Lesson 4
Steady-state Response of Transmission Lines

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Introduction

- Reasons to consider the response of TX lines to sinusoidal excitations:

  1. Power & communication signals are transmitted as (modified) sinusoids.
  2. Fourier analysis can deal with any excitation.

- Natural (transient) response will decay rapidly.
- Forced (steady-state) response is supported by the source, and will continue indefinitely.
- By using phasors, we can introduce complex reflection coefficient and line impedance to simplify the analysis.

Outline

- Phasor representations of TX lines equations and solutions
- Reflection at discontinuity:

  1. z-dependent reflection coefficient
  2. z-dependent line impedance
- Short-circuited lines
- TX lines with resistive load
- Power flow on TX lines

Sec. 4-1
Phasor Representations of the Equations & Solutions of TX Lines

1. Phasor representations of TX line equations
2. Phasor representation of voltage waves, phase velocity
3. Phasor representation of current waves, characteristic impedance

Phasor representations of TX line equations-1

$$v(z,t) = \mathrm{Re}\{\underline{V(z)} \cdot e^{j\omega t}\}, \quad i(z,t) = \mathrm{Re}\{\underline{I(z)} \cdot e^{j\omega t}\}$$

Phasor: complex function of z

$$\frac{\partial}{\partial z} v(z,t) = -L \frac{\partial}{\partial t} i(z,t)$$ (2.1)…1st-order PDE

$$\Rightarrow \frac{\partial}{\partial z} \mathrm{Re}\{V(z) \cdot e^{j\omega t}\} = -L \frac{\partial}{\partial t} \mathrm{Re}\{I(z) \cdot e^{j\omega t}\}$$

$$\Rightarrow \mathrm{Re}\left\{\underline{\left[\frac{d}{dz} V(z)\right]} e^{j\omega t}\right\} = -L \cdot \mathrm{Re}\left\{I(z)\left[\frac{\partial}{\partial t} e^{j\omega t}\right]\right\} = \underline{-L} \cdot \mathrm{Re}\{\underline{I(z) \cdot j\omega e^{j\omega t}}\}$$

$$\boxed{\frac{d}{dz} V(z) = -j\omega L \cdot I(z)}$$ (4.2)…1st-order ODE

Phasor representations of TX line equations-2

PDE's		ODE's
$\frac{\partial}{\partial z} v(z,t) = -L \frac{\partial}{\partial t} i(z,t)$ (2.1)	→	$\frac{d}{dz} V(z) = -j\omega L \cdot I(z)$ (4.2)
$\frac{\partial}{\partial z} i(z,t) = -C \frac{\partial}{\partial t} v(z,t)$ (2.2)	→	$\frac{d}{dz} I(z) = -j\omega C \cdot V(z)$ (4.3)
$\frac{\partial^2}{\partial z^2} v(z,t) = LC \frac{\partial^2}{\partial t^2} v(z,t)$ (2.3)	→	$\frac{d^2}{dz^2} V(z) = -\beta^2 V(z)$ (4.4)
		$\beta = \omega\sqrt{LC}$ (4.6)
$\frac{\partial^2}{\partial z^2} i(z,t) = LC \frac{\partial^2}{\partial t^2} i(z,t)$ (2.4)	→	$\frac{d^2}{dz^2} I(z) = -\beta^2 I(z)$ (4.5)

Phasor representation of voltage waves-1

$$\frac{d^2}{dz^2}V(z) = -\beta^2 V(z) \longrightarrow \boxed{V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}}$$

$V^+ = |V^+| e^{j\phi^+}$, $V^- = |V^-| e^{j\phi^-}$

To be determined by BC's.

$$v(z,t) = \mathrm{Re}\left\{V(z)\cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{\left(V^+ e^{-j\beta z} + V^- e^{j\beta z}\right)e^{j\omega t}\right\}$$

$$= \mathrm{Re}\left\{|V^+| e^{j\phi^+} e^{-j\beta z} e^{j\omega t} + |V^-| e^{j\phi^-} e^{j\beta z} e^{j\omega t}\right\}$$

$$= |V^+| \cos\left(\omega t - \beta z + \phi^+\right) + |V^-| \cos\left(\omega t + \beta z + \phi^-\right)$$

Phasor representation of voltage waves-2

$$v(z,t) = \underbrace{|V^+|\cos\left[\omega\left(t - \frac{z}{\omega/\beta}\right) + \phi^+\right]}_{f^+(t - z/v_p)} + \underbrace{|V^-|\cos\left[\omega\left(t + \frac{z}{\omega/\beta}\right) + \phi^-\right]}_{f^-(t + z/v_p)}$$

Phase velocity of the TX line:

$$v_p = \frac{\omega}{\beta} \xrightarrow{\beta = \omega\sqrt{LC}} v_p = \frac{1}{\sqrt{LC}} \quad (2.5)$$

Phasor representation of current waves-1

To solve the current phasor $I(z)$:

$$\begin{cases} V(z) = V^+(z) + V^-(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} \\ \dfrac{d}{dz} V(z) = -j\omega L \cdot I(z) \quad \text{(4.2)} \end{cases}$$

$$\Rightarrow -j\beta V^+ e^{-j\beta z} + j\beta V^- e^{j\beta z} = -j\omega L \cdot I(z)$$

$$\Rightarrow I(z) = \frac{\beta}{\omega L} V^+ e^{-j\beta z} - \frac{\beta}{\omega L} V^- e^{j\beta z}$$

$$I(z) = I^+(z) + I^-(z) = \frac{V^+(z)}{Z_0} - \frac{V^-(z)}{Z_0} \quad \Rightarrow Z_0 = \frac{\omega L}{\beta}$$

Phasor representation of voltage waves-2

Characteristic impedance of the TX line:

$$Z_0 = \frac{\omega L}{\beta} \xrightarrow{\beta = \omega\sqrt{LC}} Z_0 = \sqrt{\frac{L}{C}} \qquad (2.8)$$

$$I(z) = I^+(z) + I^-(z) = \frac{V^+(z)}{Z_0} - \frac{V^-(z)}{Z_0}$$

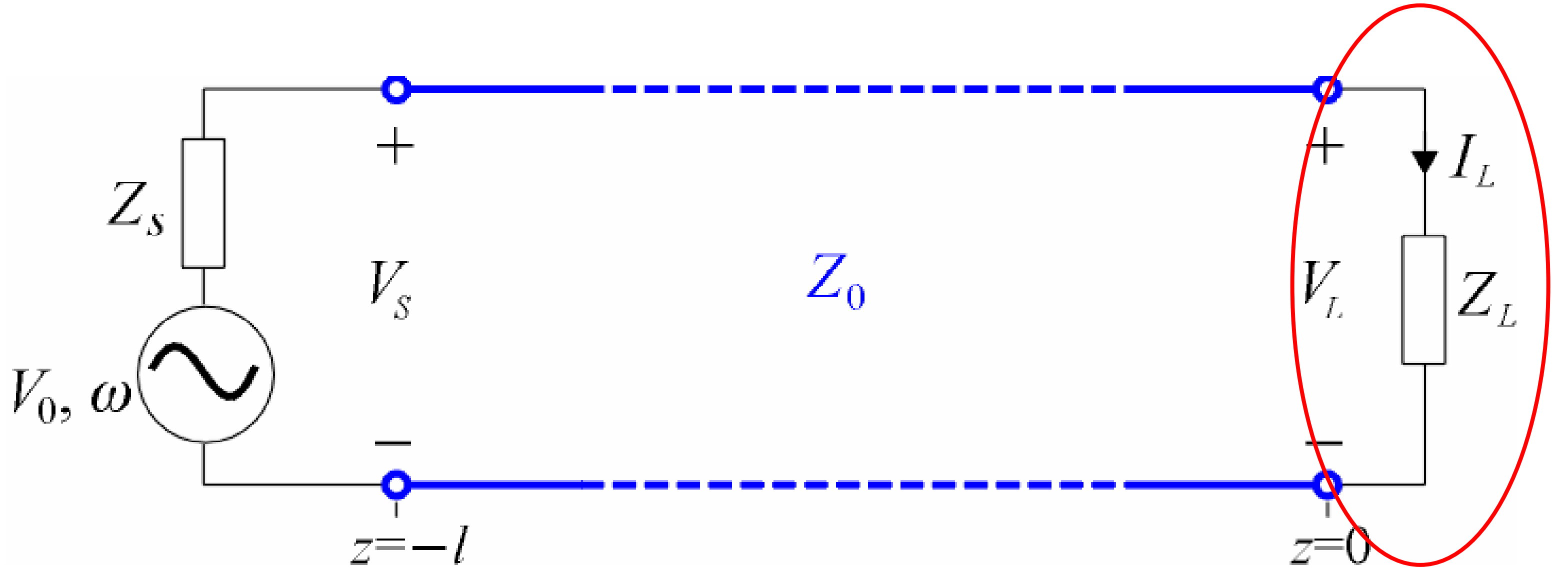
$$\boxed{Z_0 = \frac{V^+(z)}{I^+(z)} = -\frac{V^-(z)}{I^-(z)}} \longleftrightarrow Z_0 = \frac{v^+(z,t)}{i^+(z,t)} = -\frac{v^-(z,t)}{i^-(z,t)}$$

V-*I* ratio of a single wave.

Sec. 4-2
Reflection at Discontinuity

1. z-dependent reflection coefficient
2. z-dependent line impedance

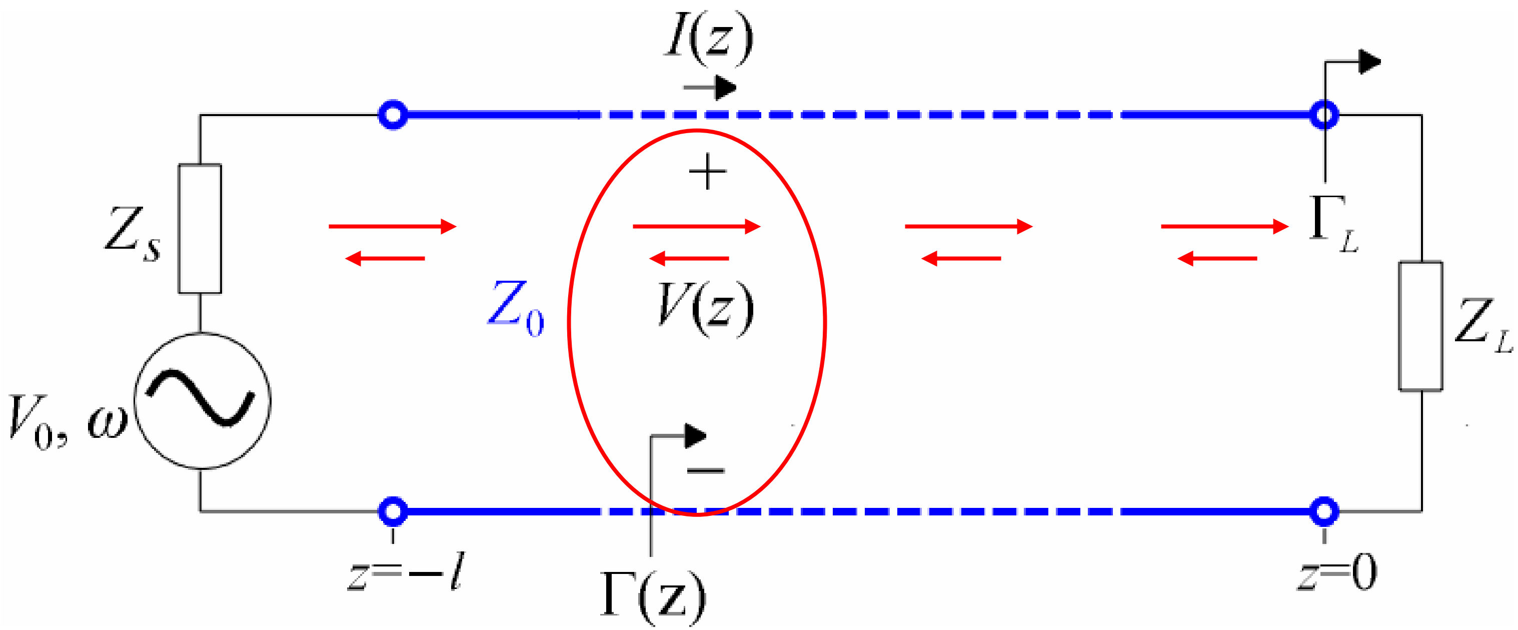
Why does reflection occur?



$\{V^-(z), I^-(z)\}$ must exist, such that the BC

$$\left.\frac{V^+(z)+V^-(z)}{I^+(z)+I^-(z)}\right|_{z=0} = Z_L (\neq Z_0)$$ can be satisfied.

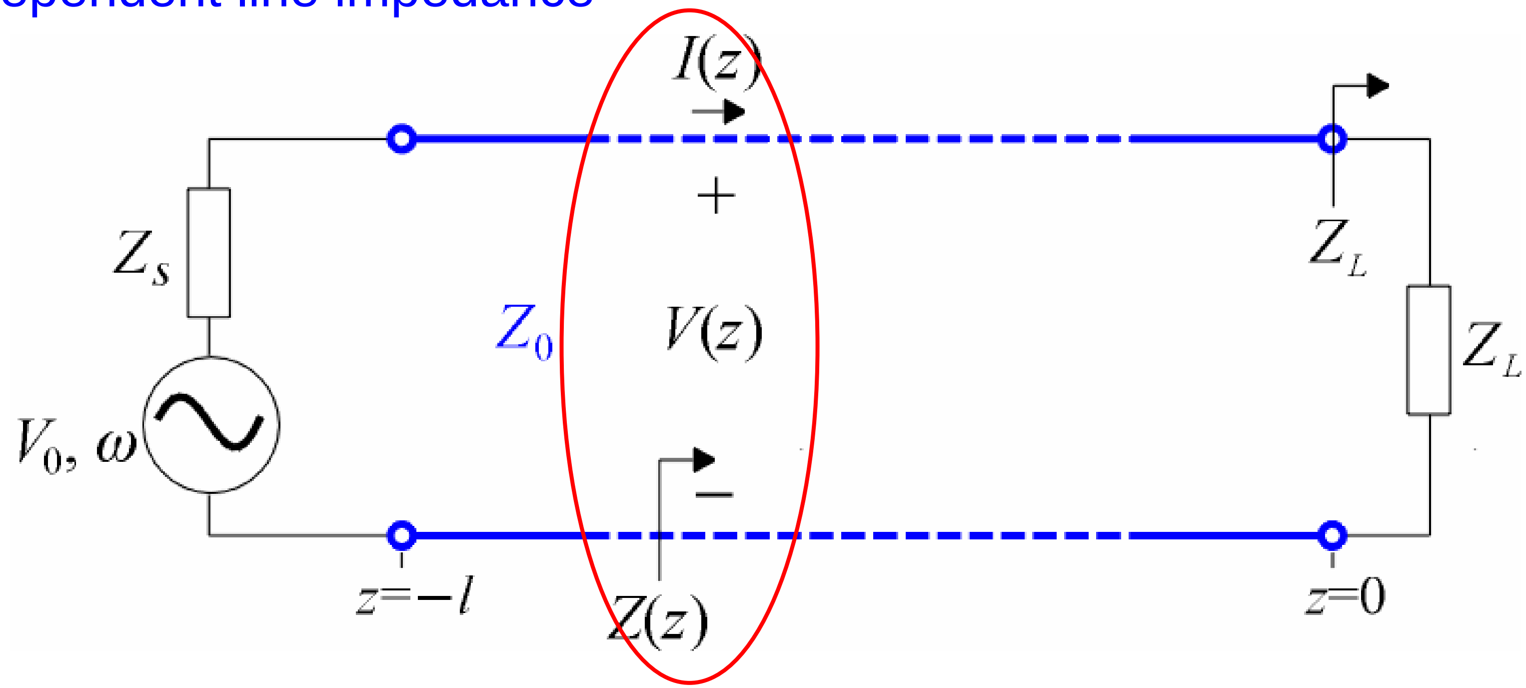
z-dependent reflection coefficient



Reflection coefficient of a loaded line:

$$\Gamma_L \equiv \frac{V^-}{V^+}, \quad \longrightarrow \quad \Gamma(\underline{z}) \equiv \frac{V^-(z)}{V^+(z)} = \frac{V^- e^{j\beta z}}{V^+ e^{-j\beta z}} = \frac{V^-}{V^+} e^{j2\beta z} = \Gamma_L e^{j2\beta z},$$

z-dependent line impedance



Line impedance of a loaded line:

$$Z(z) \equiv \frac{V(z)}{I(z)} = \frac{V^{+}(z)+V^{-}(z)}{I^{+}(z)+I^{-}(z)} = \frac{V^{+}e^{-j\beta z}+V^{-}e^{j\beta z}}{\left(V^{+}/Z_0\right)e^{-j\beta z}+\left(-V^{-}/Z_0\right)e^{j\beta z}}$$

$$= Z_0 \frac{V^{+}e^{-j\beta z}+V^{-}e^{j\beta z}}{V^{+}e^{-j\beta z}-V^{-}e^{j\beta z}}, \quad Z_L = Z(0) = Z_0 \frac{V^{+}+V^{-}}{V^{+}-V^{-}}$$

z-dependent parameters represented by constant impedances-1

Represent $\Gamma(z)$ by: $\{Z_0, Z_L\}$ $\left(\text{instead of } \{V^+, V^-\}\right)$

$$Z_L = Z_0 \frac{V^+ + V^-}{V^+ - V^-} = Z_0 \frac{1 + \left(V^- / V^+\right)}{1 - \left(V^- / V^+\right)} = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \longrightarrow \boxed{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

$$\left(\Gamma_L \equiv |\Gamma_L| e^{i\psi}\right)$$

$$\Gamma(z) = \frac{V^-}{V^+} e^{j2\beta z} = \Gamma_L e^{j2\beta z}, \longrightarrow \boxed{\Gamma(z) = \frac{Z_L - Z_0}{Z_L + Z_0} e^{j2\beta z}}$$

Periodic function of period $\lambda/2$

z-dependent parameters represented by constant impedances-2

Represent $Z(z)$ by: $\{Z_0, Z_L\}$ $\left(\text{instead of } \{V^+, V^-\}\right)$

$$Z(z) = Z_0 \frac{V^+ e^{-j\beta z} + V^- e^{j\beta z}}{V^+ e^{-j\beta z} - V^- e^{j\beta z}} = Z_0 \frac{e^{-j\beta z} + \left(V^-/V^+\right) e^{j\beta z}}{e^{-j\beta z} - \left(V^-/V^+\right) e^{j\beta z}}$$

$$= Z_0 \frac{e^{-j\beta z} + \Gamma_L e^{j\beta z}}{e^{-j\beta z} - \Gamma_L e^{j\beta z}} = Z_0 \frac{(\cos\beta z - j\sin\beta z) + \Gamma_L(\cos\beta z + j\sin\beta z)}{(\cos\beta z - j\sin\beta z) - \Gamma_L(\cos\beta z + j\sin\beta z)}$$

$$= Z_0 \frac{(1 - j\tan\beta z) + \Gamma_L(1 + j\tan\beta z)}{(1 - j\tan\beta z) - \Gamma_L(1 + j\tan\beta z)} = Z_0 \frac{(1+\Gamma_L) - j(1-\Gamma_L)\tan\beta z}{(1-\Gamma_L) - j(1+\Gamma_L)\tan\beta z}$$

by $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ $\longrightarrow$

$$\boxed{Z(z) = Z_0 \frac{Z_L - jZ_0\tan(\beta z)}{Z_0 - jZ_L\tan(\beta z)}}$$

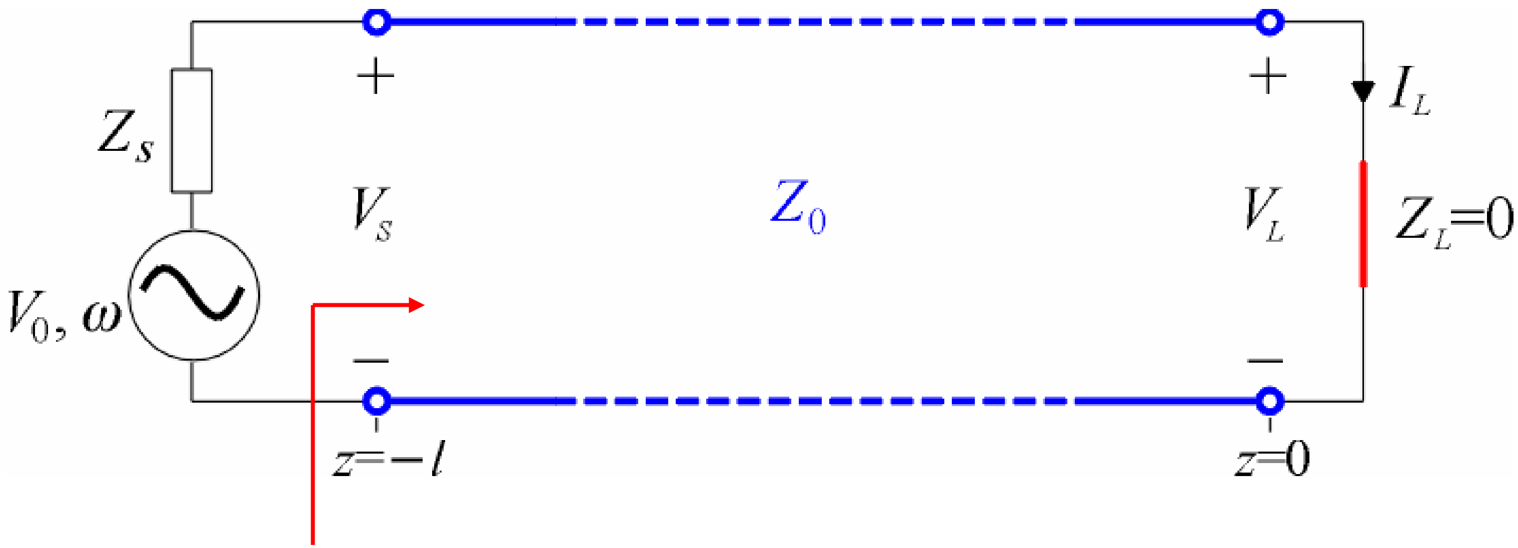
period $\lambda/2$

Sec. 4-3
Selected Examples

1. Short-circuited load, standing wave
2. Resistive load, partial standing wave

Short-circuited line-1

$$\{v(z,t), i(z,t)\} = ?$$



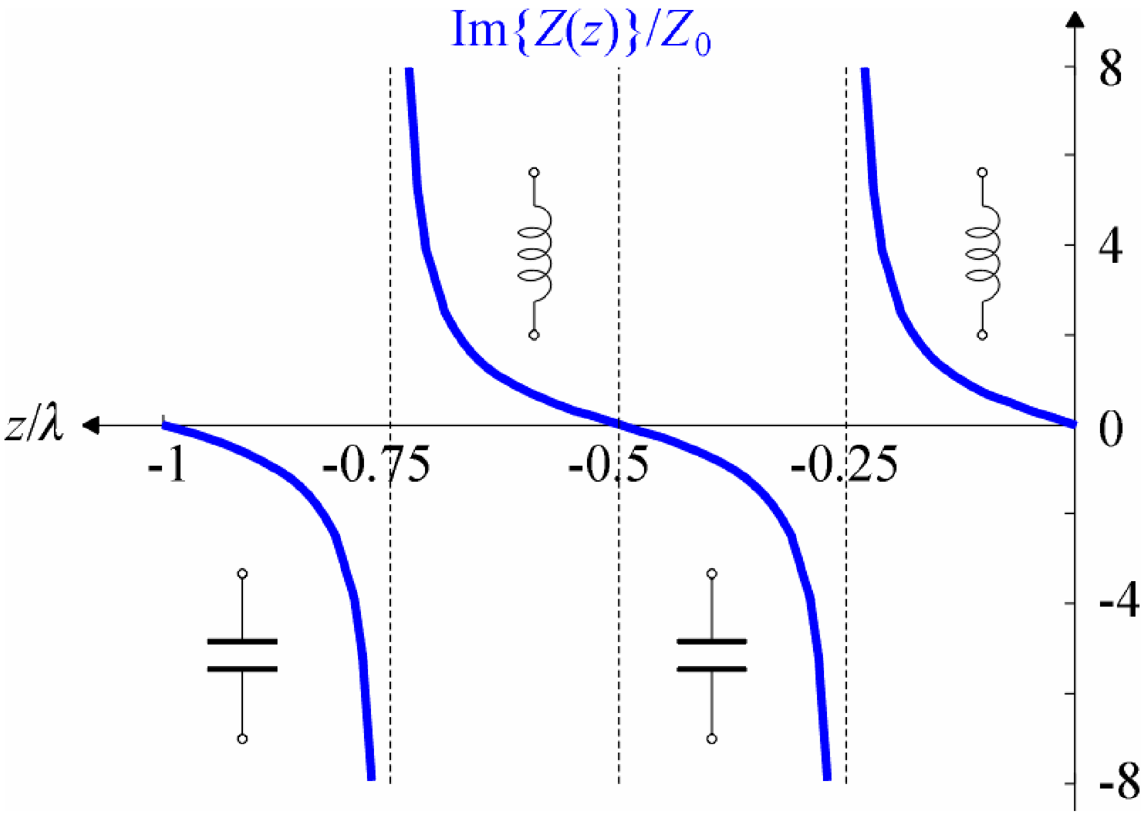
$$Z_{sc} = Z(-l) = ?$$

Short-circuited line-2

$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan(\beta z)}{Z_0 - jZ_L \tan(\beta z)}\bigg|_{Z_L=0} = -jZ_0 \tan(\beta z)$$

$$\Rightarrow Z_{sc} = Z(-l) = jZ_0 \tan(\beta l) = jX_{sc}$$ …Purely reactive

$\left(\text{if } Z_0 \in R\right)$



Short-circuited line-3

$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L=0} = -1, \qquad \Gamma_L = -1 \qquad V^+ \equiv |V^+| e^{j\phi^+}$$

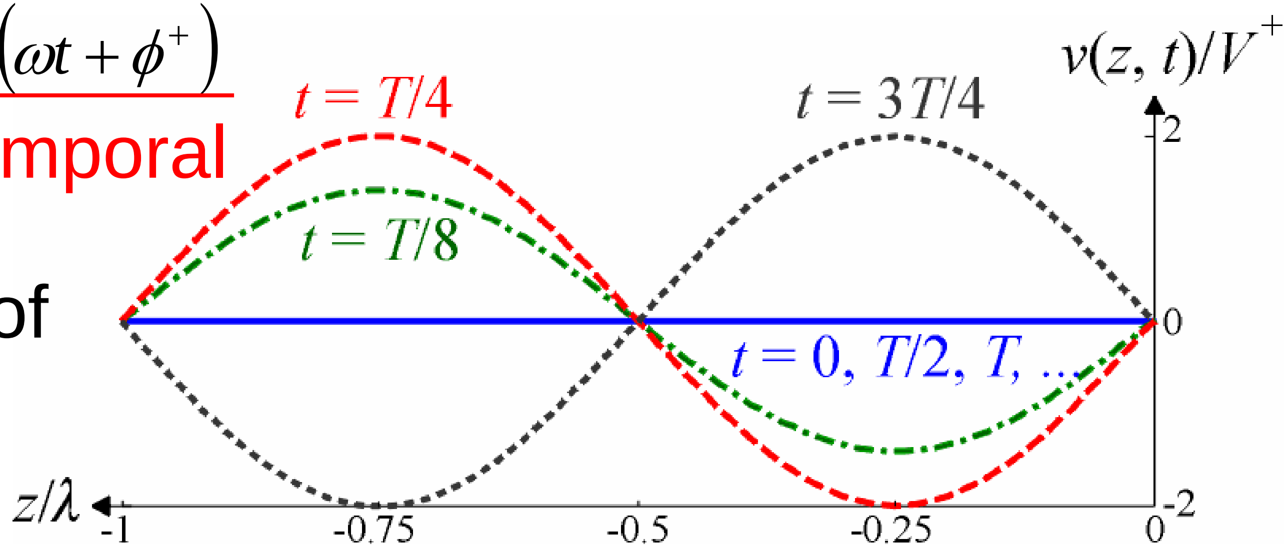
$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} = V^+\left(e^{-j\beta z} + \frac{V^-}{V^+} e^{j\beta z}\right) = -2jV^+ \sin(\beta z)$$

$$v(z,t) = \mathrm{Re}\{V(z)\cdot e^{j\omega t}\} = \mathrm{Re}\{-2j|V^+|e^{j\phi^+}\sin(\beta z)e^{j\omega t}\}$$

$$= 2|V^+|\sin(\beta z)\cdot\sin(\omega t + \phi^+)$$

spatial temporal

Not a function of $t \pm z/v_p$!



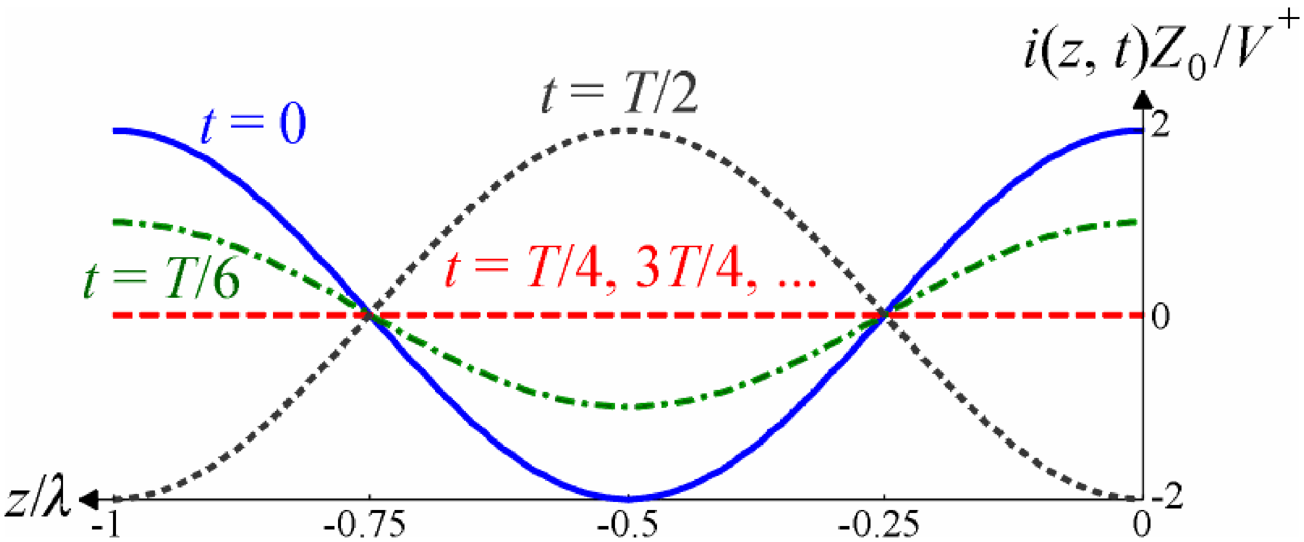
Short-circuited line-4

$$I(z)=\frac{V^{+}(z)}{Z_0}-\frac{V^{-}(z)}{Z_0}=\frac{V^{+}}{Z_0}\left(e^{-j\beta z}-\Gamma_L e^{j\beta z}\right)=\frac{2V^{+}}{Z_0}\cos(\beta z)$$

($\Gamma_L = -1$)

$$i(z,t)=\mathrm{Re}\left\{I(z)\cdot e^{j\omega t}\right\}=\frac{2\left|V^{+}\right|}{Z_0}\underbrace{\cos(\beta z)}_{\text{spatial}}\cdot\underbrace{\cos\left(\omega t+\phi^{+}\right)}_{\text{temporal}}$$

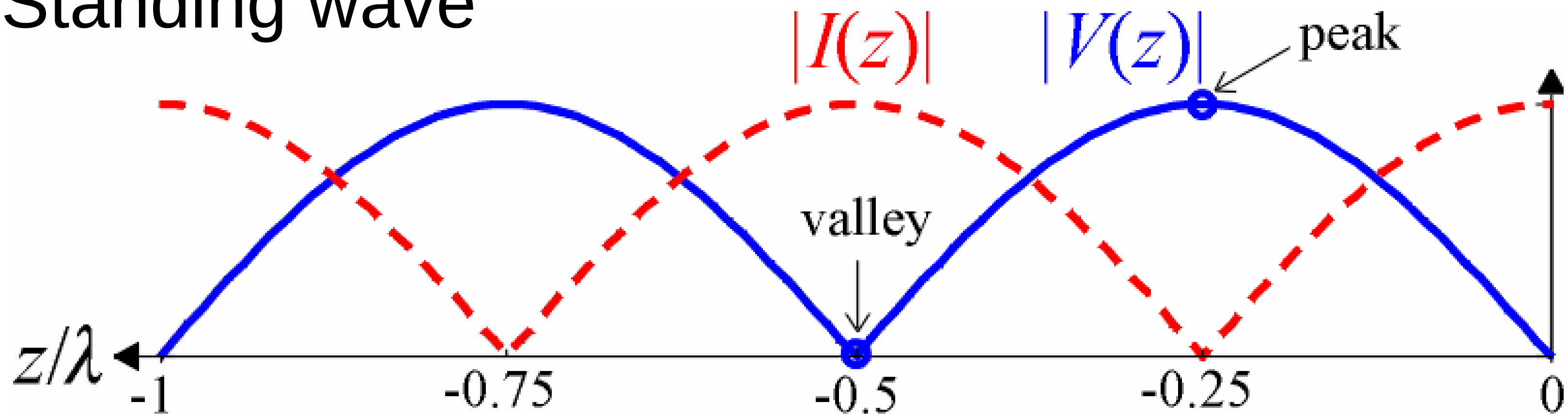
spatial temporal



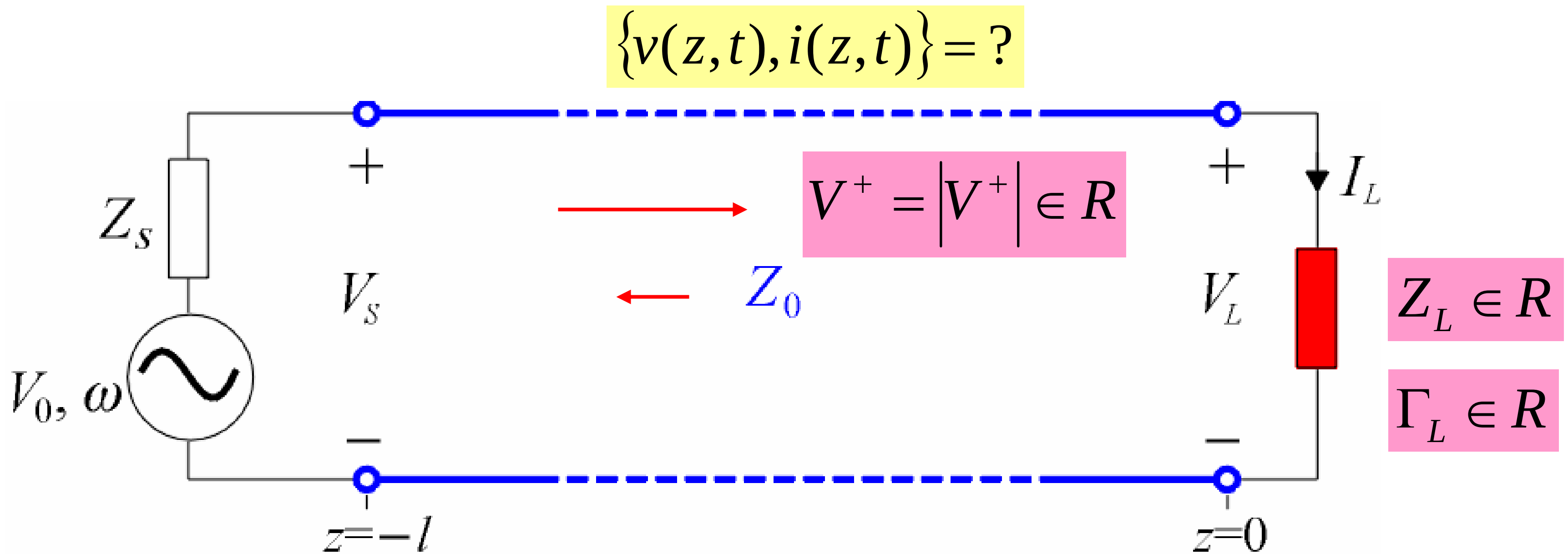
The amplitude of temporal oscillation ~ the spatial distribution of the phasor magnitude:

$$|V(z)| = 2|V^+| \cdot |\sin(\beta z)|, \qquad |I(z)| = \frac{2|V^+|}{Z_0} \cdot |\cos(\beta z)|$$

Standing wave



TX lines with resistive load-1



$$V(z) = V^+\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right) = V^+\left(e^{-j\beta z} + \Gamma_L e^{-j\beta z} - \Gamma_L e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)$$

$$= V^+\left[\left(1+\Gamma_L\right)e^{-j\beta z} + 2j\Gamma_L \sin\left(\beta z\right)\right]$$

TX lines with resistive load-2

$$|V^+|$$

$$v(z,t)=\mathrm{Re}\left\{V(z)\cdot e^{j\omega t}\right\}=\mathrm{Re}\left\{V^+\left[(1+\Gamma_L)e^{-j\beta z}+2j\Gamma_L\sin(\beta z)\right]e^{j\omega t}\right\}$$

$$=|V^+|(1+\Gamma_L)\mathrm{Re}\left\{e^{-j\beta z}e^{j\omega t}\right\}+2|V^+|\Gamma_L\sin(\beta z)\mathrm{Re}\left\{je^{j\omega t}\right\}$$

$$=\underbrace{|V^+|(1+\Gamma_L)\underline{\cos(\omega t-\beta z)}}-\underbrace{2|V^+|\Gamma_L\underline{\sin(\beta z)}\,\underline{\sin(\omega t)}}$$

Traveling wave

t and z are coupled

Standing wave

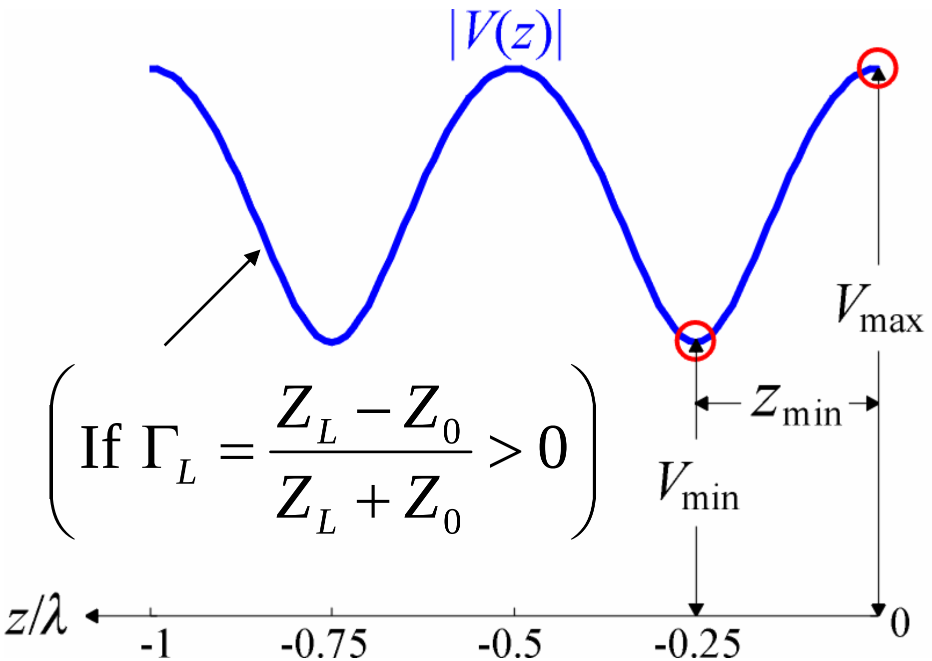
t and z are decoupled

The amplitude of temporal oscillation ~ the spatial distribution of the phasor magnitude:

$$V(z) = V^{+}\left[(1+\Gamma_L)e^{-j\beta z} + 2j\Gamma_L \sin(\beta z)\right] = V^{+}\left[(1+\Gamma_L)\cos(\beta z) - j(1-\Gamma_L)\sin(\beta z)\right]$$

$$\Rightarrow |V(z)| = |V^{+}|\sqrt{(1+\Gamma_L)^2\cos^2(\beta z) + (1-\Gamma_L)^2\sin^2(\beta z)}$$

…Period $\lambda/2$



$$\left(\text{If } \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} > 0\right)$$

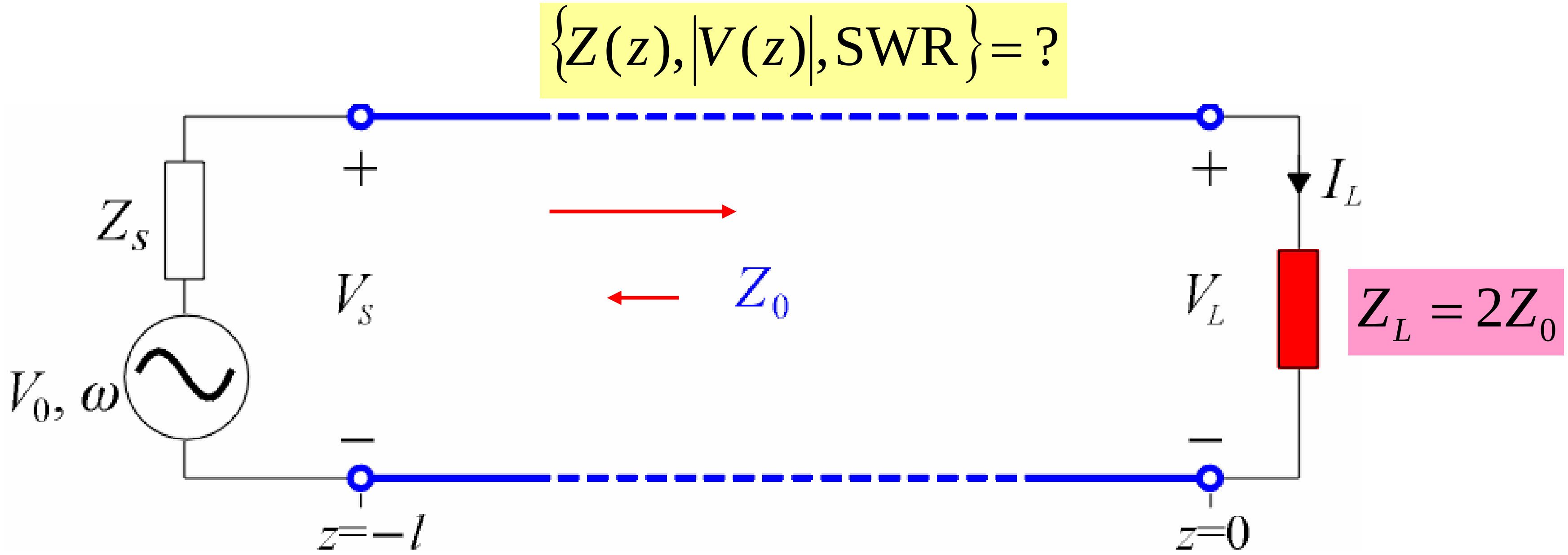
$$V_{\max} = |V^{+}|(1+|\Gamma_L|) \qquad \left(\Gamma_L \equiv |\Gamma_L| e^{i\psi}\right)$$

$$V_{\min} = |V^{+}|(1-|\Gamma_L|)$$

$$\boxed{S \equiv \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}}$$

Degree of impedance mismatch

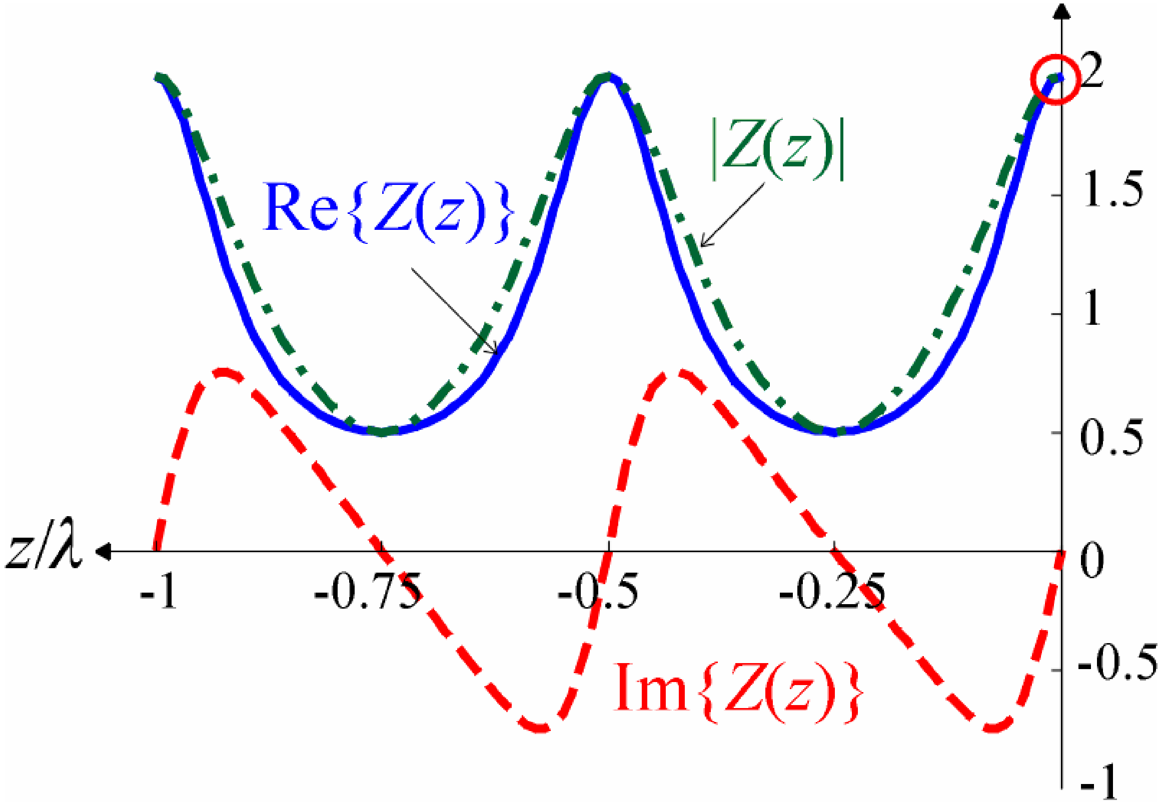
Example 4-3: TX line with a high resistive load-1



Example 4-3: TX line with a high resistive load-2

Line impedance of a resistively loaded line:

$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan(\beta z)}{Z_0 - jZ_L \tan(\beta z)}\bigg|_{Z_L = 2Z_0} = Z_0 \frac{2 - j\tan(\beta z)}{1 - j2\tan(\beta z)} \in C$$



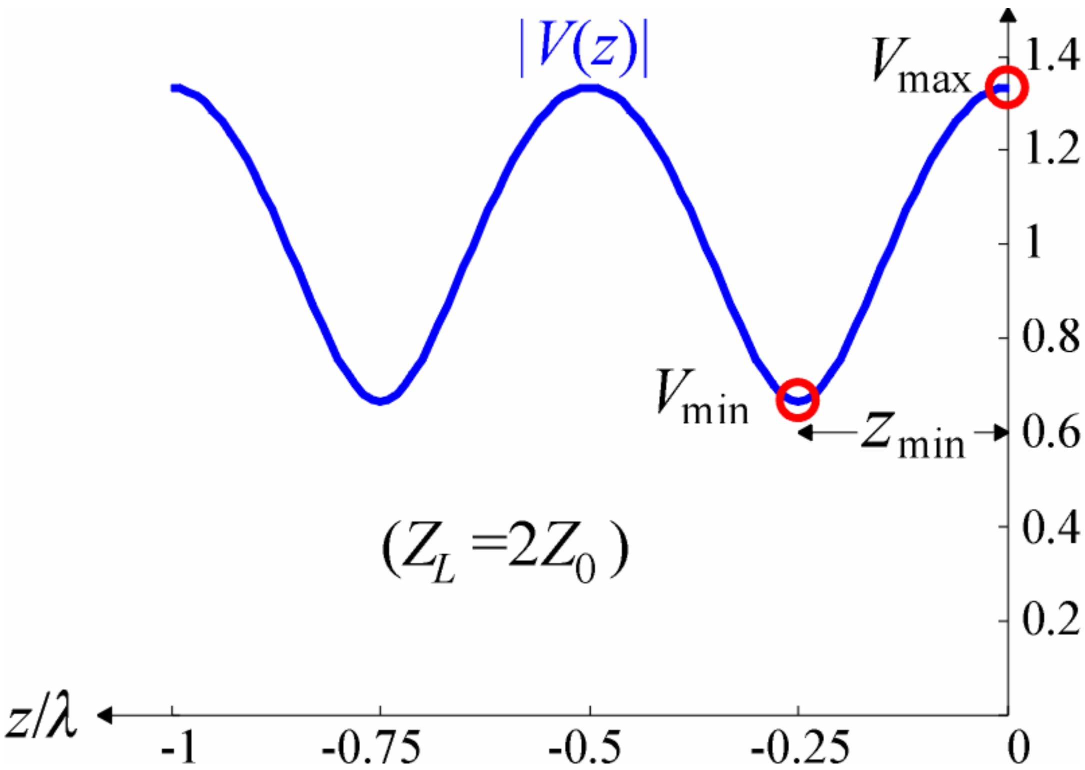
- $Z(z)$ has a period of $\lambda/2$.
- $\mathrm{Im}\{Z(z)\}$ can be positive or negative.
- $|Z(z)|$, $\mathrm{Re}\{Z(z)\}$, $\mathrm{Im}\{Z(z)\}$ vary within finite rages.

Example 4-3: TX line with a high resistive load-3

Amplitude of temporal oscillation:

$$|V(z)| = |V^+|\sqrt{(1+\Gamma_L)^2\cos^2(\beta z) + (1-\Gamma_L)^2\sin^2(\beta z)}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3}, \longrightarrow |V(z)| = \frac{2}{3}|V^+|\sqrt{4\cos^2(\beta z) + \sin^2(\beta z)}$$



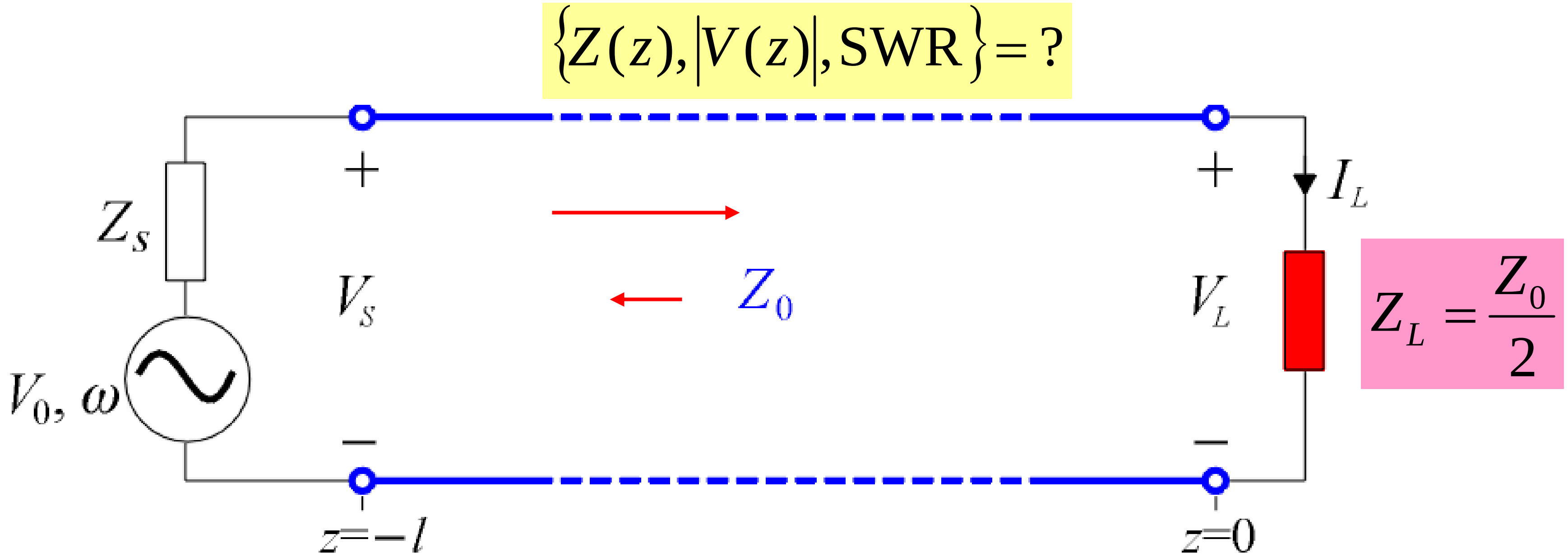
$$V_{max} = |V^+|(1+|\Gamma_L|) = \frac{4}{3}|V^+|$$

$$V_{min} = |V^+|(1-|\Gamma_L|) = \frac{2}{3}|V^+|$$

$$S \equiv \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 2$$

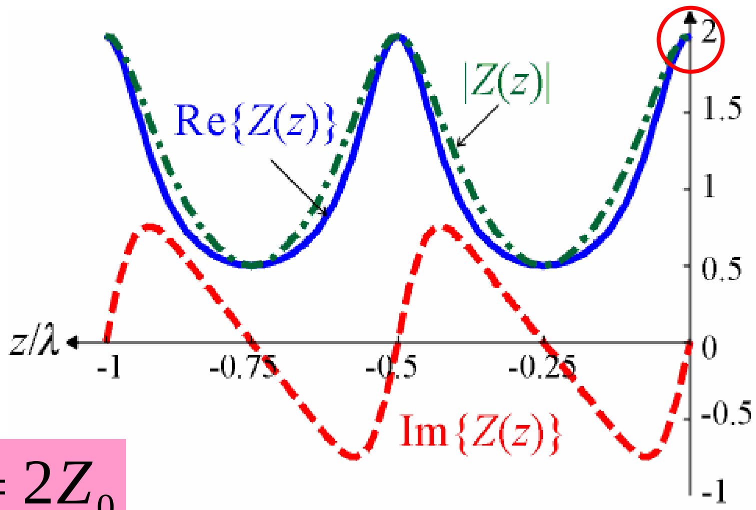
$$z_{min} = \frac{\lambda}{4}$$

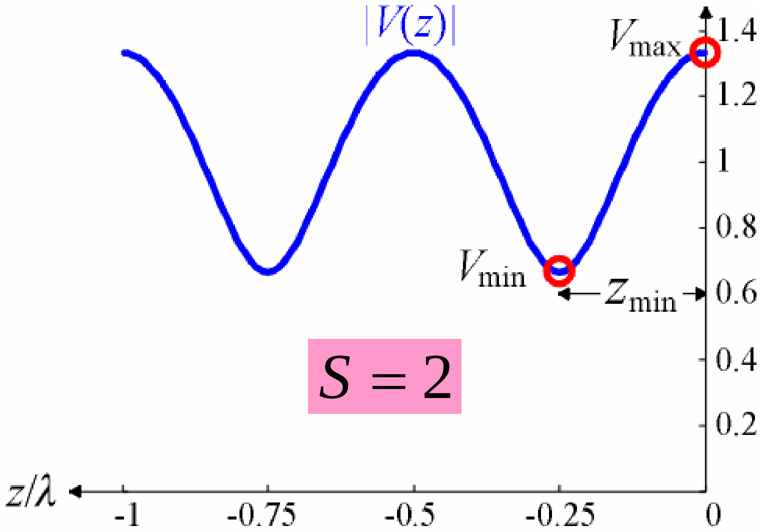
Example 4-3: TX line with a low resistive load



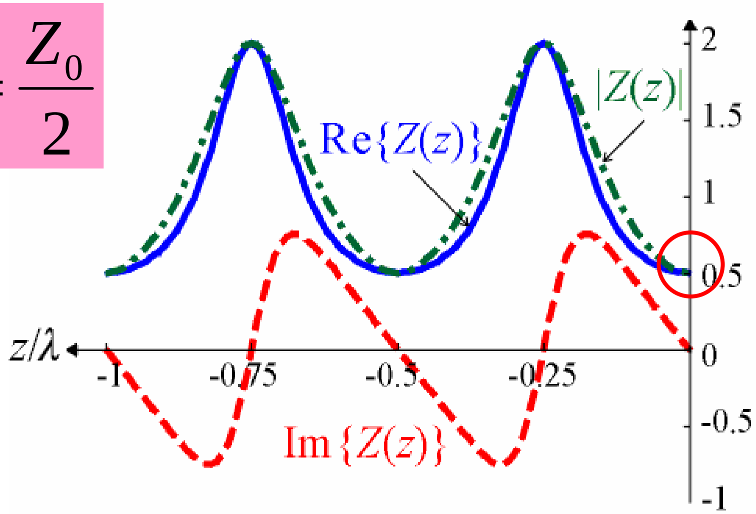
Comparison between TX lines with high & low resistive loads

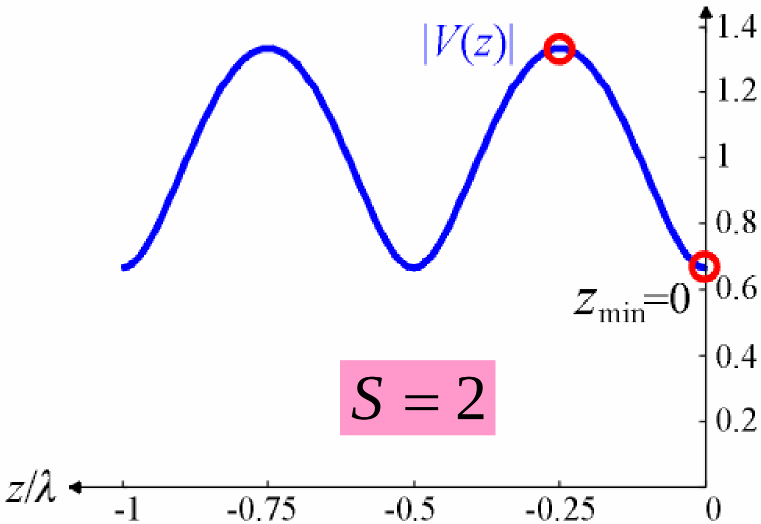
$$Z_L = 2Z_0$$



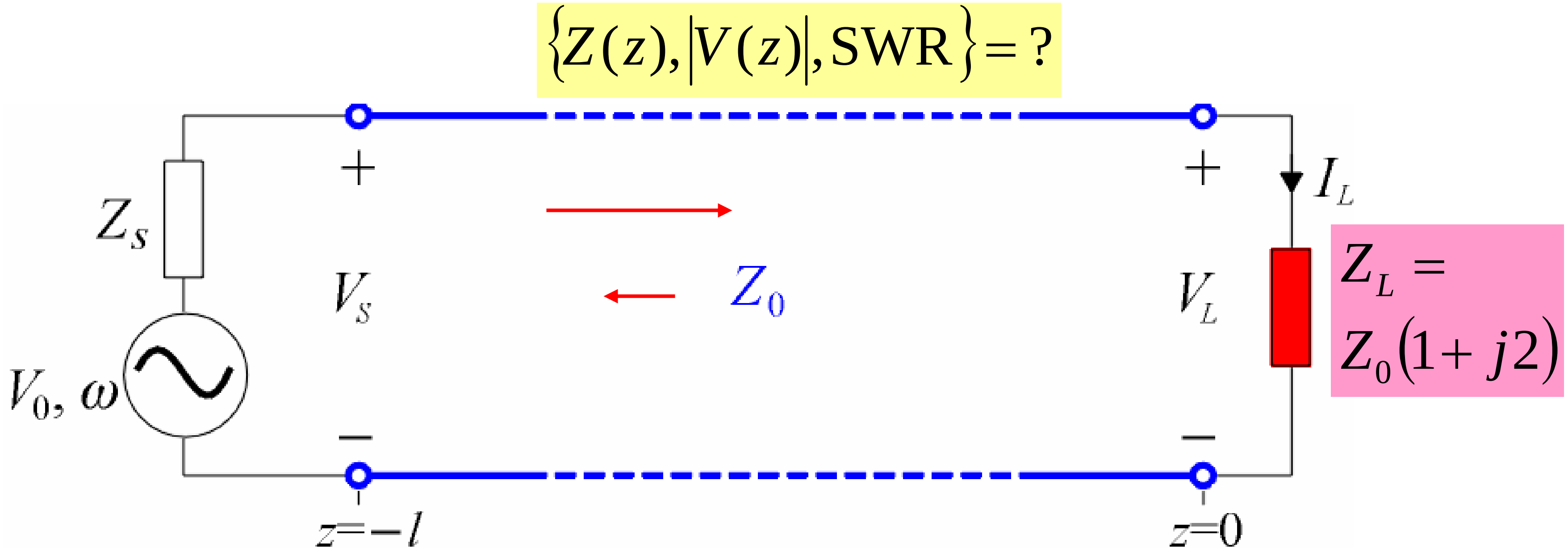


$$Z_L = \frac{Z_0}{2}$$

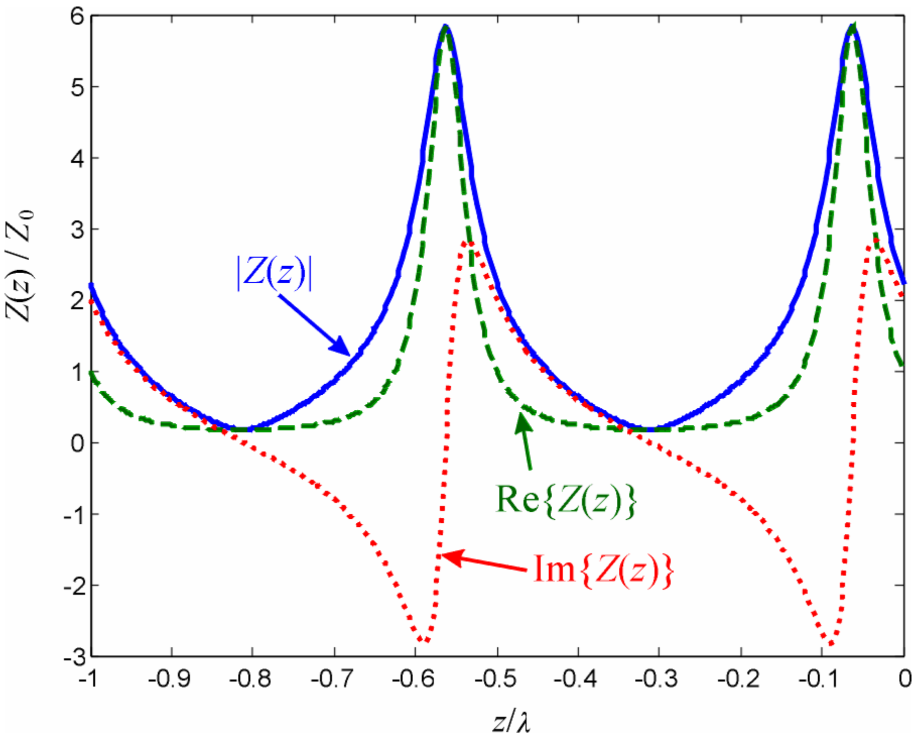


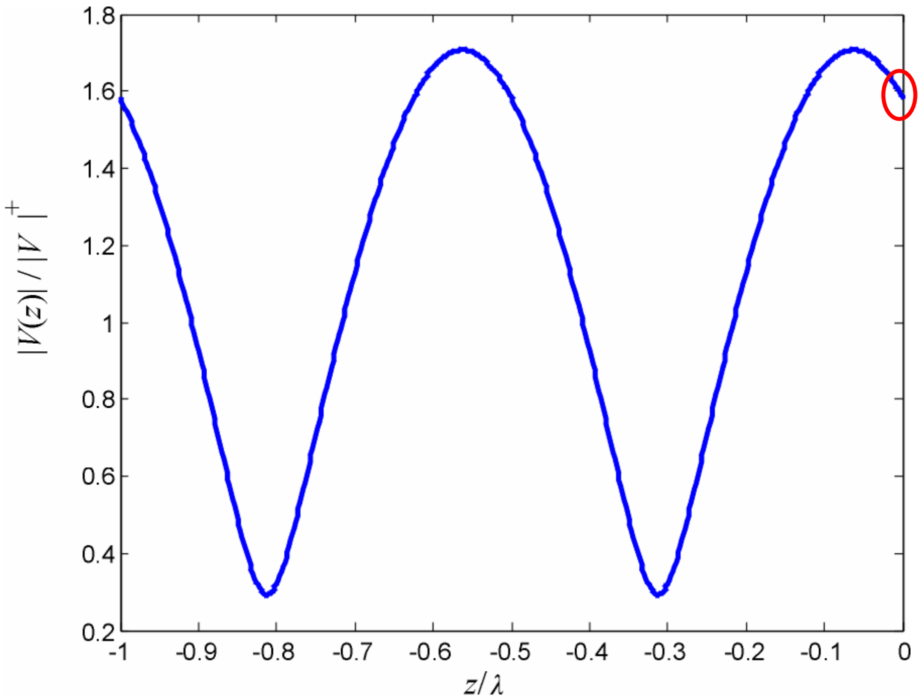


TX line with a complex load-1



TX line with a complex load-2





$$S = \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|} = \frac{Z_L - Z_0}{Z_L + Z_0} = 5.83$$

Sec. 4-4
Power Flow on TX Lines

Power flow on TX lines-1

Instantaneous power at position z:

$$p(z,t) = v(z,t) \cdot i(z,t) \longrightarrow \sim \cos(2\omega t + \phi^{+})$$

$$v(z,t) \sim \cos(\omega t + \phi_v) \qquad i(z,t) \sim \cos(\omega t + \phi_i)$$

Average power at position z:

$$P_{avg}(z) \equiv \frac{1}{T}\int_T p(z,t)dt = \boxed{\frac{1}{2}\mathrm{Re}\left\{V(z)\cdot I^*(z)\right\}}$$

$$V(z) = V^+e^{-j\beta z} + V^-e^{j\beta z} \qquad I^*(z) = \frac{1}{Z_0}\left(V^+e^{-j\beta z} - \Gamma_L V^+e^{-j\beta z}\right)^*$$

Power flow on TX lines-2

$$\Gamma_L \equiv |\Gamma_L| e^{i\psi}$$

$$P_{avg}(z) = \frac{1}{2}\mathrm{Re}\left\{ V^+\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right) \cdot \left(\frac{V^+}{Z_0}\right)^* \left(e^{-j\beta z} - \Gamma_L e^{j\beta z}\right)^* \right\}$$

$$= \frac{|V^+|^2}{2Z_0}\mathrm{Re}\left\{\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)\cdot\left(e^{j\beta z} - \Gamma_L^* e^{-j\beta z}\right)\right\} = \frac{|V^+|^2}{2Z_0}\mathrm{Re}\left\{\left(1 - |\Gamma_L|^2\right) + 2j|\Gamma_L|\sin(\beta z + \psi)\right\}$$

$$P_{avg}(z) = \frac{|V^+|^2}{2Z_0}\left(1 - |\Gamma_L|^2\right)$$

- Only valid for lossless lines $Z_0 \in R$
- Carried power is independent of z
- Max power occurs if load is matched: $Z_L = Z_0,\ \Gamma_L = 0$

Power flow on TX lines-3

Power carried by forward traveling wave:

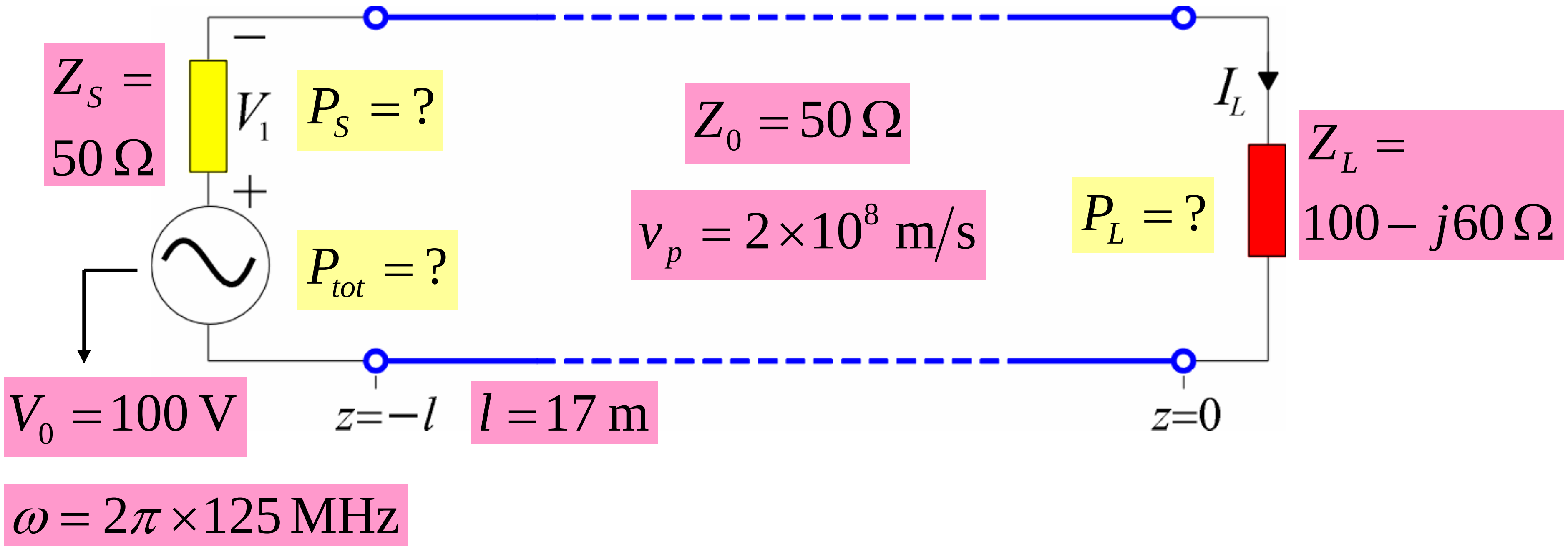
$$P^{+} = \frac{1}{2}\mathrm{Re}\left\{V^{+}(z)\cdot\left[I^{+}(z)\right]^{*}\right\} = \frac{1}{2}\mathrm{Re}\left\{V^{+}e^{-j\beta z}\left(\frac{V^{+}e^{-j\beta z}}{Z_0}\right)^{*}\right\} = \frac{\left|V^{+}\right|^{2}}{2Z_0}$$

Power carried by backward traveling wave:

$$P^{-} = \frac{1}{2}\mathrm{Re}\left\{V^{-}(z)\cdot\left[I^{-}(z)\right]^{*}\right\} = \frac{1}{2}\mathrm{Re}\left\{\Gamma_L V^{+}e^{j\beta z}\left(-\frac{\Gamma_L V^{+}e^{j\beta z}}{Z_0}\right)^{*}\right\} = -\left|\Gamma_L\right|^{2}P^{+}$$

$$\Rightarrow P_{tot} = P^{+} + P^{-} = \frac{\left|V^{+}\right|^{2}}{2Z_0}\left(1-\left|\Gamma_L\right|^{2}\right)$$

Example 4-4: Powers of a TX line with a resistive load-1



Example 4-4: Powers of a TX line with a resistive load-2

Find the equivalent circuit of the loaded TX line

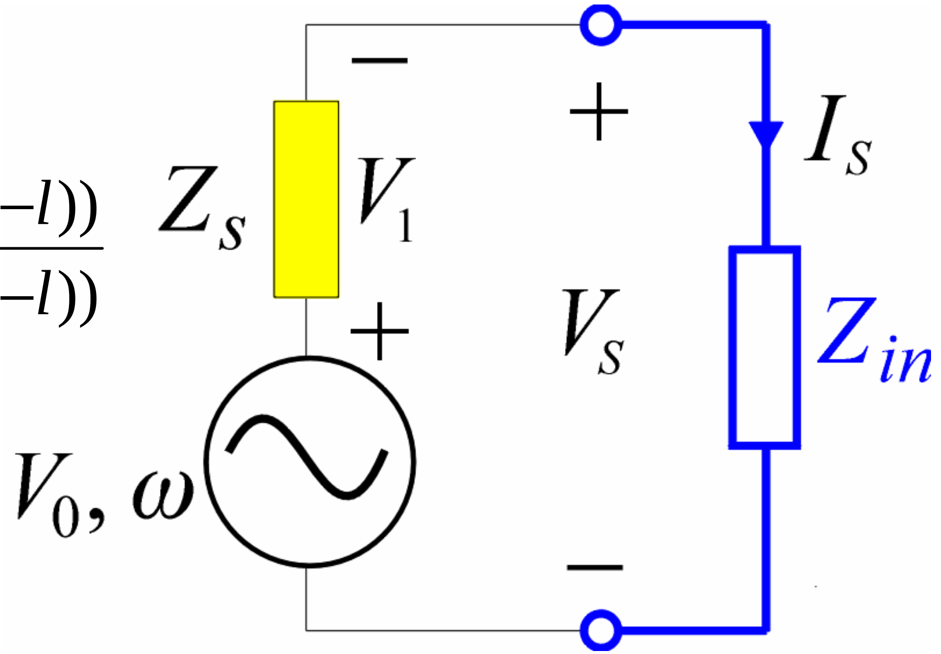
$$\lambda = \frac{v_p}{f} = \frac{2\times10^8 \text{ m/s}}{125 \text{ MHz}} = 1.6 \text{ m}, \Rightarrow \beta l = \frac{2\pi}{1.6 \text{ m}} \cdot (17 \text{ m}) = 21.25\pi,$$

$$\tan \beta l = \tan\frac{\pi}{4} = 1,$$

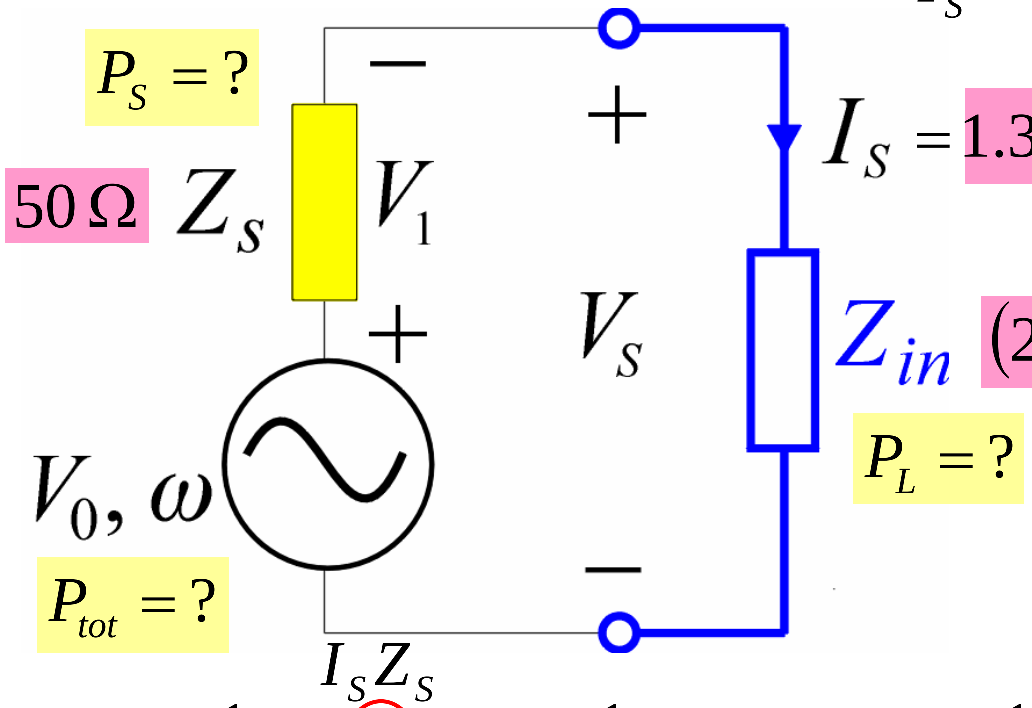
$$Z_{in} = Z(-l) = Z_0 \frac{Z_L - jZ_0 \tan(\beta(-l))}{Z_0 - jZ_L \tan(\beta(-l))}$$

$$= 50\frac{(100 - j60) + j50\cdot 1}{50 + j(100 - j60)\cdot 1}$$

$$= (22.6 - j25.1)\Omega$$



Example 4-4: Powers of a TX line with a resistive load-3



$$I_S = \frac{V_0}{Z_S + Z_{in}} = \frac{100}{50 + (22.6 - j25.1)}$$

$$I_S = 1.30e^{j0.333}\ \text{A}$$

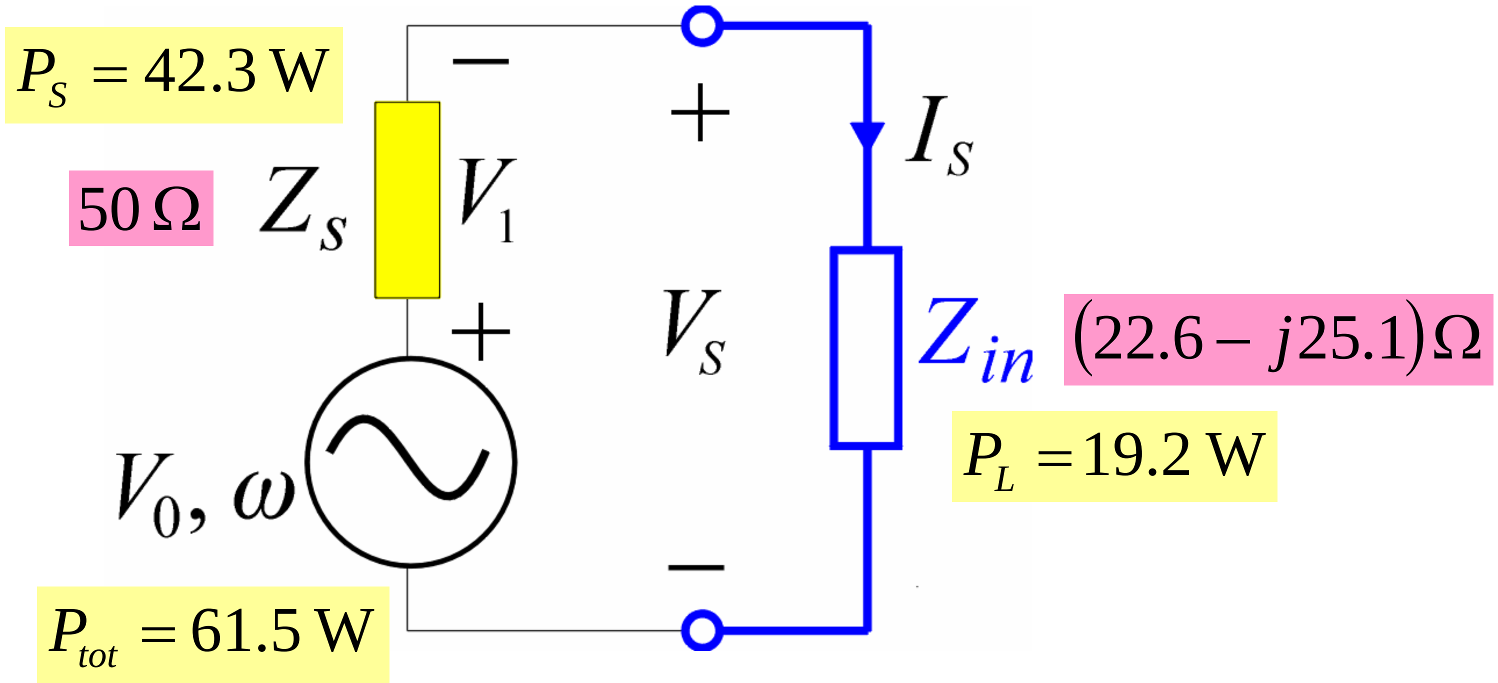
$$P_S = \frac{1}{2}\text{Re}\{V_1 \cdot I_S^*\} = \frac{1}{2}|I_S|^2\,\text{Re}\{Z_s\} = \frac{1}{2}(1.30)^2(50) = 42.3\ \text{W}$$

$$P_L = \frac{1}{2}\text{Re}\{V_S \cdot I_S^*\} = \frac{1}{2}|I_S|^2\,\text{Re}\{Z_{in}\} = \frac{1}{2}(1.30)^2(22.6) = 19.2\ \text{W}$$

$$P_{tot} = \frac{1}{2}\text{Re}\{V_0 \cdot I_S^*\} = \frac{1}{2}\text{Re}\{(V_1 + V_S)\cdot I_S^*\} = P_S + P_L = 61.5\ \text{W}$$

How to maximize the transmitted power P_L ?

When the load impedance Z_L is not matched to Z_0, a fraction of the source power would be consumed by the internal impedance Z_S.



Impedance match by a short-circuited stub

Add a shunt short-circuited line with proper length l at a proper distance d from Z_L, s.t. the equivalent impedance (admittance) at BB' happens to be Z_0 (R_0).