Homework Solutions #7

(Due date: 2011/04/25)

This problem set covers materials of Lesson 9. The full score is 40 points.

1) (15%) Fig. 1 shows a simplified model of *pn*-junction. A depletion region is formed during $-d_p < x < d_n$, where net charge densities $-eN_A$ (C/m^3) and eN_D (C/m^3) exist for $-d_p < x < 0$ and $0 < x < d_n$, respectively. Note that $N_A d_p = N_D d_n$ holds to maintain the electric neutrality. Let the potential $V(-d_p) = 0$, find the potential distribution $V(x)$ for $-d_p < x < d_n$ by solving Poisson's equation (with suitable boundary conditions).

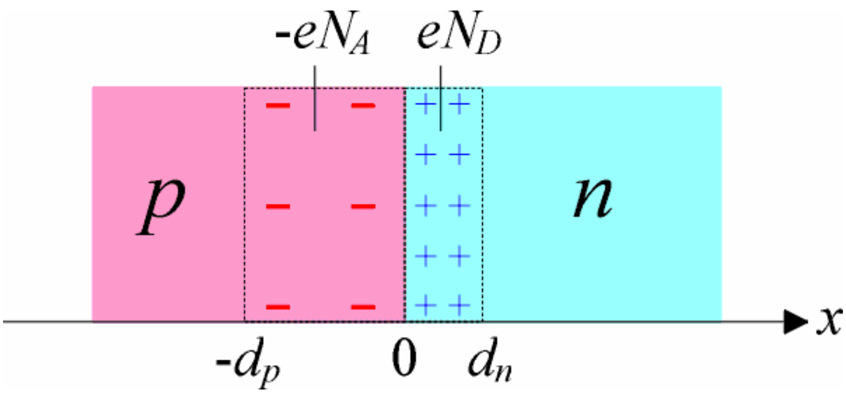

Fig. 1. *pn*-junction.

Answer:

We discuss the potential distribution during (i) $-d_p < x < 0$ and (ii) $0 < x < d_n$ separately, due to the discontinuous charge distribution at the interface.

(i) $-d_p < x < 0$:

According to the Poisson's equation, we have:

$$\frac{d^2V_A}{dx^2} = \frac{eN_A}{\varepsilon_A}, \Rightarrow V_A(x) = \frac{eN_A}{2\varepsilon_A}x^2 + c_1x + c_2.$$

Based on the BC: $V_A(-d_p) = 0$, we get $\frac{eN_A}{2\varepsilon_A}d_p{}^2 - c_1 d_p + c_2 = 0$.

Another BC is that the potential is constant (E-field is zero) near $x = -d_p$, otherwise, free electrons would keep on moving (not in steady state).

$$\Rightarrow V_A'(-d_p) = 0, \Rightarrow c_1 = \frac{d_p eN_A}{\varepsilon_A}; \Rightarrow c_2 = \frac{eN_A}{2\varepsilon_A}d_p{}^2.$$

So, $V_A(x) = \frac{eN_A}{2\varepsilon_A}x^2 + \frac{eN_A d_p}{\varepsilon_A}x + \frac{eN_A}{2\varepsilon_A}d_p{}^2$, which is a quadratic function of x.

(ii) $0 < x < d_n$:

Similar with the procedure we just perform:

$$\frac{d^2V_D}{dx^2} = -\frac{eN_D}{\varepsilon_D}, \Rightarrow V_D(x) = -\frac{eN_D}{2\varepsilon_D}x^2 + c_3x + c_4.$$

And BC: $V_D'(d_n) = 0, \Rightarrow -\frac{eN_D}{\varepsilon_D}d_n + c_3 = 0, \Rightarrow c_3 = \frac{eN_D}{\varepsilon_D}d_n$.

At the interface $x = 0$, the potential has to be continuous:

$$V_A(0) = V_D(0) \Rightarrow \frac{eN_A}{2\varepsilon_A}d_p{}^2 = c_4.$$

So, $V_D(x) = -\frac{eN_D}{2\varepsilon_D}x^2 + \frac{eN_D d_n}{\varepsilon_D}x + \frac{eN_A}{2\varepsilon_A}d_p{}^2$

2) Consider a coaxial cable capacitor with the same geometry as that shown in Fig. 9-4 in the lecture notes, while the space between the conducting surfaces ($a < r < b$) is filled with two dielectric media of permittivities ε_1, ε_2 according to:

$$\varepsilon = \begin{cases} \varepsilon_1, \text{for } \phi \in (0, \pi/2) \\ \varepsilon_2, \text{for } \phi \in (\pi/2, 2\pi) \end{cases} \quad (r, \phi \text{ are with cylindrical coordinates}).$$

2a) (10%) Deposit charges of $+Q$ and $-Q$ on the inner and outer conducting surfaces, respectively. Find the electric flux density $\vec{D}$ in the dielectric region $a < r < b$.

(*Hint*: Use boundary condition.)

Answer:

Let Q_1, Q_2 represent the charges on the inner conducting surface in contact with Medium 1 and Medium 2, respectively. We also know that $Q_1 + Q_2 = Q$ …(1).

$$\oint \vec{D} \cdot d\vec{s} = \int_{V'} \rho_f dv' \Rightarrow \vec{D} = \begin{cases} \vec{a}_r \dfrac{2}{\pi r L} Q_1, \text{at medium 1} \left(\varepsilon = \varepsilon_1\right) \\ \vec{a}_r \dfrac{2}{3\pi r L} Q_2, \text{at medium 2} \left(\varepsilon = \varepsilon_2\right) \end{cases}.$$

$$\Rightarrow \vec{E} = \begin{cases} \vec{a}_r \dfrac{2}{\pi r \varepsilon_1 L} Q_1, \text{at medium 1} \left(\varepsilon = \varepsilon_1\right) \\ \vec{a}_r \dfrac{2}{3\pi r \varepsilon_2 L} Q_2, \text{at medium 2} \left(\varepsilon = \varepsilon_2\right) \end{cases}$$

By the boundary condition between Medium 1 and 2, $E_{1t} = E_{2t}$,

$$\Rightarrow \frac{2}{\varepsilon_1} Q_1 = \frac{2}{3\varepsilon_2} Q_2 \quad \ldots(2)$$

With (1) and (2), we can derive that $Q_1 = \dfrac{\varepsilon_1}{\varepsilon_1 + 3\varepsilon_2} Q$, and $Q_2 = \dfrac{3\varepsilon_2}{\varepsilon_1 + 3\varepsilon_2} Q$.

$$\vec{D} = \begin{cases} \vec{a}_r \dfrac{2}{\pi r L} \dfrac{\varepsilon_1}{\varepsilon_1 + 3\varepsilon_2} Q, \text{at medium 1} \left(\varepsilon = \varepsilon_1\right) \\ \vec{a}_r \dfrac{2}{\pi r L} \dfrac{\varepsilon_2}{\varepsilon_1 + 3\varepsilon_2} Q, \text{at medium 2} \left(\varepsilon = \varepsilon_2\right) \end{cases}$$

2b) (5%) Find the corresponding capacitance *C*, and the effective permittivity ε_{eff} if

$$\varepsilon_{eff} \equiv \frac{C\ln(b/a)}{2\pi L}$$ (justified by the result of Example 9-3 in the lecture notes).

Answer:

$$V = \int_{+}^{-} \vec{E} \cdot d\vec{l} = \int_{a}^{b} \vec{a}_r \frac{2}{\pi r L} \frac{1}{\varepsilon_1 + 3\varepsilon_2} Q \cdot \vec{a}_r dr = \frac{2Q}{\pi L(\varepsilon_1 + 3\varepsilon_2)} \int_{a}^{b} \frac{1}{r} dr = \frac{2Q}{\pi L(\varepsilon_1 + 3\varepsilon_2)} \ln\left(\frac{b}{a}\right),$$

$$C = \frac{Q}{V} = \frac{\pi L}{2}(\varepsilon_1 + 3\varepsilon_2)\frac{1}{\ln(b/a)},$$

$$\varepsilon_{eff} \equiv \frac{C\ln(b/a)}{2\pi L} = \frac{\pi L}{2}(\varepsilon_1 + 3\varepsilon_2)\frac{1}{\ln(b/a)}\frac{\ln(b/a)}{2\pi L} = \frac{\varepsilon_1 + 3\varepsilon_2}{4}.$$

3) (10%) Problem **P.3–32** of the textbook.

Answer:

Assumed that there are total +*Q* charge uniformly distributed on the surface of the inner metal. The positive charge will induce –Q deposit on the surface outer metal as shown by the figure.

For $r_i < r < b$:

From the Guess law $\oint \vec{E}_1 \cdot d\vec{s} = \vec{a}_r E_1 \cdot 2\pi r \cdot L = \frac{Q}{\varepsilon_0 \varepsilon_{r1}} \Rightarrow \vec{E}_1 = \vec{a}_r \frac{Q}{2\pi r L \cdot \varepsilon_0 \varepsilon_{r1}}$, where *L* is length of the transmission line.

$$V_1 = -\int_b^{r_i} \vec{E}_1 \cdot d\vec{r} = \vec{a}_r \frac{Q}{2\pi L \varepsilon_0 \varepsilon_{r1}} \ln\left(\frac{b}{r_i}\right),$$

Then the capacitance per unit length is $\frac{C_1}{L} = \frac{Q}{V_1} = \frac{2\pi\varepsilon_0\varepsilon_{r1}}{\ln\left(\frac{b}{r_i}\right)}$.

For $b < r < r_o$:

From the same procedure, the capacitance per unit length is $\frac{C_2}{L} = \frac{Q}{V_2} = \frac{2\pi\varepsilon_0\varepsilon_{r2}}{\ln\left(\frac{r_o}{b}\right)}$.

The total capacitance is the series connection of the two sub-capacitances, therefore $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}, \Rightarrow$ $C = \frac{2\pi\varepsilon_0\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r2}\ln\left(\frac{b}{r_i}\right) + \varepsilon_{r2}\ln\left(\frac{r_o}{b}\right)}$