Lesson 2
Transmission Lines Fundamentals

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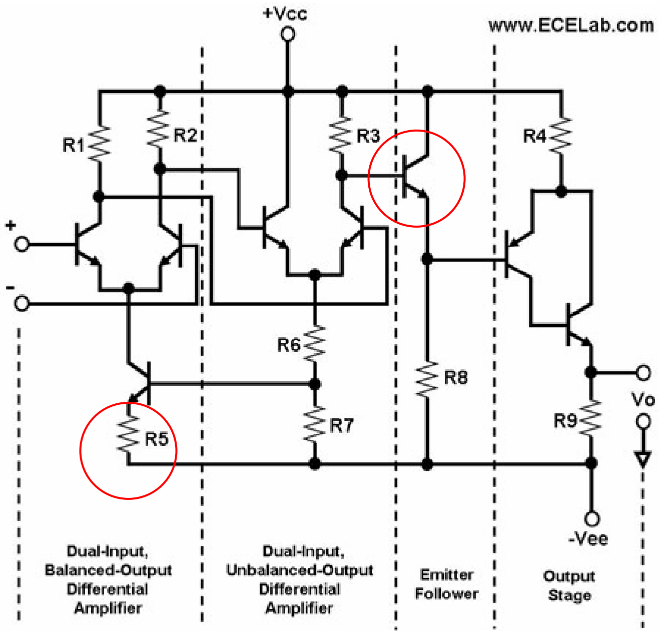
Sec. 2-1 Introduction

1. Why to discuss TX lines (distributed circuits)?
2. Criteria of using distributed circuits
3. Background knowledge

Why to discuss transmission lines-1

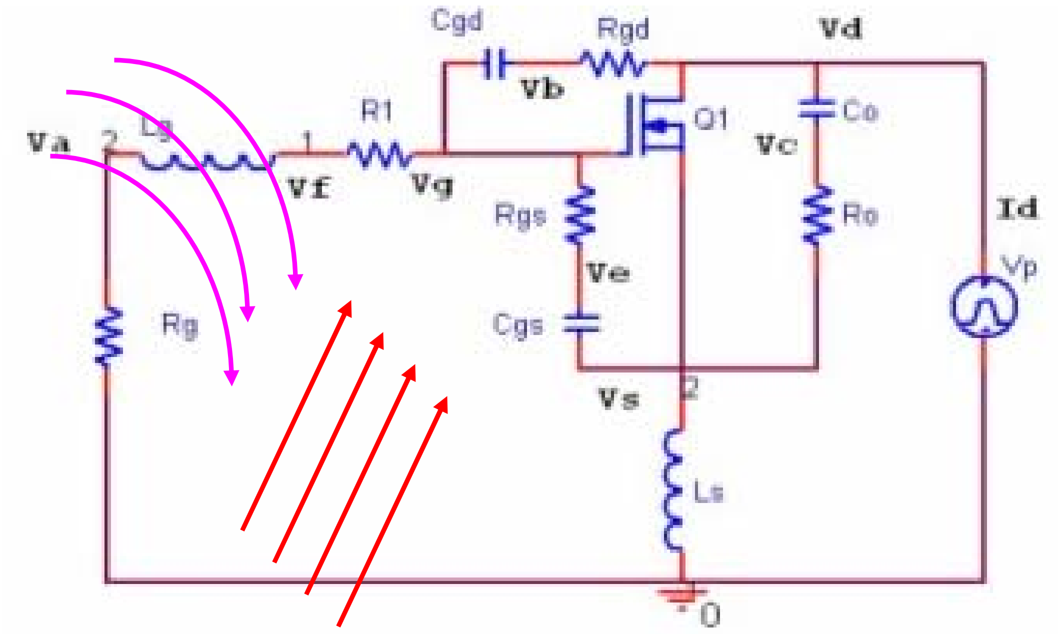
- Current electronic technology is based on lumped circuit theory (simple, powerful):

1. Lumped elements (R, C, L, dependent sources) are connected in series or shunt
2. Conducting wires play no role



Why to discuss transmission lines-2

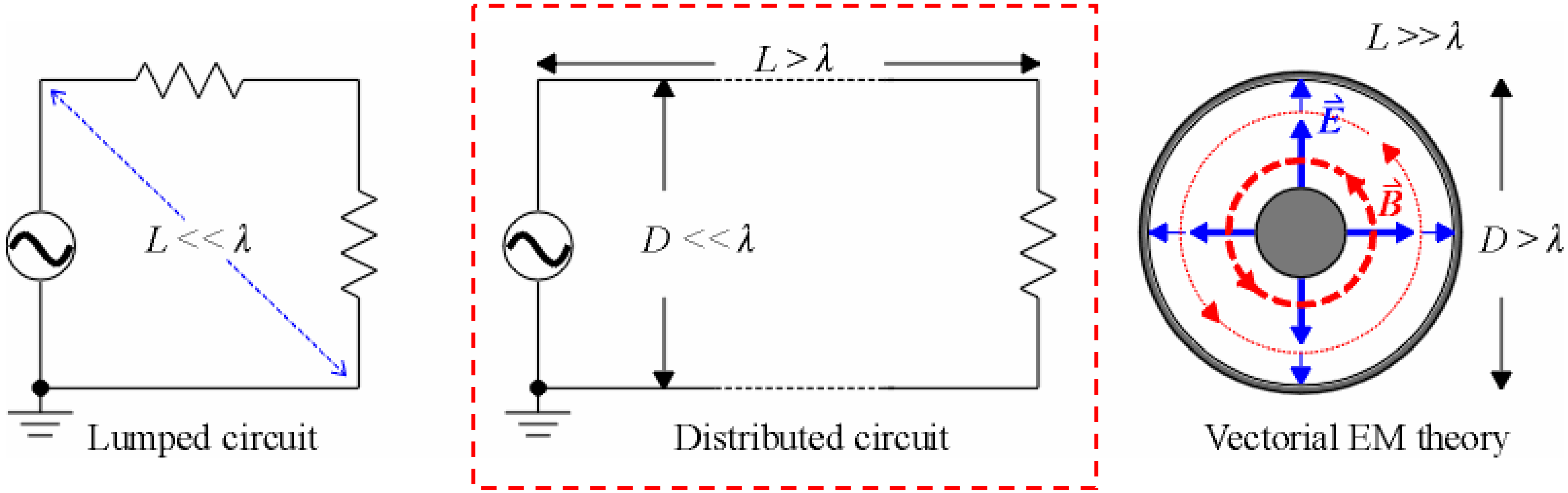
- In fact, elements and wires provide a framework over which electric charges can move and set up the (total) EM vector fields. The behavior of circuit is thus determined.



Why to discuss transmission lines-3

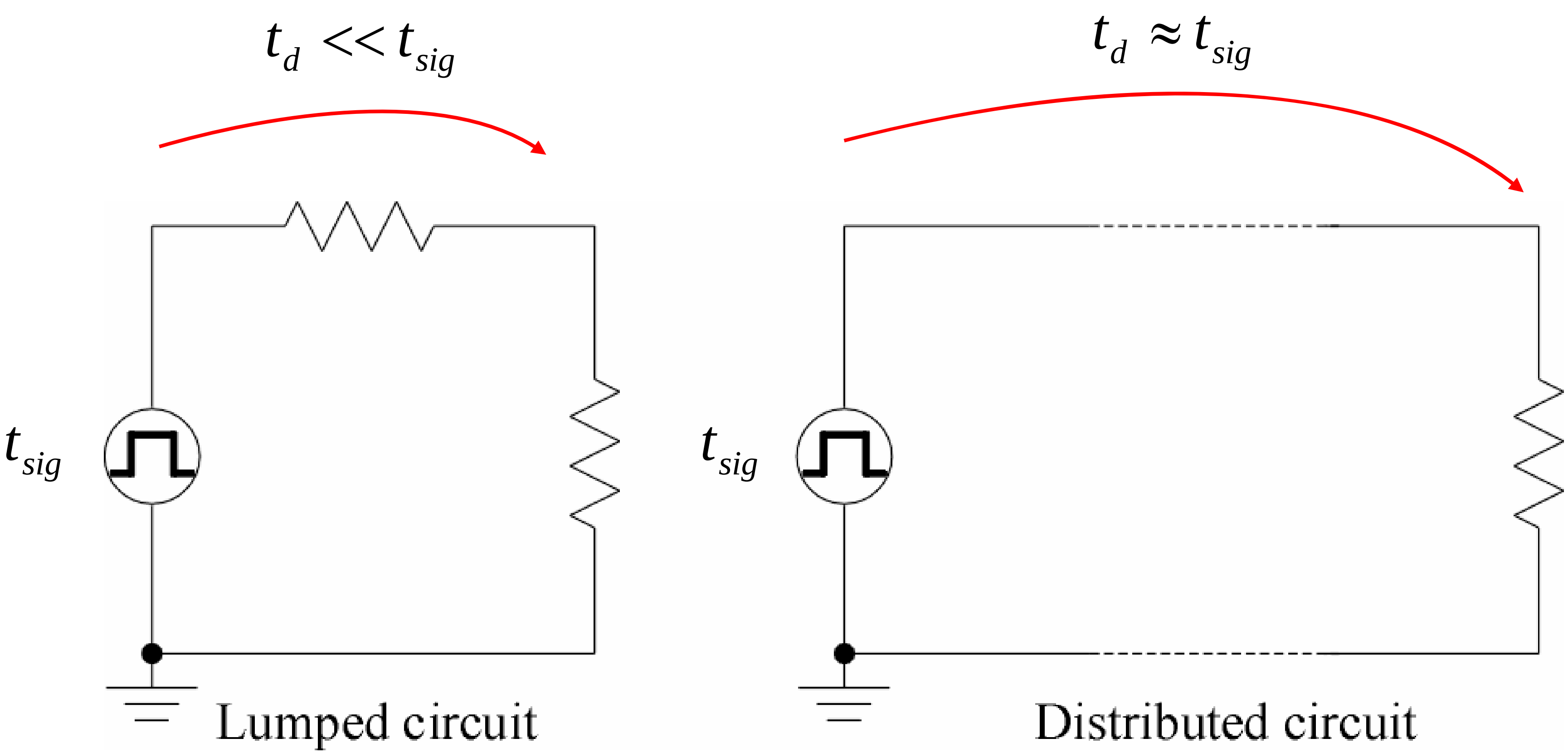
- Distributed circuit (transmission lines) theory is somewhere in between:

1. Can describe some wave properties (wavelength, phase velocity, reflection, …), which are crucial in power transmission, high-speed IC, ….
2. Only deal with scalar quantities (v, i), require no complicated vector analysis



Criteria of using distributed circuits

- Finite speed of propagation $v = c/n$

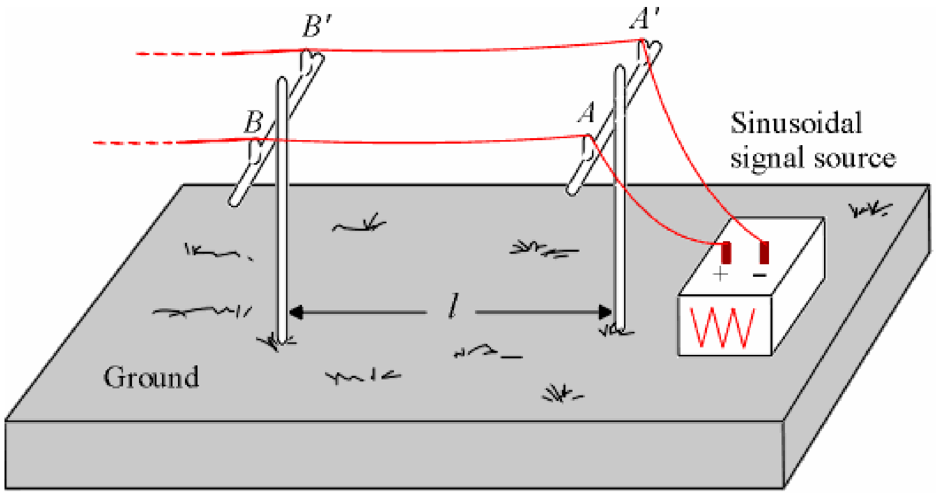


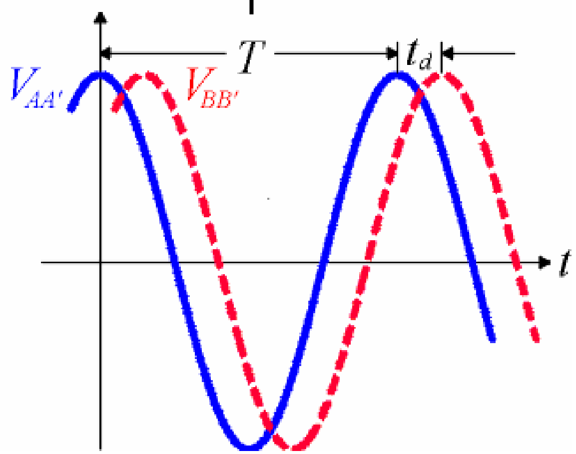
Example 2-1: Power lines

$$V_{AA'}(t) = V_0 \cos(2\pi \cdot 60 \cdot t)$$

$$V_{BB'}(t) = V_0 \cos[2\pi \cdot 60 \cdot (t - t_d)] \quad t_d = l/v$$

$$T = \frac{1}{60} \text{ sec}$$





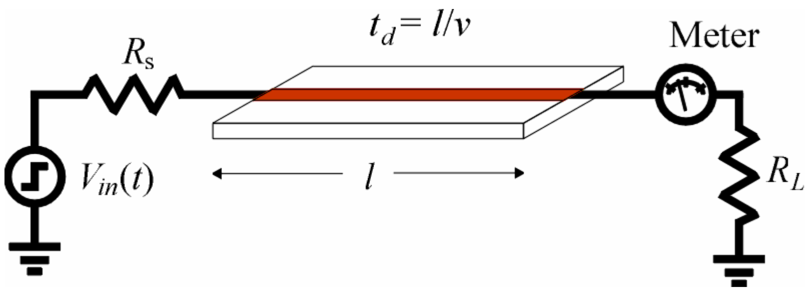
Lumped circuit is inadequate when $t_d > 0.01T$

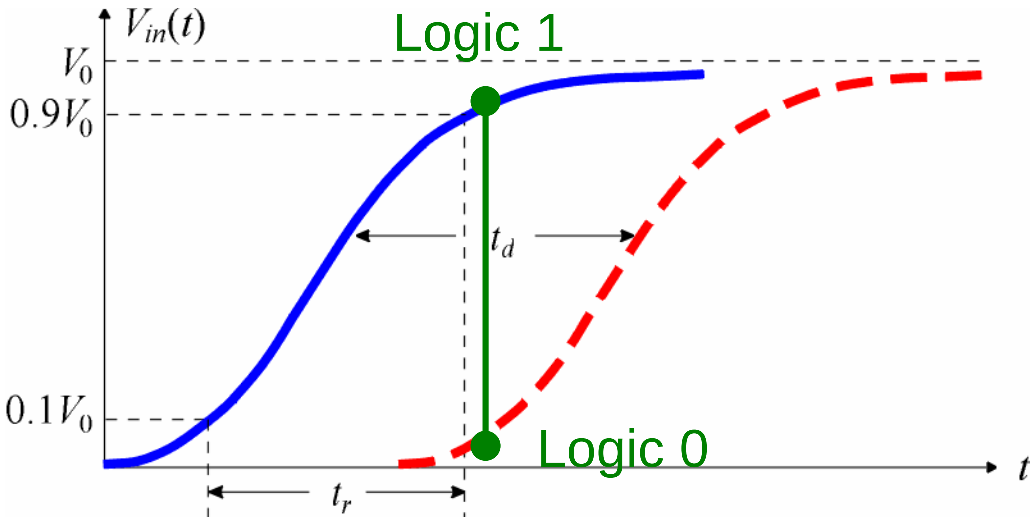
(rule of thumb), $\Rightarrow \ l > 50 \text{ km}$

Example 2-2: Interconnection of ICs

1-cm silica interconnection

$$t_d = l/v \approx 67 \text{ ps}$$



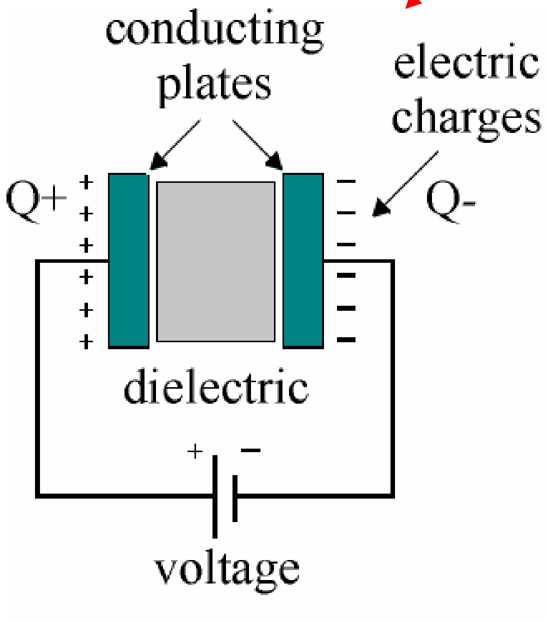


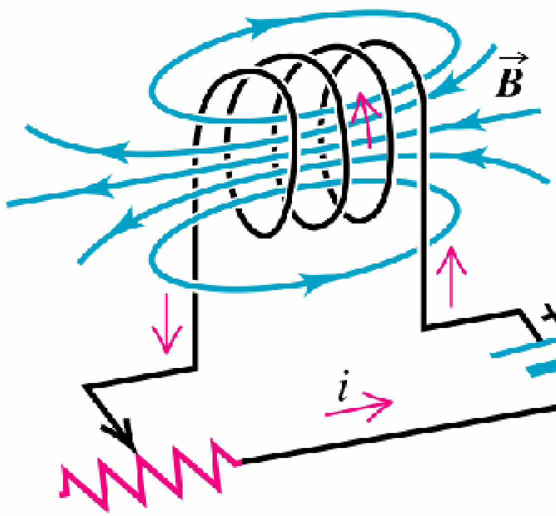
rise time

Lumped circuit is inadequate when $t_r < 2.5t_d \approx 165 \text{ ps}$ (rule of thumb). Fast CMOS can have $t_r \approx 100 \text{ ps}$!

- Models of linear circuit elements:

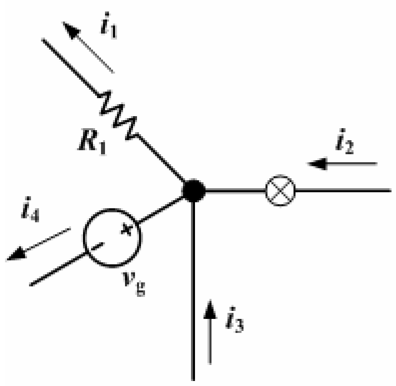
$$\boxed{v = R \cdot i} \qquad \boxed{i = C \cdot \frac{d}{dt} v} \qquad \boxed{v = L \cdot \frac{d}{dt} i}$$



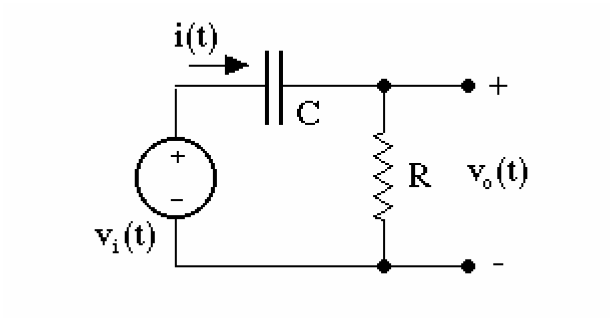


- Kirchhoff's laws :

$$\boxed{\sum_k i_k = 0}$$



$$\boxed{\sum_k v_k = 0}$$



Background knowledge-3

- Phasors of sinusoidal functions:

$$v(t) = V_0 \cos(\omega t + \phi) = \mathrm{Re}\left\{ V \cdot e^{j\omega t} \right\}$$

$$V = V_0 e^{j\phi} = V_0 (\cos\phi + j \sin\phi)$$

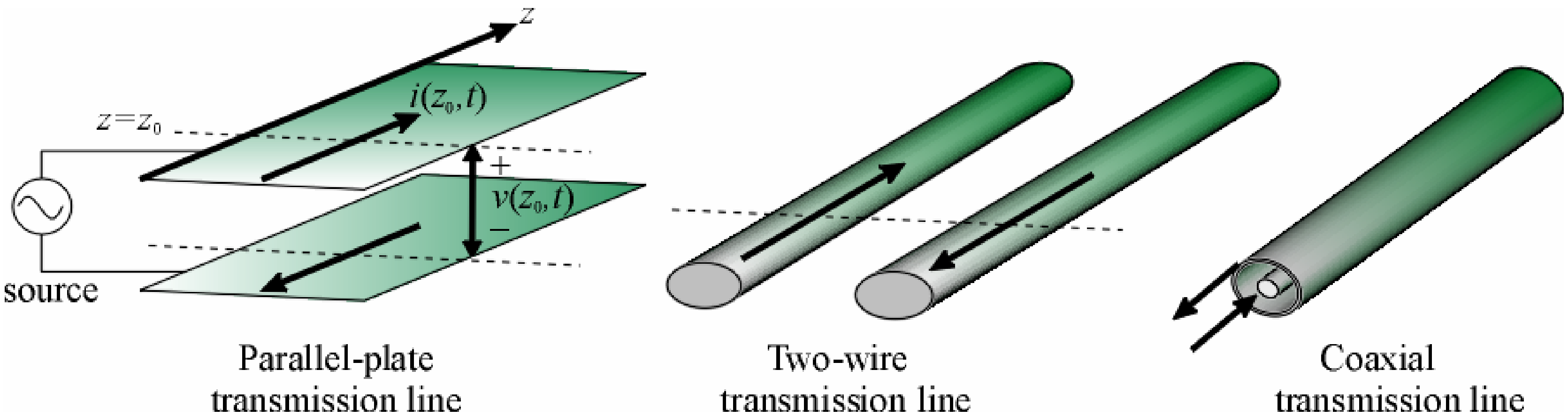
$$\frac{d}{dt} v(t) = \mathrm{Re}\left\{ V \cdot \frac{d}{dt} e^{j\omega t} \right\} = \mathrm{Re}\left\{ j\omega V \cdot e^{j\omega t} \right\}$$

$\dfrac{d^n}{dt^n} \rightarrow (j\omega)^n$ Derivatives $\rightarrow$ Algebraic multiplication

Sec. 2-2
Equivalent Circuit and Equations of Transmission Lines

1. Geometry of typical TX lines
2. Equivalent circuit
3. TX line equations
4. Solutions

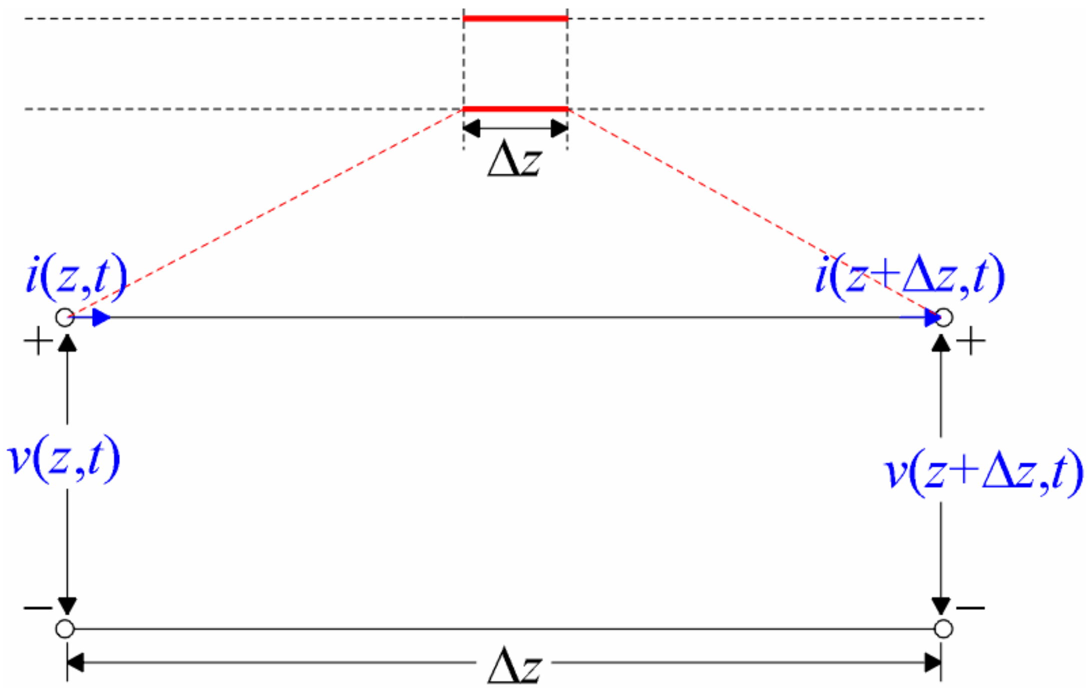
Geometry of typical transmission lines



- Two long conductors, separated by some insulating material
- $v(z_0,t)$ across, $\pm i(z_0,t)$ along the conductors

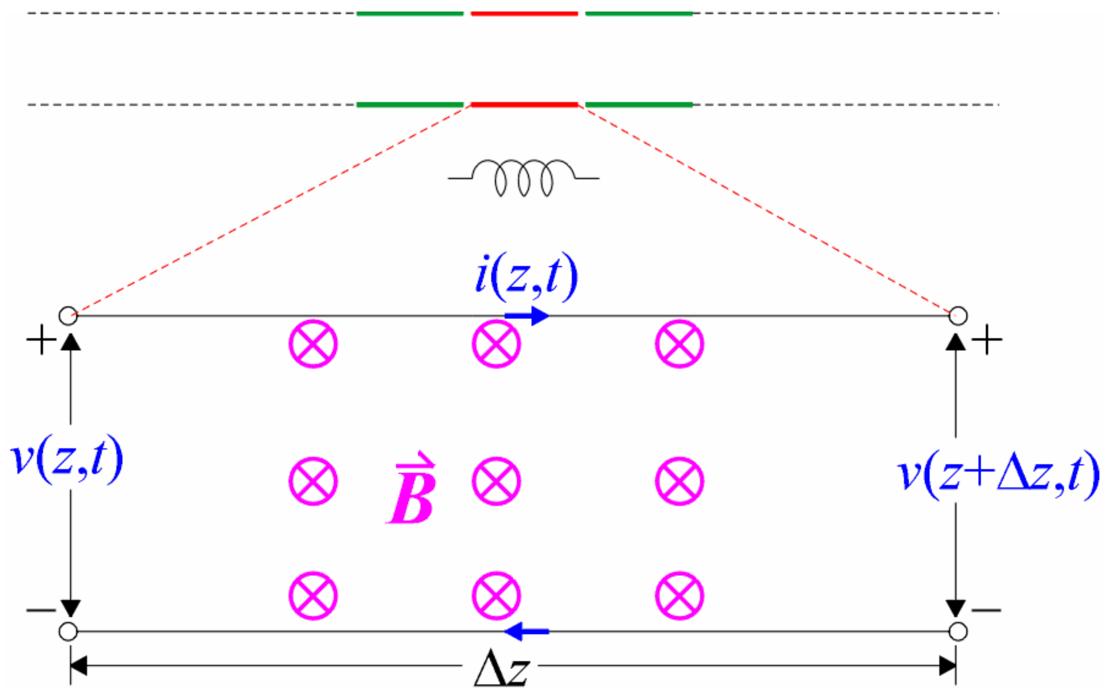
Equivalent circuit-1

- Since the voltage, current can vary with z, $\Rightarrow$ use of distributed circuit model, i.e. a TX line consists of infinitely many coupled lines of infinitesimal length Δz



Equivalent circuit-2

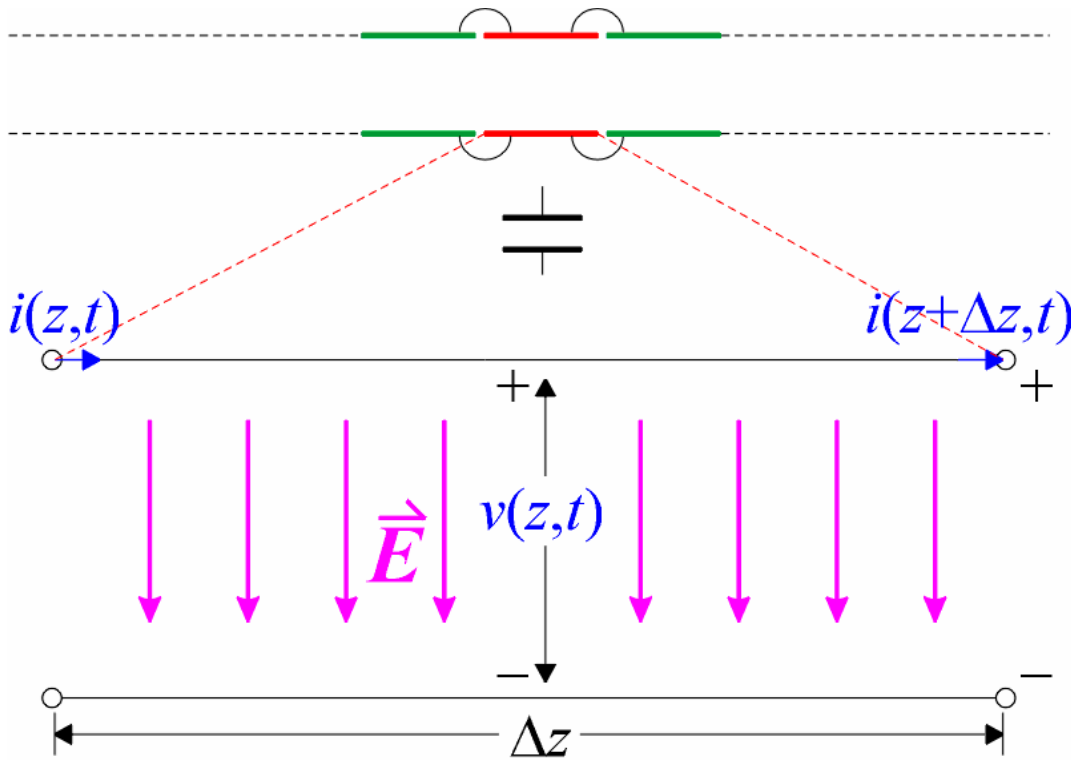
- Currents on a short line set up magnetic field (Ampere's law), $\Rightarrow$ magnetic flux
- Time-varying current (flux), $\Rightarrow$ voltage changes along the line (Faraday's law) to counter the change of current (Lenz's law), $\Rightarrow$ series inductor



$$\Delta v = L \cdot \frac{d}{dt} i$$

Equivalent circuit-3

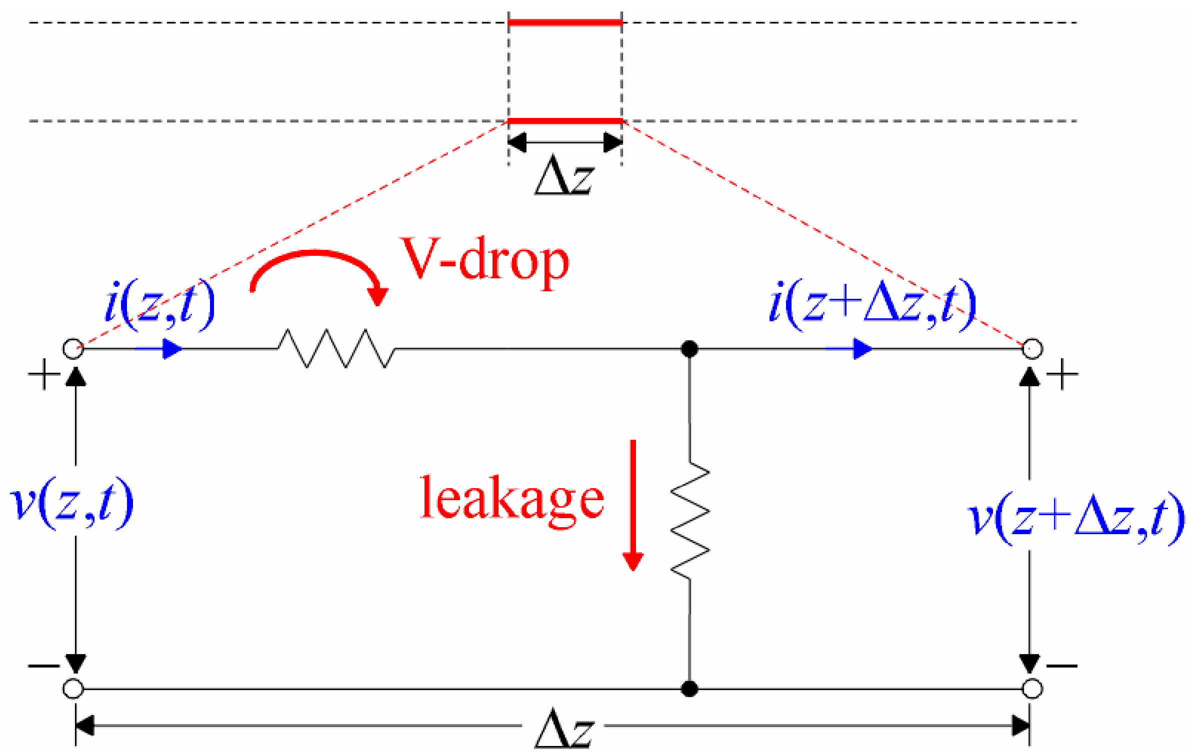
- The upper and lower conductors of the adjacent short lines are connected respectively, $\Rightarrow$ shunt capacitor



$$\boxed{\Delta i = C \cdot \frac{d}{dt} v}$$

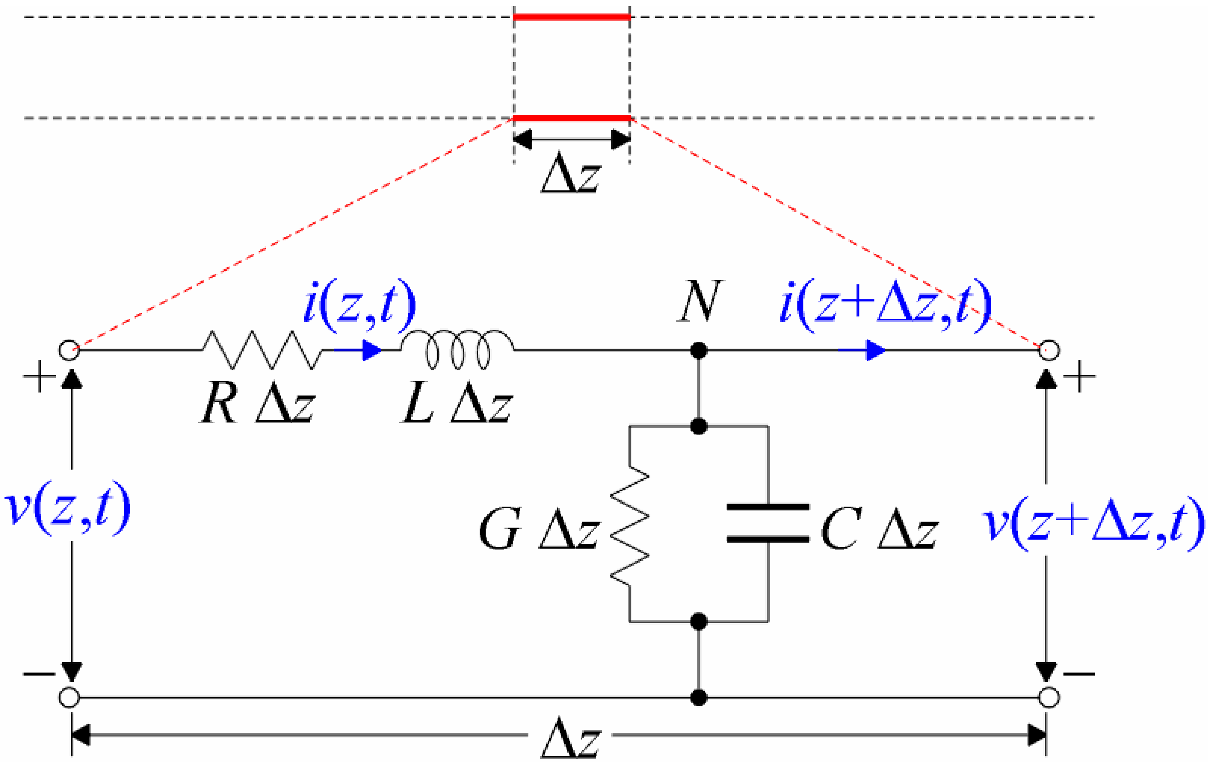
Equivalent circuit-4

- Imperfect conducting materials, $\Rightarrow$ voltage drop along the conducting line, $\Rightarrow$ series resistor
- Imperfect insulating materials, $\Rightarrow$ leakage current between conductors, $\Rightarrow$ shunt conductor



Equivalent circuit-5

- Complete model: {R, L, G, C} as {resistance, inductance, conductance, capacitance} per unit length



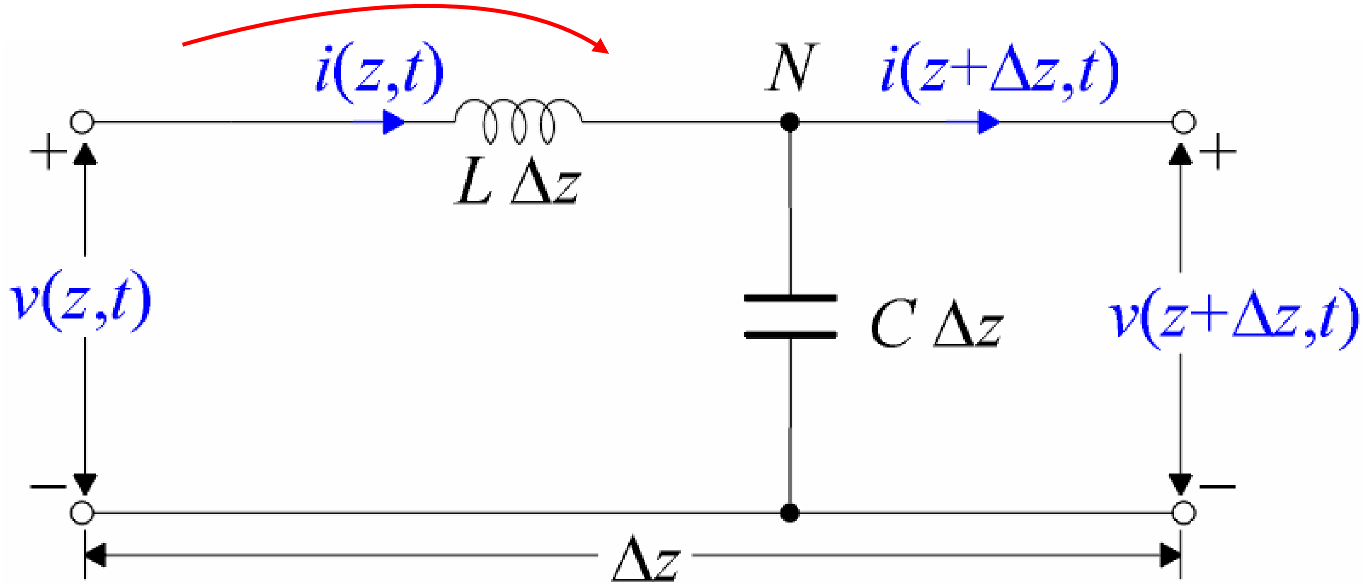
Equivalent circuit-6

- By the equivalent circuit, the behavior of vector EM fields can be described by scalar voltage and current.
- Values of *R*, *L*, *G*, *C* depend on the geometry and materials of the transmission line.

Transmission line equations-1

- Assume *R*=0, *G*=0: by KVL

$$v_L(t) = (L\Delta z)\cdot\frac{d}{dt} i_L(t)$$



$$\boxed{v(z+\Delta z,t) = v(z,t) - (L\Delta z)\frac{\partial}{\partial t} i(z,t)}$$

Transmission line equations-2

$$v(z+\Delta z,t) = v(z,t) - (L\Delta z)\frac{\partial}{\partial t}i(z,t)$$

$$\Rightarrow v(z+\Delta z,t) - v(z,t) = -(L\Delta z)\frac{\partial}{\partial t}i(z,t)$$

$$\Rightarrow \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -(L)\frac{\partial}{\partial t}i(z,t)$$

Let $\Delta z \rightarrow 0$,

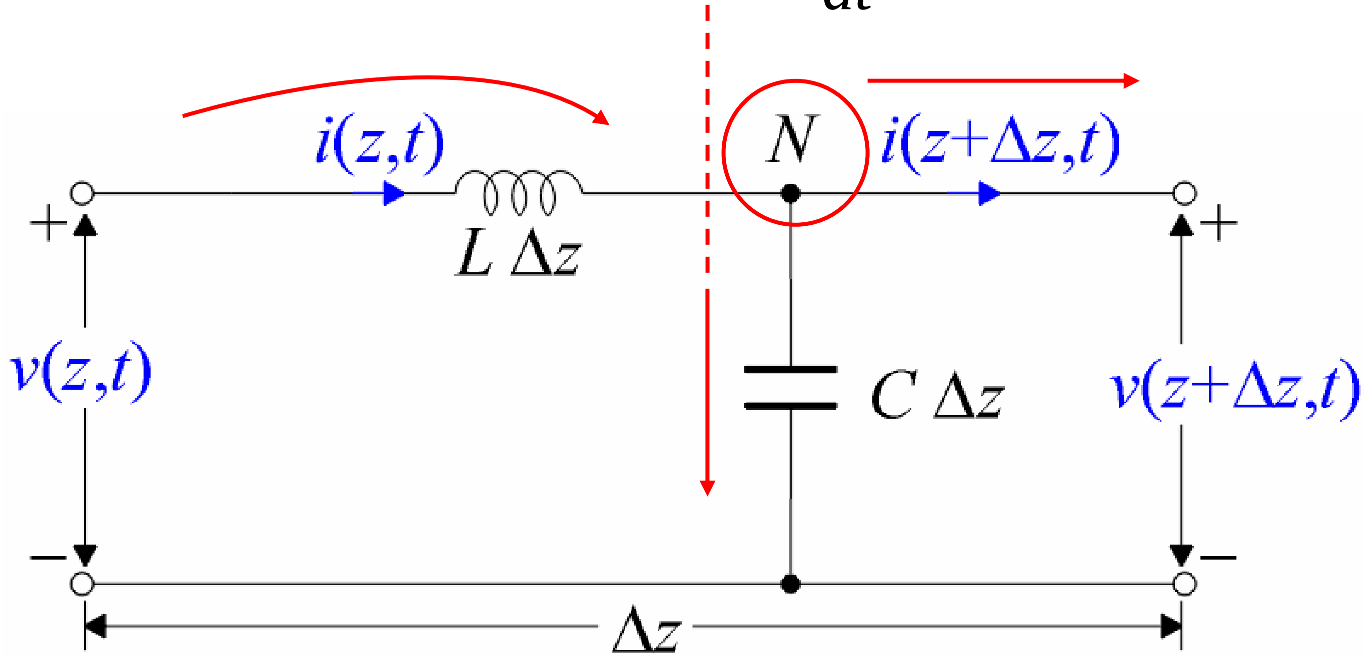
$$\Rightarrow \boxed{\frac{\partial}{\partial z}v(z,t) = -L\frac{\partial}{\partial t}i(z,t)} \quad (2.1)$$

1st-order PDE with 2 unknown functions v and i

Transmission line equations-3

- By KCL:

$$i_C(t) = (C\Delta z)\cdot\frac{d}{dt}v_C(t)$$



$$\boxed{i(z,t) = i(z+\Delta z,t) + (C\Delta z)\frac{\partial}{\partial t}v(z+\Delta z,t)}$$

Transmission line equations-4

$$i(z,t) = i(z+\Delta z,t) + (C\Delta z)\frac{\partial}{\partial t}v(z+\Delta z,t)$$

$$\Rightarrow i(z,t) - i(z+\Delta z,t) = (C\Delta z)\frac{\partial}{\partial t}v(z+\Delta z,t)$$

$$\Rightarrow \frac{i(z,t) - i(z+\Delta z,t)}{\Delta z} = (C)\frac{\partial}{\partial t}v(z+\Delta z,t)$$

Let $\Delta z \to 0,$

$$\Rightarrow \boxed{\frac{\partial}{\partial z}i(z,t) = -C\frac{\partial}{\partial t}v(z,t)} \quad (2.2)$$

1st-order PDE with 2 unknown functions v and i

Transmission line equations-5

Taking $\dfrac{\partial}{\partial z}$ for both sides of eq. (2.1),

$$\frac{\partial}{\partial z}v(z,t)=-L\frac{\partial}{\partial t}i(z,t),\Rightarrow \frac{\partial^2}{\partial z^2}v(z,t)=-L\frac{\partial^2}{\partial z\partial t}i(z,t)$$

Taking $\dfrac{\partial}{\partial t}$ for both sides of eq. (2.2),

$$\frac{\partial}{\partial z}i(z,t)=-C\frac{\partial}{\partial t}v(z,t),\Rightarrow \frac{\partial^2}{\partial z\partial t}i(z,t)=-C\frac{\partial^2}{\partial t^2}v(z,t)$$

Transmission line equations-6

Combine the two equations:

$$\Rightarrow \frac{\partial^2}{\partial z^2} v(z,t) = LC \frac{\partial^2}{\partial t^2} v(z,t)$$

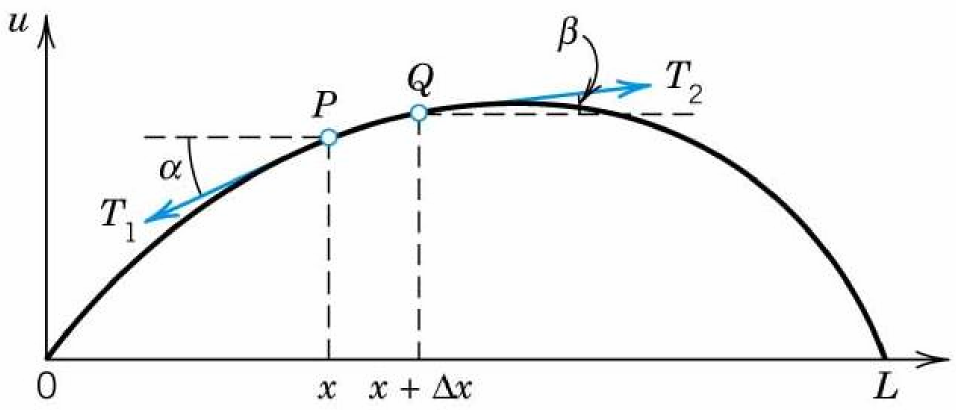
2nd-order PDE with 1 unknown function v

$$\boxed{\begin{aligned} \frac{\partial^2}{\partial z^2} v(z,t) &= LC \frac{\partial^2}{\partial t^2} v(z,t) \quad (2.3) \\ \frac{\partial^2}{\partial z^2} i(z,t) &= LC \frac{\partial^2}{\partial t^2} i(z,t) \quad (2.4) \end{aligned}}$$

Solutions to the transmission line equations-1

Compare $$\frac{\partial^2}{\partial z^2} v(z,t) = LC \frac{\partial^2}{\partial t^2} v(z,t)$$ (TX line eq.)

with $$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t)$$ (1-D wave eq.)

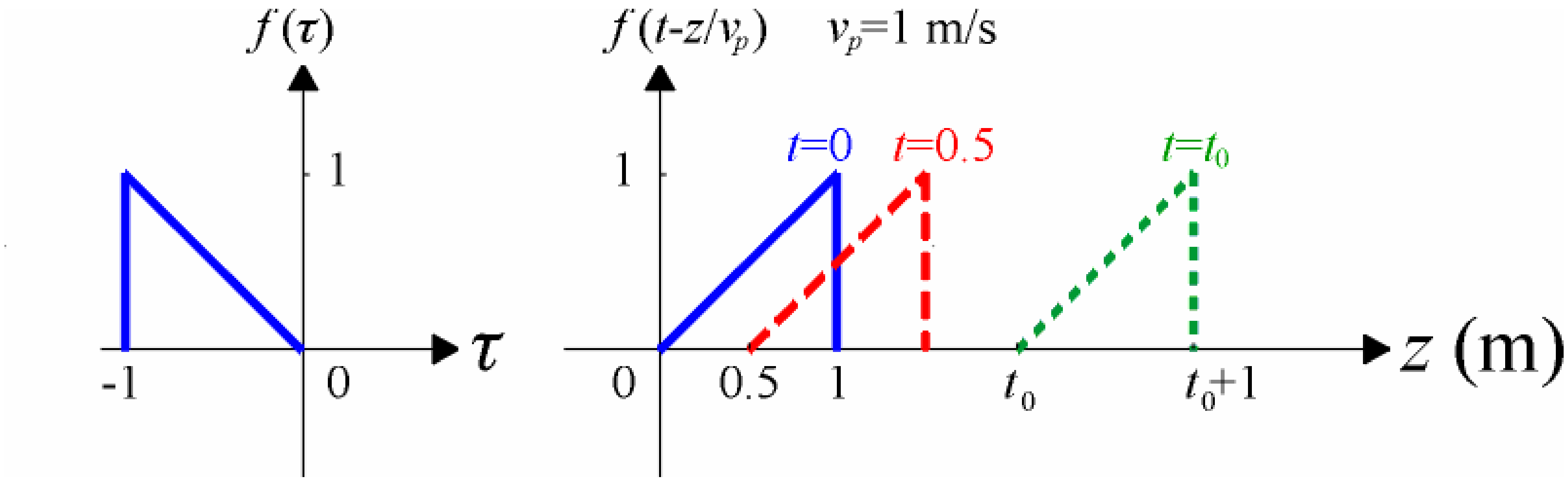


$v(z,t)$ ~ a wave propagating with velocity $\boxed{v_p \equiv \frac{1}{\sqrt{LC}}}$

D'Alembart's solution to wave equation:

any function $f(\cdot)$ of variable $\tau = t \pm \dfrac{z}{v_p}$

$f(t - z/v_p)$ ~ distortion-free wave traveling in the +z direction with velocity v_p



Solutions to the transmission line equations-3

$f\left(t+z/v_p\right)$ ~ distortion-free wave traveling in the -z direction with velocity v_p

General solution to the voltage (superposition):

$$\boxed{v(z,t)=f^{+}\left(t-z/v_p\right)+f^{-}\left(t+z/v_p\right)} \quad (2.6)$$

Can be totally different functions determined by BCs and ICs.

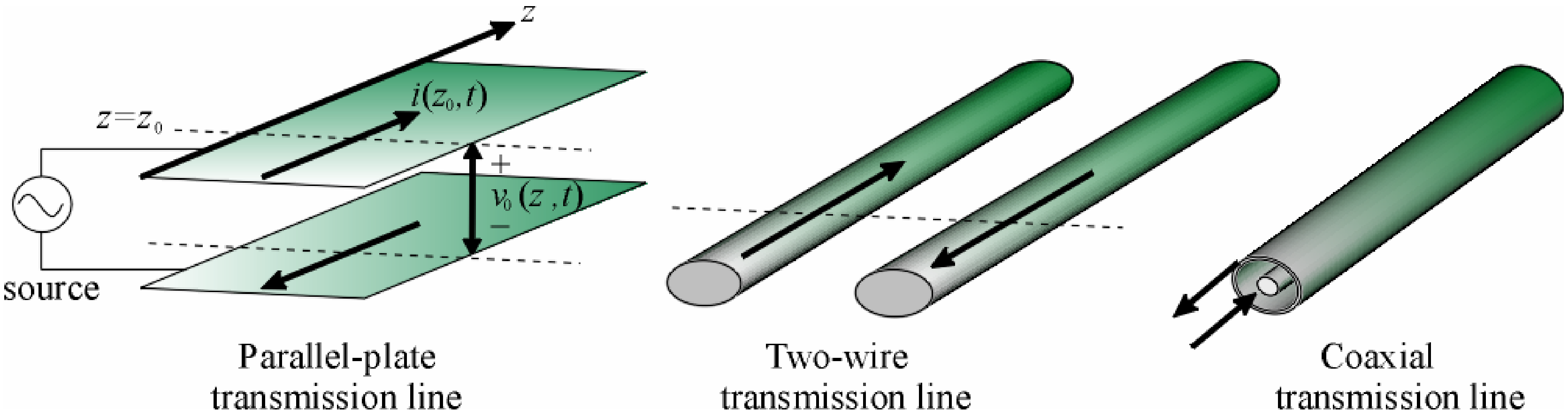
Their superposition may have "distortion".

Comments

- Phase velocity $v_p = \dfrac{1}{\sqrt{LC}}$ only depends on

  the insulating materials,

  though *L*, *C* also depend on the geometry of the lines



Solutions to the transmission line equations-4

Substitute voltage solution $v(z,t) = f^{+}\left(t - z/v_p\right) + f^{-}\left(t + z/v_p\right)$

into the 1st-order PDE: $\dfrac{\partial}{\partial z} v(z,t) = -L \dfrac{\partial}{\partial t} i(z,t)$

$$\Rightarrow \frac{\partial v}{\partial z} = -\frac{1}{v_p}\frac{df^{+}(\tau)}{d\tau} + \frac{1}{v_p}\frac{df^{-}(\tau)}{d\tau} = -L\frac{\partial i}{\partial t}$$

$$\Rightarrow i(z,t) = -\frac{1}{L}\int \frac{\partial v}{\partial z}\partial t = \frac{1}{Lv_p}\int\left[\frac{df^{+}(\tau)}{d\tau} - \frac{df^{-}(\tau)}{d\tau}\right]\partial t$$

$$\because \frac{\partial \tau}{\partial t} = 1, \Rightarrow \quad i(z,t) = \sqrt{\frac{C}{L}}\int\left[\frac{df^{+}(\tau)}{\cancel{d\tau}} - \frac{df^{-}(\tau)}{\cancel{d\tau}}\right]\cancel{d\tau}$$

Solutions to the transmission line equations-5

$$v(z,t) = f^{+}\left(t - z/v_p\right) + f^{-}\left(t + z/v_p\right) = v^{+}(z,t) + v^{-}(z,t)$$

$$\Rightarrow i(z,t) = \frac{1}{Z_0}\left[f^{+}\left(t - z/v_p\right) - f^{-}\left(t + z/v_p\right)\right] = i^{+}(z,t) + i^{-}(z,t)$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

Characteristic impedance

(not the resistance of the conductor or insulator)

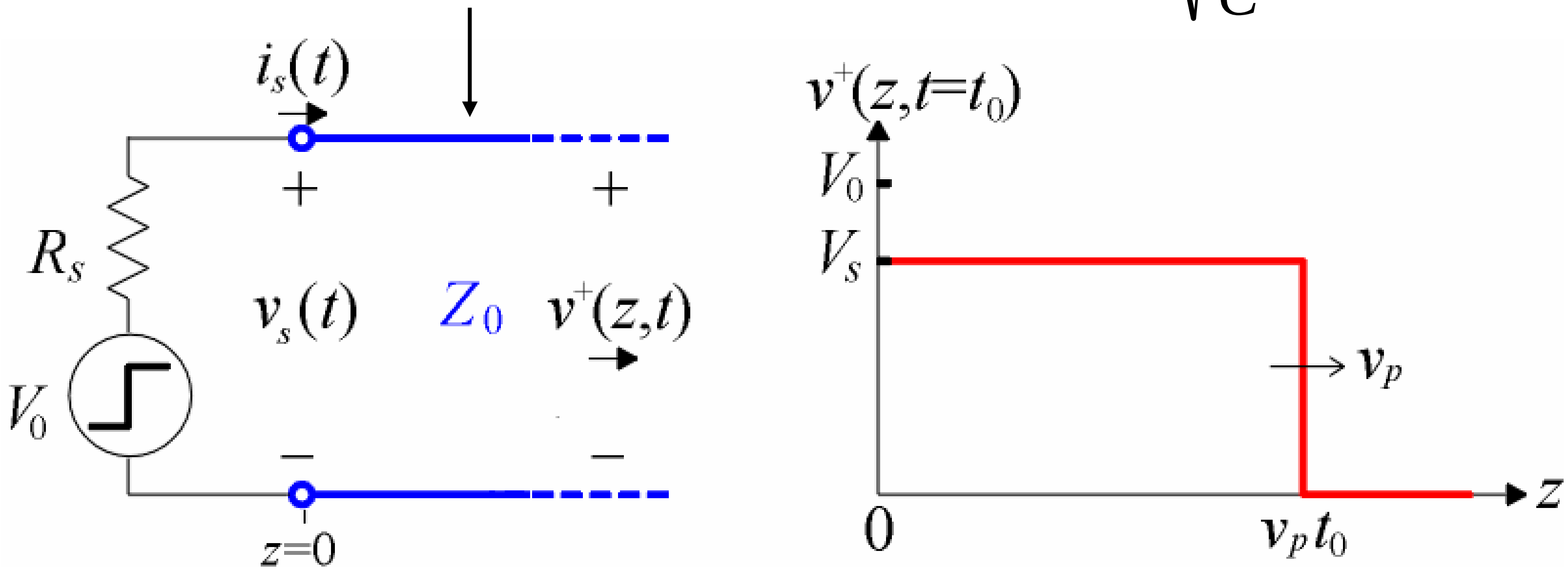
$$Z_0 = \frac{v^{+}(z,t)}{i^{+}(z,t)} = -\frac{v^{-}(z,t)}{i^{-}(z,t)} \neq \frac{v(z,t)}{i(z,t)}$$

Physical meaning

Example 2-3:

Infinitely long line, no reflected wave $v^-(z,t)$

TX line ~ a load of resistance $Z_0 = \sqrt{\dfrac{L}{C}}$



$$v_s(t=0^+) = V_s = \frac{Z_0}{Z_0 + R_s} V_0$$