Homework 12 Solutions

(Due date: 2011/06/08)

The full score is 50 points.

1) (10%) Consider a copper strip of length L on the xy-plane, where a uniform magnetic flux density $\vec{B} = \vec{a}_z B_0$ exists. One of its end is pivoted at the origin, while the entire strip is rotating counterclockwisely about the z-axis with an angular velocity ω.What is the voltage difference between the two ends (from the pivot to the tip) of the strip?

Answer:

The voltage difference:

$$V_0 = \oint \left(\vec{u} \times \vec{B}\right) \cdot d\vec{l} = -\int_0^L \left[\left(\vec{a}_\phi r\omega\right) \times \vec{a}_z B_0\right] \cdot \vec{a}_r dl = -\omega B_0 \int_0^L r dr = -\frac{\omega B_0 L^2}{2}$$

2) (5%) Consider a time-harmonic EM wave in some material with $\varepsilon = \varepsilon_0$, σ =5.70×10^7 (S/m). What is the ratio of magnitude of the conduction current density to the displacement current density at 1 GHz frequency?

Answer:

The conduction current density: $\vec{J} = \sigma\vec{E}$

The displacement current density: $\dfrac{\partial \vec{D}}{\partial t} = j\omega\varepsilon\vec{E}$

The ratio of magnitude at 10-GHz frequency is

$\dfrac{\sigma}{\omega\varepsilon} = \dfrac{5.7\times10^{7}}{2\pi\times10^{9}\cdot8.854\times10^{-12}}$ =1.025×10^9>>1.

We can find the displacement current density can be ignored in the material at 1 GHz.

3) Nonhomogeneous wave equations of fields in the frequency domain.

3a) (5%) Starting from the Maxwell's equations in the time domain, i.e. eq's (14.1), (7.8) (14.12), (11.2) in the lecture notes, write down the (phasor) Maxwell's equations of $\vec{E}$ and $\vec{H}$ in the frequency domain in a simple medium with charge and current sources.

Answer:

We can write a time-harmonic E field referring to $\cos\omega t$ as

$$\vec{E}(x,y,z,t) = \mathrm{Re}\left[\vec{E}(x,y,z)e^{j\omega t}\right]$$

So we can know that $\dfrac{\partial}{\partial t}\vec{E}(x,y,z,t) = j\omega\vec{E}(x,y,z,t)$

In other words, $\dfrac{\partial}{\partial t}\vec{E} = j\omega\vec{E}$

Similarly, we can know that $\dfrac{\partial}{\partial t}\vec{H} = j\omega\vec{H}$

We also know that $\vec{D} = \varepsilon\vec{E}$, $\vec{H} = \dfrac{\vec{B}}{\mu}$

Obviously, we can find that

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \quad (14.1) \quad \Rightarrow \nabla\times\vec{E} = -\frac{\partial}{\partial t}\mu\vec{H} = -j\omega\mu\vec{H}$$

$$\nabla\cdot\vec{D} = \rho \quad (7.8) \quad \Rightarrow \nabla\cdot\vec{E} = \nabla\cdot\frac{\vec{D}}{\varepsilon} = \frac{\rho}{\varepsilon}$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D} \quad (14.12) \quad \Rightarrow \nabla\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}\varepsilon\vec{E} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$\nabla\cdot\vec{B} = 0 \quad (11.2) \quad \Rightarrow \nabla\cdot\vec{H} = \nabla\cdot\frac{\vec{B}}{\mu} = 0$$

In conclusion,

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

$$\nabla\times\vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$\nabla\cdot\vec{H} = 0$$

3b) (10%) By the result of problem 3a, derive the nonhomogeneous (phasor) wave equations of $\vec{E}$ and $\vec{H}$, respectively.

Answer:

$$\nabla\times\nabla\times\vec{E}=\nabla\times\left(-j\omega\mu\vec{H}\right)=-j\omega\mu\left(\nabla\times\vec{H}\right)=-j\omega\mu\left(\vec{J}+j\omega\varepsilon\vec{E}\right)=-j\omega\mu\vec{J}+\omega^2\mu\varepsilon\vec{E}\quad,$$

$$\nabla\times\nabla\times E=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=\frac{\nabla\rho}{\varepsilon}-\nabla^2\vec{E}\quad,$$

$$\Rightarrow\frac{\nabla\rho}{\varepsilon}-\nabla^2\vec{E}=-j\omega\mu\vec{J}+\omega^2\mu\varepsilon\vec{E}$$

$\nabla^2\vec{E}+k^2\vec{E}=\dfrac{\nabla\rho}{\varepsilon}+j\omega\mu\vec{J}$, where $k=\omega\sqrt{\mu\varepsilon}$

$$\nabla\times\nabla\times\vec{H}=\nabla\times\left(\vec{J}+j\omega\varepsilon\vec{E}\right)=\nabla\times\vec{J}+j\omega\varepsilon\left(\nabla\times\vec{E}\right),$$

$$=\nabla\times\vec{J}+j\omega\varepsilon\left(-j\omega\mu\vec{H}\right)=\nabla\times\vec{J}+\omega^2\varepsilon\mu\vec{H}$$

$$\nabla\times\nabla\times\vec{H}=\nabla\left(\nabla\cdot\vec{H}\right)-\nabla^2\vec{H}=-\nabla^2\vec{H}\ ,$$

$$\Rightarrow\nabla\times\vec{J}+\omega^2\varepsilon\mu\vec{H}=-\nabla^2\vec{H}$$

$\nabla^2\vec{H}+k^2\vec{H}=-\nabla\times\vec{J}$, where $k=\omega\sqrt{\mu\varepsilon}$

4) Consider a short conducting wire of length *dl* carrying a spatially uniform current $i(t) = I_0 \cos\omega t$ and placed along the *z*-axis at the origin (Fig. 1).

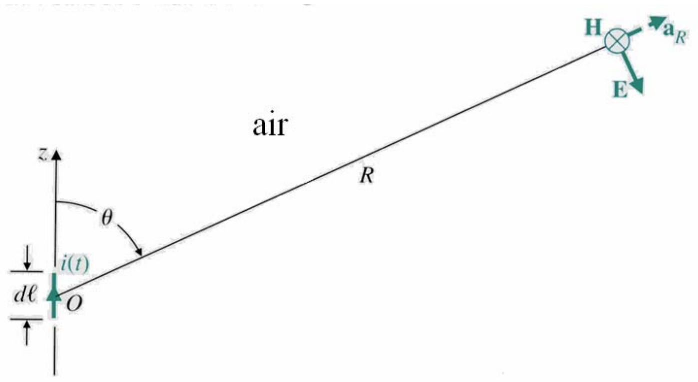

Fig. 1 A small current source creates EM fields.

4a) (5%) By eq. (15.28) in the lecture notes, find the phasor representation of resulting vector potential $\vec{A}(\vec{r})$ in spherical coordinates ($\vec{r} = \vec{a}_R R$, $R >> \lambda = 2\pi c/\omega$).

Answer:

By eq. (15.28) in the lecture notes, $\vec{A}(\vec{r}) = \dfrac{\mu}{4\pi}\int_{V''} \dfrac{\vec{J}e^{-jkR}}{R}dv' = \vec{a}_z \dfrac{\mu_0 I d\ell}{4\pi}\dfrac{e^{-jkR}}{R}$

Since $\vec{a}_z = \vec{a}_R \cos\theta - \vec{a}_\theta \sin\theta$,

$$\vec{A}(\vec{r}) = \vec{a}_R \frac{\mu_0 Idl}{4\pi}\frac{e^{-jkR}}{R}\cos\theta - \vec{a}_\theta \frac{\mu_0 Idl}{4\pi}\frac{e^{-jkR}}{R}\sin\theta,\ k = \frac{\omega}{c}.$$

4b) (5%) By the result of Problem 4a, find the "approximated" phasor representation of magnetic field intensity $\vec{H}(\vec{r})$ by neglecting the "higher order terms". [E.g. $(kR)^{-1} + (kR)^{-2} \approx (kR)^{-1}$.]

Answer:

By eq. (11.7), $\vec{H}(\vec{r}) = \dfrac{1}{\mu_0}\nabla \times \vec{A} =$

$$\frac{1}{\mu_0}\frac{1}{R^2\sin\theta}\begin{vmatrix} \vec{a}_R & \vec{a}_\theta R & \vec{a}_\phi R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ \dfrac{\mu_0 Idl}{4\pi}\dfrac{e^{-jkR}}{R}\cos\theta & -R\dfrac{\mu_0 Idl}{4\pi}\dfrac{e^{-jkR}}{R}\sin\theta & 0 \end{vmatrix}$$

$$= \vec{a}_\phi \frac{1}{\mu_0 R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right] = -\vec{a}_\phi \frac{Id\ell}{4\pi} k^2 \sin\theta \left[\frac{1}{jkR} + \frac{1}{(jkR)^2}\right] e^{-jkR}$$

$$\approx \vec{a}_\phi jk \frac{Idl}{4\pi} \frac{e^{-jkR}}{R} \sin\theta$$

4c) (5%) By the result of Problem 4b, find the "approximated" phasor representation of electric field intensity $\vec{E}(\vec{r})$ by neglecting the "higher order terms".

Answer:

By eq. (15.11), $\vec{E}(\vec{r}) = \dfrac{\nabla \times \vec{H}}{j\omega\varepsilon_0}$

$$= \frac{1}{j\omega\varepsilon_0} \frac{1}{R^2 \sin\theta} \begin{vmatrix} \vec{a}_R & \vec{a}_\theta R & \vec{a}_\phi R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ 0 & 0 & (R\sin\theta) jk \dfrac{Idl}{4\pi} \dfrac{e^{-jkR}}{R} \sin\theta \end{vmatrix}.$$

From which, we can obtain:

$$E_R = -\frac{Id\ell}{4\pi} \eta_0 k^2 2\cos\theta \left[\frac{1}{(jkR)^2} + \frac{1}{(jkR)^3}\right] e^{-jkR} \text{ , where } \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}.$$

$$E_\theta = -\frac{Id\ell}{4\pi} \eta_0 k^2 \sin\theta \left[\frac{1}{(jkR)} + \frac{1}{(jkR)^2} + \frac{1}{(jkR)^3}\right] e^{-jkR}$$

$$E_\phi = 0.$$

If we neglect the "higher order terms", $\vec{E}(\vec{r}) \approx \vec{a}_\theta jk\eta \dfrac{Idl}{4\pi} \dfrac{e^{-jkR}}{R} \sin\theta$.

4d) (5%) Plot the EM power distribution $\left|\vec{E}(R_0, \theta, \phi_0)\right|^2$ relative to its maximum in spherical coordinates, where R_0, ϕ_0 are arbitrary constants.

Answer:

When R_0, ϕ_0 are arbitrary constants, the value of $\left|\vec{E}(R_0, \theta, \phi_0)\right|^2$ is only dependent on the value of θ. From problem 4d $\left|\vec{E}(R_0,\theta,\phi_0)\right|^2 \approx \left|\vec{a}_\theta\, jk\eta_0 \frac{Id\ell}{4\pi}\frac{e^{-jkR_0}}{R_0}\sin\theta\right|^2$ can be normalized to $\left|\vec{E}(R_0,\theta,\phi_0)\right|^2 \approx \sin^2\theta$.