Homework Solutions #9

(Due date: 2011/05/18)

This problem set covers materials of Lesson 11. The full score is 35 points.

1) (20 points) P.6-12 (a-c) of the textbook (p298).

Answer:

By using eq. (6.38) in the textbook, at an arbitrary position along the x-axis, the magnetic field can be described as:

$$B_x = \frac{N\mu_0 I b^2}{2}\left\{\frac{1}{\left[\left(\frac{d}{2}+x\right)^2+b^2\right]^{3/2}} - \frac{1}{\left[\left(\frac{d}{2}-x\right)^2+b^2\right]^{3/2}}\right\}$$

(a) At $x=0$, $B_x = \frac{N\mu_0 I b^2}{\left[\left(\frac{d}{2}\right)^2+b^2\right]^{3/2}}$

(b) $\frac{dB_x}{dx} = \frac{N\mu_0 I b^2}{2}\left\{\frac{-3\left(\frac{d}{2}+x\right)}{\left[\left(\frac{d}{2}+x\right)^2+b^2\right]^{5/2}} + \frac{3\left(\frac{d}{2}-x\right)}{\left[\left(\frac{d}{2}-x\right)^2+b^2\right]^{5/2}}\right\}$

At the midpoint, $x=0 \Rightarrow \frac{dB_x}{dx}=0$

(c) $\frac{d^2B_x}{d^2x} = -\frac{3N\mu_0 Ib^2}{2} \times$

$$\left\{ \frac{1}{\left[\left(\frac{d}{2}+x\right)^2+b^2\right]^{5/2}} - \frac{5\left(\frac{d}{2}+x\right)^2}{\left[\left(\frac{d}{2}+x\right)^2+b^2\right]^{7/2}} + \frac{1}{\left[\left(\frac{d}{2}-x\right)^2+b^2\right]^{5/2}} - \frac{5\left(\frac{d}{2}-x\right)^2}{\left[\left(\frac{d}{2}-x\right)^2+b^2\right]^{7/2}} \right\}$$

At the midpoint, $x = 0 \Rightarrow \frac{d^2B_x}{d^2x} = -3N\mu_0 Ib^2 \left\{ \frac{b^2 - 4(d/2)^2}{\left[(d/2)^2 + b^2\right]^{7/2}} \right\} \to 0$, if $b = d$.

2) In Example 11-2 of the lecture notes, we applied the Ampere's circuital law [eq. (11.5)] to arrive at the magnetic flux density $\vec{B}$ created by an infinitely long hollow solenoid oriented along the z-axis, with n turns per unit length, and carrying a steady current I:

$$\vec{B} = \vec{a}_z \mu_0 nI \tag{1}$$

Here we show that eq. (1) can also be derived by the Biot-Savart law [eq. (11.13)].

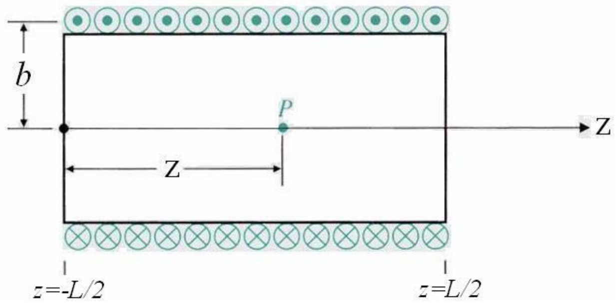

Fig. 1.

2a) (10 points) Consider a solenoid of radius b, length L, carrying a current I in its N turns of closely wound coil ($n = N/L$, Fig. 1). Use the result of Example 11-4 of the lecture notes to evaluate the magnetic flux density $\vec{B}$ at a point $P(0,0,z)$ on the axis.

Answer:

$$d\vec{l}\,' = \vec{a}_\phi' b d\phi', \quad \vec{R} = \vec{a}_z(z-z') - \vec{a}_r' b, \quad R = \sqrt{(z-z')^2 + b^2},$$

$$d\vec{B} = \frac{\mu_0 nI}{4\pi}\int_0^{2\pi} \frac{d\vec{l}\,' \times \vec{a}_R}{R^2} = \vec{a}_z \frac{\mu_0 nI}{2b}\left[1 + \left(\frac{z-z'}{b}\right)^2\right]^{-3/2},$$

$$\vec{B}(0,0,z) = \vec{a}_z \frac{\mu_0 nI}{2b}\int_{z'=-L/2}^{L/2}\left[1 + \left(\frac{z-z'}{b}\right)^2\right]^{-3/2} dz' = \left(u = \frac{z-z'}{b}\right)\vec{a}_z \frac{\mu_0 nI}{2}\int_{(z-L/2)/b}^{(z+L/2)/b}\left[1+u^2\right]^{-3/2} du$$

$$= (\text{let } u = \tan\theta)\,\vec{a}_z \frac{\mu_0 nI}{2}\left[\frac{z+L/2}{\sqrt{b^2 + (z+L/2)^2}} - \frac{z-L/2}{\sqrt{b^2 + (z-L/2)^2}}\right].$$

2b) (5 points) What are the magnetic flux densities $\vec{B}$ at the center ($z=0$), and the ends ($z=\pm L/2$), respectively? How are they simplified when $L\to\infty$?

Answer:

$$\vec{B}(0,0,z=0)=\vec{a}_z\frac{\mu_0 nI}{2}\frac{L}{\sqrt{b^2+(L/2)^2}}=\vec{a}_z\frac{\mu_0 nI}{\sqrt{1+(2b/L)^2}},$$

$$\vec{B}\left(0,0,z=\pm\frac{L}{2}\right)=\vec{a}_z\frac{\mu_0 nI}{2}\frac{L}{\sqrt{b^2+L^2}}=\vec{a}_z\frac{\mu_0 nI}{2\sqrt{1+(b/L)^2}}.$$

When $L\to\infty$, $\vec{B}(0,0,z=0)=\vec{a}_z\mu_0 nI$, and $\vec{B}\left(0,0,z=\pm\frac{L}{2}\right)=\vec{a}_z\frac{\mu_0 nI}{2}$.