Lesson 03 Transient Response of Transmission Lines

■ Introduction

Partial reflection and transmission occur whenever a wave meets with a "discontinuity", i.e., an interface between different materials. Multiple discontinuities cause successive counter-propagating waves, and the superposition of all component waves will determine the exact signal along the line. In this lesson, we will quantitatively analyze transient response of a terminated transmission line or cascaded lines excited by a step-like voltage source, which is important in digital integrated electronics and computer communications.

■ Reflection at discontinuity

Example 3-1: A step voltage source of amplitude V_0 and internal resistance R_S drives a lossless transmission line of characteristic impedance Z_0, length *l*, phase velocity v_p (one-way signal traveling time $t_d = l/v_p$), and is terminated by a load resistance R_L (Fig. 3-1).

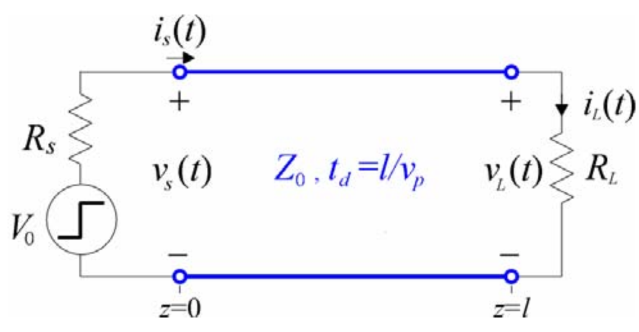

Fig. 3-1 Finite lossless transmission line driven by a step voltage source.

As in Example 2-3, a voltage signal starts to propagates in the $+z$ direction with velocity v_p at $t=0$, $\Rightarrow$ $v_1^+(z,t) = \begin{cases} V_1^+, \text{if } z < v_p t \\ 0, \text{otherwise} \end{cases}$ (valid for all $t>0$, but is only interested during

$z \in [0,l]$), where $V_1^+ = \dfrac{Z_0}{Z_0 + R_S} V_0$. When the disturbance $v_1^+(z,t)$ arrives at the load ($z = l$) at $t = t_d$, a reflected voltage wave $v_1^-(z,t) = \begin{cases} V_1^-, \text{if } z > l - v_p(t - t_d) \\ 0, \text{otherwise} \end{cases}$ (valid for all $t > t_d$) will be generated and propagate in the $-z$ direction. The total voltage at the load at $t = t_d^+$ is equal to their superposition:

$$v_L(t) = v_1^+(l,t) + v_1^-(l,t) \tag{3.1}$$

By eq. (2.9), the total current at $z = l$, $t = t_d^+$ is:

$$i_L(t) = \frac{v_1^+(l,t)}{Z_0} - \frac{v_1^-(l,t)}{Z_0} \tag{3.2}$$

Substituting eq's (3.1), (3.2) into the boundary condition $v_L(t) = i_L(t) \cdot R_L(t)$ (imposed by the load resistance) gives:

$$v_1^+(l,t) + v_1^-(l,t) = \frac{R_L}{Z_0}\left[v_1^+(l,t) - v_1^-(l,t)\right].$$

Dividing the above equation by $v_1^+(l,t)$ for both sides of equality, and defining the load voltage reflection coefficient Γ_L as the ratio of the reflected voltage $v_1^-(z,t)$ to the incident voltage $v_1^+(z,t)$, we arrive at:

$$1 + \Gamma_L = \frac{R_L}{Z_0}\left(1 - \Gamma_L\right), \Rightarrow$$

$$\Gamma_L \equiv \frac{v_1^-(l,t)}{v_1^+(l,t)} = \frac{R_L - Z_0}{R_L + Z_0} \tag{3.3}$$

The reflected voltage wave $v_1^-(z,t)$ will arrive at the source ($z = 0$) at $t = 2t_d$, generating a reflected voltage $v_2^+(z,t) = \begin{cases} V_2^+, \text{if } z < v_p(t - 2t_d) \\ 0, \text{otherwise} \end{cases}$ (valid for all $t > 2t_d$) propagating in the $+z$ direction. This process can be viewed as a voltage disturbance propagating on a line of characteristic impedance Z_0 and being incident on a resistance of R_S. By eq. (3.3), the source voltage reflection coefficient Γ_S is:

$$\Gamma_S \equiv \frac{v_2^+(l,t)}{v_1^-(l,t)} = \frac{R_S - Z_0}{R_S + Z_0} \tag{3.4}$$

Note that $v_1^+(z,t)$ is created at $t = 0$, and will continue to exist "forever". The total voltage and current at the source at $t = 2t_d^+$ are formulated as:

$$v_S(t) = v_1^+(0,t) + v_1^-(0,t) + v_2^+(0,t) = v_1^+(0,t)(1 + \Gamma_L + \Gamma_L\Gamma_S)$$

$$i_S(t) = \frac{v_1^+(0,t)}{Z_0} - \frac{v_1^-(0,t)}{Z_0} + \frac{v_2^+(0,t)}{Z_0} = \frac{v_1^+(0,t)}{Z_0}(1 - \Gamma_L + \Gamma_L\Gamma_S)$$

This process will continue indefinitely. The total voltage at the source will converge to:

$$\lim_{t\to\infty} v_S(t) = v_1^+(0, t \to \infty) \cdot \left(1 + \Gamma_L + \Gamma_L\Gamma_S + \Gamma_L^2\Gamma_S + \Gamma_L^2\Gamma_S^2 + \Gamma_L^3\Gamma_S^2 + \ldots\right)$$

$$= V_1^+ \cdot \left[(1+\Gamma_L) + \Gamma_L\Gamma_S(1+\Gamma_L) + \Gamma_L^2\Gamma_S^2(1+\Gamma_L) + \ldots\right] = V_1^+ \frac{1+\Gamma_L}{1-\Gamma_L\Gamma_S}.$$

By $V_1^+ = \dfrac{Z_0}{Z_0 + R_S} V_0$, and eq's (3.3), (3.4), we have:

$$\lim_{t\to\infty} v_S(t) = \frac{R_L}{R_S + R_L} V_0 \tag{3.5}$$

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You will find that the total voltage at arbitrary position $z \in [0, l]$ will also converge to eq. (3.5). In other words, the steady state appears as if the transmission line were absent!

■ Bounce diagram

Bounce diagram is a distance vs. time plot, illustrating successive reflections along a transmission line driven by a "step voltage source" (Fig. 3-2a). It can be used conveniently to determine:

1) The spatial voltage distribution at some instant $v(z, t_0)$: Mark a point $P_0(z_0, t_0)$ on the plot (Fig. 3-2b). The solution becomes:

$$v(z,t_0)=\begin{cases}V_{left}, & \text{for } 0<z<z_0;\\ V_{right}, & \text{for } z_0<z<l\end{cases},$$

where V_{left}, V_{right} result from the superposition of proper numbers of bouncing voltage waves $v_k^+(z,t)$, $v_k^-(z,t)$ ($k=1, 2, \ldots$), respectively. For example, the case of Fig. 3-2b leads to a solution of Fig. 3-2c, where

$$V_{left}=v_1^+(z,t)+v_1^-(z,t)+v_2^+(z,t)=V_1^+(1+\Gamma_L+\Gamma_L\Gamma_S),$$

$$V_{right}=v_1^+(z,t)+v_1^-(z,t)=V_1^+(1+\Gamma_L).$$

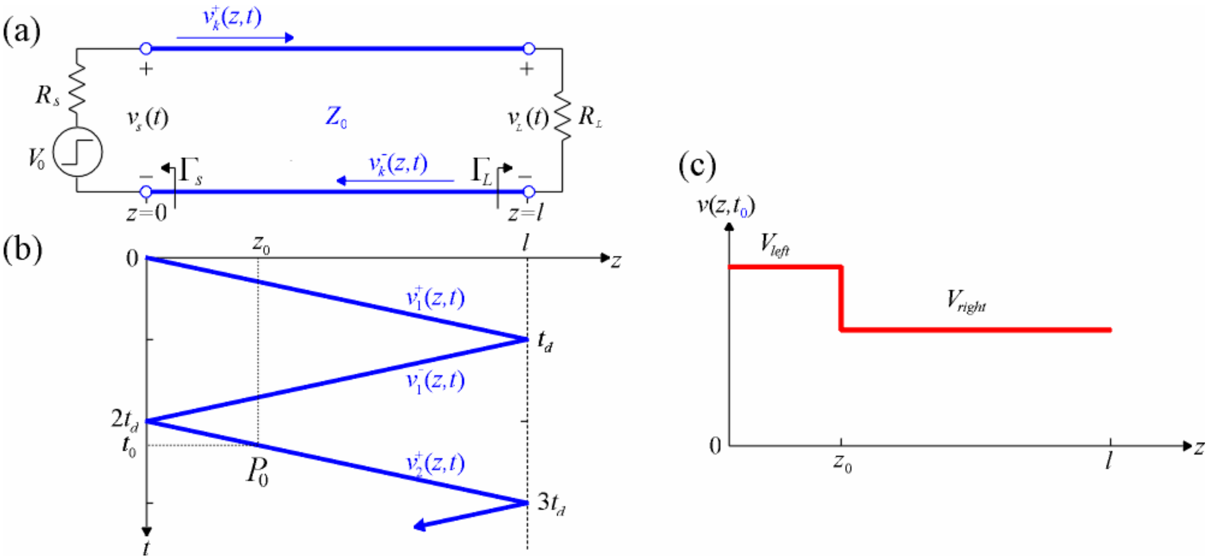

Fig. 3-2. (a) A transmission line excited by a step voltage source and terminated by a resistive load. (b) The corresponding bounce diagram. (c) The spatial voltage distribution at the time $t=t_0$.

2) The temporal voltage distribution at some position $v(z_a,t)$: Draw a vertical line $z=z_a$, intersecting with the lines of the plot at successive times $t=t_k^+$, t_k^- ($k=1, 2, \ldots$), which are the instants when the voltage wave $v_k^+(z,t)$, $v_k^-(z,t)$ arrives at the point of interest $z=z_a$. The solution becomes:

$$v(z_a,t)=\begin{cases}0, & \text{if } 0<t<t_1^+\\ V_1^+, & \text{if } t_1^+<t<t_1^-\\ V_1^+(1+\Gamma_L), & \text{if } t_1^-<t<t_2^+\\ V_1^+(1+\Gamma_L+\Gamma_L\Gamma_S), & \text{if } t_2^+<t<t_2^-\\ \ldots, & \ldots\end{cases} \tag{3.6}$$

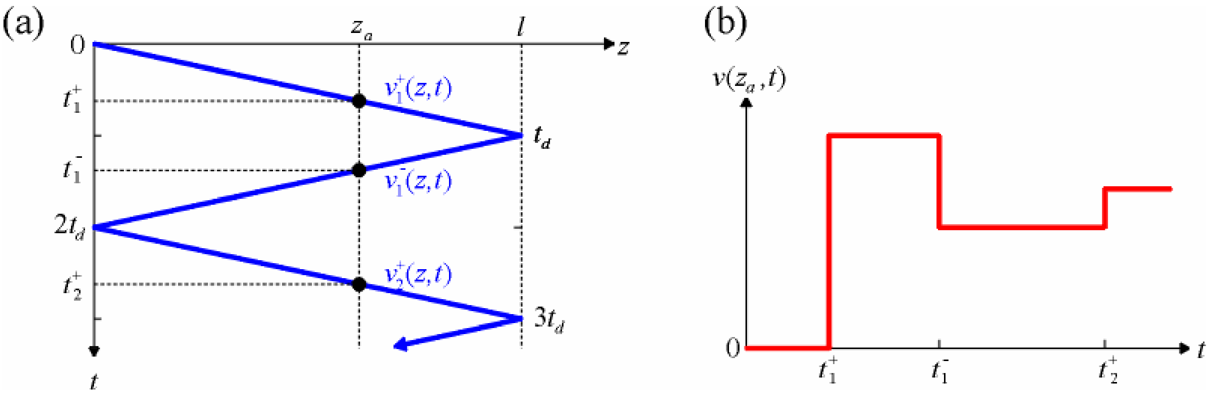

Fig. 3-3. (a) The bounce diagram. (b) The temporal voltage distribution at the position $z = z_a$.

■ Single transmission line with resistive termination

Example 3-2: Consider a system shown in Fig. 3-1 where $R_s = 0.25Z_0$, $R_L = \infty$ (open circuited load). Find the terminal voltages $v_S(t)$, $v_L(t)$.

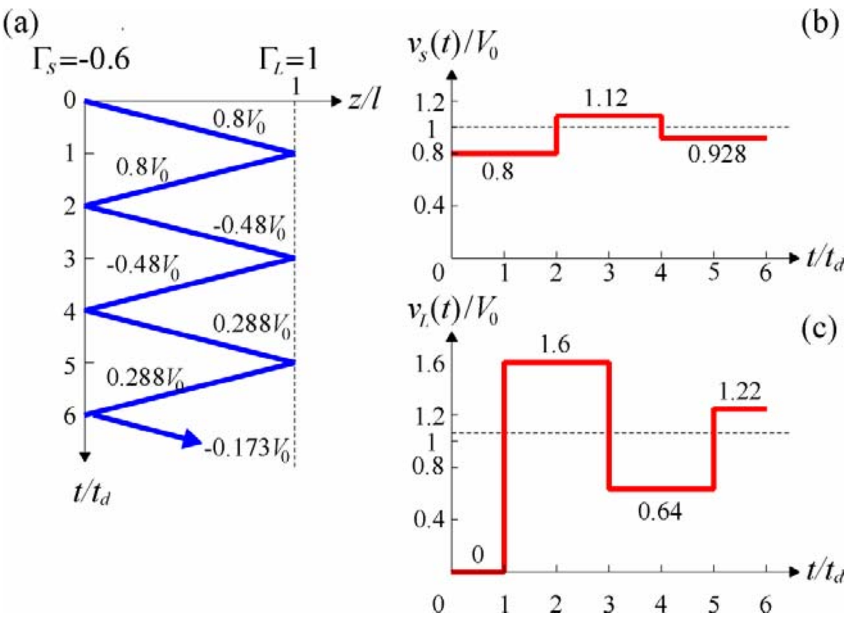

Fig. 3-4. (a) The bounce diagram. (b-c) The normalized terminal voltages of Example 3-2.

Ans: $V_1^+ = \dfrac{Z_0}{Z_0 + R_S} V_0 = 0.8V_0$. By eq's (3.3), (3.4), $\Gamma_L = 1$, $\Gamma_S = -0.6$. Draw the corresponding bounce diagram (Fig. 3-4a).

(1) For $z = z_a = 0, \Rightarrow\ t_1^+ = 0,\ t_1^- = t_2^+ = 2t_d$ (degenerate), $t_2^- = t_3^+ = 4t_d, \ldots$ By eq. (3.6),

$$v_S(t) = v(0,t) = \begin{cases} V_1^+ = 0.8V_0, & \text{if } 0 < t < 2t_d \\ V_1^+\left(1 + \Gamma_L + \Gamma_L\Gamma_S\right) = 1.12V_0, & \text{if } 2t_d < t < 4t_d \\ \ldots, & \ldots \end{cases} \quad \text{(See Fig. 3-4b).}$$

(2) For $z = z_a = l, \Rightarrow\ t_1^+ = t_1^- = t_d, \ldots,\ t_k^+ = t_k^- = (2k-1)t_d$ (degenerate). By eq. (3.6),

$$v_L(t) = v(l,t) = \begin{cases} 0, & \text{if } 0 < t < t_d \\ V_1^+\left(1 + \Gamma_L\right) = 1.6V_0, & \text{if } t_d < t < 3t_d \\ \ldots, & \ldots \end{cases} \quad \text{(See Fig. 3-4c).}$$

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1) Both $v_S(t)$ and $v_L(t)$ converge to V_0 as predicted by eq. (3.5).

2) Overshooting and ringing effects during the transient state could be harmful for circuits.

Example 3-3: Consider a system shown in Fig. 3-1 where $R_s = 4Z_0$, $R_L = Z_0$ (matched load). Find the terminal voltages $v_S(t)$, $v_L(t)$.

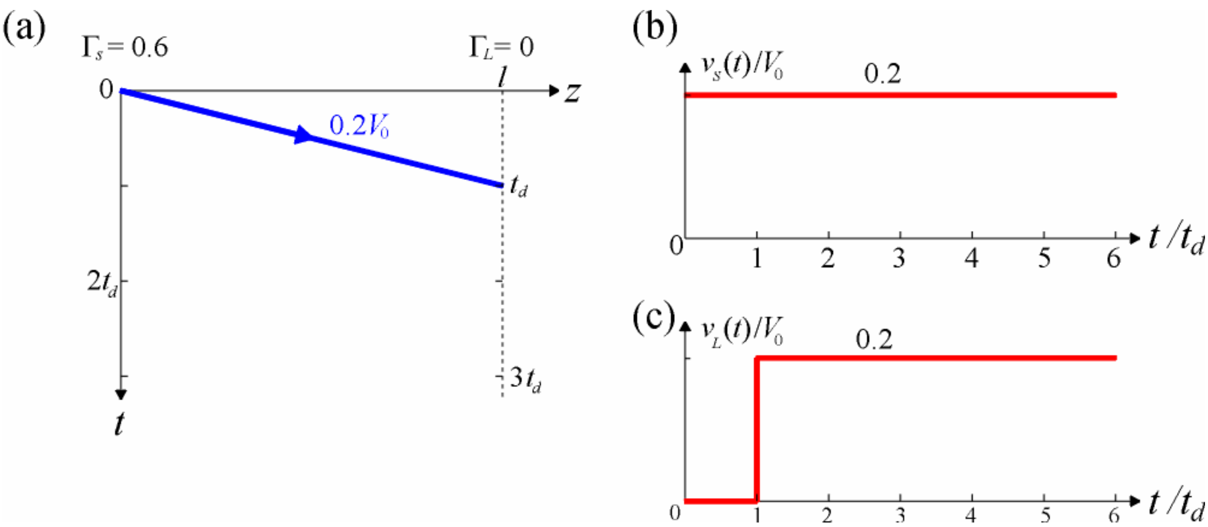

Fig. 3-5. (a) The bounce diagram. (b-c) The normalized terminal voltages of Example 3-3.

Ans: $V_1^+ = \dfrac{Z_0}{Z_0 + R_S} V_0 = 0.2V_0$. By eq's (3.3), (3.4), $\Gamma_L = 0$, $\Gamma_S = 0.6$. Draw the corresponding bounce diagram (Fig. 3-5a).

(1) For $z = z_a = 0$, $\Rightarrow$ $t_1^+ = 0$, $t_1^- = t_2^+ = 2t_d$ (degenerate), $t_2^- = t_3^+ = 4t_d$, … By eq. (3.6),

$$v_S(t) = v(0,t) = 0.8V_0, \text{ if } t > 0$$ (See Fig. 3-5b).

(2) For $z = z_a = l$, $\Rightarrow$ $t_1^+ = t_1^- = t_d$, …, $t_k^+ = t_k^- = (2k-1)t_d$ (degenerate). By eq. (3.6),

$$v_L(t) = v(l,t) = \begin{cases} 0, & \text{if } 0 < t < t_d \\ V_1^+(1+\Gamma_L) = 0.2V_0, & \text{if } t > 3t_d \end{cases}$$ (See Fig. 3-5c).

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1) Both $v_S(t)$ and $v_L(t)$ converge to V_0 as predicted by eq. (3.5).

2) No overshooting and ringing for the matched load prevents successive reflections.

■ Cascaded transmission lines

Example 3-4: Consider a system shown in Fig. 3-6a. Find the terminal voltages $v_S(t)$, $v_L(t)$.

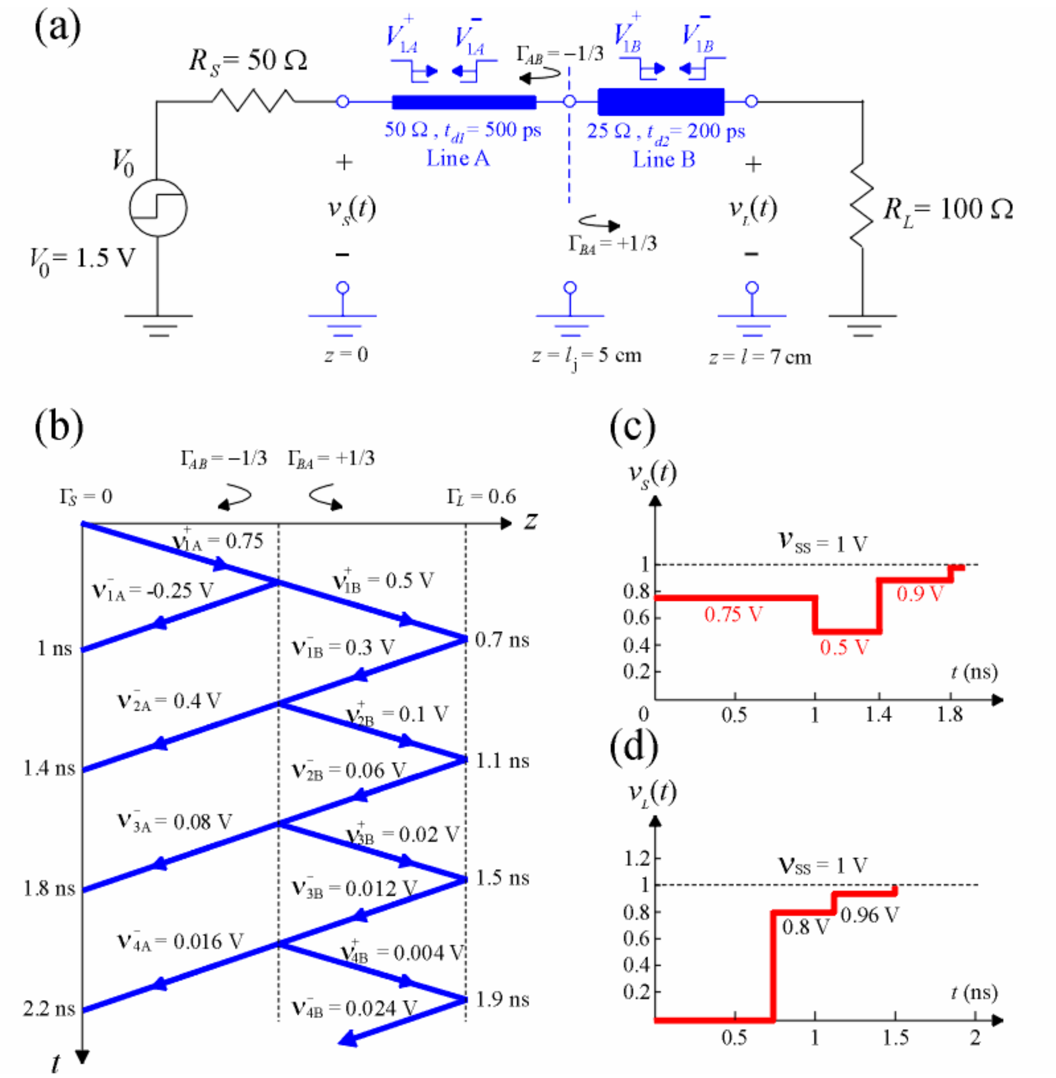

Fig. 3-6 (a) System configuration. (b) The bounce diagram. (c-d) The normalized terminal voltages of Example 3-4.

Ans: When a voltage disturbance $v^+_{1A}(z,t)$ arrives at the junction $z = l_j$ between lines A and B at time $t = t_{d1}$, a reflected wave $v^-_{1A}(z,t)$ and a transmitted wave $v^+_{1B}(z,t)$ are generated simultaneously. Boundary condition requires that the total voltages on both sides of the junction must be equal:

$$v^+_{1A}(l_j,t_{d1}) + v^-_{1A}(l_j,t_{d1}) = v^+_{1B}(l_j,t_{d1}).$$

Dividing $v_{1A}^{+}(l_j,t_{d1})$ for both sides of the equality leads to: $1+\dfrac{v_{1A}^{-}(l_j,t_{d1})}{v_{1A}^{+}(l_j,t_{d1})}=\dfrac{v_{1B}^{+}(l_j,t_{d1})}{v_{1A}^{+}(l_j,t_{d1})}$. By eq. (3.3), we have:

$$T_{AB}=\frac{v_{1B}^{+}(l_j,t_{d1})}{v_{1A}^{+}(l_j,t_{d1})}=1+\Gamma_{AB}, \tag{3.7}$$

where T_{AB} is the transmission coefficient.

$V_{1A}^{+}=\dfrac{Z_0}{Z_0+R_S}V_0=0.5V_0=0.75V$. By eq's (3.3), (3.4), $\Rightarrow$ $\Gamma_S=0$, $\Gamma_{AB}=-\dfrac{1}{3}$, $T_{AB}=\dfrac{2}{3}$, $\Gamma_{BA}=\dfrac{1}{3}$, $T_{BA}=\dfrac{4}{3}$, $\Gamma_L=0.6$. Draw the corresponding bounce diagram (Fig. 3-6b).

(1) $v_S(t)=v(0,t)=\begin{cases}V_{1A}^{+}=0.75V, & \text{if } 0<t<2t_{d1}=1\,\text{ns};\\ V_{1A}^{+}+V_{1A}^{-}=0.5V, & \text{if } 1\,\text{ns}<t<2t_{d1}+2t_{d2}=1.4\,\text{ns};\\ V_{1A}^{+}+V_{1A}^{-}+V_{2A}^{-}=0.9V, & \text{if } 1.4\,\text{ns}<t<2t_{d1}+4t_{d2}=1.8\,\text{ns};\ldots\end{cases}$, (Fig. 3-6c).

(2) $v_L(t)=v(l,t)=\begin{cases}0, & \text{if } 0<t<t_{d1}+t_{d2}=0.7\,\text{ns};\\ V_{1B}^{+}+V_{1B}^{-}=0.8V, & \text{if } 0.7\,\text{ns}<t<t_{d1}+3t_{d2}=1.1\,\text{ns};\\ V_{1B}^{+}+V_{1B}^{-}+V_{2B}^{+}+V_{2B}^{-}=0.96V, & \text{if } 1.1\,\text{ns}<t<t_{d1}+5t_{d2}=1.5\,\text{ns};\ldots\end{cases}$ (Fig. 3-6d).