Lesson 04 Steady-state Response of Transmission Lines

Introduction

Though there are infinitely many types of excitation waveforms, it is of particular importance to consider the response of transmission lines to sinusoidal excitations. The fundamental reasons are twofold: (1) The electrical power and communication signals are often transmitted as sinusoids or modified sinusoids. (2) Any non-sinusoidal signals can be treated as superposition of sinusoids of different frequencies (Fourier analysis). The initial onset of a sinusoidal excitation produces a natural (transient) response, which will decay rapidly in time (Example 3-2). In contrast, the forced (steady-state) response supported by the sinusoidal source will continue indefinitely. In this lesson, we will employ two powerful tools, i.e., phasors and complex impedances (commonly used in alternating-circuit lumped circuit analysis) to show the rich phenomena of waves.

Phasor representations of transmission line equations and solutions

When the steady-state due to a sinusoidal excitation is reached, the voltage v(z,t) and the current i(z,t) on the transmission line must also be sinusoidal waves, which can be represented by the z-dependent phasors V(z), I(z):

$$v(z,t) = \operatorname{Re}\left\{ V(z) \cdot e^{j\omega t} \right\}, \quad i(z,t) = \operatorname{Re}\left\{ I(z) \cdot e^{j\omega t} \right\}$$
(4.1)

The lossless transmission line equations, i.e., eq's (2.1-4), can be rewritten as:

$$\frac{d}{dz}V(z) = -j\omega L \cdot I(z) \tag{4.2}$$

$$\frac{d}{dz}I(z) = -j\omega C \cdot V(z) \tag{4.3}$$

$$\frac{d^{2}}{dz^{2}}V(z) = -\beta^{2}V(z)$$
(4.4)

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$$\frac{d^2}{dz^2}I(z) = -\beta^2 I(z) \tag{4.5}$$

where $\frac{\partial^n}{\partial t^n}$ is replaced by $(j\omega)^n$, and the propagation constant β is defined as:

$$\beta = \omega \sqrt{LC} \tag{4.6}$$

Eq. (4.4) is a second-order "ordinary" differential equation, whose general solution is of the form:

$$V(z) = V^{+}(z) + V^{-}(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}, \qquad (4.7)$$

where $V^+ \equiv |V^+| e^{j\phi^+}$, $V^- \equiv |V^-| e^{j\phi^-}$ are determined by the boundary conditions. By eq. (4.1), we can derive the space-time expression of eq. (4.7):

$$v(z,t) = \operatorname{Re}\left\{V(z) \cdot e^{j\omega t}\right\} = \operatorname{Re}\left\{\left(V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}\right)e^{j\omega t}\right\},$$
$$= \left|V^{+}\right|\cos\left(\omega t - \beta z + \phi^{+}\right) + \left|V^{-}\right|\cos\left(\omega t + \beta z + \phi^{-}\right),$$
$$= \left|V^{+}\right|\cos\left[\omega\left(t - \frac{z}{\omega/\beta}\right) + \phi^{+}\right] + \left|V^{-}\right|\cos\left[\omega\left(t + \frac{z}{\omega/\beta}\right) + \phi^{-}\right]$$

Compared with eq. (2.6), we find that $V^+(z)$, $V^-(z)$ of eq. (4.7) stand for the sinusoidal waves propagating in the +z and -z directions with a common phase velocity of:

$$v_p = \frac{\omega}{\beta},\tag{4.8}$$

respectively. Substituting eq. (4.6) into eq. (4.8) gives $v_p = \frac{1}{\sqrt{LC}}$, consistent with eq. (2.5).

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For a sinusoidal wave, each point (say the peak) moves a distance of one wavelength λ within a time duration of one period T, $\Rightarrow v_p = \frac{\lambda}{T}$. By examining the voltage wave $v^+(z,t) = |V^+|\cos(\omega t - \beta z + \phi^+):$ (1) $v^+(z,t_0) = |V^+|\cos(\omega t_0 - \beta z + \phi^+), \Rightarrow \lambda = \frac{2\pi}{\beta}.$ (2) $v^+(z_a,t) = |V^+|\cos(\omega t - \beta z_a + \phi^+), \Rightarrow T = \frac{2\pi}{\omega}.$ We thus have $v_p = \frac{\lambda}{T} = \frac{\omega}{\beta}$, consistent with eq. (4.8).

The z-dependent current phasor can be obtained by substituting eq. (4.7) into eq. (4.2):

$$I(z) = I^{+}(z) + I^{-}(z) = \frac{1}{Z_{0}} \Big[V^{+}(z) - V^{-}(z) \Big],$$
(4.9)

where Z_0 is the characteristic impedance of the transmission line given in eq. (2.8). Similar to eq. (2.9), the characteristic impedance is the ratio of the voltage phasor $V^+(z)$ to the current phasor $I^+(z)$ of a "single" wave propagating in the +z direction:

$$Z_0 = \frac{V^+(z)}{I^+(z)} = -\frac{V^-(z)}{I^-(z)}$$
(4.10)

Reflection at discontinuity

Consider a lossless transmission line of characteristic impedance $Z_0 \in R$, propagation constant β , driven by a sinusoidal source of angular frequency ω , and terminated by an impedance $Z_L \in C$.



Fig. 4-1. Terminated lossless transmission line driven by sinusoidal voltage source.

Eq. (4.10) gives
$$\frac{V^+(z)}{I^+(z)} = Z_0$$
, while the boundary condition requires $\frac{V^+(z) + V^-(z)}{I^+(z) + I^-(z)}\Big|_{z=0} = Z_L$.

Therefore, a reflected wave $V^{-}(z)$ must be generated if the load is not matched to the line $(Z_{L} \neq Z_{0})$. In general, the voltage reflection coefficient $\Gamma(z)$ and the line impedance Z(z) (looking toward the load at z = 0) depend on the point of observation z:

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$$\Gamma(z) = \frac{V^{-}(z)}{V^{+}(z)} = \frac{V^{-}e^{j\beta z}}{V^{+}e^{-j\beta z}} = \frac{V^{-}}{V^{+}}e^{j2\beta z}, \implies$$

$$\Gamma(z) = \Gamma_{L}e^{j2\beta z}, \qquad \Gamma_{L} = \Gamma(0) = \frac{V^{-}}{V^{+}}$$

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V^{+}(z) + V^{-}(z)}{I^{+}(z) + I^{-}(z)} = \frac{V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}}{(V^{+}/Z_{0})e^{-j\beta z} + (-V^{-}/Z_{0})e^{j\beta z}}, \implies$$
(4.11)

$$Z(z) = Z_0 \frac{V^+ e^{-j\beta z} + V^- e^{j\beta z}}{V^+ e^{-j\beta z} - V^- e^{j\beta z}}, \quad Z_L = Z(0) = Z_0 \frac{V^+ + V^-}{V^+ - V^-}$$
(4.12)

Instead of using the voltage amplitudes V^+ and V^- , it is more convenient to express $\Gamma(z)$, Z(z) by the characteristic and load impedances Z_0 and Z_L . By eq's (4.12), (4.11),

$$Z_{L} = Z_{0} \frac{V^{+} + V^{-}}{V^{+} - V^{-}} = Z_{0} \frac{1 + \Gamma_{L}}{1 - \Gamma_{L}}, \Rightarrow$$

$$\Gamma_{L} = \frac{Z_{L} - Z_{0}}{Z_{L} + Z_{0}} \equiv \left| \Gamma_{L} \right| e^{i\psi}$$
(4.13)

$$\Gamma(z) = \frac{Z_L - Z_0}{Z_L + Z_0} e^{j2\beta z}$$
(4.14)

$$Z(z) = Z_0 \frac{V^+ e^{-j\beta z} + V^- e^{j\beta z}}{V^+ e^{-j\beta z} - V^- e^{j\beta z}}.$$
 By dividing V^+ for the numerator and the denominator, \Rightarrow
$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_L e^{j\beta z}}{e^{-j\beta z} - \Gamma_L e^{j\beta z}} = Z_0 \frac{(\cos\beta z - j\sin\beta z) + \Gamma_L(\cos\beta z + j\sin\beta z)}{(\cos\beta z - j\sin\beta z) - \Gamma_L(\cos\beta z + j\sin\beta z)}.$$
 By dividing $\cos\beta z$

for the numerator and the denominator, \Rightarrow

$$Z(z) = Z_0 \frac{(1 - j \tan \beta z) + \Gamma_L(1 + j \tan \beta z)}{(1 - j \tan \beta z) - \Gamma_L(1 + j \tan \beta z)} = Z_0 \frac{(1 + \Gamma_L) - j(1 - \Gamma_L) \tan \beta z}{(1 - \Gamma_L) - j(1 + \Gamma_L) \tan \beta z}.$$
By eq. (4.13), \Rightarrow
$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan(\beta z)}{Z_0 - jZ_L \tan(\beta z)}$$
(4.15)

Both $\Gamma(z)$ and Z(z) are complex periodic functions of period $\lambda/2$.

■ Short-circuited line

Example 4-1: Consider a system shown in Fig. 4-1 where $Z_L = 0$ (short-circuited load). Find the input impedance $Z_{sc} = Z(-l)$, and the voltage and current distributions v(z,t), i(z,t).



Fig. 4-2. (a) The z-dependent reactance of a short-circuited line. (b-c) The voltage v(z,t) and current i(z,t) on the line at several time instants. (d) The Spatial distribution of the temporal oscillation amplitude of v(z,t) and i(z,t), i.e., |V(z)|, |I(z)|, of Example 4-1. $T = 2\pi/\omega$, $\lambda = 2\pi/\beta$, assume $\phi^+ = 0$.

Ans: (1) By eq. (4.15), $Z(z) = -jZ_0 \tan(\beta z)$, \Rightarrow

$$Z_{sc} = jZ_0 \tan(\beta l) = jX_{sc} \tag{4.16}$$

Eq. (4.16) means that a lossless line $(Z_0 \in R)$ with a short-circuited load is purely reactive, i.e., Z(z) is purely imaginary. It can be capacitive $(X_{sc} < 0)$ or inductive $(X_{sc} > 0)$ of arbitrary magnitude, depending on the length of the line (Fig. 4-2a).

(2) By eq's (4.7), (4.11),
$$V(z) = V^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z} \right)$$
. By eq. (4.13), $\Gamma_L = -1$. $\Rightarrow V(z) = -2jV^+ \sin(\beta z)$. By eq. (4.1), \Rightarrow

$$v(z,t) = 2 |V^+| \sin(\beta z) \cdot \sin(\omega t + \phi^+)$$
(4.17)

This means that v(z,t) oscillates with angular frequency ω at any position z, and the amplitude of temporal oscillation is described by the spatial distribution of the phasor magnitude |V(z)| (Fig. 4-2d, solid):

$$|V(z)| = 2|V^+| \cdot |\sin(\beta z)|$$
(4.18)

The peaks (maximum amplitude) and valleys (minimum amplitude) of the temporal

oscillation of v(z,t) are fixed at $z/\lambda = -0.25, -0.75, \dots$ and $0, -0.5, -1, \dots$, respectively. As a result, v(z,t) is a standing wave.

(3) By eq's (4.9), (4.11),
$$I(z) = \frac{V^+}{Z_0} \left(e^{-j\beta z} - \Gamma_L e^{j\beta z} \right) = \frac{2V^+}{Z_0} \cos(\beta z)$$
. By eq. (4.1), \Rightarrow
 $i(z,t) = \frac{2|V^+|}{Z_0} \cos(\beta z) \cdot \cos(\omega t + \phi^+)$ (4.19)

Compare Fig. 4-2b and Fig. 4-2c, i(z,t) is in space and time quadrature (i.e., 90° out of phase) with respect to v(z,t).

Example 4-2: Consider a short-circuited ($Z_L = 0$) coaxial line with characteristic impedance $Z_0 = 50 \Omega$, $v_p = 2.07 \times 10^8$ m/s. (1) Find the shortest possible length *l* if the line is designed to provide an inductance of 15 nH at the operation (non-angular) frequency f = 3 GHz. (2) What is the lumped element value of the line at f = 4 GHz?

Ans: (1) Wavelength
$$\lambda = \frac{v_p}{f} = 6.9 \text{ cm}$$
. By eq. (4.16), $Z_{sc} = jZ_0 \tan\left(\frac{2\pi}{\lambda}l\right) = j50 \tan\left(\frac{2\pi}{6.9 \text{ cm}}l\right) = j\omega L_{sc} = j2\pi \cdot (3 \text{ GHz}) \cdot (15 \text{ nH}), \Rightarrow l = 1.53 \text{ cm}.$
(2) At $f = 4$ GHz, $\lambda = \frac{v_p}{f} = 5.175$ cm. $Z_{sc} = j50 \tan\left(\frac{2\pi}{5.175 \text{ cm}}1.53 \text{ cm}\right) = -j167.4 \Omega$ (<0, capacitive load). $Z_{sc} = \frac{1}{j\omega C_{sc}} = \frac{-j}{2\pi \cdot (4 \text{ GHz}) \cdot C_{sc}}, \Rightarrow C_{sc} = 0.238 \text{ pF}.$

■ Transmission line with resistive load

Consider the system shown in Fig. 4-1 where the load is purely resistive $(Z_L \in R, \Gamma_L \in R)$. Without loss of generality, we choose a proper time reference such that $\phi^+ = 0$ (i.e., $V^+ = |V^+|$). By eq's (4.7), (4.11),

$$V(z) = V^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z} \right) = V^+ \left(e^{-j\beta z} + \Gamma_L e^{-j\beta z} - \Gamma_L e^{-j\beta z} + \Gamma_L e^{j\beta z} \right), \Longrightarrow$$

$$V(z) = V^{+} \left[\left(1 + \Gamma_L \right) e^{-j\beta z} + 2j\Gamma_L \sin\left(\beta z\right) \right]$$
(4.20)

Similarly,

$$I(z) = \frac{V^+}{Z_0} \left[\left(1 + \Gamma_L \right) e^{-j\beta z} - 2\Gamma_L \cos(\beta z) \right]$$
(4.21)

The time-space expression of eq. (4.20) becomes:

$$v(z,t) = |V^+|(1+\Gamma_L)\operatorname{Re}\left\{e^{-j\beta z}e^{j\omega t}\right\} + 2|V^+|\Gamma_L\sin(\beta z)\operatorname{Re}\left\{je^{j\omega t}\right\}, \Rightarrow$$
$$v(z,t) = |V^+|(1+\Gamma_L)\cos(\omega t - \beta z) - 2|V^+|\Gamma_L\sin(\beta z)\sin(\omega t) \tag{4.22}$$

Similarly,

$$i(z,t) = \frac{\left|V^{+}\right|}{Z_{0}} (1+\Gamma_{L}) \cos(\omega t - \beta z) - 2\frac{\left|V^{+}\right|}{Z_{0}} \Gamma_{L} \cos(\beta z) \cos(\omega t)$$
(4.23)

The first term of eq. (4.23) represents a traveling wave, where the time and space variables are coupled in the form of $\omega t - \beta z$. The peaks of the traveling wave component (e.g. $\omega t - \beta z = 0$) move with phase velocity $v_p = \frac{\omega}{\beta}$. The second term of eq. (4.23) represents a standing wave, where the time and space variables are decoupled. The peaks of the standing wave component (e.g. $\beta z = -\pi/2$) always occur at the same position.

The oscillating amplitude of v(z,t) is described by the phasor magnitude |V(z)|:

$$|V(z)| = |V^{+}| \sqrt{(1 + \Gamma_{L})^{2} \cos^{2}(\beta z) + (1 - \Gamma_{L})^{2} \sin^{2}(\beta z)}$$
(4.24)

Eq. (4.24) is a periodic function of period $\lambda/2$, whose maximum and minimum values are:

$$V_{\max} = |V^+| (1 + |\Gamma_L|), \qquad V_{\min} = |V^+| (1 - |\Gamma_L|).$$
(4.25)

The ratio of V_{max} to V_{min} , called the standing wave ratio (SWR), is a key parameter used in quantitatively describe the degree of impedance mismatch between the transmission line and the load:

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$$S \equiv \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + \left|\Gamma_L\right|}{1 - \left|\Gamma_L\right|} \tag{4.26}$$

A larger SWR corresponds to a stronger reflection and a higher weighting of standing wave component in eq. (4.22).

Example 4-3: Find the line impedance Z(z), voltage amplitude |V(z)|, and SWR if $Z_L = 2Z_0, Z_0/2 \quad (\in R)$ in Fig. 4-1, respectively.



Fig. 4-3. (a) The line impedance normalized to Z_0 for $Z_L = 2Z_0$. (b) The voltage amplitude |V(z)| normalized to $|V^+|$ for $Z_L = 2Z_0$. (c-d) The counterparts of (a-b) for $Z_L = Z_0/2$.

Ans: (1) If $Z_L = 2Z_0$: By eq. (4.15), $Z(z) = Z_0 \frac{2 - j \tan(\beta z)}{1 - j2 \tan(\beta z)}$, which is a complex periodic function of period $\lambda/2$ (Fig. 4-3a). By eq. (4.13), $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3}$. By eq. (4.24), $|V(z)| = \frac{2}{3} |V^+| \sqrt{4\cos^2(\beta z) + \sin^2(\beta z)}, \Rightarrow V_{\max} = \frac{4}{3} |V^+|, V_{\min} = \frac{2}{3} |V^+|, S = 2$ (Fig. 4-3b). (2) If $Z_L = Z_0/2$: By eq. (4.15), $Z(z) = Z_0 \frac{1 - j2 \tan(\beta z)}{2 - j \tan(\beta z)}$ (Fig. 4-3c). By eq. (4.13),

$$\Gamma_{L} = -\frac{1}{3} \quad \text{By eq. (4.24), } |V(z)| = \frac{2}{3} |V^{+}| \sqrt{\cos^{2}(\beta z) + 4\sin^{2}(\beta z)} \quad \text{,} \quad \Rightarrow \quad V_{\text{max}} = \frac{4}{3} |V^{+}| \quad \text{,}$$
$$V_{\text{min}} = \frac{2}{3} |V^{+}| \quad \text{,} \quad S = 2 \quad (\text{Fig. 4-3d}).$$

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- 1) Compare Fig. 4-3b and Fig. 4-3d, we found: (a) $|V(z=0)| = V_{\text{max}}$, $z_{\min} = \frac{\lambda}{4}$, if $Z_L (\in R) > Z_0$; $|V(z=0)| = V_{\min}$, $z_{\min} = 0$, if $Z_L (\in R) < Z_0$. (b) $Z_L = rZ_0$ and $Z_L = \frac{Z_0}{r}$ produce the same SWR.
- 2) Compare with Example 4-1, short-circuited load is a special case of resistive load where: (a) no traveling wave component exists [eq. (4.17) vs. eq. (4.22)], (b) $V_{\min} = 0$, $S = \infty$.

Power flow on a transmission line

The instantaneous power flowing into the line at position z is defined as:

$$p(z,t) = v(z,t) \cdot i(z,t) \tag{4.27}$$

Recall the case of short-circuited line where v(z,t), i(z,t) are given by eq's (4.17), (4.19),

$$p(z,t) = \frac{\left|V^{+}\right|^{2}}{Z_{0}}\sin(2\beta z)\sin(2\omega t + \phi^{+}),$$

which oscillates with angular frequency 2ω . However, the primary purpose of most steady-state transmission line applications is to maximize the carried power "averaged" over one sinusoidal period *T*:

$$P_{avg}(z) = \frac{1}{T} \int_{T} p(z,t) dt \qquad (4.28)$$

Since $\frac{1}{T} \int_{T} \sin(2\omega t + \phi^+) dt = 0$, the pure standing wave on a short-circuited line does not carry (but store) time-average power. In time-harmonic cases, time-average power can be calculated from the voltage and current phasors more conveniently (prove it!):

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$$P_{avg}(z) = \frac{1}{2} \operatorname{Re} \left\{ V(z) \cdot I^{*}(z) \right\}$$
(4.29)

For arbitrary complex load (Z_L , $\Gamma_L \in C$), substituting eq's (4.7), (4.9), (4.11) into eq. (4.28),

$$P_{avg}(z) = \frac{1}{2} \operatorname{Re} \left\{ V^{+} \left(e^{-j\beta z} + \Gamma_{L} e^{j\beta z} \right) \cdot \left(\frac{V^{+}}{Z_{0}} \right)^{*} \left(e^{-j\beta z} - \Gamma_{L} e^{j\beta z} \right)^{*} \right\}$$
$$= \frac{\left| V^{+} \right|^{2}}{2Z_{0}} \operatorname{Re} \left\{ \left(1 - \left| \Gamma_{L} \right|^{2} \right) + 2j \left| \Gamma_{L} \right| \sin(\beta z + \psi) \right\}, \Longrightarrow$$
$$P_{avg}(z) = \frac{\left| V^{+} \right|^{2}}{2Z_{0}} \left(1 - \left| \Gamma_{L} \right|^{2} \right)$$
(4.30)

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- 1) Eq. (4.30) is only valid for lossless lines ($Z_0 \in R$). In this case, the time-average power is independent of *z*, and the power delivered to the load is: $P_L \equiv P_{avg}(0) = P_{avg}(z)$.
- 2) The time-average power carried by the forward and backward traveling waves are:

$$P^{+} = \frac{1}{2} \operatorname{Re} \left\{ V^{+}(z) \cdot \left[I^{+}(z) \right]^{*} \right\} = \frac{1}{2} \operatorname{Re} \left\{ V^{+} e^{-j\beta z} \left(\frac{V^{+} e^{-j\beta z}}{Z_{0}} \right)^{*} \right]$$
$$= \frac{1}{2} \operatorname{Re} \left\{ V^{+} e^{-j\beta z} \left(\frac{V^{+} e^{-j\beta z}}{Z_{0}} \right)^{*} \right\} = \frac{\left| V^{+} \right|^{2}}{2Z_{0}},$$
$$P^{-} = \frac{1}{2} \operatorname{Re} \left\{ V^{-}(z) \cdot \left[I^{-}(z) \right]^{*} \right\} = -\left| \Gamma_{L} \right|^{2} P^{+}.$$

Therefore, the total carried power is:

$$P_{avg}(z) = P^{+} + P^{-} = \frac{\left|V^{+}\right|^{2}}{2Z_{0}} \left(1 - \left|\Gamma_{L}\right|^{2}\right),$$

consistent with eq. (4.30).

3) The load will receive a maximum power of $P_L = P^+ = \frac{|V^+|^2}{2Z_0}$ if the load is matched to the

line: $Z_L = Z_0$, $\Gamma_L = 0$.

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Example 4-4: Consider a system shown in Fig. 4-1 where $Z_0 = 50 \ \Omega$, $v_p = 2 \times 10^8 \text{ m/s}$, l = 17 m, $Z_L = 100 - j60 \ \Omega$, $V_0 = 100 \text{ V}$, $\omega = 2\pi \times 125 \text{ MHz}$, $Z_s = 50 \ \Omega$. Find the time-average powers absorbed by the source impedance P_s and the load P_L , and supplied by the source P_{tot} , respectively.



Fig. 4-4. Equivalent circuit of Example 4-4.

Ans: $\lambda = \frac{v_p}{f} = \frac{2 \times 10^8 \text{ m/s}}{125 \text{ MHz}} = 1.6 \text{ m}, \implies \beta l = \frac{2\pi}{1.6 \text{ m}} \cdot (17 \text{ m}) = 21.25\pi, \tan \beta l = \tan \frac{\pi}{4} = 1.$ By eq. (4.15), the input impedance $Z_{in} = Z(-l) = 50 \frac{(100 - j60) + j50 \cdot 1}{50 + j(100 - j60) \cdot 1} = (22.6 - j25.1) \Omega$. The

equivalent circuit is shown in Fig. 4-4.

(1) By eq. (4.29),
$$P_{s} = \frac{1}{2} \operatorname{Re} \{ V_{1} \cdot I_{s}^{*} \} = \frac{1}{2} \operatorname{Re} \{ I_{s} Z_{s} \cdot I_{s}^{*} \}, \Rightarrow$$

 $P_{s} = \frac{1}{2} |I_{s}|^{2} \operatorname{Re} \{ Z_{s} \}$
(4.31)

 $I_{s} = \frac{V_{0}}{Z_{s} + Z_{in}} = \frac{100}{50 + (22.6 - j25.1)} = 1.30e^{j0.333} \quad A. \Rightarrow P_{s} = \frac{1}{2}(1.30)^{2}(50) = 42.3 \text{ W.}$ (2) Similarly, $P_{L} = \frac{1}{2} \operatorname{Re}\{V_{s} \cdot I_{s}^{*}\} = \frac{1}{2} \operatorname{Re}\{I_{s}Z_{in} \cdot I_{s}^{*}\}, \Rightarrow$

$$P_{L} = \frac{1}{2} |I_{S}|^{2} \operatorname{Re}\{Z_{in}\}$$
(4.32)

$$\Rightarrow P_{L} = \frac{1}{2} (1.30)^{2} (22.6) = 19.2 \text{ W.}$$
(3) $P_{tot} = \frac{1}{2} \text{Re} \{ V_{0} \cdot I_{s}^{*} \} = \frac{1}{2} \text{Re} \{ (V_{1} + V_{s}) \cdot I_{s}^{*} \} = P_{s} + P_{L} = 61.5 \text{ W.}$

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The power absorbed by the load can also be calculated by eq. (4.30), where we need to find V^+ first. However, this more complicated procedure can give the spatial distribution of the voltage phasor V(z).