Homework Solutions #8

(Due date: 2011/05/02)

This problem set covers materials of Lesson 10. The full score is 30 points.

1) (10 points) The procedures of evaluating the resistance of one piece of conducting material are summarized in the lecture notes, where we need to solve for the Laplace's equation $\nabla^2 V = 0$ under the specified boundary conditions to get the potential distribution $V(\vec{r})$. Please prove that Laplace's equation $\nabla^2 V = 0$ remains valid for steady electric currents, where free charges are actually in motion.

(*Hint*: Use laws regarding current density $\vec{J}$ and equation of continuity.)

Answer:

The current density equation can be expressed as:

$$\vec{J} = \sigma\vec{E} = \sigma\left(-\nabla V\right) \Rightarrow \nabla V = -\frac{\vec{J}}{\sigma}.$$

To verify the validity of Laplace's equation, we substitute $\nabla V = -\frac{\vec{J}}{\sigma}$

$$\Rightarrow \nabla^2 V = \nabla \cdot \nabla V = \nabla \cdot \left(-\frac{\vec{J}}{\sigma}\right).$$

For steady current density, $\nabla \cdot \vec{J} = 0 \Rightarrow \nabla^2 V = -\frac{1}{\sigma}\nabla \cdot \vec{J} = 0$.

2) (10 points) Problem **P.5–14** of the textbook. (*Hint*: $V(\vec{r}) = V(r)$)

Answer:

Starting from Laplace equation $\nabla^2 V = 0$ and potential V is a function of r.

Thus, Laplace equation becomes as follow:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) = 0.$$

And the solution is :$V(r) = C_1 \ln(r) + C_2$.

Given that boundary condition :$V(a) = V_0; V(b) = 0$ we can solve for the coefficients C_1, C_2.

$$C_1 = \frac{V_0}{\ln(a/b)}, C_2 = -\frac{Vo\ln b}{\ln(a/b)}.$$

$$\therefore V(r) = \frac{V_0}{\ln(a/b)}\ln r - \frac{V_0 \ln b}{\ln(a/b)} = \frac{V_0 \ln b}{\ln(b/a)} - \frac{V_0 \ln r}{\ln(b/a)} = V_0 \frac{\ln(b/r)}{\ln(b/a)}.$$

$$\Rightarrow \vec{E}(r) = -\vec{a}_r \frac{\partial V}{\partial r} = \vec{a}_r \frac{V_0}{r\ln(b/a)}.$$

$$\Rightarrow \vec{J}(r) = \sigma\vec{E}(r) = \vec{a}_r \frac{\sigma V_0}{r\ln(b/a)}.$$

$$\Rightarrow I = \int_s \vec{J} \cdot d\vec{s} = \int_0^{\pi/2} \vec{J} \cdot (\vec{a}_r hrd\phi) = \frac{\pi\sigma h V_0}{2\ln(b/a)}.$$

$$\therefore R = \frac{V_0}{I} = \frac{2\ln(b/a)}{\pi\sigma h}.$$

3) Consider a "leaky" parallel-plate capacitor shown in Fig. 1. The top and bottom plates are rectangular [$0 < x < w, 0 < z < L$ (not shown)] and separated by a distance d. Two types of "imperfect" dielectric slabs with permittivities and conductivities of $\{\varepsilon_1, \sigma_1\}$, $\{\varepsilon_2, \sigma_2\}$ are filled between the two plates biased by a voltage difference of V_0 (>0).

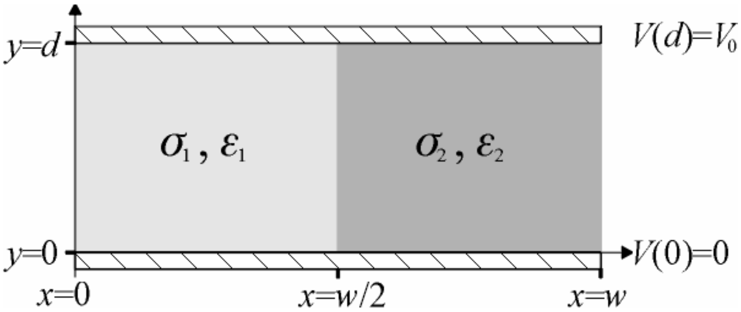

Fig. 10

3a) (5 points) Find the current densities $\vec{J}_1$, $\vec{J}_2$ in slab-1 and slab-2, respectively.

Answer:

$$E_1 = E_2 = -\hat{a}_y \frac{V_0}{d} \text{ due to the boundary condition } E_{1t} = E_{2t}$$

$$\Rightarrow \vec{J}_i = -\hat{a}_y \sigma_i \frac{V_0}{d}, \quad i = 1, 2.$$

3b) (5 points) Find the total (leakage) resistance R of the capacitor.

Answer:

$$I_i = J_i \times S_i, \quad I = I_1 + I_2 = \sigma_1 \frac{V_0}{d} \times \frac{S}{2} + \sigma_2 \frac{V_0}{d} \times \frac{S}{2} = (\sigma_1 + \sigma_2) \frac{V_0}{d} \times \frac{S}{2}, \text{ where } S = wL.$$

$$R = \frac{V_0}{I} = \frac{2}{\sigma_1 + \sigma_2} \frac{d}{S}.$$