Lesson 6
Electrostatics in Free Space

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Introduction

In this lesson, we will consider the electric field and potential due to electric charges at rest, and there is no magnetic field.

Outline

- Fundamental postulates
- Gauss’s law
- Coulomb’s law
- Electric potential
- Electric dipole

Sec. 6-1
Fundamental Postulates

1. Differential forms
2. Integral forms

Differential forms

$\vec{E}$: Force per unit charge (N/C)

Helmholtz's theorem:

$$\begin{cases} \nabla \cdot \vec{F} & \text{...flow source } g \\ \nabla \times \vec{F} & \text{...vortex source } \vec{G} \end{cases} \longrightarrow \vec{F}$$

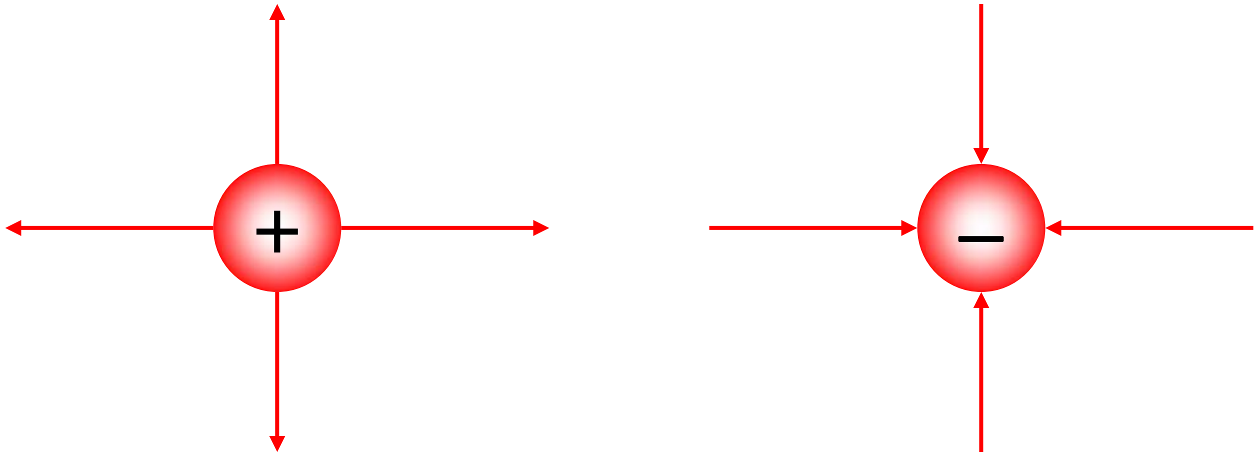
$$\Rightarrow \boxed{\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} = 0 \end{cases}}$$

ρ ...volume charge density (C/m^3)

ε_0 ...permittivity of vacuum

Physical meaning

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} & \text{...free charges are flow source of } \vec{E} \\ \nabla \times \vec{E} = 0 & \text{...no vortex source of } \vec{E} \end{cases}$$

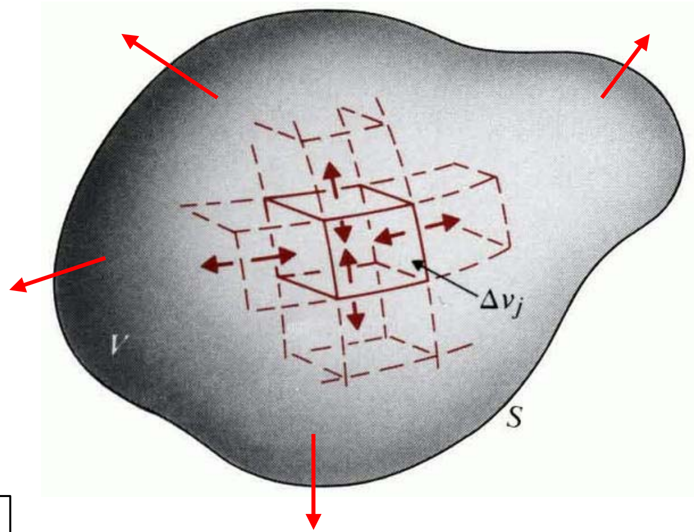


Integral forms-1

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0},\Rightarrow\int_V\left(\nabla\cdot\vec{E}\right)dv=\int_V\frac{\rho}{\varepsilon_0}dv=\frac{Q}{\varepsilon_0}$$

By the divergence theorem:

$$\oint_S\vec{A}\cdot d\vec{s}=\int_V\left(\nabla\cdot\vec{A}\right)dv$$



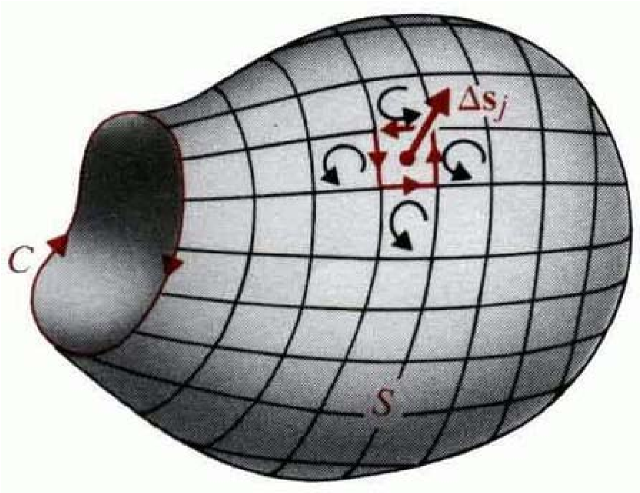
$$\Rightarrow\quad\boxed{\oint_S\vec{E}\cdot d\vec{s}=\frac{Q}{\varepsilon_0}}$$ …Gauss's law

Integral forms-2

$$\nabla \times \vec{E} = 0, \Rightarrow \int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{s} = 0$$

By the Stoke's theorem:

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{s}$$



$$\oint_C \vec{E} \cdot d\vec{l} = 0$$

Equivalent to Kirchhoff's voltage law: $\sum_k v_k = 0$

Static electric field is conservative.

Sec. 6-2
Gauss's Law

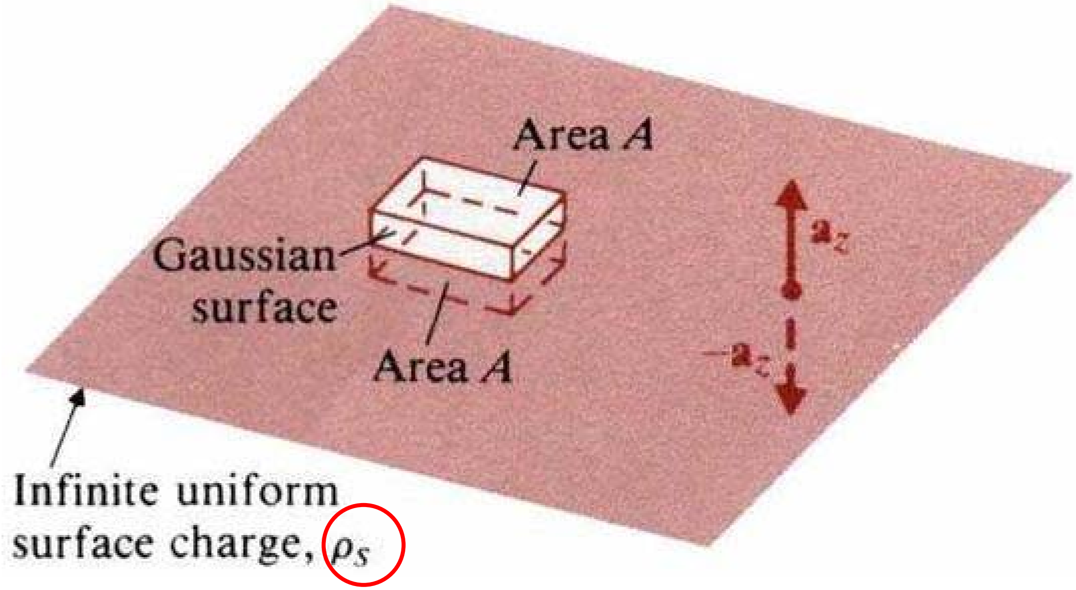
1. Definition
2. Examples

If the charge distribution has certain symmetry, such that the normal component of $\vec{E}$ is constant over an enclosed surface S (Gaussian surface), $\Rightarrow$

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

becomes convenient in determining $\vec{E}$

Example 6-1: Planar charge

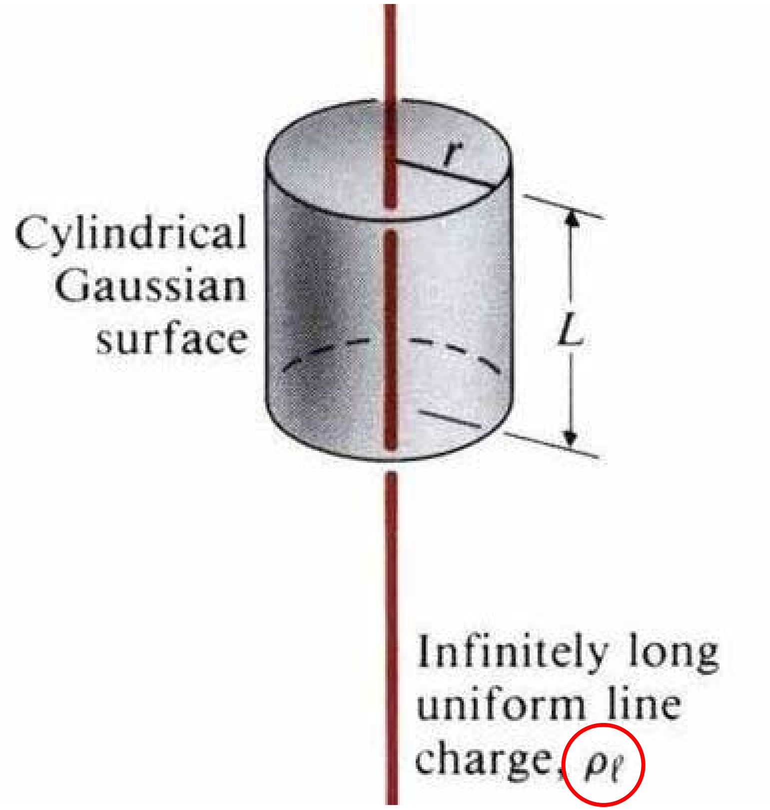


Planar symmetry, $\Rightarrow \vec{E} = \begin{cases} \vec{a}_z E_z(z), \text{if } z > 0 \\ -\vec{a}_z E_z(-z), \text{if } z < 0 \end{cases}$

$\Rightarrow \oint_S \vec{E} \cdot d\vec{s} = 2E_z(z)A = \dfrac{\rho_s A}{\varepsilon_0}$, $\quad E_z(z) = \dfrac{\rho_s}{2\varepsilon_0}$ …Independent of z!

Gaussian surface

Example 6-2: Line charge



Cylindrical symmetry, $\Rightarrow$

$$\vec{E} = \vec{a}_r E_r(r)$$

$$\oint_S \vec{E} \cdot d\vec{s} = E_r(r) \cdot (2\pi rL) = \frac{\rho_l L}{\varepsilon_0},$$

Gaussian surface

$$E_r(r) = \frac{\rho_l}{2\pi\varepsilon_0 r} \propto \frac{1}{r}$$

Comparison of different types of light source

Planar light source, minimal decay with distance

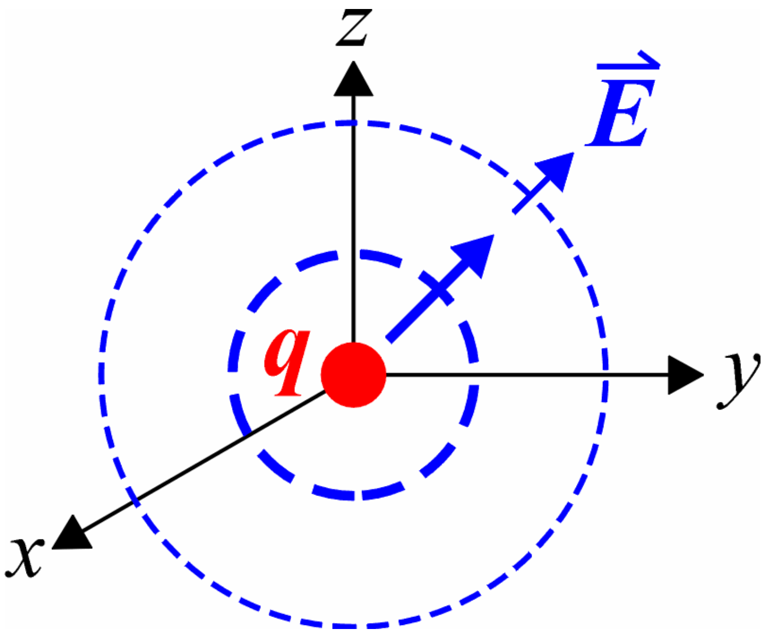
Linear light source, linear decay with distance

Point light source, quadratic decay with distance

Sec. 6-3
Coulomb's Law

1. Definition
2. Electric field due to point change
3. Electric field due to charge distributions
4. Electric sheltering

E-field due to a point charge



Spherical symmetry, $\Rightarrow$

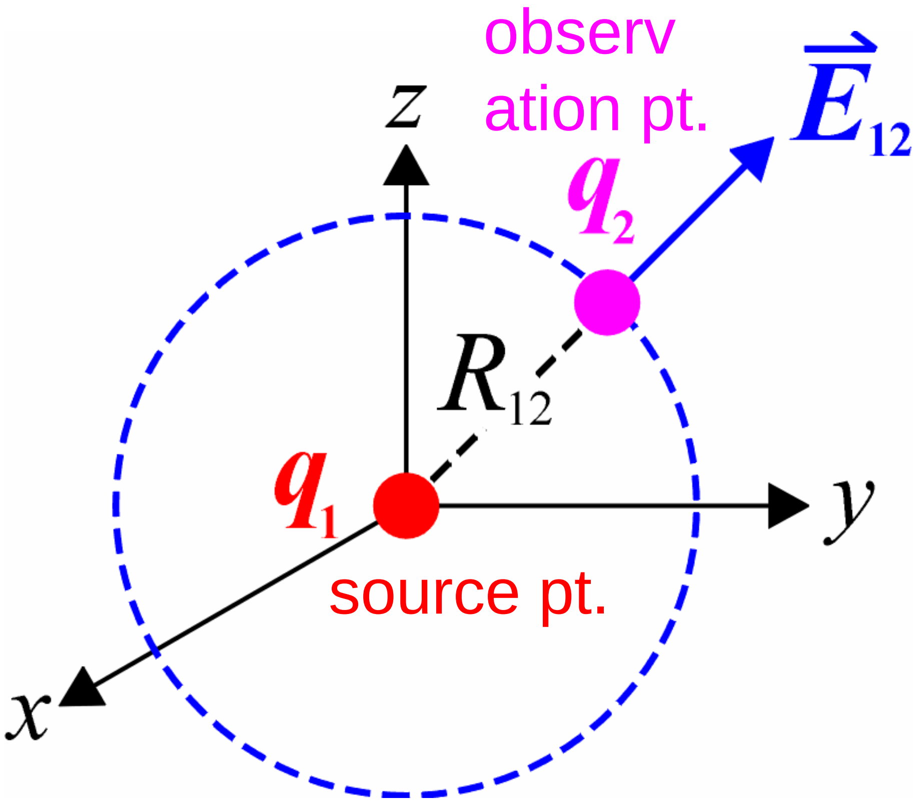
$$\vec{E} = \vec{a}_R E_R(R)$$

$$\oint_S \vec{E} \cdot d\vec{s} = E_R(R) \cdot \left(4\pi R^2\right) = \frac{q}{\varepsilon_0}$$

Gaussian surface

$$\vec{E} = \vec{a}_R \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \propto \frac{1}{R^2}$$

Coulomb's law



By $\vec{E} = \vec{F}/q$, $\Rightarrow$

$$\vec{F}_{12} = q_2 \vec{E}_{12} = \vec{a}_{R_{12}} \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{R_{12}^2}$$

…Coulomb's law, which is experimentally measurable.

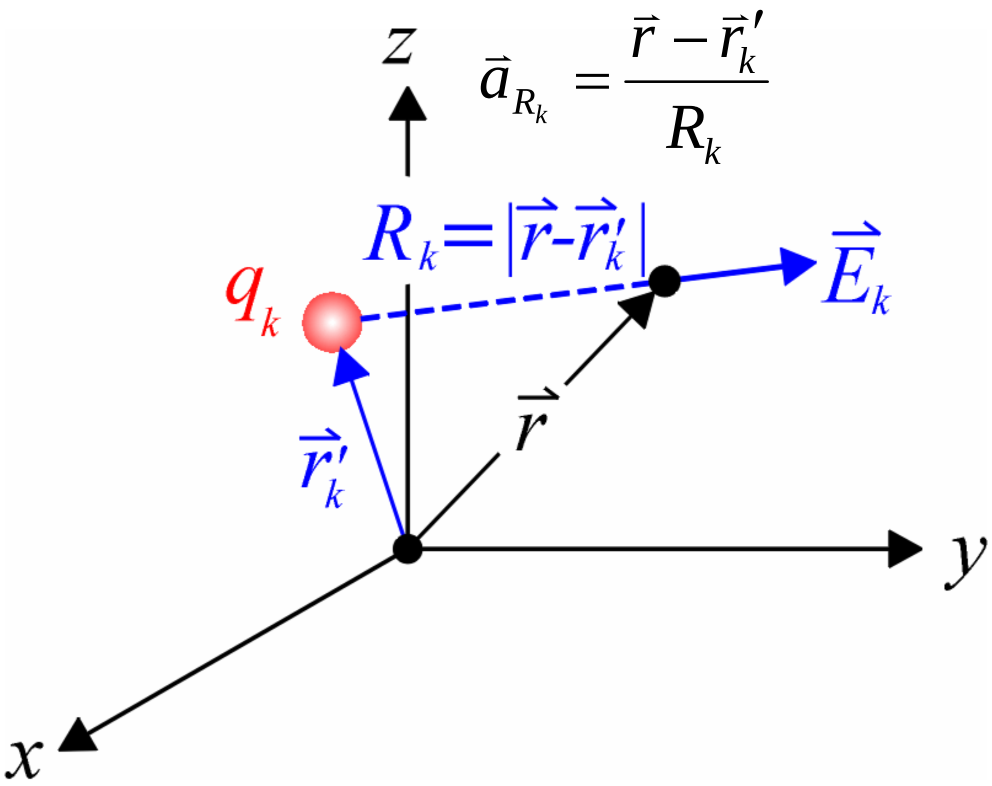
The 2 fundamental postulates are linear:

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} = 0 \end{cases} \longrightarrow \begin{cases} \nabla \cdot \vec{E}_1 = \dfrac{\rho_1}{\varepsilon_0} \\ \nabla \cdot \vec{E}_2 = \dfrac{\rho_2}{\varepsilon_0} \end{cases}$$

$$\Rightarrow \nabla \cdot \left(c_1 \vec{E}_1 + c_2 \vec{E}_2\right) = \frac{c_1 \rho_1 + c_2 \rho_2}{\varepsilon_0}$$

E-field due to charge distribution-2

For a system of discrete charges $\{q_k, k=1,2,...,n\}$:

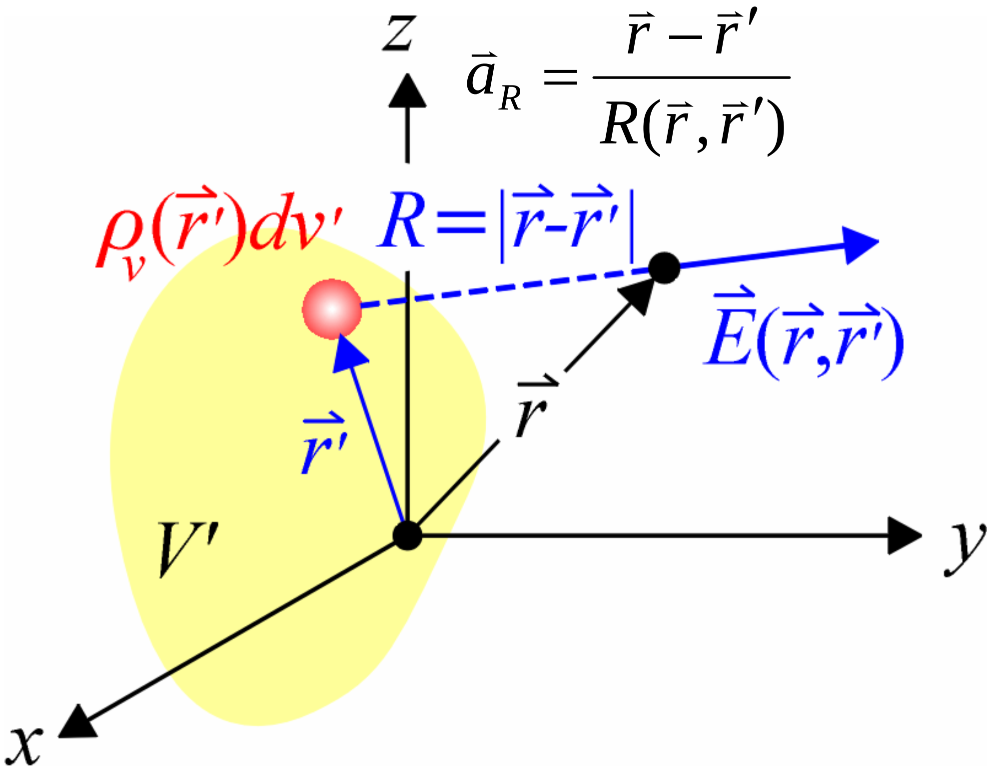


$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n} \vec{a}_{R_k}\frac{q_k}{R_k^2}$$

Principle of superposition

E-field due to charge distribution-3

For a system of continuous charge distribution $\rho_v(\vec{r}\,')$ within a volume V':

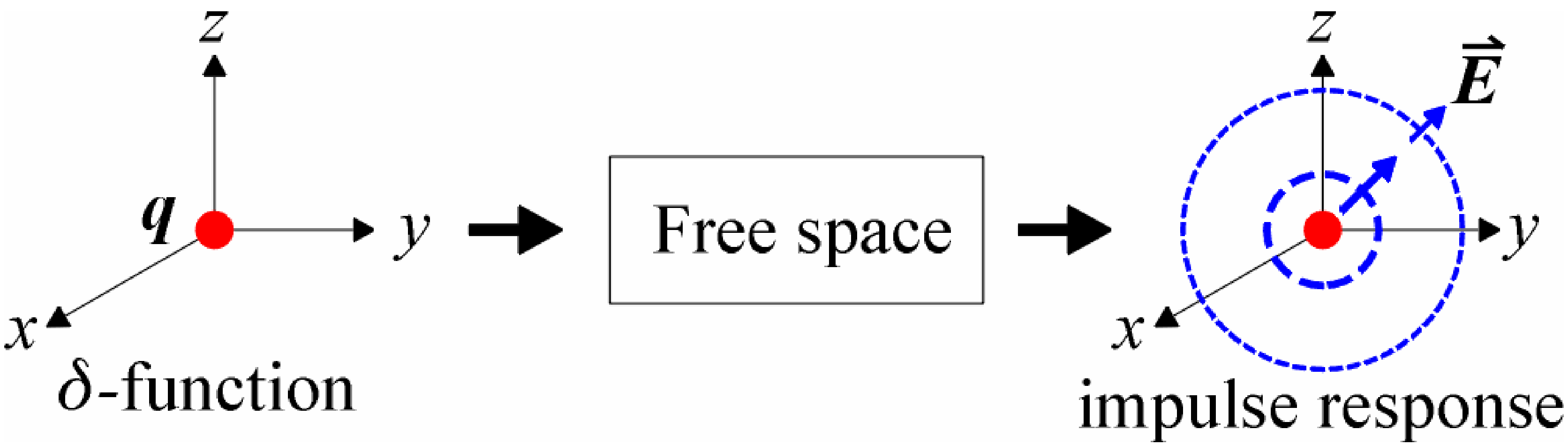


$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{a}_R \frac{\rho_v(\vec{r}\,')}{R(\vec{r}, \vec{r}\,')^2} dv'$$

Principle of superposition

Viewpoint of linear system

If the point charge at the origin is regarded as an impulse source, the resulting E-field becomes the impulse response of the system (free space).

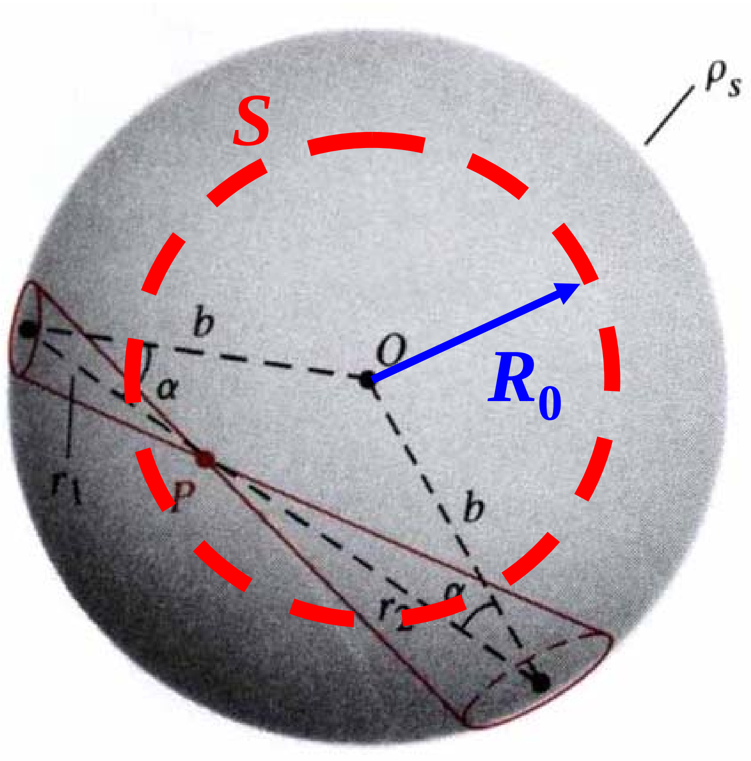


$$\rho(\vec{r}) = q\delta(x, y, z)$$

$$\vec{E}(\vec{r}) = \vec{a}_R \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

Example 6-3: Sheltering effect (1)

Consider a thin spherical shell with uniform surface charge distribution $\rho_s (\mathrm{C/m^2})$



(M1) Spherical symmetry,

$$\Rightarrow \vec{E} = \vec{a}_R E_R(R)$$

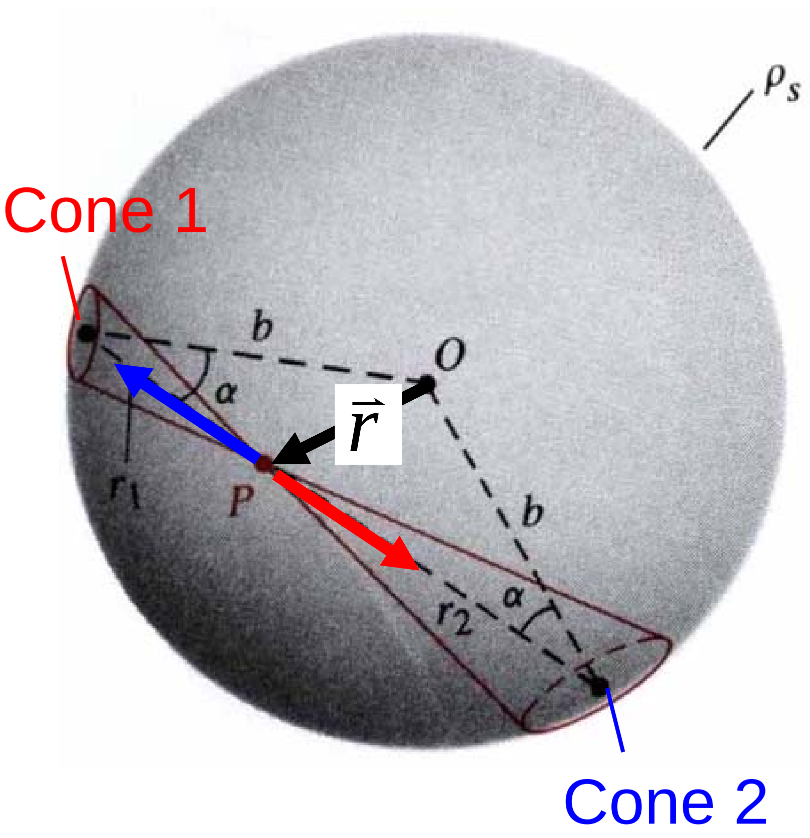
No charge inside:

$$\oint_S \vec{E} \cdot d\vec{s} = E_R(R) \cdot \left(4\pi R^2\right) = 0$$

for all Gaussian surfaces S: $R = R_0 < b$

$$\Rightarrow \vec{E} = 0 \quad \text{…for } R < b$$

Example 6-3: Sheltering effect (2)



(M2) By source integration:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{a}_R \frac{\rho_v(\vec{r}\,')}{R(\vec{r},\vec{r}\,')^2}\,dv',$$

Contributions from a pair of elementary cones cancel with each other.

$\Rightarrow$ $\vec{E} = 0$ for any point P inside the shell

Sec. 6-4
Electric Potential

1. Definition
2. Electric potential due to point change
3. Electric potential due to charge distributions
4. Procedures to determine electric field

Definition

$$\begin{cases} \nabla \times (\nabla V) = 0 & \text{... null identity} \\ \nabla \times \vec{E} = 0 & \text{... fundamental postulate} \end{cases} \longrightarrow \vec{E} = -\nabla V$$

Vector E-field $\vec{E}$ can be represented as the gradient of a scalar potential field V

Physical meaning

The work that has to be done to move a charge q from P_1 to P_2 in an electric field $\vec{E}$:

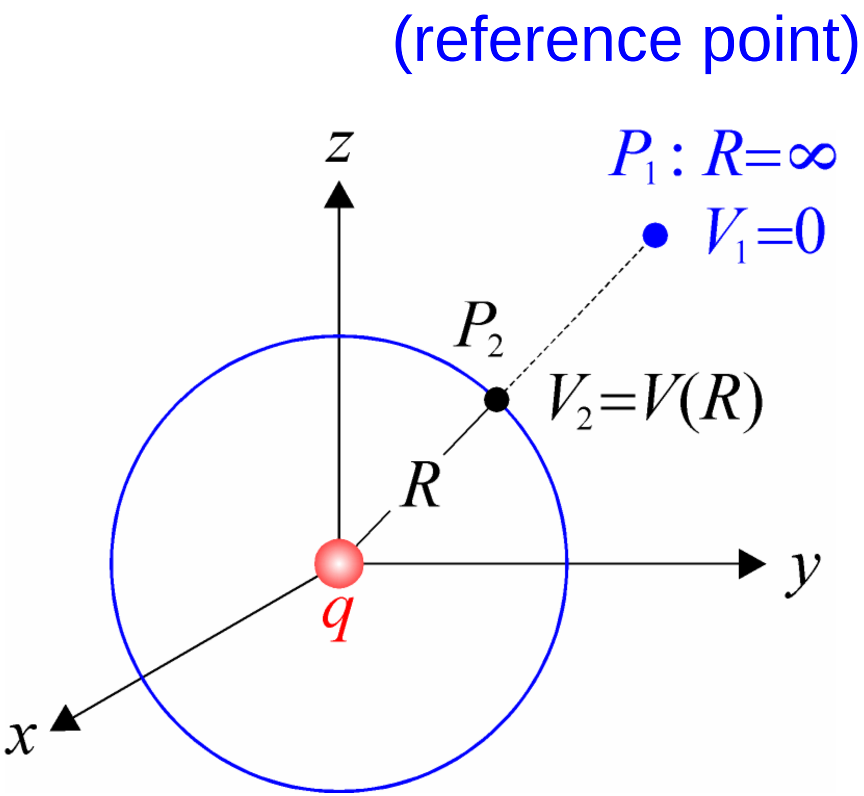
$$W_{12} = -q\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} = q\int_{P_1}^{P_2} \nabla V \cdot d\vec{l}$$

$$\vec{E} = -\nabla V$$

$$\Rightarrow \frac{W_{12}}{q} = \int_{P_1}^{P_2}\left[\vec{a}_x\frac{\partial V}{\partial x} + \vec{a}_y\frac{\partial V}{\partial y} + \vec{a}_z\frac{\partial V}{\partial z}\right]\cdot\left[\vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz\right]$$

$$= \int_{P_1}^{P_2} \Delta V_x + \Delta V_y + \Delta V_z = V_2 - V_1$$ …independent of path

Point charge



$$V_2 - V_1 = V(R) - 0$$

$$= \int_{P_1}^{P_2} \left(-\vec{E}\right) \cdot d\vec{l}$$

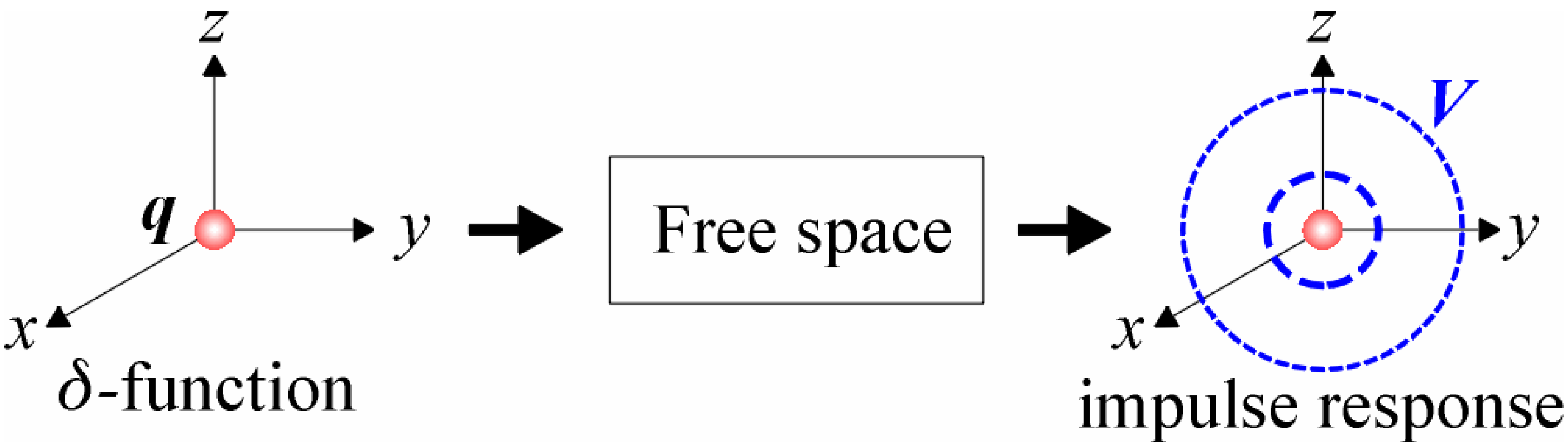
$$= \int_{\infty}^{R} \left(-\vec{a}_R \frac{1}{4\pi\varepsilon_0} \frac{q}{R'^2} \right) \cdot \left(\vec{a}_R dR' \right)$$

$$= \frac{q}{4\pi\varepsilon_0} \int_{R}^{\infty} \frac{dR'}{R'^2}$$

$$\Rightarrow V(R) = \frac{q}{4\pi\varepsilon_0 R} \propto \frac{1}{R}$$

Viewpoint of linear system

If the point charge at the origin is regarded as an impulse source:

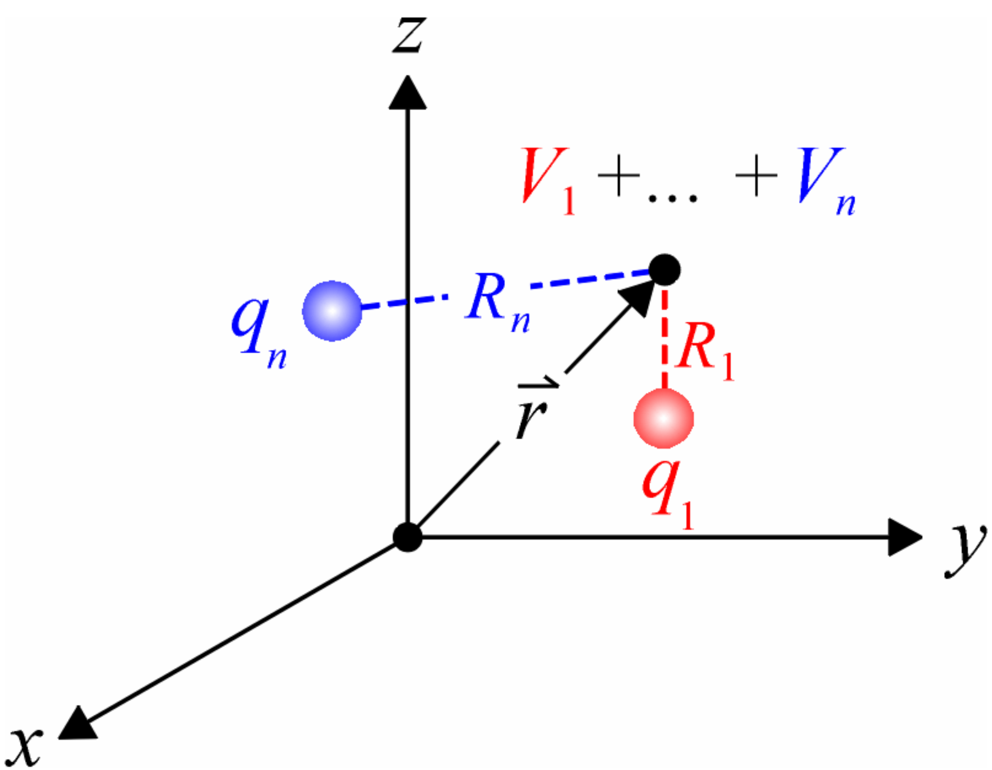


$$\rho(\vec{r}) = q\delta(x, y, z)$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 R}$$

Electrical potential due to charges-1

For a system of discrete charges $\{q_k, k = 1, 2, ..., n\}$:

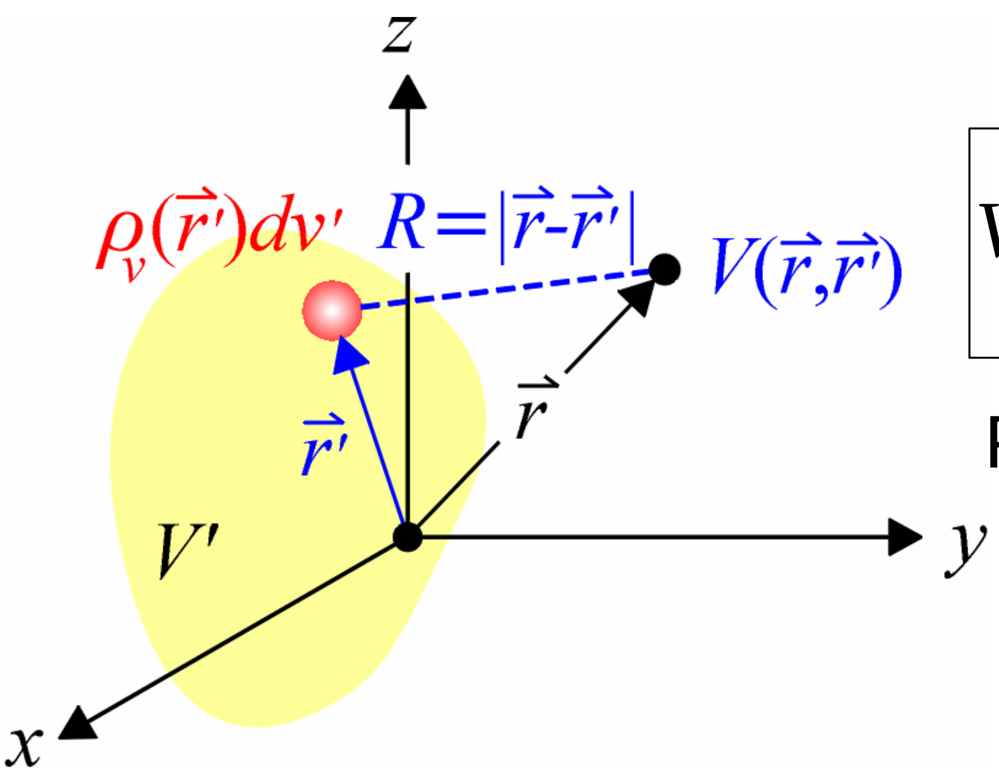


$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{R_k}$$

Principle of superposition

For a system of continuous charge distribution

$\rho_v(\vec{r}\,')$ within a volume V':



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_v(\vec{r}\,')}{R(\vec{r},\vec{r}\,')}dv'$$

Principle of superposition

How to derive the E-field?

Given charge distribution:

1. By Gauss's law whenever possible

2. Evaluate potential field V first (scalar integration), then $\vec{E} = -\nabla V$

3. Directly determine E-field by vector integration

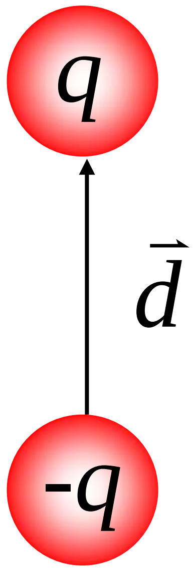
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \vec{a}_R \frac{\rho_v(\vec{r}\,')}{R(\vec{r},\vec{r}\,')^2} dv'$$

Sec. 6-5
Electric Dipole

1. Definition
2. Far electric field of a dipole
3. Far electric potential of a dipole
4. Comparison between point charge and dipole

Definition

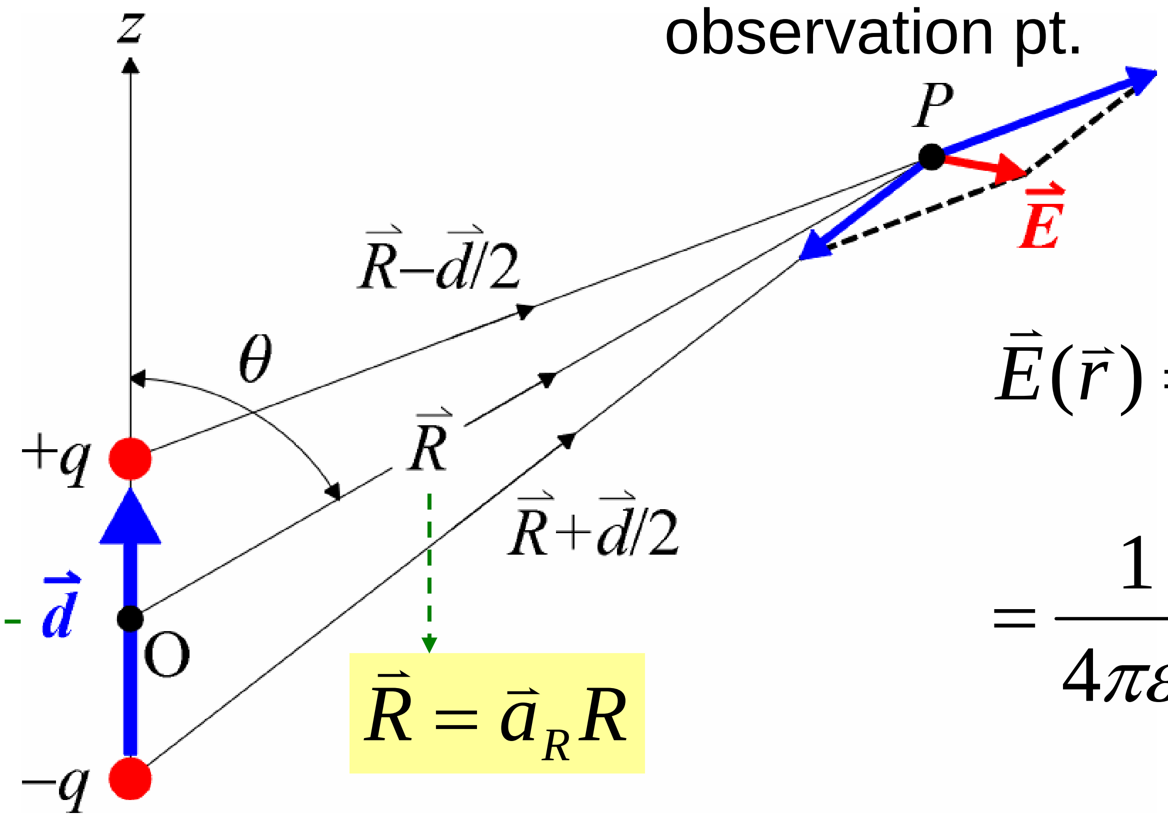
Essential in modeling the interaction between non-conducting material and external electric field (Lesson 7)



Dipole moment:

$$\vec{p} \equiv q\vec{d}$$

Far E-field-1



$$\vec{R} = \vec{a}_R R$$

$$\vec{d} = \vec{a}_z d$$

$$\vec{a}_{R_k} = \frac{\vec{r} - \vec{r}_k'}{R_k}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \vec{a}_{R_k} \frac{q_k}{R_k^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} q_k \frac{\vec{r} - \vec{r}_k'}{R_k^3}$$

$$\Rightarrow \vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{\vec{R} - \vec{d}/2}{\left|\vec{R} - \vec{d}/2\right|^3} - \frac{\vec{R} + \vec{d}/2}{\left|\vec{R} + \vec{d}/2\right|^3} \right\}$$

Far E-field-2

Since $R >> d$

$$\left|\vec{R}-\vec{d}/2\right|^2=\left(\vec{R}-\vec{d}/2\right)\cdot\left(\vec{R}-\vec{d}/2\right)=R^2-\left(\vec{R}\cdot\vec{d}\right)+\frac{d^2}{4}$$

$$=R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}+\frac{d^2}{4R^2}\right)\approx R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}\right)$$

$Rd\cos\theta$

$$\Rightarrow\left|\vec{R}-\vec{d}/2\right|^{-3}\approx\left[R^2\left(1-\frac{\vec{R}\cdot\vec{d}}{R^2}\right)\right]^{-3/2}\approx R^{-3}\left(1+\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right)$$

$$(1-x)^r\approx 1-rx$$

$$\text{Similarly,}\left|\vec{R}+\vec{d}/2\right|^{-3}\approx R^{-3}\left(1-\frac{3}{2}\frac{Rd\cos\theta}{R^2}\right)$$

Far E-field-3

$$\vec{a}_R R - \vec{a}_z \frac{d}{2} \qquad\qquad \vec{a}_R R + \vec{a}_z \frac{d}{2}$$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\left|\vec{R}-\vec{d}/2\right|^{-3}\left(\vec{R}-\vec{d}/2\right) - \left|\vec{R}+\vec{d}/2\right|^{-3}\left(\vec{R}+\vec{d}/2\right)\right]$$

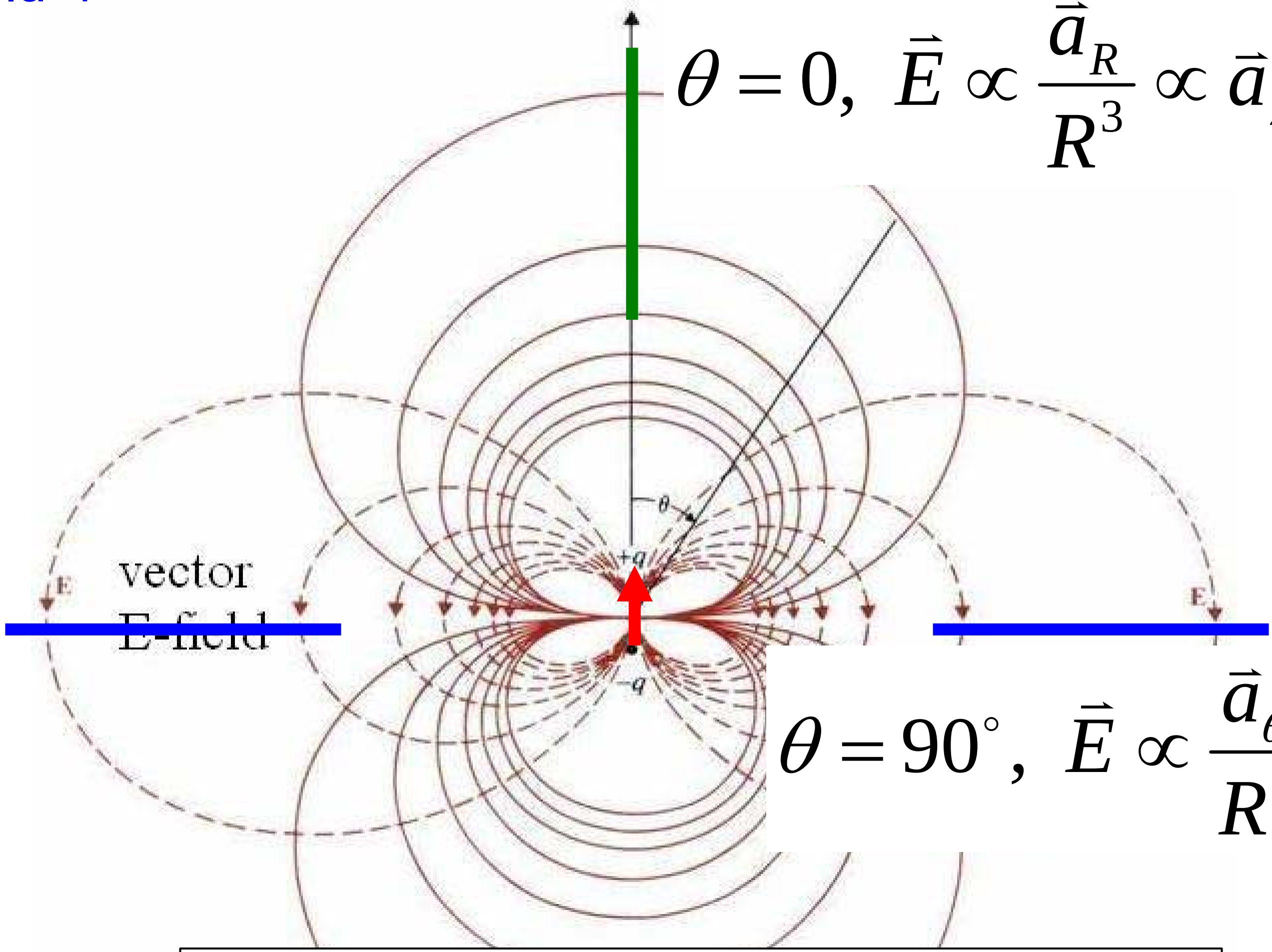
$$\left|\vec{R}-\vec{d}/2\right|^{-3} \approx R^{-3}\left(1+\frac{3}{2}\frac{Rd\cos\theta}{R^2}\right) \qquad \left|\vec{R}+\vec{d}/2\right|^{-3} \approx R^{-3}\left(1-\frac{3}{2}\frac{Rd\cos\theta}{R^2}\right)$$

$$\vec{a}_z = \vec{a}_R\cos\theta - \vec{a}_\theta \sin\theta, \qquad p = qd$$

$$\Rightarrow \vec{E}(\vec{r}) \approx \frac{qd}{4\pi\varepsilon_0 R^3}\cdot\left(\vec{a}_R 3\cos\theta - \vec{a}_z\right)$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) \approx \frac{p}{4\pi\varepsilon_0 R^3}\left(\vec{a}_R 2\cos\theta + \vec{a}_\theta \sin\theta\right)} \quad (\text{if } R >> d)$$

Far E-field-4

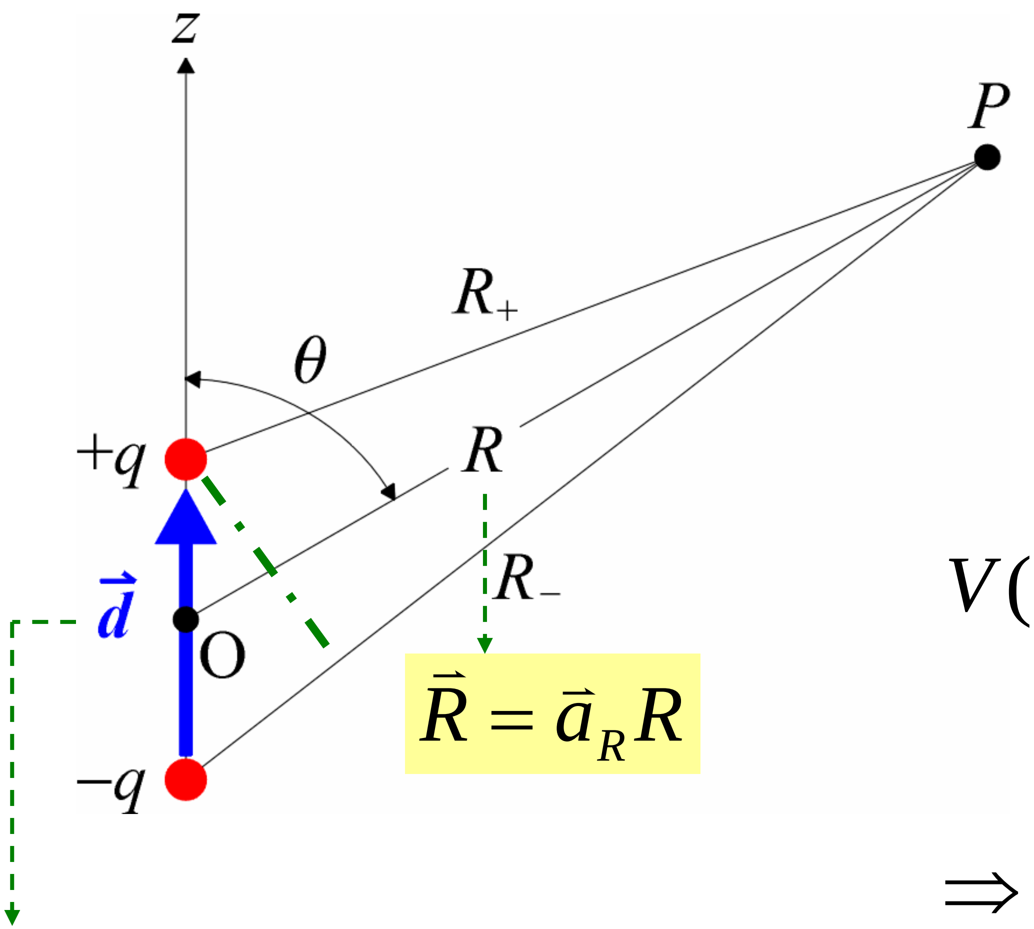


$$\theta = 0,\ \vec{E} \propto \frac{\vec{a}_R}{R^3} \propto \vec{a}_z$$

$$\theta = 90^\circ,\ \vec{E} \propto \frac{\vec{a}_\theta}{R^3} \propto -\vec{a}_z$$

$$\boxed{\vec{E}(\vec{r}) \approx \frac{p}{4\pi\varepsilon_0 R^3}\left(\vec{a}_R\, 2\cos\theta + \vec{a}_\theta \sin\theta\right)} \quad (\text{if } R >> d)$$

Far field electric potential-1



$$\vec{R} = \vec{a}_R R$$

$$\vec{d} = \vec{a}_z d$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{R_k}$$

$$\Rightarrow V(R) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

Far field electric potential-2

Since $R >> d$

$$R_+ \approx R - \frac{d}{2}\cos\theta = R\left(1 - \frac{d}{2R}\cos\theta\right)$$

$$\Rightarrow \frac{1}{R_+} \approx R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$R_- \approx R + \frac{d}{2}\cos\theta = R\left(1 + \frac{d}{2R}\cos\theta\right)$$

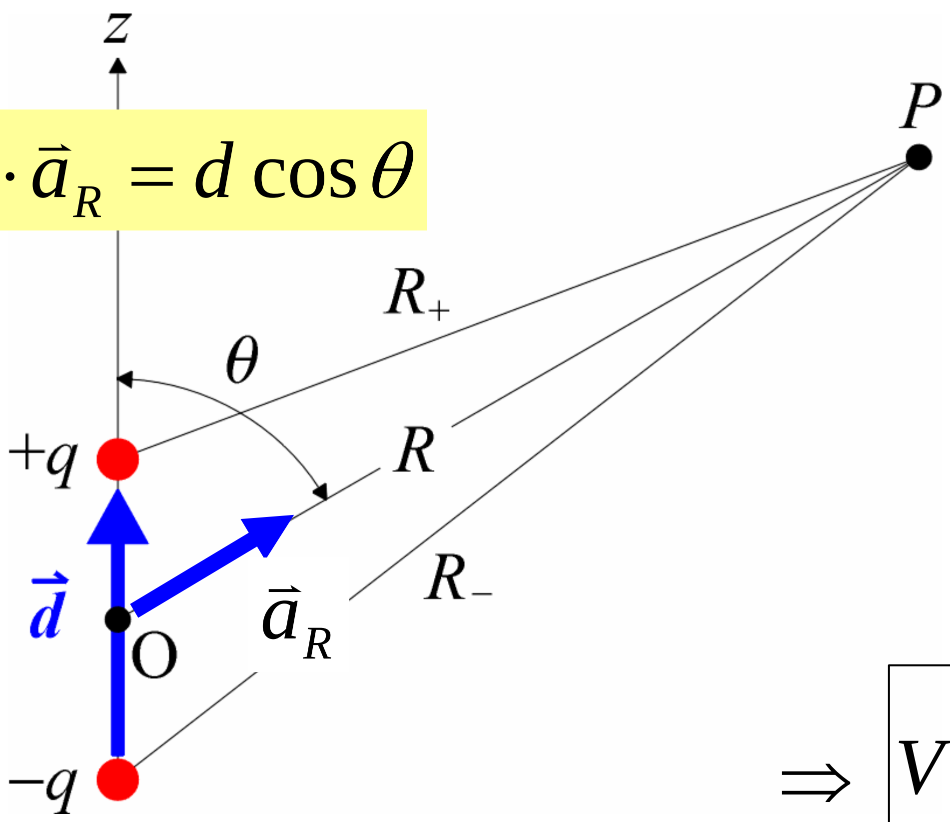
$$\Rightarrow \frac{1}{R_-} \approx R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right),$$

$$\Rightarrow \frac{1}{R_+} - \frac{1}{R_-} \approx R^{-1}\left(\frac{d}{R}\cos\theta\right)$$

$$= d\cos\theta / R^2$$

Far field electric potential-3

$$\vec{d}\cdot\vec{a}_R = d\cos\theta$$



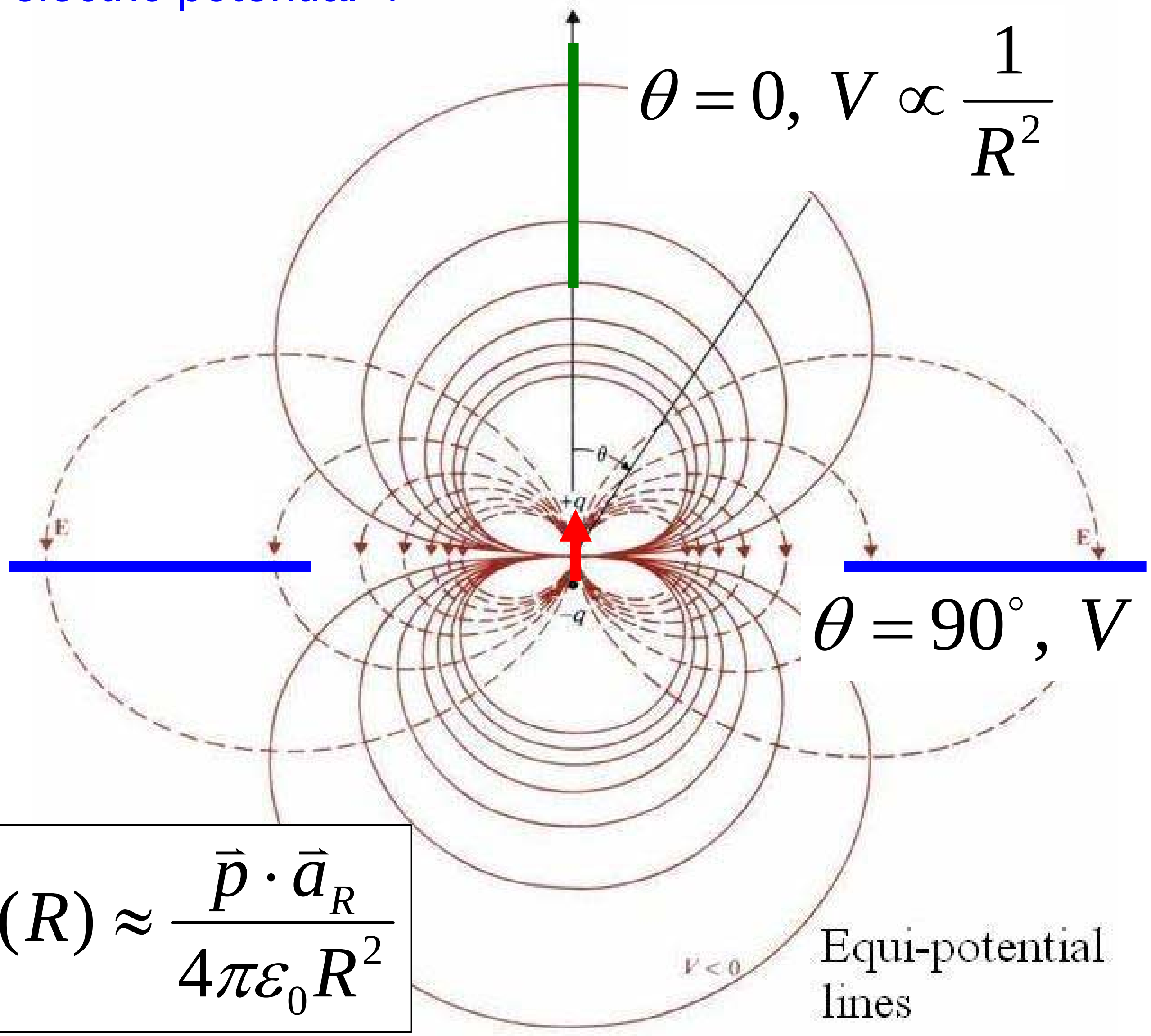
$$V(R) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

$$\approx \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{q\vec{d}\cdot\vec{a}_R}{4\pi\varepsilon_0 R^2}$$

$$\Rightarrow \boxed{V(R) \approx \frac{\vec{p}\cdot\vec{a}_R}{4\pi\varepsilon_0 R^2}} \quad (\text{if } R >> d)$$

$$\Rightarrow \vec{E} = -\nabla V \approx \frac{p}{4\pi\varepsilon_0 R^3}\left(\vec{a}_R\, 2\cos\theta + \vec{a}_\theta \sin\theta\right)$$

Far field electric potential-4



$$\theta = 0,\ V \propto \frac{1}{R^2}$$

$$\theta = 90^\circ,\ V = 0$$

$$V(R) \approx \frac{\vec{p} \cdot \vec{a}_R}{4\pi\varepsilon_0 R^2}$$

Dipole vs. point charge

$$\begin{cases} \vec{E}(\vec{r}) \approx \dfrac{p}{4\pi\varepsilon_0 R^3}\left(\vec{a}_R\, 2\cos\theta + \vec{a}_\theta \sin\theta\right) \\ \vec{E}(\vec{r}) = \vec{a}_R \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{R^2} \end{cases}$$

$$\begin{cases} V(R) \approx \dfrac{\vec{p}\cdot\vec{a}_R}{4\pi\varepsilon_0 R^2} \\ V(R) = \dfrac{q}{4\pi\varepsilon_0 R} \end{cases}$$