Lesson 5
Vector Analysis

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Introduction

- Physical quantities in EM could be:

  1. Scalar: charge, current, energy

  2. Vector: EM fields

- EM laws are independent of coordinate system of use
- Vector analysis is not necessary for EM, but can lead to elegant formulations

Outline

- Vector algebra
- Orthogonal coordinate systems

    1. Cartesian
    2. Cylindrical
    3. Spherical

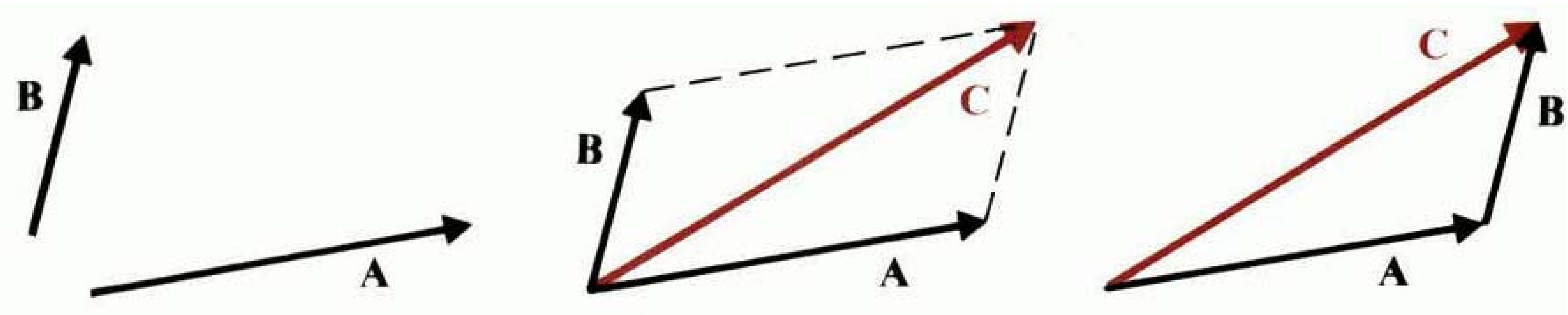
- Vector calculus

    1. Gradient
    2. Divergence
    3. Curl
    4. Laplacian

Sec. 5-1
Vector Algebra

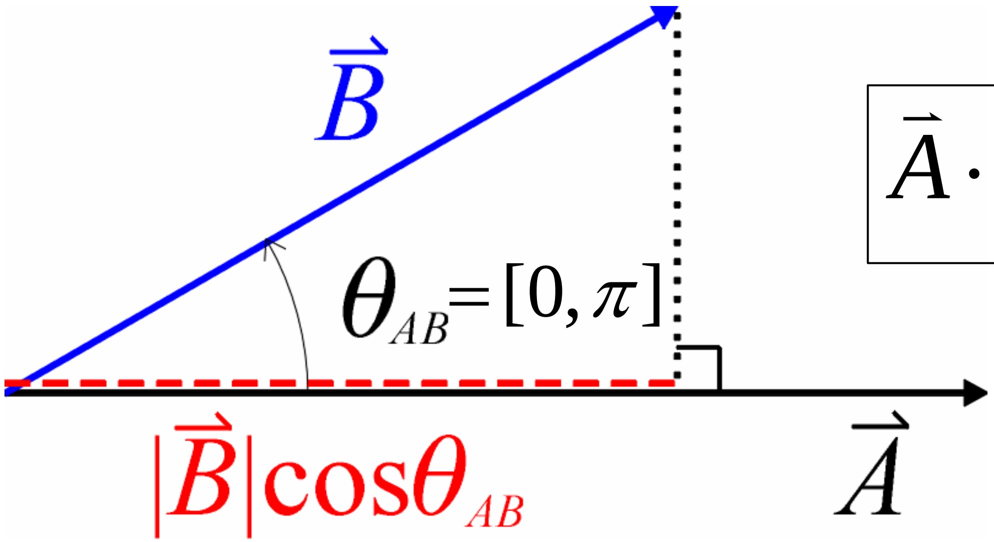
1. Addition
2. Products of two vectors
3. Products of three vectors

Vector addition/subtraction



$$\vec{C} = \vec{A} + \vec{B}$$

Dot (inner) product

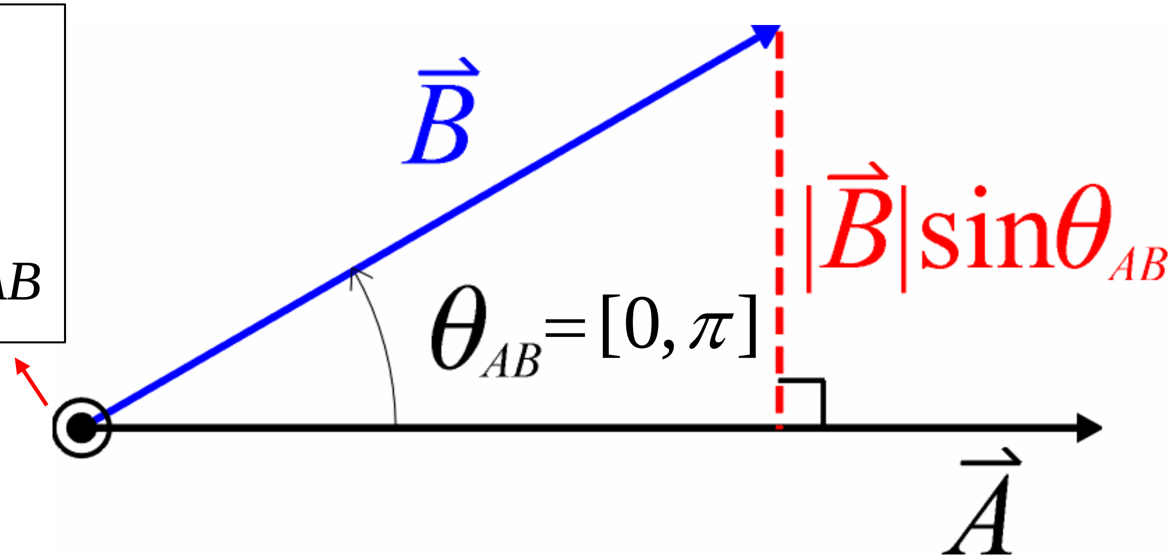


$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta_{AB}$$

- Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

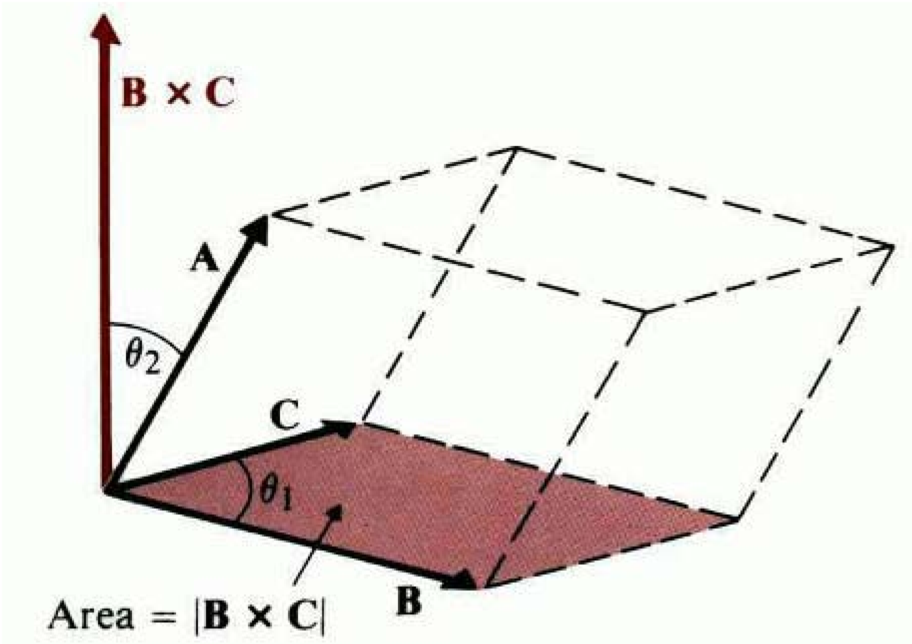
Cross (outer) product

$$\vec{A}\times\vec{B} = \vec{a}_n\left|\vec{A}\right|\cdot\left|\vec{B}\right|\cdot\sin\theta_{AB}$$



- Not commutative: $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$
- Not associative: $\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$
- Distributive: $\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

Scalar triple product



$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$$

Volume of the parallelepiped

Vector triple product

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\cdot\left(\vec{A}\cdot\vec{B}\right)$$

vector scalar vector scalar

Sec. 5-2
Orthogonal Coordinate Systems

1. Cartesian
2. Cylindrical
3. Spherical

Definition

A point is located as the intersection of 3 mutually perpendicular surfaces $u_i = \text{consant}$

3 base vectors $\{\vec{a}_{u_i}\}$ satisfy with:

$$\begin{cases} \vec{a}_{u_i} \times \vec{a}_{u_j} = \vec{a}_{u_k}, \quad (i, j, k) = (1,2,3), (2,3,1), (3,1,2) \\ \vec{a}_{u_i} \cdot \vec{a}_{u_j} = \delta_{ij} = \begin{cases} 0, \text{if } i \neq j \\ 1, \text{if } i = j \end{cases} \end{cases}$$

Application-1

$$\vec{A} = \vec{a}_{u_1} A_1 + \vec{a}_{u_2} A_2 + \vec{a}_{u_3} A_3, \qquad \vec{B} = \vec{a}_{u_1} B_1 + \vec{a}_{u_2} B_2 + \vec{a}_{u_3} B_3$$

$$\vec{A} \cdot \vec{B} = \left(\vec{a}_{u_1} A_1 + \vec{a}_{u_2} A_2 + \vec{a}_{u_3} A_3\right) \cdot \left(\vec{a}_{u_1} B_1 + \vec{a}_{u_2} B_2 + \vec{a}_{u_3} B_3\right)$$

$$= \underbrace{\left(\vec{a}_{u_1} \cdot \vec{a}_{u_1}\right)}_{1} A_1 B_1 + \left(\vec{a}_{u_1} \cdot \vec{a}_{u_2}\right) A_1 B_2 + \ldots \underbrace{\left(\vec{a}_{u_3} \cdot \vec{a}_{u_3}\right)}_{1} A_3 B_3$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\vec{A} \cdot \vec{B} = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \times \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \sum_{i=1}^{3} A_i B_i$$

vs.

$$\boxed{\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \cdot \left|\vec{B}\right| \cos\theta_{AB}}$$

Application-2

$$\vec{A} = \vec{a}_{u_1} A_1 + \vec{a}_{u_2} A_2 + \vec{a}_{u_3} A_3, \qquad \vec{B} = \vec{a}_{u_1} B_1 + \vec{a}_{u_2} B_2 + \vec{a}_{u_3} B_3$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_{u_1} & \vec{a}_{u_2} & \vec{a}_{u_3} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

vs.

$$\vec{A} \times \vec{B} = \vec{a}_n \left|\vec{A}\right| \cdot \left|\vec{B}\right| \cdot \sin\theta_{AB}$$

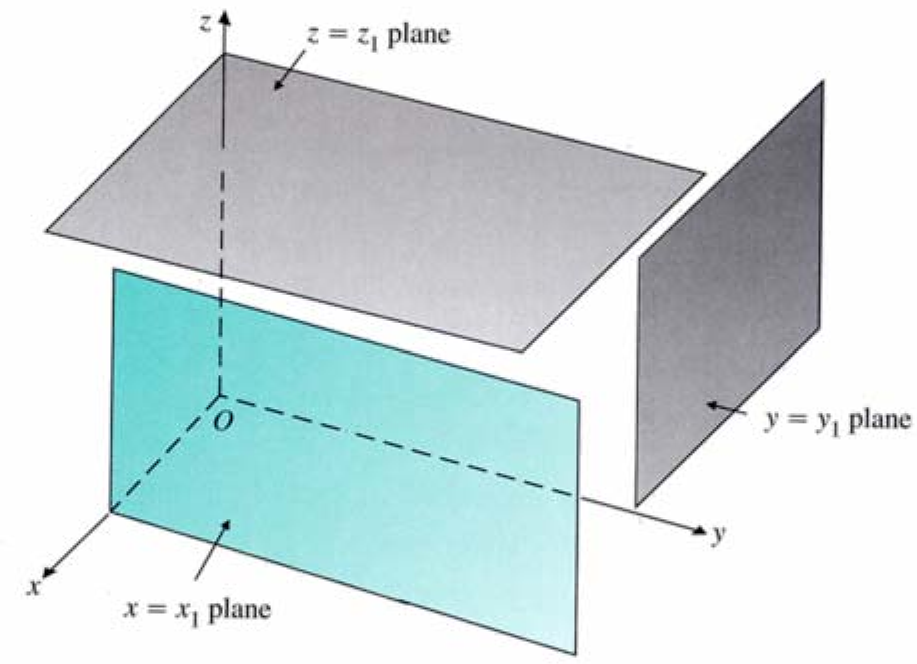
Cartesian

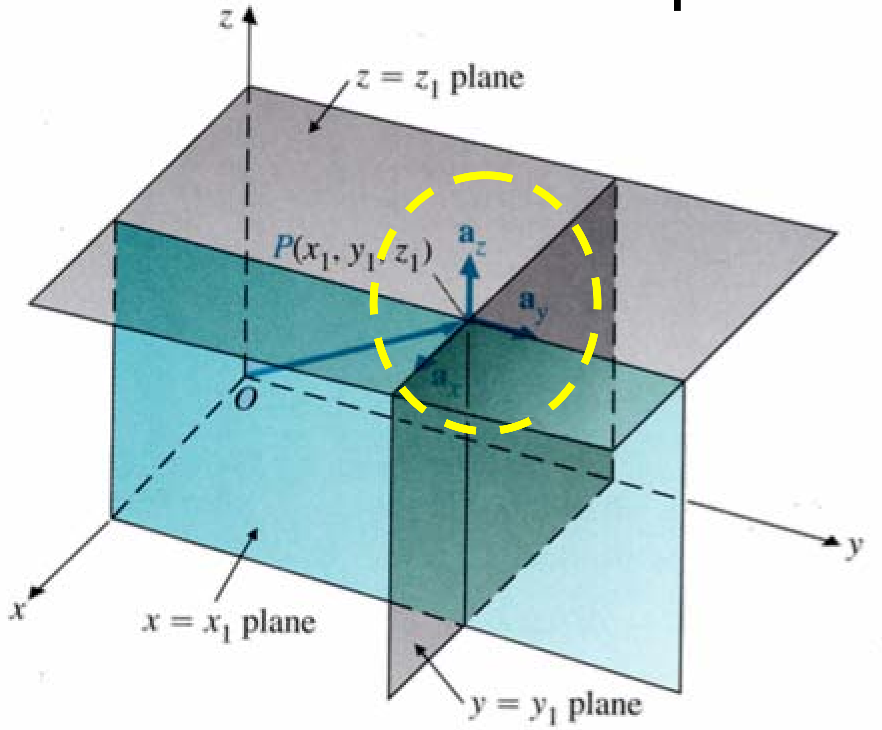
$$\left(u_1, u_2, u_3\right) = \left(x, y, z\right)$$

All base vectors

$$\left\{ \vec{a}_{u_i},\ i = 1, 2, 3 \right\}$$

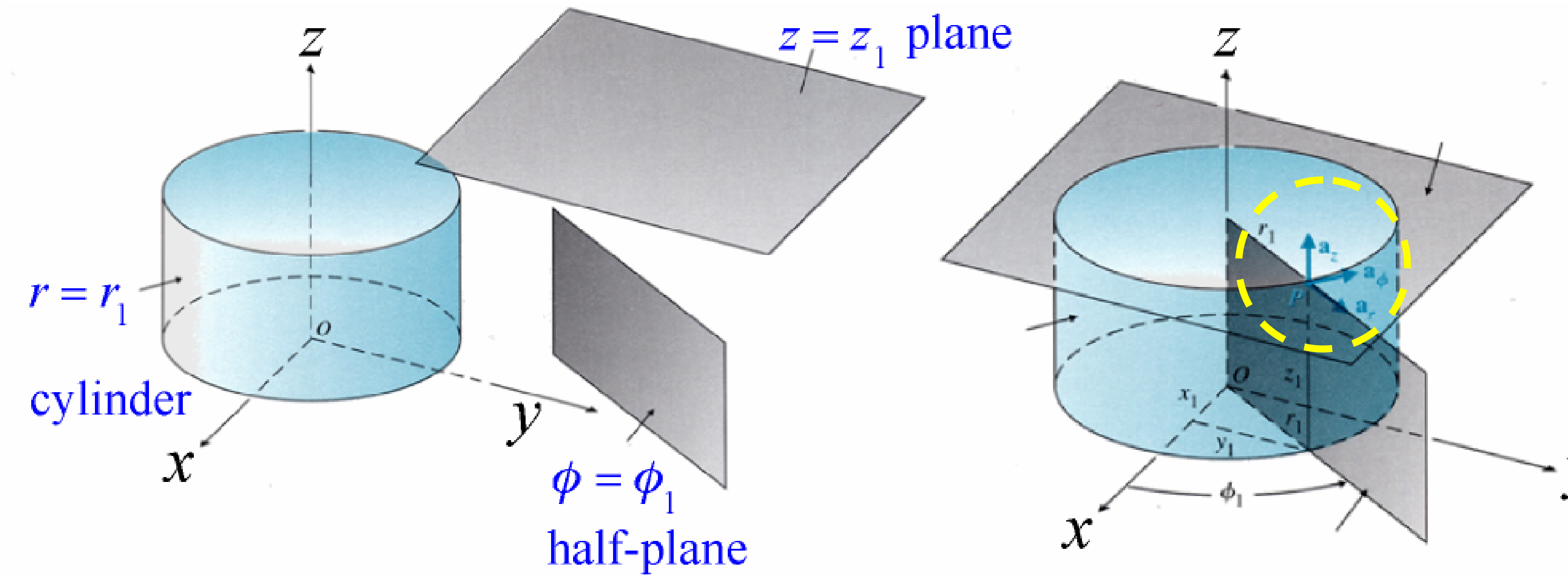
are independent of the observation point





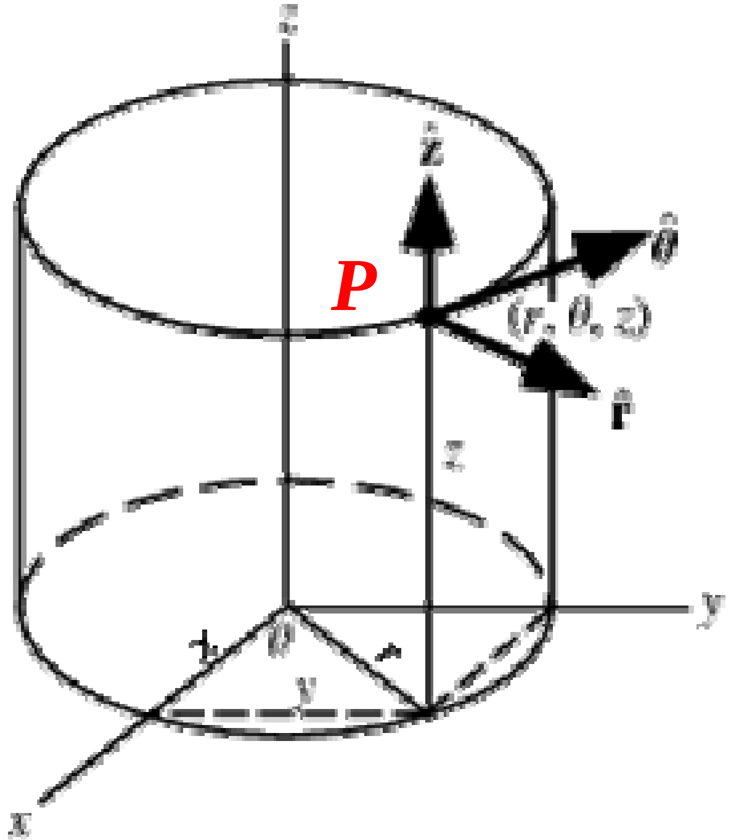
Cylindrical-1

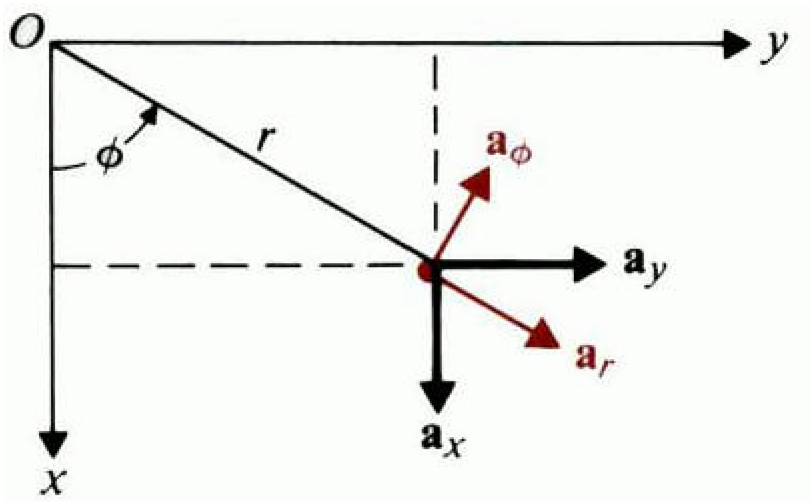
$$\left(u_1, u_2, u_3\right) = \left(r, \phi, z\right)$$



Cylindrical-2

2 base vectors $\{\vec{a}_r, \vec{a}_\phi\}$ vary with ϕ of obs. point





$$\vec{a}_r = \vec{a}_x \cos\phi + \vec{a}_y \sin\phi$$

$$\vec{a}_\phi = -\vec{a}_x \sin\phi + \vec{a}_y \cos\phi$$

Cylindrical-3

What does ϕ-dependent base vector mean?

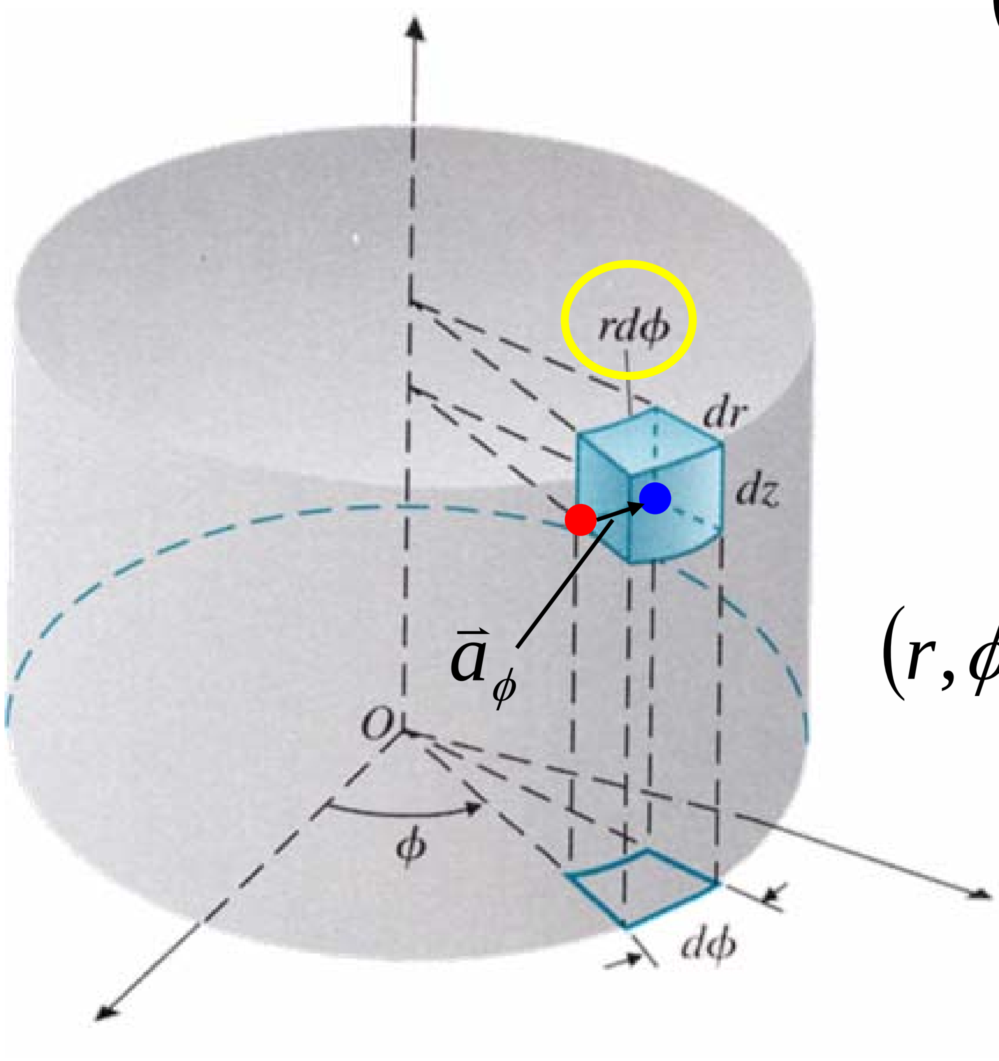
e.g. $\vec{a}_r = \vec{a}_r(\phi) = \vec{a}_x \cos\phi + \vec{a}_y \sin\phi$

$$\Rightarrow \frac{\partial}{\partial\phi}\vec{a}_r A_r(r,\phi,z) = \left(\frac{\partial}{\partial\phi}\vec{a}_r\right)A_r + \vec{a}_r\left(\frac{\partial}{\partial\phi}A_r\right)$$

$$-\vec{a}_x \sin\phi + \vec{a}_y \cos\phi = \vec{a}_\phi$$

In contrast, $\frac{\partial}{\partial x}\vec{a}_x A_x(x,y,z) = \vec{a}_x\left(\frac{\partial}{\partial x}A_x\right)$

Cylindrical-4



$$(r, \phi, z) \rightarrow (r, \phi + d\phi, z)$$

Observation point moves along $\vec{a}_\phi$ by $dl_\phi = r \cdot d\phi$

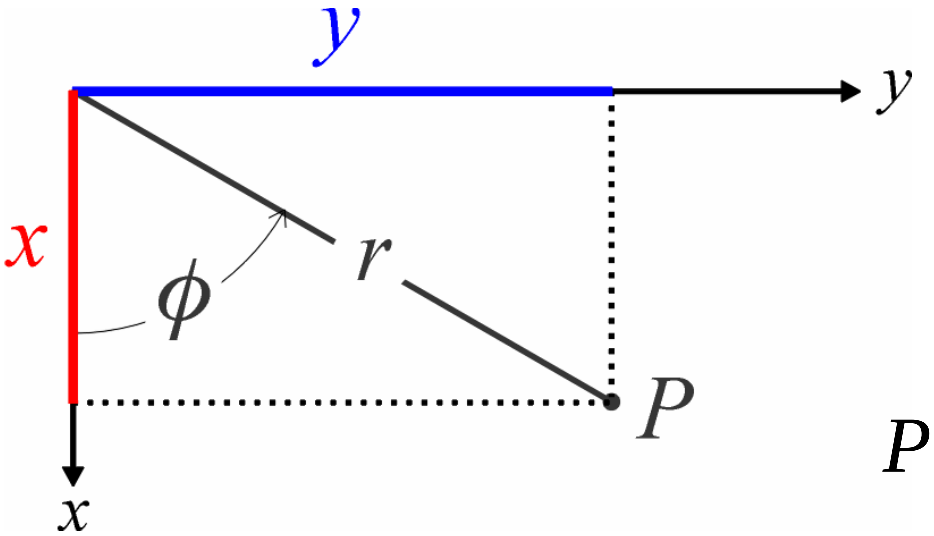
Metric coefficient of ϕ

$$(r, \phi, z) \rightarrow (r + dr, \phi + d\phi, z + dz)$$

$$dv = r dr d\phi dz$$

…Differential volume

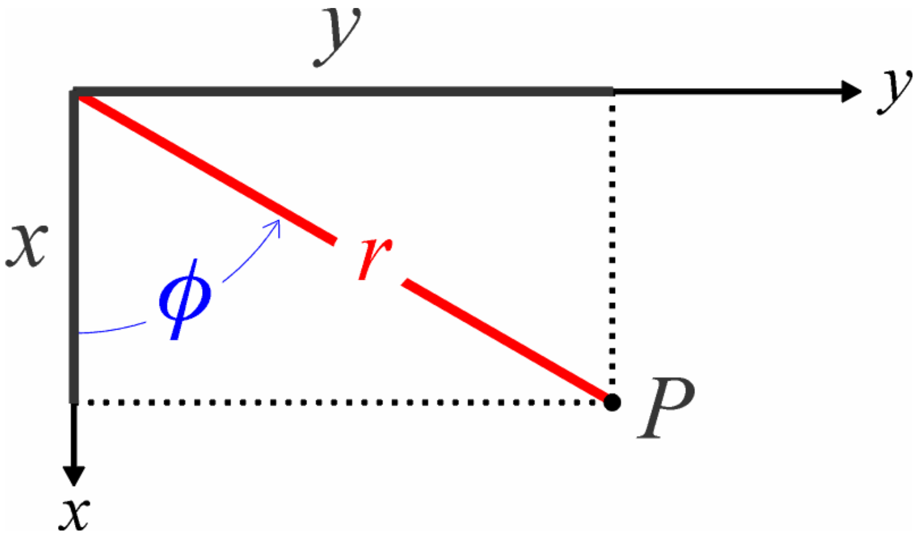
Transformation of position representation-1



$$P(r,\phi,z) \rightarrow P(x,y,z)$$

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

Transformation of position representation-2



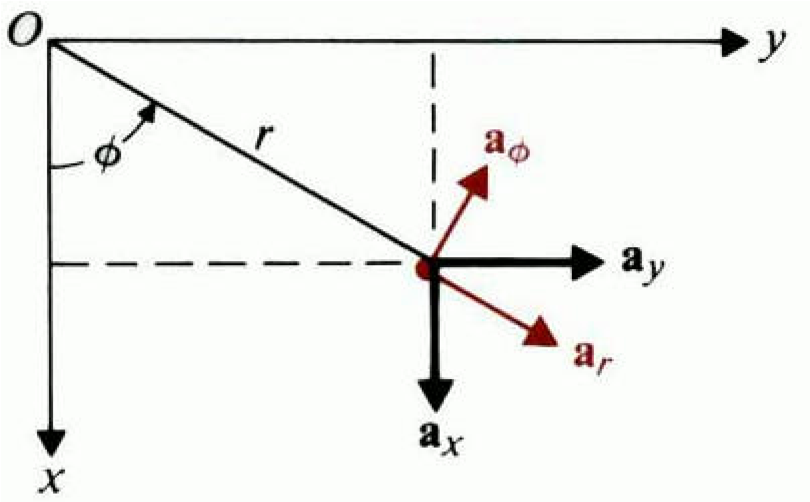
$$P(x, y, z) \rightarrow P(r, \phi, z)$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

Transformation of vector components

$$\vec{V} = \vec{a}_r A_r + \vec{a}_\phi A_\phi + \vec{a}_z A_z = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$



$$A_x = \vec{V} \cdot \vec{a}_x = \left(\vec{a}_r A_r + \vec{a}_\phi A_\phi + \vec{a}_z A_z\right) \cdot \vec{a}_x$$

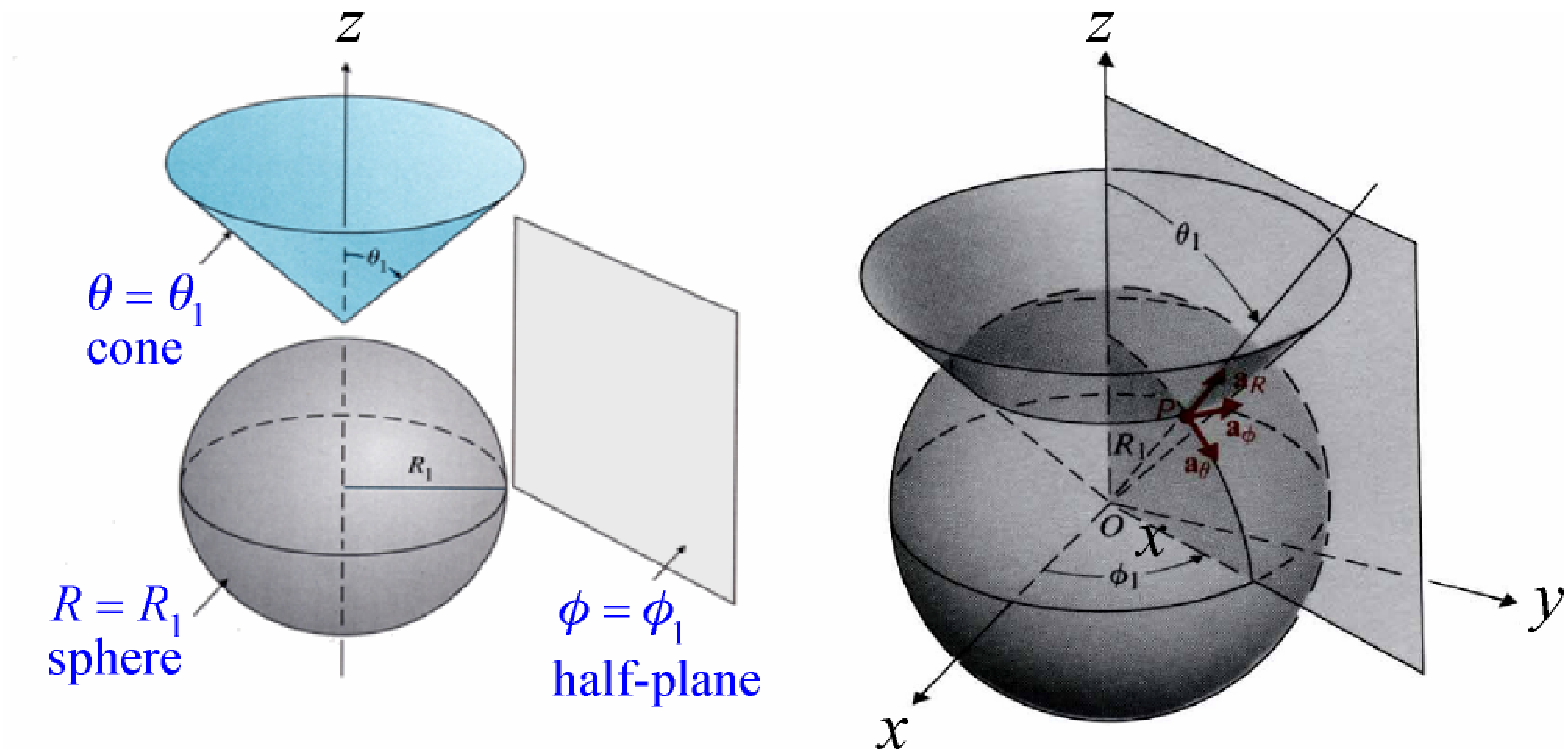
$$= (\vec{a}_r \cdot \vec{a}_x) A_r + (\vec{a}_\phi \cdot \vec{a}_x) A_\phi + (\vec{a}_z \cdot \vec{a}_x) A_z$$

$$\cos\left(90^\circ + \phi\right) = -\sin\phi$$

Spherical-1

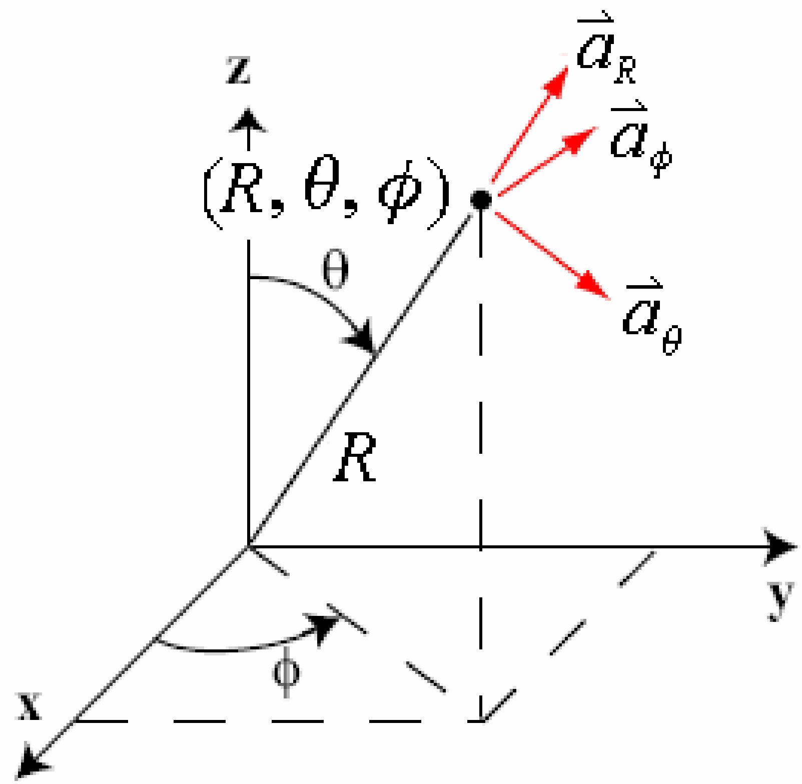
azimuthal polar

$$\left(u_1, u_2, u_3\right) = \left(R, \theta, \phi\right)$$



Spherical-2

3 base vectors $\{\vec{a}_R, \vec{a}_\theta, \vec{a}_\phi\}$ vary with $\{\theta, \phi\}$ of the observation point

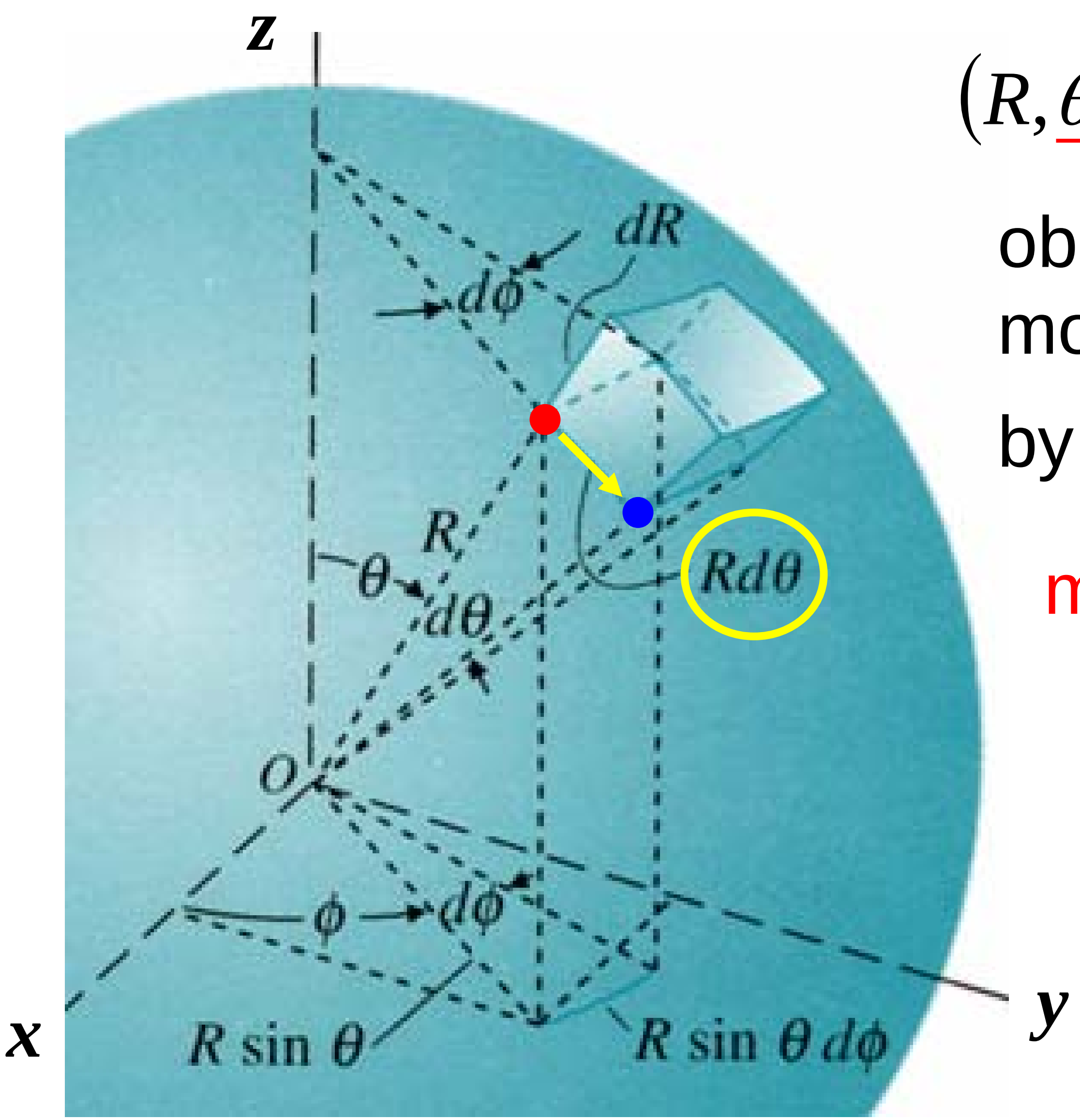


$$\vec{a}_R = \vec{a}_x \sin\theta\cos\phi + \vec{a}_y \sin\theta\sin\phi + \vec{a}_z \cos\theta$$

$$\vec{a}_\theta = \vec{a}_x \cos\theta\cos\phi + \vec{a}_y \cos\theta\sin\phi - \vec{a}_z \sin\theta$$

$$\vec{a}_\phi = -\vec{a}_x \sin\phi + \vec{a}_y \cos\phi$$

Spherical-3



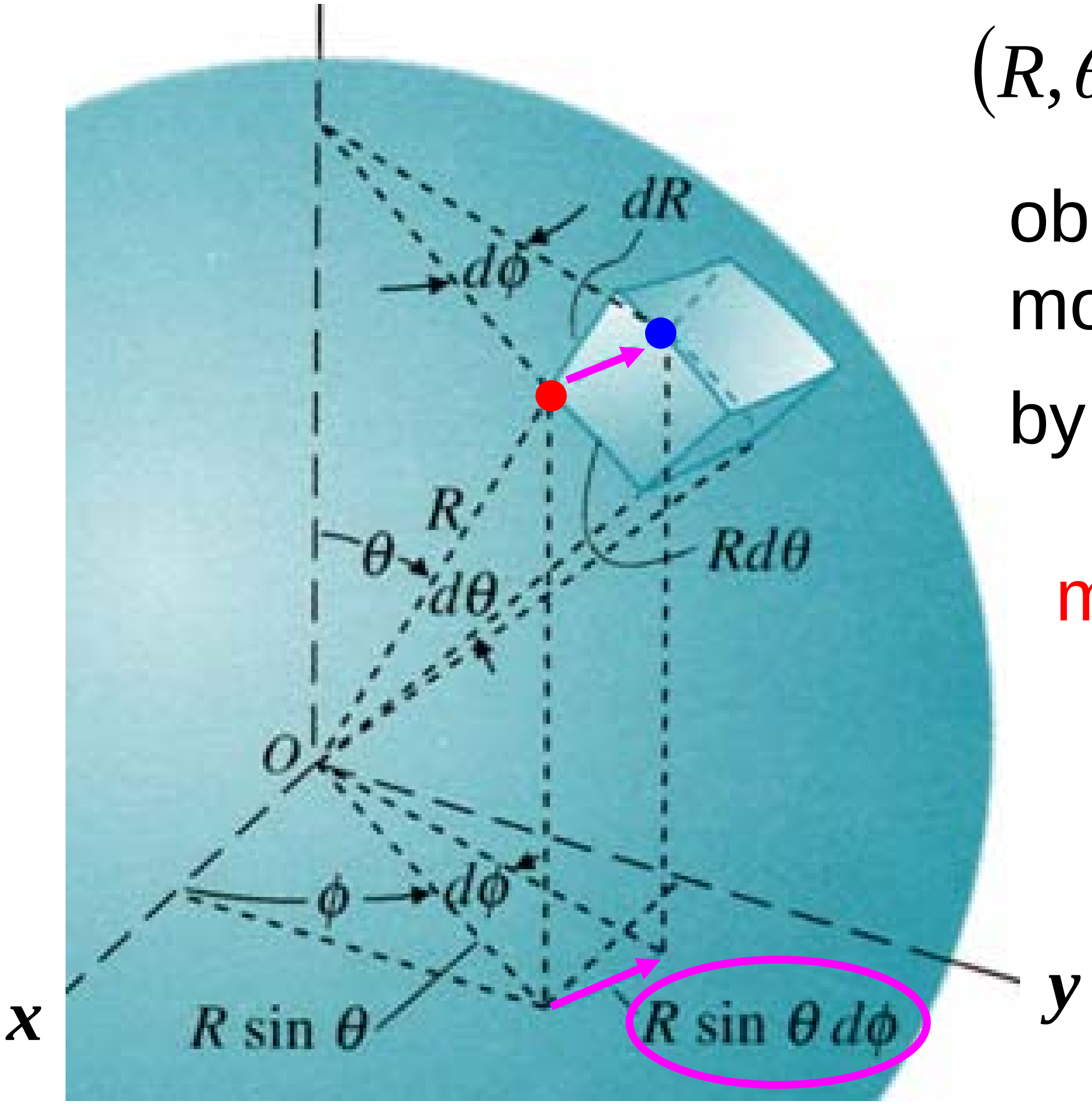
$$(R,\underline{\theta},\phi)\rightarrow(R,\underline{\theta+d\theta},\phi)$$

observation point moves along $\vec{a}_\theta$

by $dl_\theta = R \cdot d\theta$

metric coefficient of θ

Spherical-4



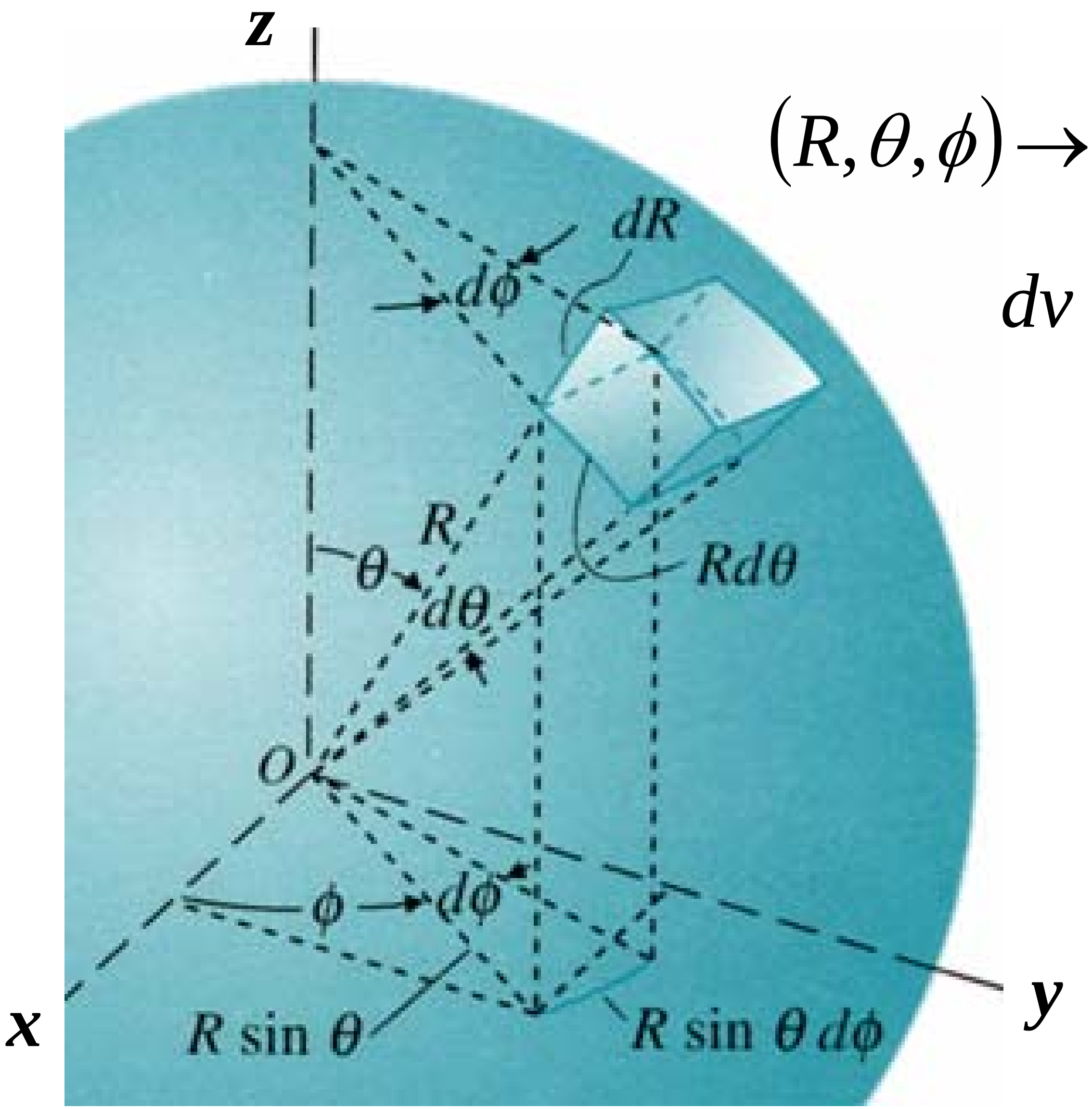
$$(R,\theta,\phi)\rightarrow(R,\theta,\phi+d\phi)$$

observation point moves along $\vec{a}_\phi$

by $dl_\phi = R\sin\theta \cdot d\phi$

metric coefficient of ϕ

Spherical-5

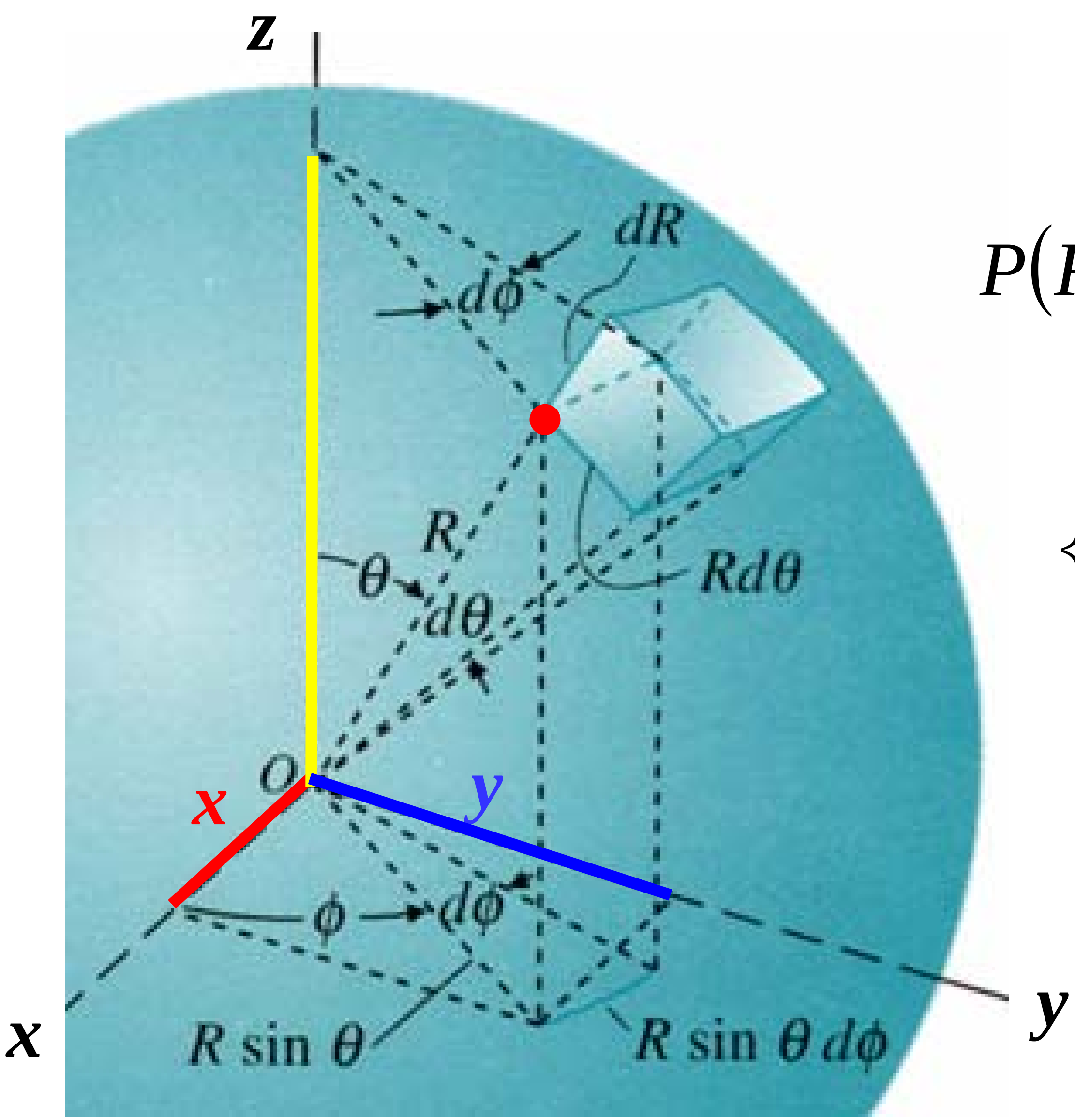


$$(R,\theta,\phi)\rightarrow(R+dR,\theta+d\theta,\phi+d\phi)$$

$$dv = R^2 \sin\theta \cdot dR d\theta d\phi$$

…differential volume

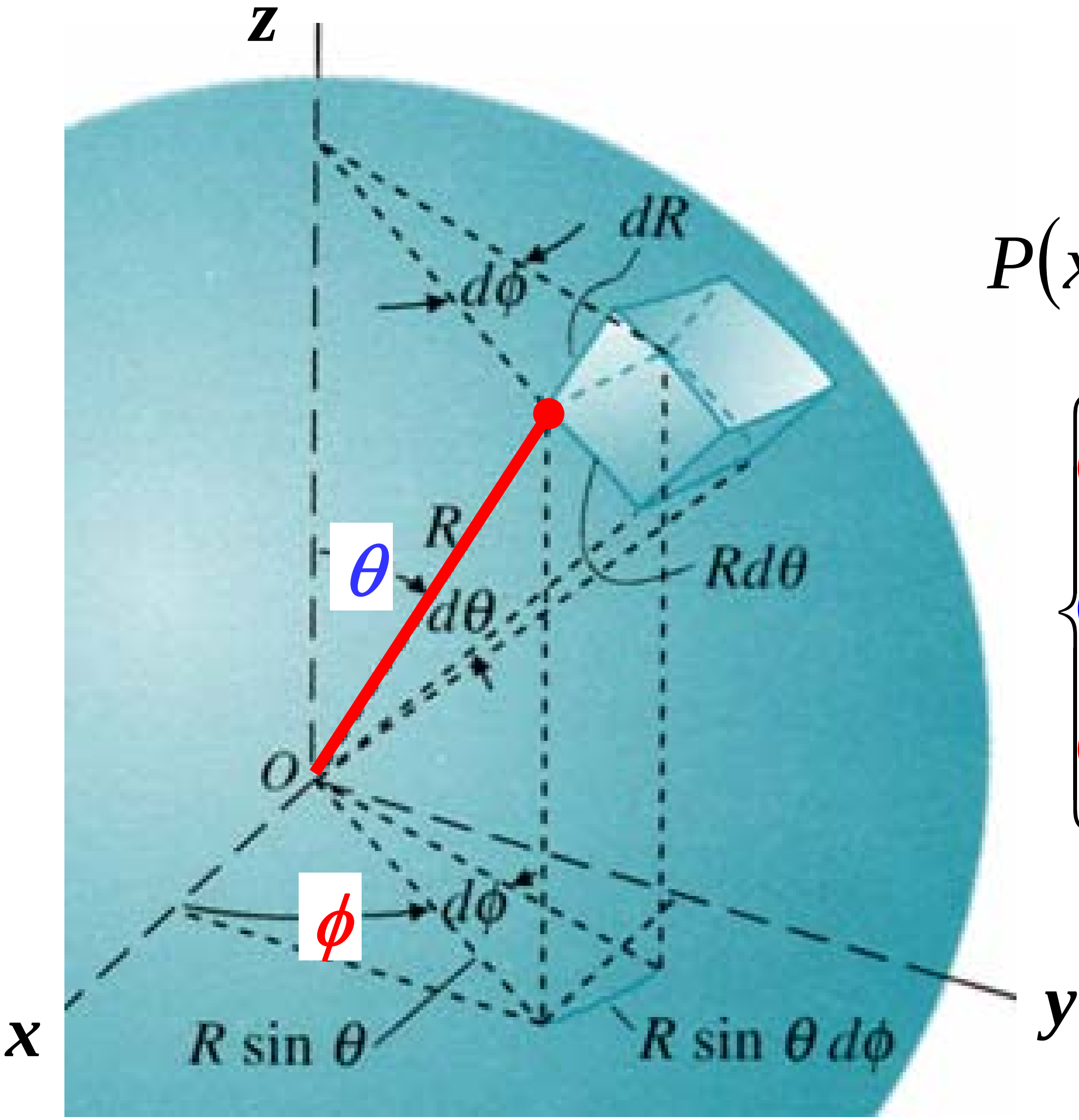
Transformation of position representation-1



$$P(R,\theta,\phi) \rightarrow P(x,y,z)$$

$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$$

Transformation of position representation-2



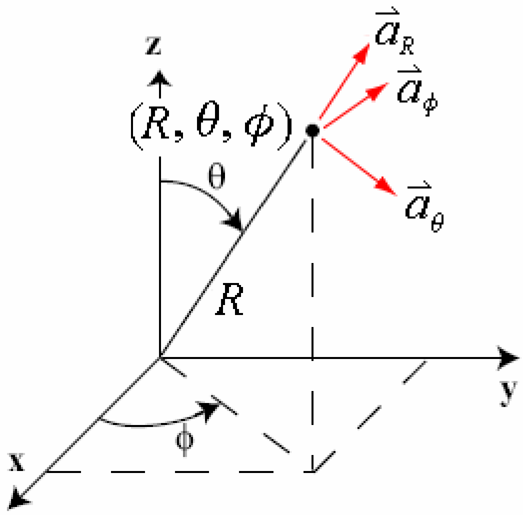
$$P(x, y, z) \rightarrow P(R, \theta, \phi)$$

$$\begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2} / z\right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

Transformation of vector components

$$\vec{V} = \vec{a}_R A_R + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$



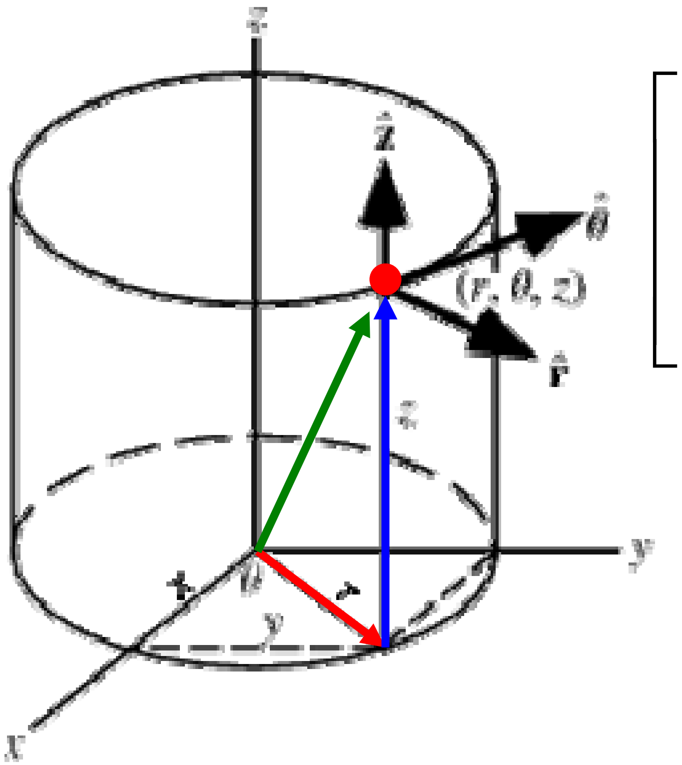
$$\vec{a}_R = \vec{a}_x \sin\theta\cos\phi + \vec{a}_y \sin\theta\sin\phi + \vec{a}_z \cos\theta$$

$$A_x = \vec{V}\cdot\vec{a}_x = \left(\vec{a}_R A_R + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi\right)\cdot\vec{a}_x$$

$$= (\vec{a}_R \cdot \vec{a}_x) A_R + (\vec{a}_\theta \cdot \vec{a}_x) A_\theta + (\vec{a}_\phi \cdot \vec{a}_x) A_\phi$$

Transformation of position vector (cylindrical)

$$P(r,\phi,z) \text{ means } \vec{P} = \vec{a}_r r + \vec{a}_z z, \Rightarrow \left(A_r, A_\phi, A_z\right) = (r, 0, z)$$

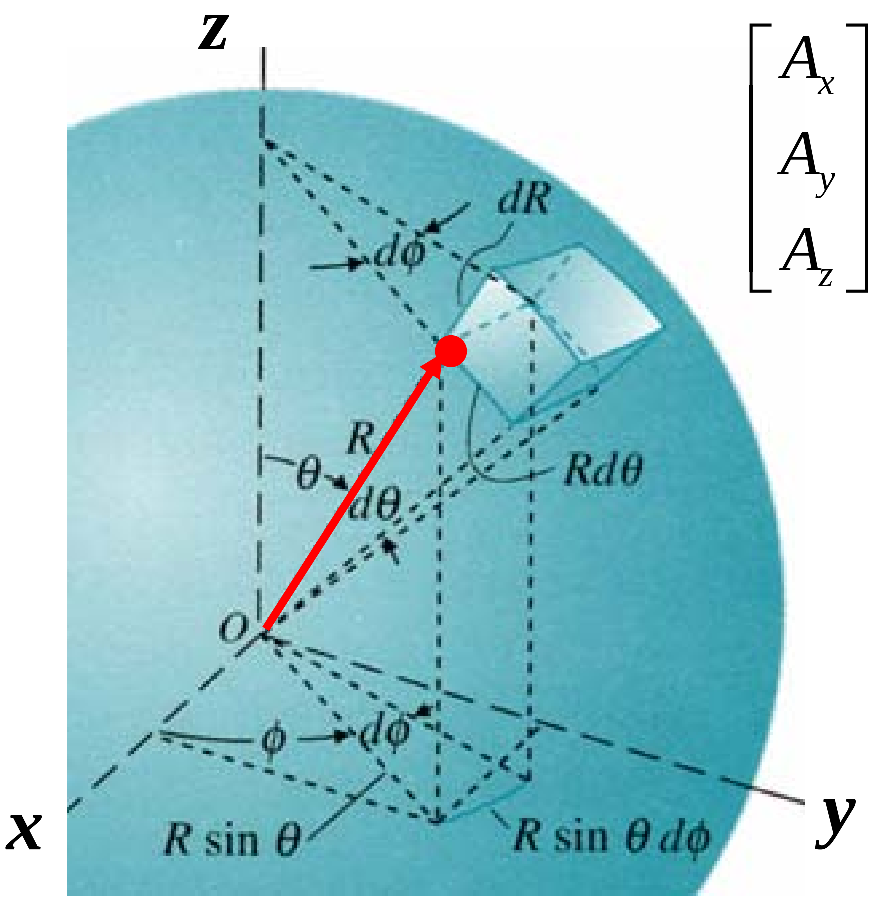


$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \\ z \end{bmatrix}$$

consistent with $\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$

Transformation of position vector (spherical)

$$P(R,\theta,\phi) \text{ means } \vec{P} = \vec{a}_R R, \Rightarrow (A_R, A_\theta, A_\phi) = (R,0,0)$$



$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

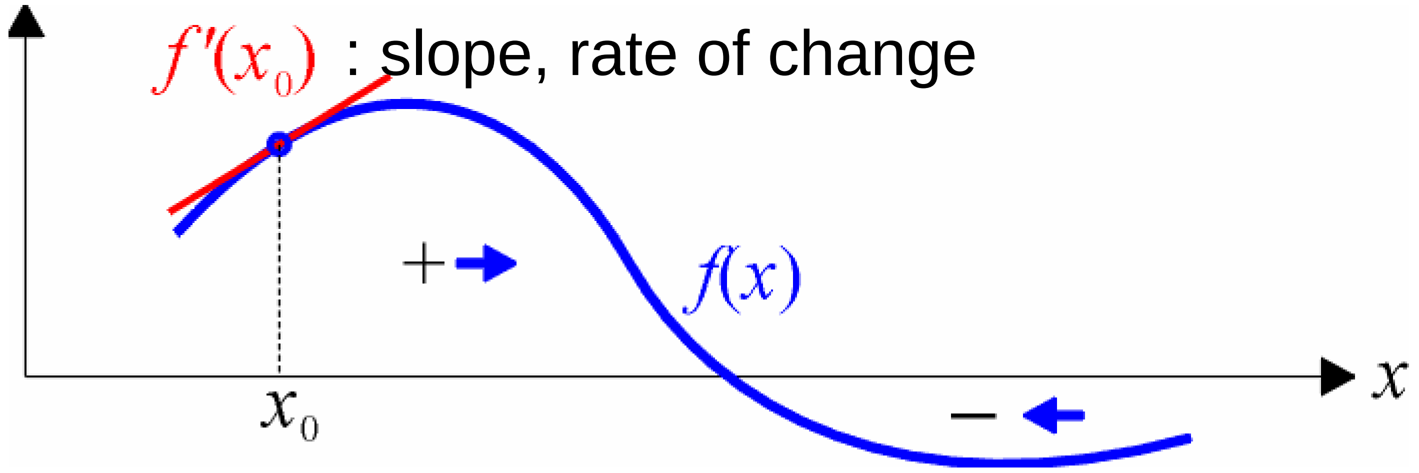
$$= \begin{bmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{bmatrix}$$

consistent with
$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$$

Sec. 5-3 Vector Calculus

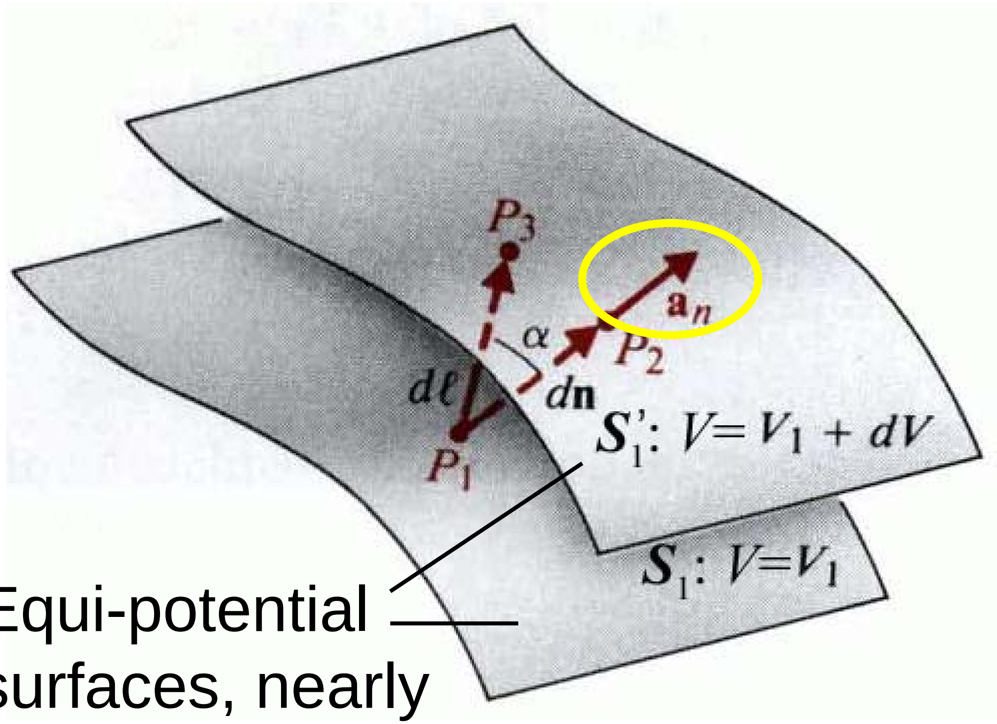
1. Gradient
2. Divergence
3. Curl
4. Laplacian

Overview: Derivative of functions



- 1D only has 1 variable and 2 directions ($\pm x$), can be represented by the sign of function values (e.g. the force function exerted on an object).
- ≥2D has multiple variables and infinitely many directions:
  - Slope of scalar function depends on the choice of differential path.
  - "Slope" of vector function is even complicated, for the change of function "value" is a vector.

The gradient of a scalar field V is a vector field whose magnitude and direction describes the max. space rate of increase of V



Equi-potential surfaces, nearly parallel if $dV \to 0$

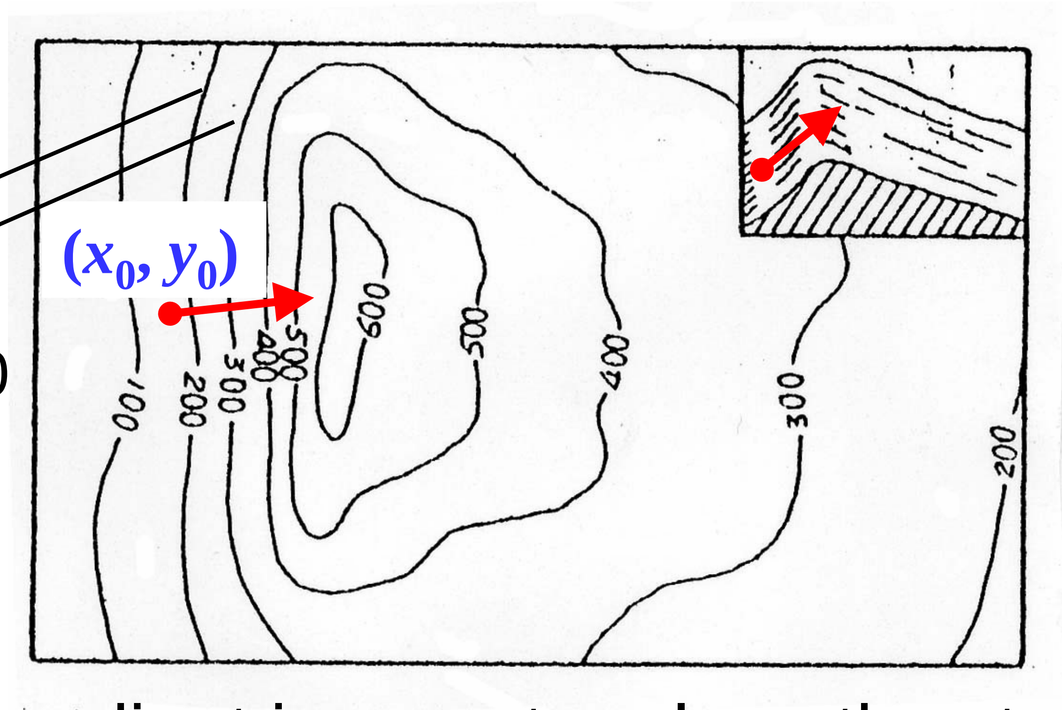
$$\nabla V \equiv \vec{a}_n \frac{dV}{dn}$$

Max. change of V occurs along the normal direction of the equi-potential surface (zero change)

Gradient - 2D example

A scalar function of two variables $V(x,y)$ can be fully represented by a contour plot:

Equi-potential lines, nearly parallel if $dV \to 0$



At (x_0,y_0), gradient is a vector along the steepest ascent direction, and the magnitude is the slope of $V(x,y)$ in that direction.

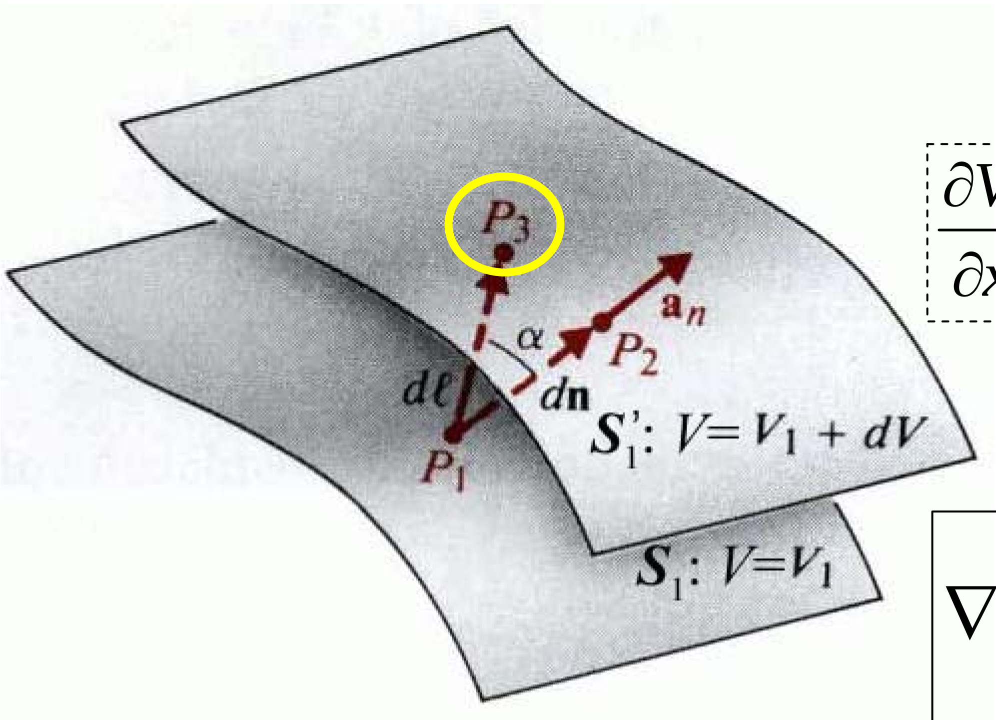
Gradient - Proof of formula

$$\overline{P_1P_3} = dl = \frac{dn}{\cos\alpha}, \Rightarrow \frac{dV}{dl} = \frac{dV}{dn}\cos\alpha$$

$$= \left|\nabla V\right|\left(\vec{a}_n \cdot \vec{a}_l\right) = \left(\nabla V\right)\cdot \vec{a}_l, \Rightarrow dV = \left(\nabla V\right)\cdot d\vec{l}$$

$$\vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz$$

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

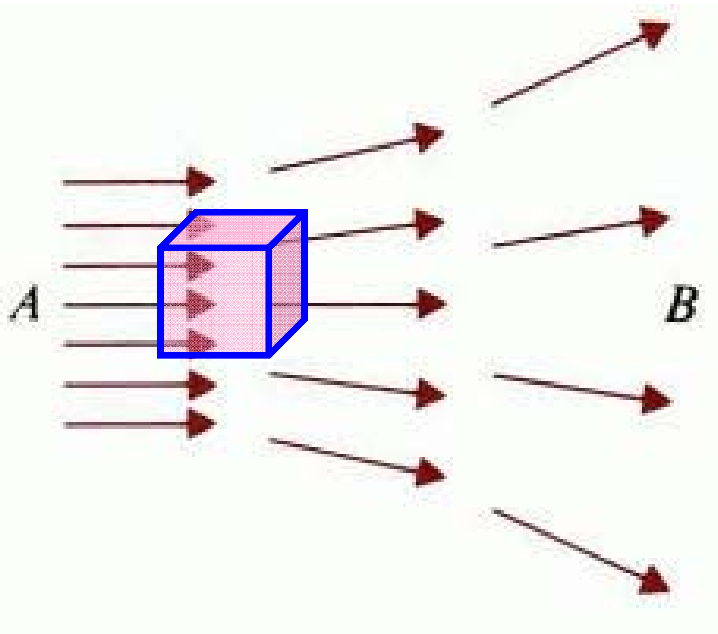


By comparison, $\Rightarrow$

$$\nabla V = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$$

Divergence - Definition & meaning

The divergence of a vector field $\vec{A}$ is a scalar field describing the net outward flux per unit volume



$$\nabla \cdot \vec{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$

$\left(\nabla \cdot \vec{A}\right) > 0$ …flow source

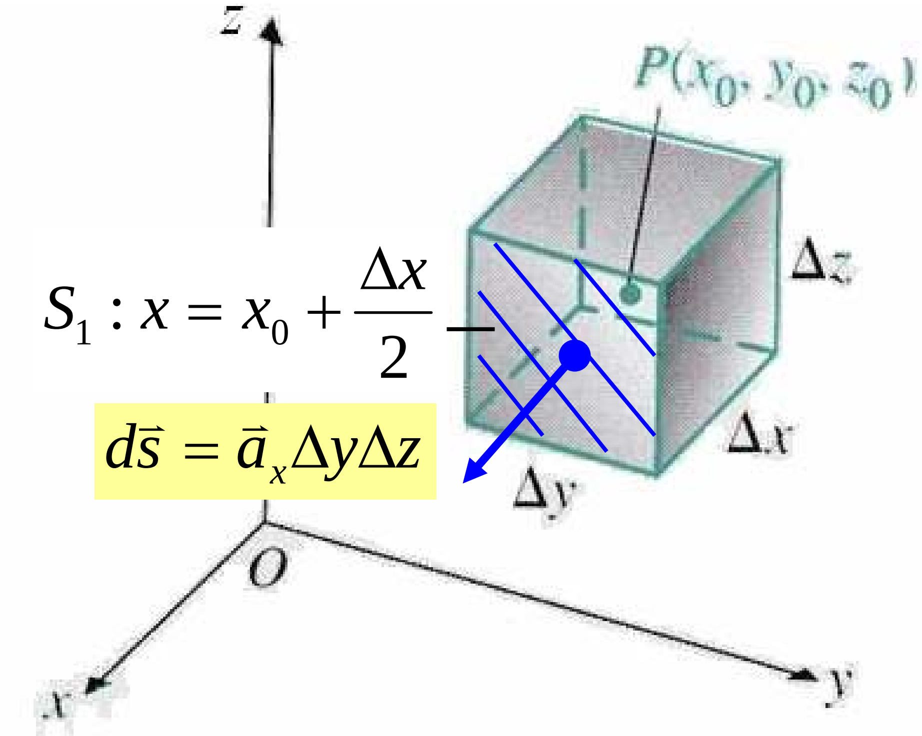
$\left(\nabla \cdot \vec{A}\right) < 0$ …flow sink

Divergence - Proof of formula (1)

$$\vec{A}(x,y,z) = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

$$S_1 : x = x_0 + \frac{\Delta x}{2}$$

$$d\vec{s} = \vec{a}_x \Delta y \Delta z$$



$$d\vec{s} = \vec{a}_x \Delta y \Delta z$$

$$F_1 = \iint_{S_1} \vec{A}(x,y,z) \cdot d\vec{s}$$

$$\approx \vec{A}(x_0 + \Delta x/2, y_0, z_0)$$

$$F_1 \approx A_x(x_0 + \Delta x/2, y_0, z_0) \cdot (\Delta y \Delta z)$$

$$\approx A_x(x_0, y_0, z_0) + \left[\left.\frac{\partial A_x}{\partial x}\right|_P\right] \frac{\Delta x}{2}$$

$$F_1 \approx A_x(x_0, y_0, z_0) \cdot (\Delta y \Delta z) + \left[\left.\frac{\partial A_x}{\partial x}\right|_P\right] \frac{\Delta x \Delta y \Delta z}{2}$$

$$\Delta x \Delta y \Delta z = \Delta v$$

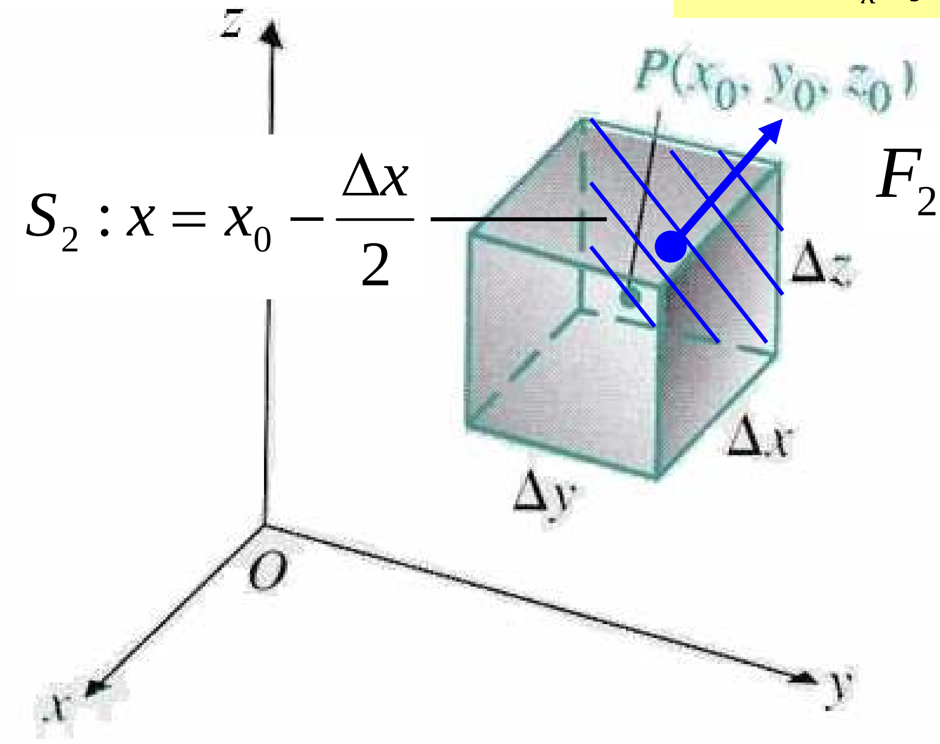
Divergence - Proof of formula (2)

$$d\vec{s} = -\vec{a}_x \Delta y \Delta z$$

$$F_2 = \iint_{S_2} \vec{A}(x,y,z) \cdot d\vec{s}$$

$$\approx \vec{A}(x_0 - \Delta x/2, y_0, z_0)$$

$d\vec{s} = -\vec{a}_x \Delta y \Delta z$



$$S_2 : x = x_0 - \frac{\Delta x}{2}$$

$$F_2 \approx A_x(x_0 - \Delta x/2, y_0, z_0) \cdot (-\Delta y \Delta z)$$

$$\approx A_x(x_0, y_0, z_0) - \left[\left. \frac{\partial A_x}{\partial x} \right|_P \right] \frac{\Delta x}{2}$$

$$F_2 \approx -A_x(x_0, y_0, z_0) \cdot (\Delta y \Delta z)$$

$$+ \left[\left. \frac{\partial A_x}{\partial x} \right|_P \right] \frac{\Delta v}{2}$$

Divergence - Proof of formula (3)

$$\Rightarrow F_1 + F_2 = \left[\left. \frac{\partial A_x}{\partial x} \right|_{P(x_0, y_0, z_0)} \right] \Delta v$$

Total flux of the cuboid:

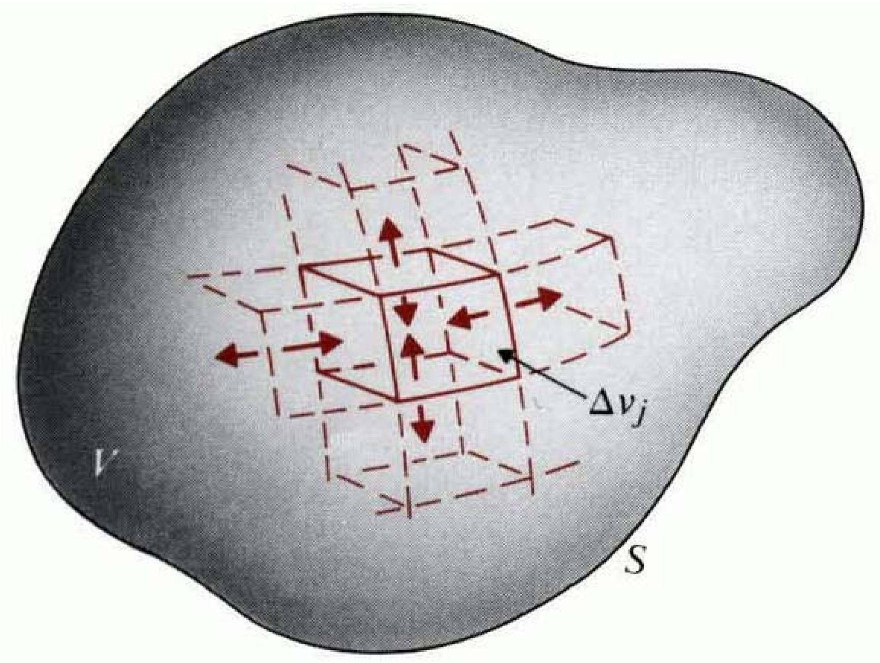
$$\oint_S \vec{A} \cdot d\vec{s} = \sum_{n=1}^{6} F_n = \left[\left. \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right|_{P(x_0, y_0, z_0)} \right] \Delta v$$

$$\nabla \cdot \vec{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}, \Rightarrow \boxed{\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Divergence theorem

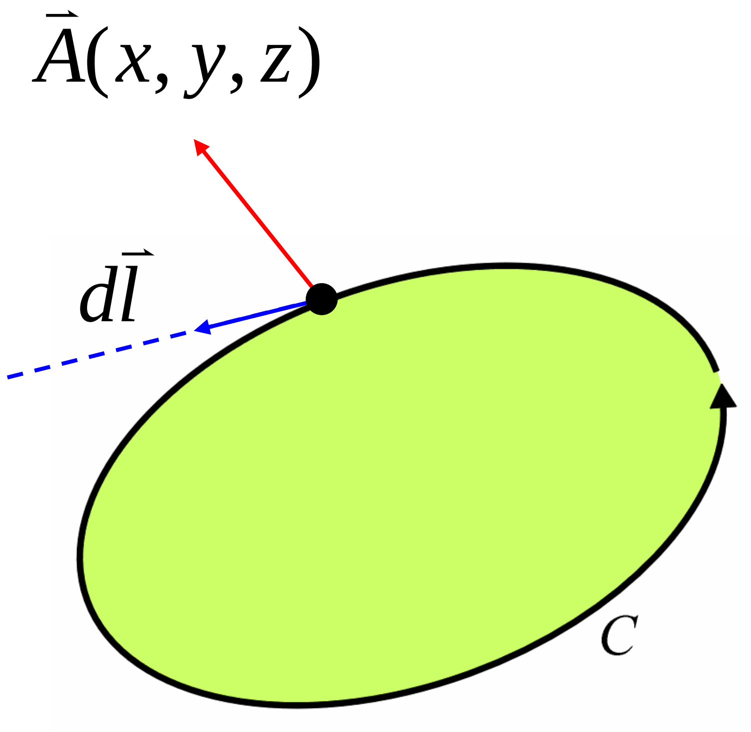
$$\nabla \cdot \vec{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v} \text{ implies } \boxed{\oint_S \vec{A} \cdot d\vec{s} = \int_V \left(\nabla \cdot \vec{A}\right) dv}$$

microscopic　　　　　macroscopic



Contributions of flux from the internal surfaces will cancel with one another.

Circulation: the work done by the force $\vec{A}$ in moving some object (energy obtained by the object) around a contour C



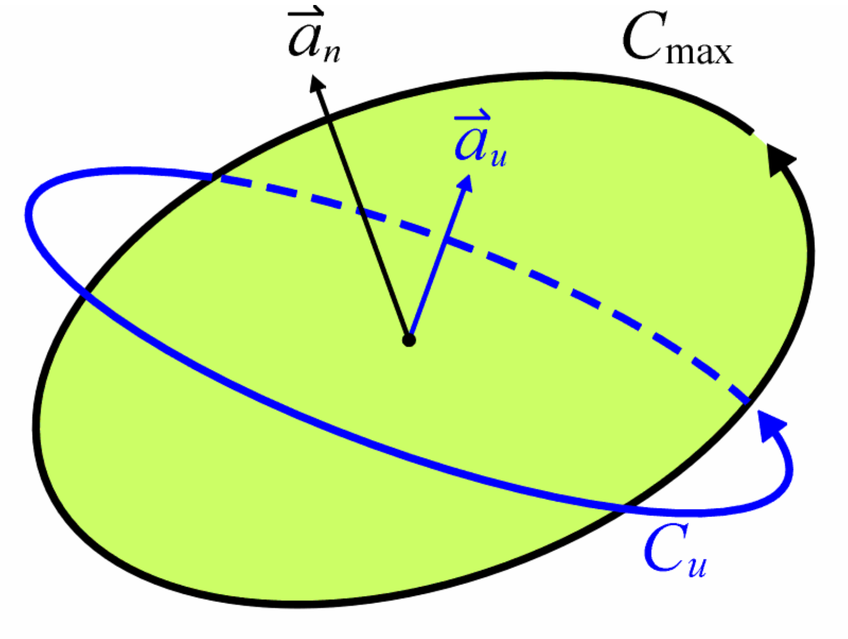
$$\text{Circulation} \equiv \oint_C \vec{A} \cdot d\vec{l}$$

$\oint_C \vec{A} \cdot d\vec{l} \neq 0$ at point P,

$\Rightarrow$ $\vec{A}$ is a non-conservative force, forming a vortex source at P that drives flow circularly.

Curl - Definition & meaning (2)

The curl of a vector field $\vec{A}$ is a vector field whose:
(1) magnitude: the net circulation per unit area,
(2) direction: normal direction of a contour that maximizes the circulation.

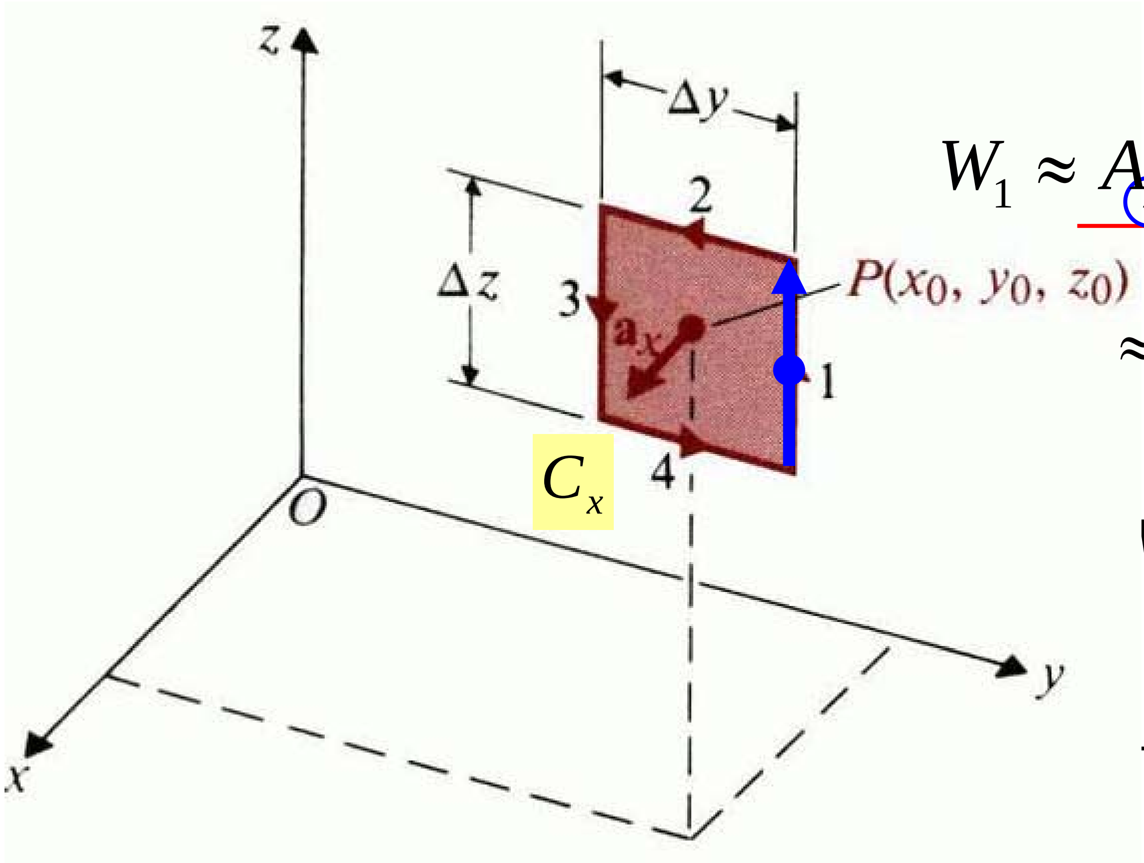


$$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\vec{a}_n \left(\oint_{C_{\max}} \vec{A} \cdot d\vec{l} \right)}{\Delta s}$$

$$\Rightarrow \lim_{\Delta s_u \to 0} \frac{\oint_{C_u} \vec{A} \cdot d\vec{l}}{\Delta s_u} = \left(\nabla \times \vec{A}\right) \cdot \vec{a}_u$$

Curl - Proof of formula (1)

$$\vec{A}(x,y,z)$$
$$= \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$



$$d\vec{l} = \vec{a}_z \Delta z$$

$$W_1 = \int_1 \vec{A}(x,y,z)\cdot d\vec{l}$$

$$\approx \vec{A}(x_0, y_0 + \Delta y/2, z_0)$$

$$W_1 \approx A_z(x_0, y_0 + \Delta y/2, z_0)\cdot(\Delta z)$$

$$\approx A_z(x_0, y_0, z_0) + \left[\left.\frac{\partial A_z}{\partial y}\right|_P\right]\frac{\Delta y}{2}$$

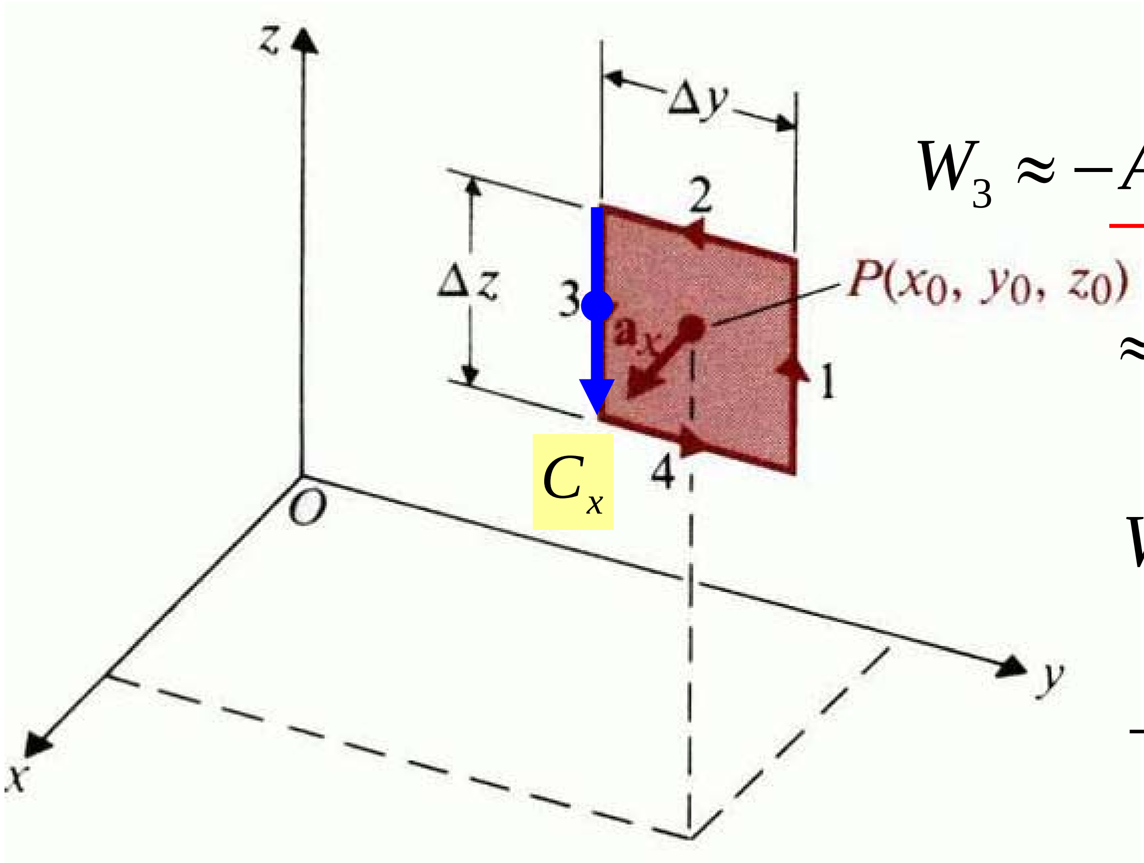
$$W_1 \approx A_z(x_0, y_0, z_0)\cdot(\Delta z)$$
$$+\left[\left.\frac{\partial A_z}{\partial y}\right|_P\right]\frac{\Delta y \Delta z}{2} \qquad \Delta y \Delta z = \Delta s$$

Curl - Proof of formula (2)

$$d\vec{l} = -\vec{a}_z \Delta z$$

$$W_3 = \int_3 \vec{A}(x,y,z)\cdot d\vec{l}$$

$$\approx \vec{A}(x_0, y_0 - \Delta y/2, z_0)$$



$$W_3 \approx -A_z\left(x_0, y_0 - \Delta y/2, z_0\right)\cdot\left(\Delta z\right)$$

$$\approx A_z\left(x_0, y_0, z_0\right) - \left[\left.\frac{\partial A_z}{\partial y}\right|_P\right]\frac{\Delta y}{2}$$

$$W_3 \approx -A_z\left(x_0, y_0, z_0\right)\cdot\left(\Delta z\right)$$

$$+\left[\left.\frac{\partial A_z}{\partial y}\right|_P\right]\frac{\Delta s}{2}$$

Curl - Proof of formula (3)

$$\Rightarrow W_1 + W_3 = \left[\left. \frac{\partial A_z}{\partial y} \right|_{P(x_0, y_0, z_0)} \right] \Delta s$$

Total circulation of the contour C_x :

$$\int_{1234} \vec{A} \cdot d\vec{l} = \left[\left. \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right|_{P(x_0, y_0, z_0)} \right] \Delta s$$

$$\Rightarrow \left(\nabla \times \vec{A} \right) \cdot \vec{a}_x = \lim_{\Delta s \to 0} \frac{\oint_{C_x} \vec{A} \cdot d\vec{l}}{\Delta s} = \left[\left. \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right|_{P(x_0, y_0, z_0)} \right]$$

Curl - Proof of formula (4)

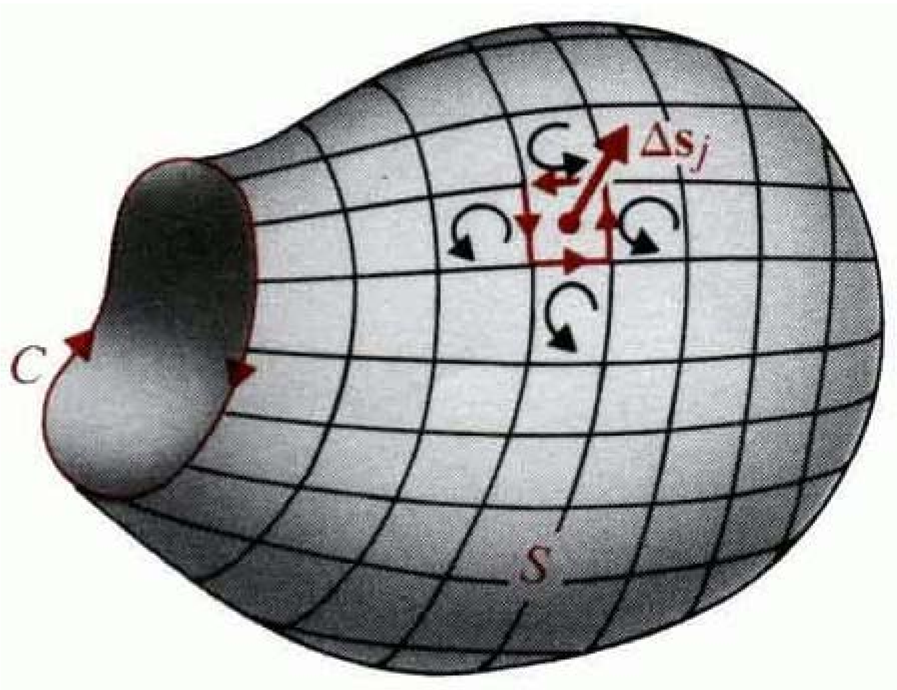
Find y-, z- components by evaluating the total circulation over contour C_y, C_z:

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

Stokes' theorem

$$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\vec{a}_n \left(\oint_{C_{\max}} \vec{A} \cdot d\vec{l} \right)}{\Delta s} \text{ implies } \boxed{\oint_C \vec{A} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{A} \right) \cdot d\vec{s}}$$

microscopic macroscopic

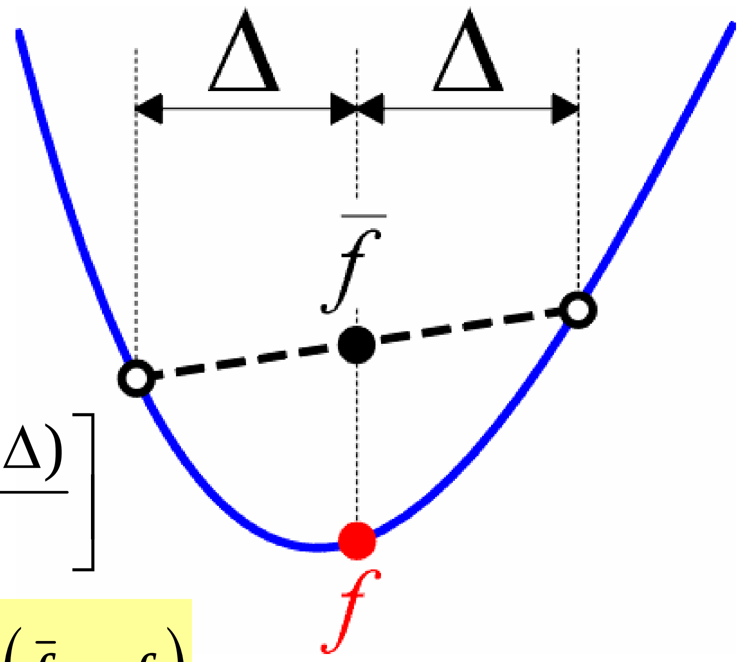


Contributions of work from the internal boundaries will cancel with one another.

2nd-order derivative

Consider a scalar function of single variable $f(x)$:

$$\frac{df}{dx} = \lim_{\Delta \to 0} \frac{f(x+\Delta/2) - f(x-\Delta/2)}{\Delta}$$



$$\frac{d^2 f}{dx^2} = \lim_{\Delta \to 0} \frac{f'(x+\Delta/2) - f'(x-\Delta/2)}{\Delta}$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta}\left[\frac{f(x+\Delta) - f(x)}{\Delta} - \frac{f(x) - f(x-\Delta)}{\Delta}\right]$$

$$= \lim_{\Delta \to 0} \frac{2}{\Delta^2}\left[\frac{f(x+\Delta) + f(x-\Delta)}{2} - f(x)\right] \propto \left(\bar{f} - f\right)$$

Scalar Laplacian

$$\nabla V = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$$

$$\boxed{\nabla^2 V \equiv \nabla \cdot (\nabla V)}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

In Cartesian coordinate system:

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

Vector Laplacian

$$\nabla^2 \vec{A} \equiv \nabla\left(\nabla \cdot \vec{A}\right) - \nabla \times \nabla \times \vec{A}$$

In Cartesian coordinate system:

$$\nabla^2 \vec{A} = \vec{a}_x\left(\nabla^2 A_x\right) + \vec{a}_y\left(\nabla^2 A_y\right) + \vec{a}_z\left(\nabla^2 A_z\right)$$

Null identities

$$\nabla \times (\nabla V) = 0$$

A conservative (curl-free) vector field can be expressed as the gradient of a scalar field (electrostatic potential)

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

A solenoidal (divergence-free) vector field can be expressed as the curl of another vector field (magnetostatic potential)

Helmholtz's theorem-1

A vector field is uniquely determined if both its divergence and curl are specified everywhere

$$\begin{cases} \nabla \cdot \vec{E} = g(\vec{r}) \\ \nabla \times \vec{E} = \vec{G}(\vec{r}) \end{cases} \longrightarrow \vec{E}(\vec{r})$$

In other words, a vector field is completely determined by its flow source $g(\vec{r})$ and vortex source $\vec{G}(\vec{r})$.

Helmholtz's theorem-2

A vector field $\vec{F}$ can be decomposed into:

- Curl-free (irrotational) component $\vec{F}_i$, with:

$$\begin{cases} \nabla \cdot \vec{F}_i = g \\ \nabla \times \vec{F}_i = 0 \end{cases} \xrightarrow{\nabla \times (\nabla V) = 0} \vec{F}_i = -\nabla V$$

V: scalar potential of $\vec{F}$

- Divergence-free (solenoidal) component $\vec{F}_s$, with:

$$\begin{cases} \nabla \cdot \vec{F}_s = 0 \\ \nabla \times \vec{F}_s = \vec{G} \end{cases} \xrightarrow{\nabla \cdot (\nabla \times \vec{A}) = 0} \vec{F}_s = \nabla \times \vec{A}$$

$\vec{A}$: vector potential of $\vec{F}$

$$\boxed{\vec{F} = \vec{F}_i + \vec{F}_s = -\nabla V + \nabla \times \vec{A}}$$