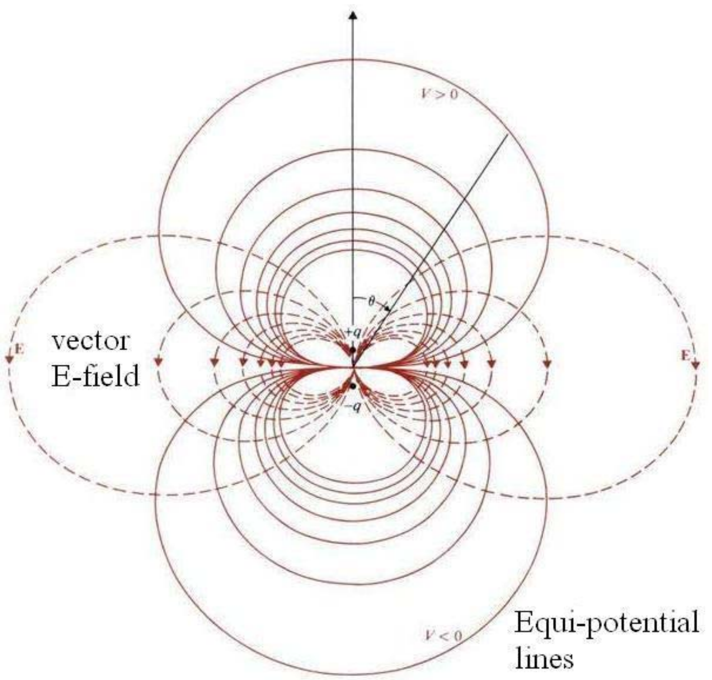

Fig. 6-6. The electric field and equi-potential lines created by an electric dipole (after DKC).

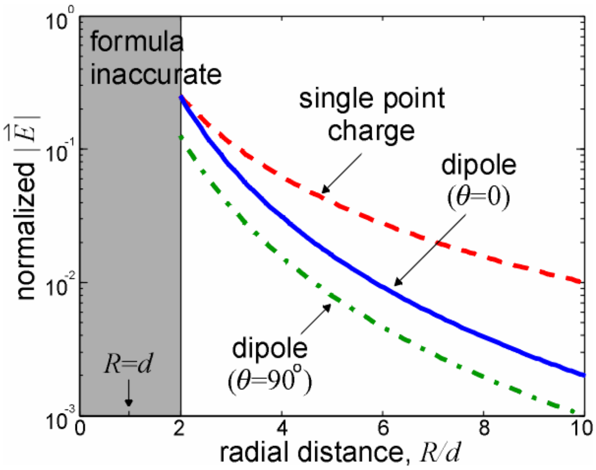

Fig. 6-7. Magnitude of the electric field versus the radial distance due to an electric dipole (solid and dash-dot) and a point charge (dashed).

<Comment>

1) The dipole-induced V and $|\vec{E}|$ [eq's (6.18), (6.16)] decay with the radial distance R one-order faster than those of single point charge [eq's (6.13), (6.7)]. This is attributed to the fact that the effects of two opposite charges tend to cancel (weaken) with each other.
2) Electric dipole is essential in modeling the interaction between non-conducting material and external electric field (Lesson 7).