Lesson 05 Vector Analysis

■ Introduction

1) Physical quantities in EM could be scalar (charge, current, energy) or vector (EM fields).
2) Specifying a vector in a 3-D space requires three numbers, depending on the choice of a coordinate system. However, EM laws are independent of coordinate system of use.
3) The use of vector analysis in EM is not necessary (e.g., James Maxwell's original work), but will lead to elegant formulations.

5.1 Vector Algebra

■ Vector addition and subtraction

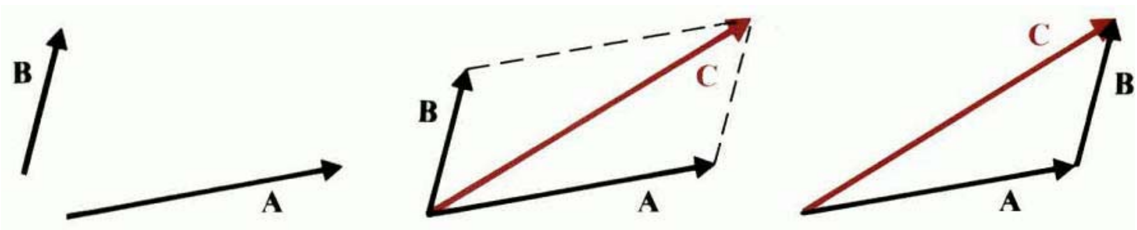

Fig. 5-1. Illustration of vector addition: $\vec{C} = \vec{A} + \vec{B}$ (after DKC).

■ Dot (inner) product

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is a scalar equal to the product of $|\vec{A}|$ and the "projection" of $\vec{B}$ on $\vec{A}$:

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta_{AB}, \tag{5.1}$$

where $\theta_{AB} = [0, \pi]$ is the smaller included angle. Dot product is commutative ($\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$) and distributive [$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$].

■ Cross (outer) product

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector:

$$\vec{A}\times\vec{B}=\vec{a}_n\left|\vec{A}\right|\cdot\left|\vec{B}\right|\cdot\sin\theta_{AB}, \tag{5.2}$$

where $\vec{a}_n$ is a unit vector normal to $\vec{A}$ and $\vec{B}$ (in the direction of the right thumb when the four fingers rotate from $\vec{A}$ to $\vec{B}$), $\theta_{AB}=[0,\pi]$ is the smaller included angle. Cross product is neither commutative ($\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$) nor associative [$\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$], but is distributive [$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$].

■ Product of three vectors

1) Scalar triple product:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right) \tag{5.3}$$

Its magnitude represents the volume of the parallelepiped (Fig. 5-2).

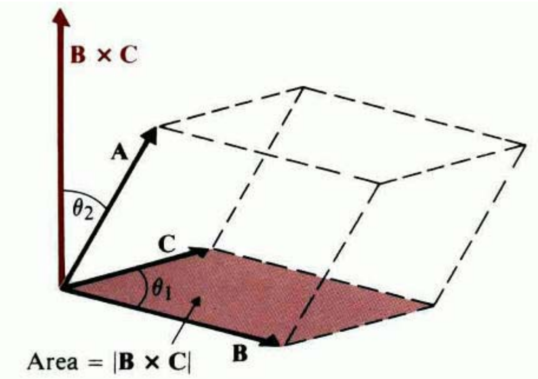

Fig. 5-2. Illustration of scalar triple product: $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$ (after DKC).

2) Vector triple product:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\cdot\left(\vec{A}\cdot\vec{B}\right) \tag{5.4}$$

5.2 Orthogonal Coordinate Systems

■ Definition and basic properties

A point in a 3-D orthogonal coordinate system can be located as the intersection of three "curved", mutually perpendicular surfaces represented by $\{u_i = \text{constant},\ i = 1,\ 2,\ 3\}$. The three base vectors (unit vectors in the directions of coordinate axes) $\{\vec{a}_{u_i}\}$ satisfy with:

$$\vec{a}_{u_i} \times \vec{a}_{u_j} = \vec{a}_{u_k},\quad (i, j, k) = (1,2,3),\ (2,3,1),\ (3,1,2); \tag{5.5}$$

$$\vec{a}_{u_i} \cdot \vec{a}_{u_j} = \delta_{ij} = \begin{cases} 0, \text{if } i \neq j \\ 1, \text{if } i = j \end{cases}, \tag{5.6}$$

Eq's (5.1), (5.2) can also be evaluated by linear algebra formulas if the involved vectors are represented by the linear combination of the base vectors of some orthogonal coordinate system: $\vec{A} = \vec{a}_{u_1} A_1 + \vec{a}_{u_2} A_2 + \vec{a}_{u_3} A_3$, $\vec{B} = \vec{a}_{u_1} B_1 + \vec{a}_{u_2} B_2 + \vec{a}_{u_3} B_3$:

$\vec{A} \cdot \vec{B} = \left(\vec{a}_{u_1} A_1 + \vec{a}_{u_2} A_2 + \vec{a}_{u_3} A_3\right) \cdot \left(\vec{a}_{u_1} B_1 + \vec{a}_{u_2} B_2 + \vec{a}_{u_3} B_3\right)$, by eq's (5.5), (5.6),

$\vec{A} \cdot \vec{B} = \left(\vec{a}_{u_1} \cdot \vec{a}_{u_1}\right) A_1 B_1 + \left(\vec{a}_{u_1} \cdot \vec{a}_{u_{12}}\right) A_1 B_2 + \ldots \left(\vec{a}_{u_3} \cdot \vec{a}_{u_3}\right) A_3 B_3 = A_1 B_1 + A_2 B_2 + A_3 B_3, \Rightarrow$

$$\vec{A} \cdot \vec{B} = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \times \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \sum_{i=1}^{3} A_i B_i \tag{5.7}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_{u_1} & \vec{a}_{u_2} & \vec{a}_{u_3} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{5.8}$$

■ Metric coefficients

When the coordinate changes from (u_1, u_2, u_3) to $(u_1 + du_1, u_2, u_3)$, the observation point P moves along the direction of $\vec{a}_{u_1}$ by a differential length of $dl_1 = h_1 \cdot du_1$, where h_1 denotes the metric coefficient of u_1. The same rule applies to u_2 and u_3 as well. Note that $h_i = 1$ only if u_i represents a quantity of "length".

When the coordinate changes from (u_1, u_2, u_3) to $(u_1 + du_1, u_2 + du_2, u_3 + du_3)$, the observation point P moves by a differential displacement of $d\vec{l} = \sum_{i=1}^{3} \vec{a}_{u_1} dl_i$, defining an enclosed differential volume of $dv = \prod_{i=1}^{3} dl_i$.

■ Cartesian coordinate system

1) $(u_1, u_2, u_3) = (x, y, z)$ (Fig. 5-3).

2) The base vectors $\{\vec{a}_{u_i},\ i = 1, 2, 3\}$ are independent of the observation point.

3) Since $\{x, y, z\}$ are all quantities of "length", the metric coefficients and differential volume are:

$$\{h_1 = h_2 = h_3 = 1\}, \quad dv = dxdydz \tag{5.9}$$

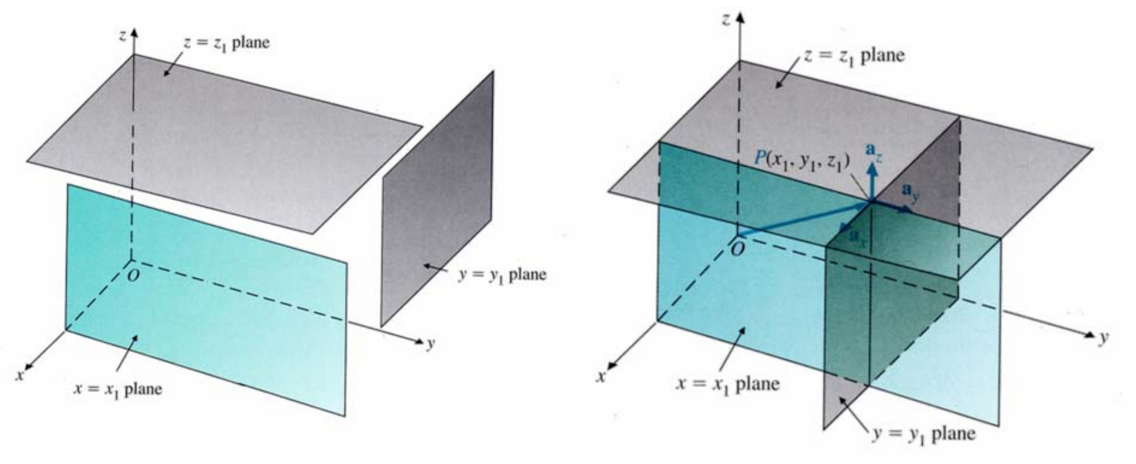

Fig. 5-3. Positioning of one point in the Cartesian coordinate system (after DKC).

■ Cylindrical coordinate system

1) $(u_1, u_2, u_3) = (r, \phi, z)$ (Fig. 5-4a).

2) Two of the base vectors $\{\vec{a}_r, \vec{a}_\phi\}$ change with the polar angle ϕ of the observation point (Fig. 5-4b).

3) When the coordinate changes from (r,ϕ,z) to $(r,\phi+d\phi,z)$, the observation point P moves along the direction of $\vec{a}_\phi$ by a differential length of $dl_\phi = r\cdot d\phi$, $\Rightarrow$ $h_2 = r$. The metric coefficients and differential volume are (Fig. 5-4c):

$$\{h_1=1,\ h_2=r, h_3=1\},\quad dv = r dr d\phi dz \tag{5.10}$$

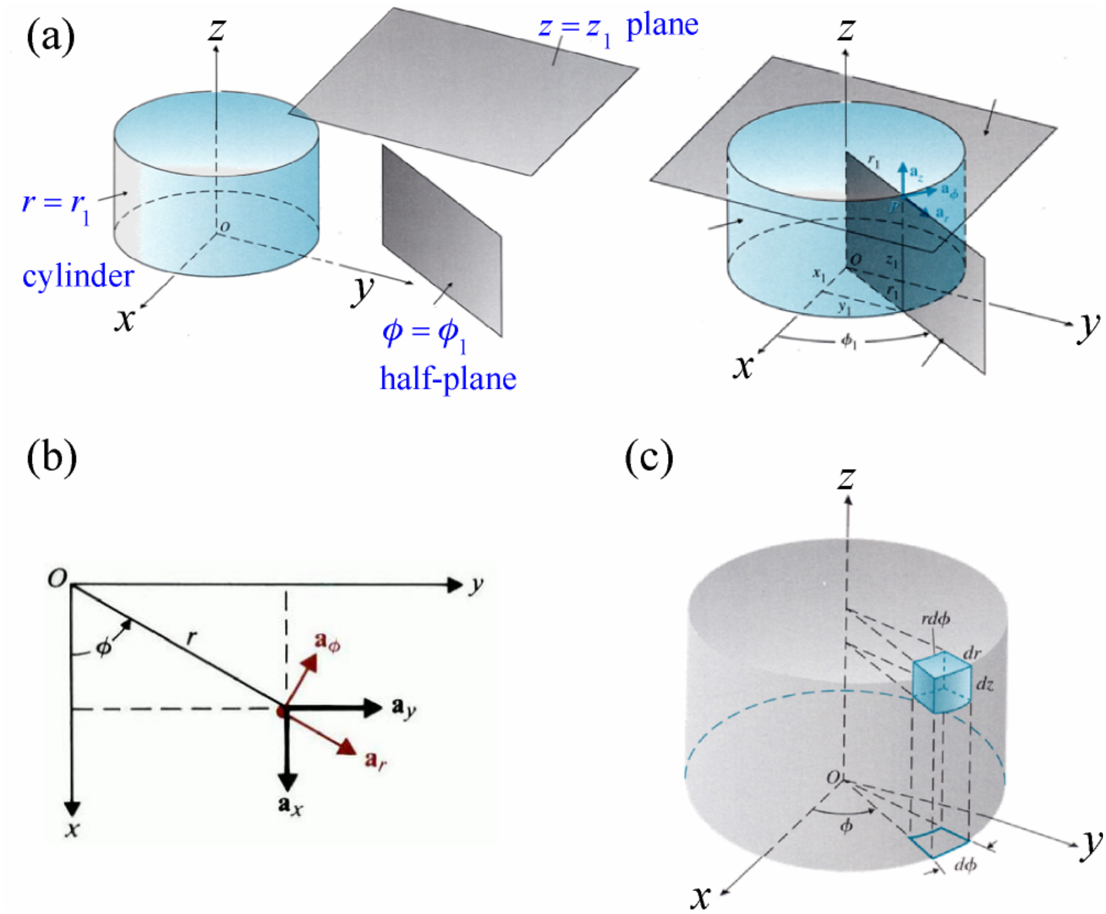

Fig. 5-4. (a) Positioning of one point, (b) the two $\phi-$dependent base vectors, (c) the differential volume of the cylindrical coordinate system (after DKC).

■ Cylindrical coordinate transformation

1) Transformation of position representation (Fig. 5-4b):

(a) Cylindrical $P(r,\phi,z)$ $\to$ Cartesian $P(x,y,z)$:

$$x = r\cos\phi,\quad y = r\sin\phi,\quad z = z \tag{5.11}$$

(b) Cartesian $P(x,y,z)$ $\to$ cylindrical $P(r,\phi,z)$:

$$r = \sqrt{x^2 + y^2}\,, \quad \phi = \tan^{-1}(y/x), \quad z = z \tag{5.12}$$

2) Transformation of the vector components:

A vector $\vec{V}$ can be represented in the cylindrical coordinate system $\vec{V} = \vec{a}_r A_r + \vec{a}_\phi A_\phi + \vec{a}_z A_z$ or in the Cartesian coordinate system $\vec{V} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$, where

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \tag{5.13}$$

Proof: $A_x = \vec{V} \cdot \vec{a}_x = \left(\vec{a}_r A_r + \vec{a}_\phi A_\phi + \vec{a}_z A_z\right) \cdot \vec{a}_x = (\vec{a}_r \cdot \vec{a}_x) A_r + (\vec{a}_\phi \cdot \vec{a}_x) A_\phi + (\vec{a}_z \cdot \vec{a}_x) A_z$. By observing Fig. 5-4b, $A_x = A_r \cos\phi - A_\phi \sin\phi$.

■ Spherical coordinate system

1) $(u_1, u_2, u_3) = (R, \theta, \phi)$ (Fig. 5-5a).

2) All of the base vectors $\{\vec{a}_R, \vec{a}_\theta, \vec{a}_\phi\}$ change with the azimuthal angle θ and the polar angle ϕ of the observation point (Fig. 5-5a).

3) When the coordinate changes from (R, θ, ϕ) to $(R, \theta + d\theta, \phi)$, the observation point P moves along the direction of $\vec{a}_\theta$ by a differential length of $dl_\theta = R \cdot d\theta$, $\Rightarrow$ $h_2 = R$. When the coordinate changes from (R, θ, ϕ) to $(R, \theta, \phi + d\phi)$, the observation point P moves along the direction of $\vec{a}_\phi$ by a differential length of $dl_\phi = R\sin\theta \cdot d\phi$, $\Rightarrow$ $h_3 = R\sin\theta$. The metric coefficients and differential volume are (Fig. 5-5b):

$$\{h_1 = 1,\ h_2 = R, h_3 = R\sin\theta\}, \qquad dv = R^2 \sin\theta \cdot dR\,d\theta\,d\phi \tag{5.14}$$

Spherical coordinate system is useful when the observer is very far away from the source region.

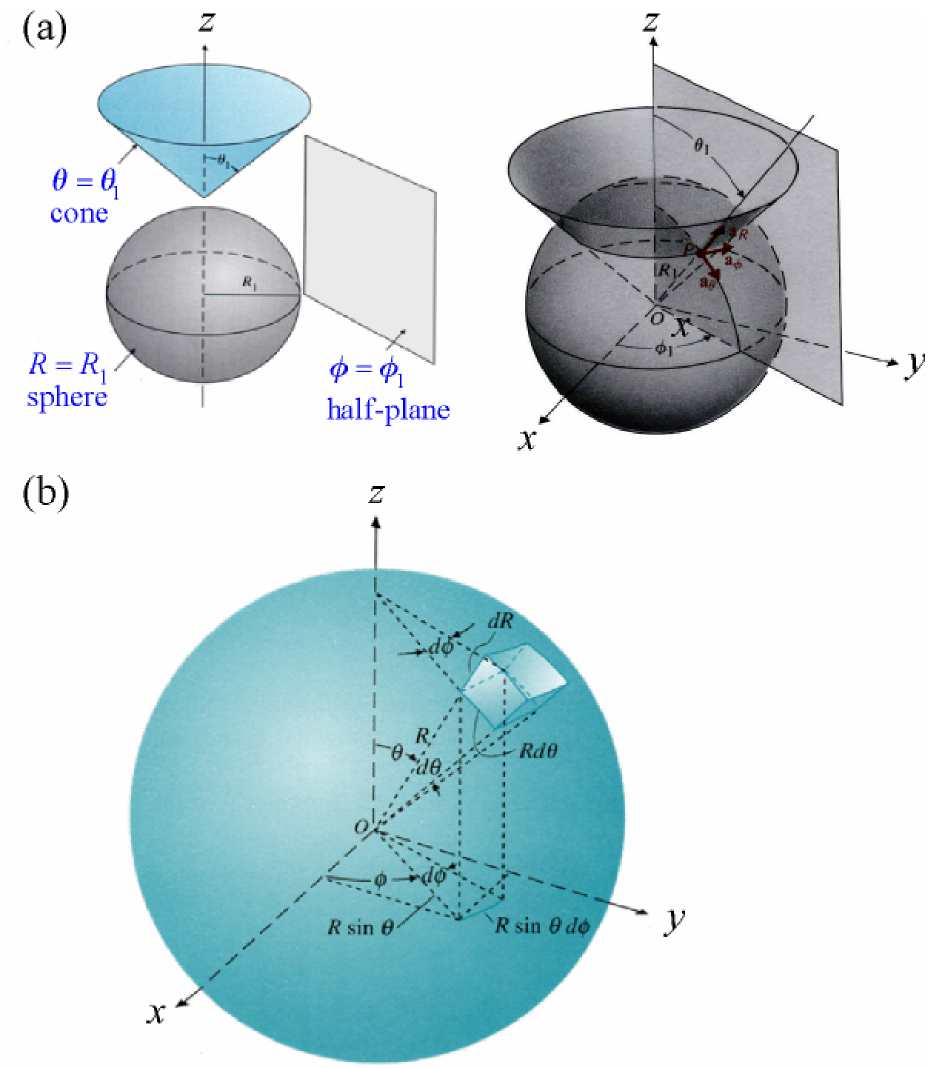

Fig. 5-5. (a) Positioning of one point, (b) the differential volume of the spherical coordinate system (after DKC).

■ Spherical coordinate transformation

1) Transformation of position representation (Fig. 5-5a):

(a) Spherical $P(R,\theta,\phi)$ $\rightarrow$ Cartesian $P(x,y,z)$:

$$x = R\sin\theta\cos\phi\,,\quad y = R\sin\theta\sin\phi\,,\quad z = R\cos\theta \tag{5.15}$$

(b) Cartesian $P(x,y,z)$ $\rightarrow$ cylindrical $P(R,\theta,\phi)$:

$$R = \sqrt{x^2+y^2+z^2}\,,\quad \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)\quad \phi = \tan^{-1}(y/x) \tag{5.16}$$

2) Transformation of the vector components:

A vector $\vec{V}$ can be represented in the spherical coordinate system $\vec{V} = \vec{a}_R A_R + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi$ or in the Cartesian coordinate system $\vec{V} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$,

where

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} \tag{5.17}$$

Proof: $A_x = \vec{V}\cdot\vec{a}_x = \left(\vec{a}_R A_R + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi\right)\cdot\vec{a}_x = (\vec{a}_R\cdot\vec{a}_x)A_R + (\vec{a}_\theta\cdot\vec{a}_x)A_\theta + (\vec{a}_\phi\cdot\vec{a}_x)A_\phi$.

By observing Fig. 5-5b, $A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$.

<Comment>

1) By eq's (5.13), (5.17), an arbitrary vector $\vec{V}$ in the cylindrical (spherical) coordinate system is uniquely specified only if the position of observation, i.e., the variable ϕ (variables ϕ, θ), is fixed.

2) Transformation of position representation can be regarded as transformation of "position vector" components.

(a) Cylindrical: $P(r,\phi,z)$ means $\vec{P} = \vec{a}_r r + \vec{a}_z z$, $\Rightarrow$ $\left(A_r, A_\phi, A_z\right) = (r,0,z)$. By eq. (5.13),

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \\ z \end{bmatrix}$$

, consistent with eq. (5.11). The dependence on ϕ comes from the base vector $\vec{a}_r$.

(b) Spherical: $P(R,\theta,\phi)$ means $\vec{P} = \vec{a}_R R$, $\Rightarrow$ $\left(A_R, A_\theta, A_\phi\right) = (R,0,0)$. By eq. (5.17),

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{bmatrix}$$

, consistent with eq. (5.15).

The dependence on θ and ϕ comes from the base vector $\vec{a}_R$.

5.3 Vector Calculus

■ Gradient: definition and physical meaning

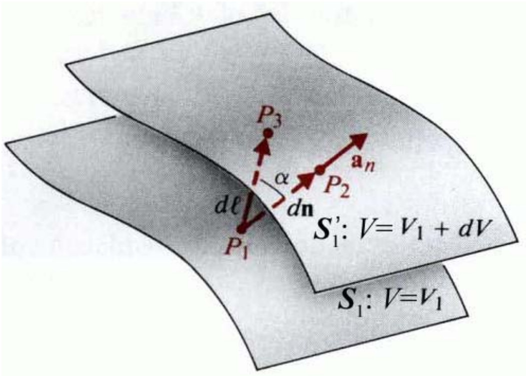

Fig. 5-6. Illustration of gradient of a scalar field *V* (after DKC).

For a scalar field $V(u_1,u_2,u_3)$, consider two “equi-potential” surfaces $S_1 : V = V_1$ and $S_1' : V = V_1 + dV$ (Fig. 5-6). The shortest distance between an observation point $P_1 \in S_1$ and the surface S_1' would be $\overline{P_1P_2} = |d\vec{n}|$, where $d\vec{n} = \vec{a}_n dn$ is a vector pointing to S_1' (thus $dn > 0$) and normal to S_1 at P_1. The space rate of change of the scalar field V along some arbitrary direction $d\vec{l}$, i.e., $|dV|/dl$, is maximized when $d\vec{l} \,//\, d\vec{n}$. The gradient of a scalar field V is a vector field, whose magnitude and direction characterize the maximum space rate of increase of V:

$$\nabla V \equiv \vec{a}_n \frac{dV}{dn} \tag{5.18}$$

<Comment>

If $dV > 0$ (the field increases along $\vec{a}_n$), $\nabla V \,//\, \vec{a}_n$. If $dV < 0$ (the field decreases along $\vec{a}_n$), $\nabla V \,//(-\vec{a}_n)$. $\Rightarrow$ ∇V always points along the direction of field “increase”.

■ Gradient: formulas of evaluation

Consider an arbitrary point $P_3 \in S_1'$ (Fig. 5-6), $\overline{P_1P_3} = dl = \dfrac{dn}{\cos\alpha}$, $\Rightarrow$ $\dfrac{dV}{dl} = \dfrac{dV}{dn}\cos\alpha = |\nabla V|(\vec{a}_n \cdot \vec{a}_l)$. $\Rightarrow$ The space rate of increase of the scalar field V along any direction $d\vec{l} = \vec{a}_l dl$ is:

$$\frac{dV}{dl} = (\nabla V)\cdot \vec{a}_l \tag{5.19}$$

In a 3-D coordinate system (u_1, u_2, u_3), the change of the scalar field dV due to a displacement $d\vec{l} = \vec{a}_{u_1} dl_1 + \vec{a}_{u_2} dl_2 + \vec{a}_{u_3} dl_3$ can be represented by:

$$dV = \frac{\partial V}{\partial l_1} dl_1 + \frac{\partial V}{\partial l_2} dl_2 + \frac{\partial V}{\partial l_3} dl_3 .$$

Eq. (5.19) is equivalent to $dV = (\nabla V)\cdot d\vec{l}$. By the representations of dV and $d\vec{l}$, we have:

$$\frac{\partial V}{\partial l_1} dl_1 + \frac{\partial V}{\partial l_2} dl_2 + \frac{\partial V}{\partial l_3} dl_3 = (\nabla V)\cdot\left(\vec{a}_{u_1} dl_1 + \vec{a}_{u_2} dl_2 + \vec{a}_{u_3} dl_3\right).$$

By comparison both sides of the equality, ∇V must be $\nabla V = \vec{a}_{u_1}\dfrac{\partial V}{\partial l_1} + \vec{a}_{u_2}\dfrac{\partial V}{\partial l_2} + \vec{a}_{u_3}\dfrac{\partial V}{\partial l_3}$.

Since the differential length dl_i due to the change of variable u_i by a small amount of du_i is $dl_i = h_i du_i$ (h_i is the metric coefficient of u_i), $\Rightarrow$

$$\nabla V = \vec{a}_{u_1}\frac{\partial V}{h_1 \partial u_1} + \vec{a}_{u_2}\frac{\partial V}{h_2 \partial u_2} + \vec{a}_{u_3}\frac{\partial V}{h_3 \partial u_3} \tag{5.20}$$

In Cartesian coordinates, $(u_1, u_2, u_3) = (x, y, z)$, $\{h_1 = h_2 = h_3 = 1\}$ [eq. (5.9)],

$$\nabla V = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z} \tag{5.21}$$

■ Divergence: definition and physical meaning

Vector field $\vec{A}$ can be illustrated by "flux lines", such that the field magnitude $|\vec{A}|$ is measured by the number of flux lines passing through a unit surface normal to the vector.

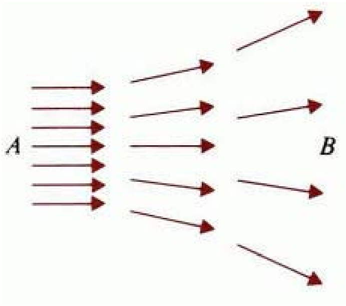

Fig. 5-7. Flux lines of a vector field (after DKC).

If $\vec{A}$ represents the directed flow density, $\iint_S \vec{A}\cdot d\vec{s}$ represents the total flow over an open surface S. For a volume V enclosed by a closed surface S, the outward flux $\oint_S \vec{A}\cdot d\vec{s}$ will be positive(negative) only if the volume contains a "flow source(sink)". The divergence of a vector field $\vec{A}$ is a scalar field characterizing the **net outward flux per unit volume**:

$$\nabla\cdot\vec{A} \equiv \lim_{\Delta v\to 0}\frac{\oint_S \vec{A}\cdot d\vec{s}}{\Delta v}, \tag{5.22}$$

which is used to characterize the flow source(sink) quantitatively.

■ Divergence: formulas of evaluation

In the Cartesian coordinate system, consider a vector field $\vec{A}(x,y,z)=\vec{a}_x A_x+\vec{a}_y A_y+\vec{a}_z A_z$ and an infinitesimal cuboid Δv centered at $P(x_0,y_0,z_0)$ with side lengths Δx, Δy, Δz, respectively (Fig. 5-8).

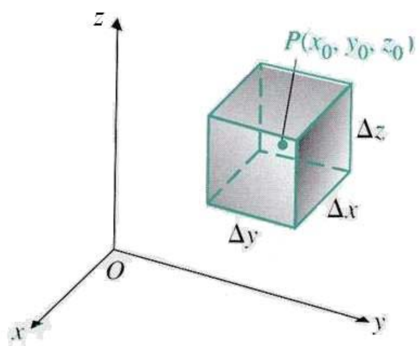

Fig. 5-8. A differential volume in the Cartesian coordinate system used to derive eq. (5.23) (after DKC).

1) On the front face $S_1: x = x_0 + \frac{\Delta x}{2}$, the outward flux is $F_1 = \iint_{S_1} \vec{A} \cdot d\vec{s}$, where $d\vec{s} = \vec{a}_x \Delta y \Delta z$. Although the vector field $\vec{A}(x, y, z)$ is a "function" of position, we can approximate it by a "constant" vector $\vec{A}(x_0 + \Delta x/2, y_0, z_0)$ on the infinitesimal surface S_1 of the cuboid. $\Rightarrow$ $F_1 \approx A_x(x_0 + \Delta x/2, y_0, z_0) \cdot (\Delta y \Delta z)$. By the first-order Taylor series approximation, $A_x(x_0 + \Delta x/2, y_0, z_0) \approx A_x(x_0, y_0, z_0) + \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \frac{\Delta x}{2}$, $\Rightarrow$

$$F_1 \approx A_x(x_0, y_0, z_0) \cdot (\Delta y \Delta z) + \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \frac{\Delta x \Delta y \Delta z}{2}.$$

Since $\Delta x \Delta y \Delta z$ is the volume of the cuboid (denoted by Δv), we have:

$$F_1 \approx A_x(x_0, y_0, z_0) \cdot (\Delta y \Delta z) + \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \frac{\Delta v}{2}$$

2) On the back face $S_2: x = x_0 - \frac{\Delta x}{2}$, the outward flux is $F_2 = \iint_{S_2} \vec{A} \cdot d\vec{s}$, where $d\vec{s} = -\vec{a}_x \Delta y \Delta z$. We approximate $\vec{A}(x, y, z)$ by a "constant" vector $\vec{A}(x_0 - \Delta x/2, y_0, z_0)$ on the infinitesimal surface S_2 of the cuboid. $\Rightarrow$ $F_2 \approx A_x(x_0 - \Delta x/2, y_0, z_0) \cdot (-\Delta y \Delta z)$.

By the first-order Taylor series approximation,

$$A_x(x_0 - \Delta x/2, y_0, z_0) \approx A_x(x_0, y_0, z_0) - \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \frac{\Delta x}{2}, \Rightarrow$$

$$F_2 \approx -A_x(x_0, y_0, z_0) \cdot (\Delta y \Delta z) + \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \frac{\Delta v}{2}.$$

The total outward flux for the front and back surfaces S_1 and S_2 becomes:

$$F_1 + F_2 = \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \Delta v.$$

3) The same strategy can be used for the remaining four surfaces of the cuboid. The total outward flux for the cuboid becomes: $\oint_S \vec{A}\cdot d\vec{s} = \sum_{n=1}^{6} F_n = \left[\left.\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right|_{P(x_0,y_0,z_0)}\right]\Delta v$.

By eq. (5.22), the divergence of $\vec{A}(x,y,z)$ at $P(x_0,y_0,z_0)$ is $\lim_{\Delta v\to 0}\frac{\oint_S \vec{A}\cdot d\vec{s}}{\Delta v}$, $\Rightarrow$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \tag{5.23}$$

For other orthogonal coordinate systems, eq. (5.23) is generalized to eq. (2-110) of the textbook.

■ Divergence theorem

The definition of divergence [eq. (5.22)] implies that the total outward flux of a vector field over a closed surface S is equal to the volume integral of the divergence of the vector field over the volume V enclosed by S:

$$\oint_S \vec{A}\cdot d\vec{s} = \int_V \left(\nabla\cdot\vec{A}\right)dv \tag{5.24}$$

This fact can be shown by subdividing the volume V into many small areas, where the contributions of flux from the internal surfaces of adjacent small elements will cancel with one another (Fig. 5-9).

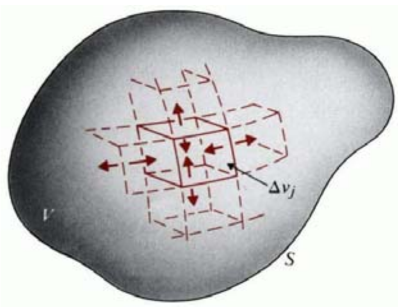

Fig. 5-9. Subdivided volumes for proof of the divergence theorem (after DKC).

■ Curl: definition and physical meaning

If the vector field $\vec{A}$ represents a field of force, the "work" done by the force in moving some object around a closed path (contour) C (i.e., the energy obtained by the object when traveling along C) is:

$$\text{Circulation} \equiv \oint_C \vec{A} \cdot d\vec{l} \tag{5.25}$$

A "conservative force" $\vec{A}$ produces no circulation, because $\oint_C \vec{A} \cdot d\vec{l} = 0$ for any contour C. In other words, a conservative force does not drive objects circularly. If a non-conservative force $\vec{A}$ has nonzero circulation for an infinitesimal contour C around a point *P*, it forms a vortex source at *P* that drives circulating flows. To quantitatively measure the strength and direction of a vortex source, we define the curl of a vector field $\vec{A}$ as a vector field, whose (1) magnitude represents the **net circulation per unit area**, and (2) direction is the **normal** direction $\vec{a}_n$ of the differential contour $C_{\max}$ (with area Δs) which is oriented to maximize the circulation.

$$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\vec{a}_n \left(\oint_{C_{\max}} \vec{A} \cdot d\vec{l} \right)}{\Delta s} \tag{5.26}$$

■ Curl: formulas of evaluation

The circulation per unit area of a vector field $\vec{A}$ along an arbitrarily oriented contour C_u (with area Δs_u and unit normal vector $\vec{a}_u$) is:

$$\lim_{\Delta s_u \to 0} \frac{\oint_{C_u} \vec{A} \cdot d\vec{l}}{\Delta s_u} = \left(\nabla \times \vec{A}\right) \cdot \vec{a}_u \tag{5.27}$$

In the Cartesian coordinate system, we can derive the *x*-component of $\nabla \times \vec{A}$ by considering a vector field $\vec{A}(x,y,z)=\vec{a}_x A_x+\vec{a}_y A_y+\vec{a}_z A_z$ and an infinitesimal rectangular contour C_x centered at $P(x_0,y_0,z_0)$ with a unit normal vector $\vec{a}_x$ and side lengths Δy, Δz, respectively (Fig. 5-10).

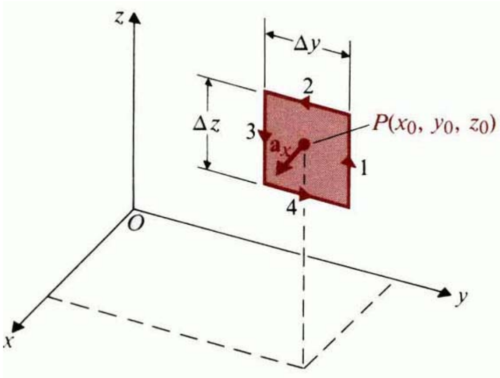

Fig. 5-10. A differential area in Cartesian coordinates used to derive *x*-component of eq. (5.22) (after DKC).

1) On the path 1, i.e., $\{(x_0, y_0+\Delta y/2, z),\ z=[z_0-\Delta z/2, z_0+\Delta z/2]\}$, the work done by the force is $W_1=\int_1 \vec{A}\cdot d\vec{l}$, where $d\vec{l}=\vec{a}_z\Delta z$. Although the vector field $\vec{A}(x,y,z)$ is a "function" of position, we can approximate it by a "constant" vector $\vec{A}(x_0, y_0+\Delta y/2, z_0)$ on the infinitesimal path 1. $\Rightarrow$ $W_1\approx A_z(x_0,y_0+\Delta y/2,z_0)\cdot(\Delta z)$. By the first-order Taylor series approximation, $A_z(x_0,y_0+\Delta y/2,z_0)\approx A_z(x_0,y_0,z_0)+\left[\left.\frac{\partial A_z}{\partial y}\right|_{P(x_0,y_0,z_0)}\right]\frac{\Delta y}{2}$, $\Rightarrow$

$$W_1\approx A_z(x_0,y_0,z_0)\cdot(\Delta z)+\left[\left.\frac{\partial A_z}{\partial y}\right|_{P(x_0,y_0,z_0)}\right]\frac{\Delta y\Delta z}{2}.$$

Since $\Delta y\Delta z$ is the area of the rectangle (denoted by Δs), we have:

$$W_1\approx A_z(x_0,y_0,z_0)\cdot(\Delta z)+\left[\left.\frac{\partial A_z}{\partial y}\right|_{P(x_0,y_0,z_0)}\right]\frac{\Delta s}{2}$$

2) On the path 3, i.e., $\{(x_0, y_0-\Delta y/2, z),\ z=[z_0-\Delta z/2, z_0+\Delta z/2]\}$, the work done by the force is $W_3=\int_3 \vec{A}\cdot d\vec{l}$, where $d\vec{l}=-\vec{a}_z\Delta z$. We approximate $\vec{A}(x,y,z)$ by a "constant"

vector $\vec{A}(x_0, y_0 - \Delta y/2, z_0)$ on the infinitesimal path 3. $\Rightarrow$

$W_3 \approx -A_z(x_0, y_0 - \Delta y/2, z_0)\cdot(\Delta z)$. By the first-order Taylor series approximation,

$$A_z(x_0, y_0 - \Delta y/2, z_0) \approx A_z(x_0, y_0, z_0) - \left[\left.\frac{\partial A_z}{\partial y}\right|_{P(x_0, y_0, z_0)}\right]\frac{\Delta y}{2}, \Rightarrow$$

$$W_3 \approx -A_z(x_0, y_0, z_0)\cdot(\Delta z) + \left[\left.\frac{\partial A_z}{\partial y}\right|_{P(x_0, y_0, z_0)}\right]\frac{\Delta s}{2}.$$

The total work done along the path 1 and path 3 becomes:

$$W_1 + W_3 = \left[\left.\frac{\partial A_z}{\partial y}\right|_{P(x_0, y_0, z_0)}\right]\Delta s$$

3) The same strategy can be applied to the remaining path 2 and path 4. The total circulation due to contour C_x becomes: $\int_{1234} \vec{A}\cdot d\vec{l} = \left[\left.\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right|_{P(x_0, y_0, z_0)}\right]\Delta s$. By eq. (5.27), the x-component of $\nabla\times\vec{A}$ is:

$$(\nabla\times\vec{A})\cdot\vec{a}_x = \lim_{\Delta s\to 0}\frac{\oint_{C_x}\vec{A}\cdot d\vec{l}}{\Delta s} = \left[\left.\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right|_{P(x_0, y_0, z_0)}\right].$$

We can further derive the y- and z-component of $\nabla\times\vec{A}$ by examining the circulation due to contour C_y and C_z, respectively. As a result, $\nabla\times\vec{A}$ in the Cartesian coordinate system can be formulated as:

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \tag{5.28}$$

For other orthogonal coordinate systems, eq. (5.28) is generalized to eq. (2-137) of the textbook.

■ Stokes' theorem

The definition of curl [eq. (5.26)] implies that the total circulation of a vector field over a contour C is equal to the surface integral of the curl of the vector field over the open surface S bounded by C:

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S \left(\nabla\times\vec{A}\right)\cdot d\vec{s} \tag{5.29}$$

This fact can be shown by subdividing the open surface S into many small areas, where the contributions from the internal boundaries of adjacent small elements will cancel with one another (Fig. 5-11).

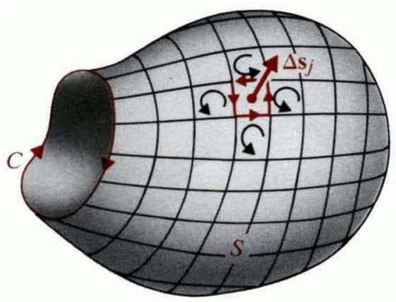

Fig. 5-11. Subdivided areas for proof of Stokes' theorem (after DKC).

<Comment>

Gradient, divergence, and curl are all "point" functions, describing "local" field behaviors.

■ Laplacian: definition and physical meaning

Gradient, divergence, and curl are all first-order differential operators. In EM theory, however, we need to deal with second-order derivatives of scalar and vector fields. Laplacian of a scalar field V is another scalar field defined as:

$$\nabla^2 V \equiv \nabla\cdot\left(\nabla V\right) \tag{5.30}$$

To show the meaning of Laplacian, take a scalar function of single variable $f(x)$ as an example.

$$\frac{df}{dx}=\lim_{\Delta\to 0}\frac{f(x+\Delta/2)-f(x-\Delta/2)}{\Delta};$$

$$\frac{d^2f}{dx^2}=\lim_{\Delta\to 0}\frac{f'(x+\Delta/2)-f'(x-\Delta/2)}{\Delta}=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{f(x+\Delta)-f(x)}{\Delta}-\frac{f(x)-f(x-\Delta)}{\Delta}\right]$$

$$=\lim_{\Delta\to 0}\frac{2}{\Delta^2}\left[\frac{f(x+\Delta)+f(x-\Delta)}{2}-f(x)\right]\propto\left(\bar{f}-f\right).$$

This means that the second-order derivative of $f(x)$, i.e., $\frac{d^2f}{dx^2}$, describes the difference between the "field value f" and the "average field value $\bar{f}$" of its surrounding points. As a result, the scalar Laplacian of a scalar field of multiple variables V, i.e., $\nabla^2 V$, has the similar meaning.

Laplacian of a vector field $\vec{A}$ is another vector field defined as:

$$\nabla^2\vec{A}\equiv\nabla\left(\nabla\cdot\vec{A}\right)-\nabla\times\nabla\times\vec{A} \tag{5.31}$$

■ Laplacian: formulas of evaluation

In the Cartesian coordinate system, we can substitute eq's (5.21), (5.23) into eq. (5.30) to obtain:

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} \tag{5.32}$$

Similarly, we can substitute eq's (5.21), (5.23), (5.28) into eq. (5.31) to obtain:

$$\nabla^2\vec{A}=\vec{a}_x\left(\nabla^2 A_x\right)+\vec{a}_y\left(\nabla^2 A_y\right)+\vec{a}_z\left(\nabla^2 A_z\right) \tag{5.33}$$

Laplacian formulas for cylindrical and spherical coordinates can be found in the inside of back cover of the textbook.

■ Null identities

Two identities involving with repeated del (∇) operations are important for the concept of potential functions (DKC Ch3, Ch6):

$$\nabla \times (\nabla V) = 0 \tag{5.34}$$

$\Rightarrow$ A conservative (curl-free) vector field can be expressed as the gradient of a scalar field (electrostatic potential).

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{5.35}$$

$\Rightarrow$ A solenoidal (divergence-free) vector field can be expressed as the curl of another vector field (magnetostatic potential). Eq's (5.34), (5.35) can be easily proven in the Cartesian coordinate system.

■ Helmholtz's theorem (decomposition)

A vector field $\vec{F}$ is uniquely determined if both its divergence and curl are specified everywhere. As a result, we will introduce electric and magnetic vector fields by specifying their divergence and curl (fundamental postulates) first.

A vector field $\vec{F}$ can be decomposed into:

1) The curl-free (irrotational) component $\vec{F}_i$, with

$$\begin{cases} \nabla \cdot \vec{F}_i = g \\ \nabla \times \vec{F}_i = 0 \end{cases},$$

where g represents the flow source generating $\vec{F}$. By eq. (5.34), $\vec{F}_i = -\nabla V$, where V represents the scalar potential of $\vec{F}$.

2) The divergence-free (solenoidal) component $\vec{F}_s$, with

$$\begin{cases} \nabla \cdot \vec{F}_s = 0 \\ \nabla \times \vec{F}_s = \vec{G} \end{cases},$$

where $\vec{G}$ represents the vortex source generating $\vec{F}$. By eq. (5.35), $\vec{F}_s = \nabla \times \vec{A}$, where $\vec{A}$ represents the vector potential of $\vec{F}$.

As a result,

$$\vec{F} = \vec{F}_i + \vec{F}_s = -\nabla V + \nabla \times \vec{A}, \tag{5.36}$$

i.e., a vector field can also be determined by specifying its scalar and vector potentials.